Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 05–04

DESCENT FOR PRIESTLEY SPACES

MARGARIDA DIAS AND MANUELA SOBRAL

Abstract: A characterization of descent morphism in the category of Priestley spaces, as well as necessary and sufficient conditions for such morphisms to be effective are given. For that we embed this category in suitable categories of preordered topological spaces were descent and effective morphisms are described using the monadic description of descent.

Keywords: ordered (preordered) topological spaces, Priestley space, Stone space, regular and universal regular epimorphisms, (effective) descent morphisms, monadic categories.
AMS Subject Classification (2000): 18D30, 18A20, 18C20.

0. Introduction

A *preordered topological space (a preordered space)* is a triple $(X, \tau, \leq)$ where X is a set, τ is a topology and $\leq$ is a preorder (i.e. a reflexive and transitive relation) on X. When $\leq$ is also antisymmetric, then $(X, \tau, \leq)$ is called an *ordered topological space (an ordered space).* They are the objects of the category $\mathcal{T}opPreord$ ($\mathcal{T}opOrd$) whose morphisms are the continuous maps which preserve the preorder (the order, respectively).

An ordered space $(X, \tau, \leq)$ is said to be *totally order-disconnected* if given $x \not\leq x'$ in X there exists a closed and open (clopen, for short) decreasing subset U of X such that $x' \in U$ and $x \notin X$. The compact totally order-disconnected spaces are called the Priestley spaces. The full subcategory of $\mathcal{T}opOrd$ whose objects are the Priestley spaces will be denoted by $\mathcal{P}sp$.

The category $\mathcal{P}sp$ is dually equivalent to the category of bounded distributive lattices $\mathcal{D}Lat$, the well-known Priestley duality. Since $\mathcal{D}Lat$ is monadic over $\mathcal{S}et$, it is easy to describe descent there with respect to the codomain fibration $\mathcal{D}Lat^2 \to \mathcal{D}Lat$: the effective descent morphisms are the descent morphisms and they are exactly the regular epimorphisms. Therefore, we conclude that, in $\mathcal{P}sp$, the classes of effective codescent morphisms, of codescent morphisms and of regular monomorphisms coincide.

Received March 24, 2005.
The authors gratefully acknowledge financial support from FCT/Centro de Matemática da Universidade de Coimbra.

The descent morphisms with respect to the codomain fibration $\mathcal{P}sp^2 \to \mathcal{P}sp$ form a class which is strictly contained in the one of the regular epimorphisms. Also the class of effective descent morphisms in $\mathcal{P}sp$ is a proper subclass of the one of the descent morphisms. To prove that we consider $\mathcal{S}tonePreord$, the full subcategory of $\mathcal{T}opPreord$ with objects all $(X, \tau, \leq)$ such that (X, τ) is a Stone space, as well as its full subcategory $\mathcal{P}Preord$ with objects all totally preordered-disconnected Stone spaces, and study the reflective embeddings of $\mathcal{P}sp$ in $\mathcal{P}Preord$ and of $\mathcal{P}Preord$ in $\mathcal{S}tonePreord$.

Necessary and sufficient conditions for descent morphisms in $\mathcal{P}Preord$ and in $\mathcal{S}tonePreord$ to be effective are obtained by embedding these categories into the category $\mathcal{S}toneRel$, with objects all triples (X, τ, R_X) where (X, τ) is a Stone space and R_X is an arbitrary binary relation on X. An explicit description of the effective descent morphisms in $\mathcal{S}tonePreord$ is given, using the one presented in [3] for the effective descent morphisms in $\mathcal{P}reord$. Finally, we prove that a $\mathcal{P}sp$-morphism is an effective descent morphism in $\mathcal{P}sp$ if and only if it is an effective descent morphism in $\mathcal{P}Preord$.

For a comprehensive description of descent theory see [5].

1. Fundamentals of Monadic Descent

Let $\mathcal{C}$ be a category with pullbacks. The fibres with respect to the codomain functor $\mathcal{C}^2 \to \mathcal{C}$ are the slice categories $\mathcal{C} \downarrow B$, for each $B \in \mathcal{C}$

For every $\mathcal{C}$-morphism $p : E \to B$, the pullback functor $p^* : \mathcal{C} \downarrow B \to \mathcal{C} \downarrow E$ has a left adjoint $p!$ which is defined by composition with p on the left.

For bifibrations satisfying Beck-Chevalley condition, descent data can be interpreted as structure maps for a monad, a fact first proved by Bénabou and Roubaud in [1]. This is the case of the bifibration above: the category $\mathcal{D}es(p)$, of objects equipped with descent data and morphisms preserving it, is equivalent to the the category $(\mathcal{C} \downarrow E)^{\mathbb{T}}$ of $\mathbb{T}$-algebras for the monad $\mathbb{T}$ induced in $\mathcal{C} \downarrow E$ by the adjunction $p! \dashv p^*(\eta, \epsilon)$.

Let $\Phi : \mathcal{C} \downarrow B \to (\mathcal{C} \downarrow E)^{\mathbb{T}}$ be the Eilenberg-Moore comparison functor. A morphism $p : E \to B$ is a *descent morphism* if Φ is full and faithful and it is an *effective descent morphism* if Φ is an equivalence of categories.

Proposition 1.1. *For the monad $\mathbb{T}$ induced by $p! \dashv p^*(\eta, \varepsilon)$ in $\mathcal{C} \downarrow E$ we have that:*

(i) *Φ is full and faithful if and only if ε is pointwise a regular epimorphism.*

(ii) Φ *has a left adjoint if and only if, for each* $\mathbb{T}$*-algebra* $(C, \gamma : C \to E, \xi : E \times_B C \to C)$, $\mathcal{C} \downarrow B$ *has coequalizers of the pair* (π_2, ξ).

A morphism p is called a *universal regular epimorphism* if its pullback along any morphism is a regular epimorphism.

For each (A, α) in $\mathcal{C} \downarrow B$, $\varepsilon_{(A,\alpha)} = \pi_2 : (E \times_B A, p \cdot \pi_1) \to (A, \alpha)$, where the diagram

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

is the pullback of p along α. Then the following is an immediate consequence of 1.1(i):

Corollary 1.2. *A morphism* p *is a descent morphism if and only if it is a universal regular epimorphism.*

We are going to describe descent in categories $\mathcal{C}$ that, other than pullbacks, have all coequalizers. Then $\mathcal{C} \downarrow B$ has coequalizers and they are constructed at the level of $\mathcal{C}$. Therefore, the corresponding comparison functor is always part of an adjunction $L \dashv \Phi(\alpha, \beta)$.

We first look for a characterization of the universal regular epimorphism in $\mathcal{C}$ because the following holds:

p descent morphism $\Longleftrightarrow$ β is an isomorphism $\Longleftrightarrow$ p is a universal regular epimorphism.

For a $\mathbb{T}$-algebra (C, γ, ξ) we consider the diagram

$$\begin{tikzcd} E \times_B C \arrow[r, shift left, "\pi_2"] \arrow[r, shift right, "\xi"'] & C \arrow[rr, "q"] \arrow[dr, "\alpha"] \arrow[dd, "\gamma"'] & & Q \arrow[dd, "\delta"] \\ & & E \times_B Q \arrow[ur, "\pi_2"'] \arrow[dl, "\pi_1"'] & \\ & E \arrow[rr, "p"'] & & B \end{tikzcd} \tag{1}$$

where q is the coequalizer of the pair (π_2, ξ) and $\alpha = \alpha_{(C,\gamma,\xi)}$, the component of the unit of the comparison adjunction, is the unique morphism for which the upper triangles commute.

Proposition 1.3. *Let $\mathcal{C}$ be a concrete category over $\mathcal{S}et$ with pullbacks and coequalizers which are preserved by the forgetful functor. Then $\alpha_{(C,\gamma,\xi)}$ is a bimorphism for every $\mathbb{T}$-algebra.*

Proof: If $\alpha(c) = (\gamma(c), q(c)) = (\gamma(c'), q(c') = \alpha(c')$ then $\gamma(c) = \gamma(c')$ and $q(c) = q(c')$. Since (see e.g. 5.A in [4])

$$q(c) = q(c') \Longleftrightarrow \xi(\gamma(c), c') = c \Longleftrightarrow \xi(\gamma(c'), c) = c'$$

and $\xi(\gamma(c), c) = c$ then

$$c = \xi(\gamma(c), c) = \xi(\gamma(c'), c) = c'.$$

Hence $\alpha_{(C,\gamma,\xi)}$ is an injective map.

Since $\alpha_{(C,\gamma,\xi)} \cdot \xi = p^*(q)$ and $p^*(q) = 1 \times_B q$ is surjective, we conclude that $\alpha_{(C,\gamma,\xi)}$ is an epimorphism. ∎

Corollary 2.8 in [6] holds for these categories, as observed there, giving a criterion that will be useful in the sequel.

Theorem 1.4. *Let $\mathcal{C}$ be a concrete category over $\mathcal{S}et$ with coequalizers and pullbacks. If pullbacks are preserved by the forgetful functor then, for a morphism p, the following are equivalent:*

(i) *p is an effective descent morphism;*
(ii) *for each $\mathbb{T}$-algebra (C, γ, ξ) the coequalizer of (π_2, ξ) is a universal regular epimorphism;*
(iii) *for each $\mathbb{T}$-algebra (C, γ, ξ) the square in diagram (1) is a pullback.*

This follows from general results (see [4] and [6]). They are presented here in the appropriated form for the context we are interested in.

2. Regular and universal regular epimorphisms

Let $\mathcal{P}reord$ be the category with objects $X = (X, \leq)$ where X is a set and $\leq$ is a preorder on X and with morphisms the preorder preserving maps. We denote by R_X the subset $\{(x, x')|x \leq x'\}$ of $X \times X$.

Proposition 2.1. *Let $f : X \to Y$ be a morphism in $\mathcal{P}reord$.*

(i) f is a regular epimorphism if and only if $f(X) = Y$ and R_Y is the transitive closure of $f \times f(R_X)$;

(ii) f is a universal regular epimorphism if and only if $R_Y = f \times f(R_X)$.

For details see 2.2 and 2.3 in [3].

Let $\mathcal{C}Preord$ be the full subcategory of $\mathcal{T}opPreord$ with objects $(X, \tau, \leq)$ such that (X, τ) belongs to a full subcategory $\mathcal{C}$ of $\mathcal{T}op$ closed under pullbacks and finite subspaces. Then the forgetful functors $U : \mathcal{C}Preord \to \mathcal{C}$ and $V : \mathcal{C}Preord \to \mathcal{P}reord$ preserve pullbacks.

Proposition 2.2. *A morphism f in $\mathcal{C}Preord$ is a (universal) regular epimorphism if and only if its underlying maps in $\mathcal{C}$ and in $\mathcal{P}reord$ are (universal) regular epimorphisms.*

Proof: Let $f : X \to Y$ be a morphism in $\mathcal{C}Preord$ and (π_1, π_2) be its kernel pair.

If f is a regular epimorphism then the coequalizer diagram

$$X \times_Y X \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} X \xrightarrow{f} Y$$

is preserved by the forgetful functor $U : \mathcal{C}Preord \to \mathcal{C}$ because U has both a left and a right adjoint defined on objects by $F(C) = (C, \Delta_C)$ and $G(C) = (C, C \times C)$, respectively. Furthermore, assuming that R_Y strictly contains the transitive closure of $f \times f(R_X)$, let Y' have the same underlying topological space as Y and $R_{Y'}$ be the transitive closure of $f \times f(R_X)$. The morphism $f' : X \to Y'$, defined by $f'(x) = f(x)$ for each $x \in X$, is such that $f' \cdot \pi_1 = f' \cdot \pi_2$ but does not factor through f. Consequently, f would not be the coequalizer of (π_1, π_2) in $\mathcal{C}Preord$.

Conversely, if the underlying morphisms of f in $\mathcal{C}$ and in $\mathcal{P}reord$ are regular epimorphisms and $g \cdot \pi_1 = g \cdot \pi_2$ in $\mathcal{C}Preord$, then both structures produce a unique factorization of g through f, say h and h' and, since f is surjective we have that $h = h'$ is the unique morphism h in $\mathcal{C}Preord$ such that $h \cdot f = g$.

If f is a universal regular epimorphism in $\mathcal{C}$ and in $\mathcal{P}reord$ then, as every pullback in $\mathcal{C}Preord$

$$\begin{array}{ccc} X \times_Y A & \xrightarrow{\pi_2} & A \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle \alpha} \\ X & \xrightarrow{f} & Y \end{array}$$

is preserved by U and by V, $U(\pi_2)$ and $V(\pi_2)$ are regular epimorphisms and so π_2 is a regular epimorphism in $\mathcal{C}Preord$.

Let us assume now that f is a universal regular epimorphism in $\mathcal{C}Preord$. Then Uf is a universal regular epimorphism in $\mathcal{C}$ because the pullback of Uf along a morphism $\alpha' : C \to U(Y)$ in $\mathcal{C}$ is the image by U of a pullback in $\mathcal{C}Preord$ where $A = (C, \Delta_C)$.

For $y \leq y'$ in Y, the pullback of $V(f)$ along $\alpha' : \{y \leq y'\} \to V(Y)$ is the image by V of the pullback in $\mathcal{C}Preord$ of f and $\alpha : A \to Y$ where A is the ordered set $\{y \leq y'\}$ with the subspace topology. Then π_2 is a regular epimorphism in $\mathcal{C}Preord$ and so in $\mathcal{P}reord$. Since $X \times_Y A = f^{-1}(y) \times \{y\} \cup f^{-1}(y') \times \{y'\}$, there exist $(x, y) \leq (x', y')$ in $X \times_Y A$ which gives $x \leq x'$ such that $f(x) = y$ and $f(x') = y'$. Thus $V(f)$ is a universal regular epimorphism in $\mathcal{P}reord$. ■

Proposition 2.3. *Let $f : X \to Y$ be a morphism in $\mathcal{S}tonePreord$.*

(i) *f is a regular epimorphism if and only if $f(X) = Y$ and R_Y is the transitive closure of $f \times f(R_X)$;*

(ii) *f is a universal regular epimorphism if and only if $R_Y = f \times f(R_X)$.*

Proof: (i) and (ii) follow from the previous proposition and the fact that in the category $\mathcal{S}tone$ of Stone spaces the regular epimorphisms are universal and they are the surjective maps. ■

The category $\mathcal{S}tonePreord$ has a factorization system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E}$ the class of regular epimorphisms and $\mathcal{M}$ the class of monomorphisms. Indeed,the $(\mathcal{E}, \mathcal{M})$-factorization of a morphism $f : X \to Y$ is obtained by considering the $(RegularEpi, Mono)$-factorization $f = m \cdot q$ in $\mathcal{S}tone$, and endowing the codomain of q with the preorder which is the transitive closure of $q \times q(R_X)$.

Proposition 2.4. *The category $\mathcal{P}Preord$ is an epireflective subcategory of $\mathcal{S}tonePreord$.*

Proof: For $(X, \tau, \leq)$ in $\mathcal{S}tonePreord$ let $I(X)$ be $(X, \tau, \leq^1)$ with $x \leq^1 x'$ if

$$\begin{array}{l} x \leq x' \text{ in } X \text{ or} \\ x' \in U \in DClopen(X) \Longrightarrow x \in U, \end{array}$$

where $DClopen(X)$ denotes the set of decreasing clopen subsets of X. Then $\leq^1$ is a preorder on $I(X)$ which is an object of $\mathcal{P}Preord$. The morphism $r_X : X \to I(X)$ defined by $r_X(x) = x$ is the reflection of X in $\mathcal{P}Preord$ as we show next. Given $g : X \to Y$ with $Y \in \mathcal{P}Preord$ the unique continuous function $g' : I(X) \to Y$ such that $g' \cdot r_X = g$ is a preorder preserving map. Indeed, if $x \not\leq x'$, $x \leq^1 x'$ and $g(x) \not\leq g(x')$ there exists $U \in DClopen(Y)$

which contains $g(x')$ but not $g(x)$. Hence, $x' \in g^{-1}(U)$ which is a decreasing clopen subset of X and $x \notin g^{-1}(U)$, a contradiction.

Furthermore, for each X, r_X, being a surjective map, is an epimorphism. ■

Proposition 2.5. *Let $f : X \to Y$ be a morphism in $\mathcal{P}Preord$ and (π_1, π_2) its kernel pair.*

(i) *f is a regular epimorphism if and only if, up to isomorphism, $f = r_{Y'} \cdot q$ where $q : X \to Y'$ is the coequalizer of (π_1, π_2) in $\mathcal{S}tonePreord$ and $r_{Y'}$ is the reflection of Y' in $PPreord$.*

(ii) *f is an universal regular epimorphism if and only if $R_Y = f \times f(R_X)$.*

Proof: (i) follows from the way colimits are constructed in full replete reflective subcategories of categories where these colimits exist.

(ii) The "if" part is clear. We prove the "only if" part. If $y \leq y'$ in Y let $A = \{y, y'\}$ be the subspace of Y in $\mathcal{P}Preord$ and consider the pullback along the inclusion $i : A \to Y$. Then, since π_2 is a regular epimorphism, $\pi_2 = r_{A'} \cdot q$.

$$\begin{tikzcd}
X \times_Y A \arrow[rr, "\pi_2"] \arrow[dd, "\pi_1"'] \arrow[dr, "q"] & & A \arrow[dd, "i"] \\
& A' \arrow[ur, "r_{A'}"'] & \\
X \arrow[rr, "f"] & & Y
\end{tikzcd}$$

But finite discrete spaces are compact and totally preordered-disconnected for every preorder. Hence A' belongs to $\mathcal{P}Preord$ and so $r_{A'}$ is an isomorphism. Now, like in the proof of 2.2, we conclude that there exist $x \leq x'$ in X such that $f(x) = y$ and $f(x') = y'$. ■

The category $\mathcal{P}Preord$ also has a factorization system $(\mathcal{E}, \mathcal{M})$ with $\mathcal{E}$ is the class of regular epimorphisms and $\mathcal{M}$ is the class of monomorphisms where the factorization of a morphism $f \in \mathcal{P}Preord$ is obtained by first considering the factorization $f = m \cdot q$ in $\mathcal{S}tonePreord$ and then taking $f = m' \cdot r_Q \cdot q$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "q"'] & & Y \\ & Q \arrow[ur, "m"'] \arrow[dr, "r_Q"'] & \\ & & R(Q) \arrow[uu, "m'"'] \end{tikzcd}$$

with m' the unique morphism such that $m' \cdot r_Q = m$. Then m' is a monomorphism, since it is injective map, and $r_Q \cdot q$ is a coequalizer in $\mathcal{P}Preord$.

Proposition 2.6. *$\mathcal{P}sp$ is a regular-epireflective subcategory of $\mathcal{P}Preord$.*

Proof: For $X = (X, \tau, \leq) \in \mathcal{P}Preord$ we consider the binary relation

$$x \sim x' \text{ if } x \leq x' \text{ and } x' \leq x$$

which is an equivalence relation on X.

Let $I(X)$ be the quotient set, $X/\sim$, equipped with the quotient topology with respect to the canonical projection $r_X : X \to I(X)$ and the preorder $R_{I(X)}$ obtained by transitive closure of $r_X \times r_X(R_X)$. Then we have that

$$x \leq y \Leftrightarrow [x] \leq [y]$$

where $[x]$ denotes the equivalence classe of x.

Being a continuous image of the compact space X, $I(X)$ is also compact. It remains to prove that it is totally order-disconnected. If $[x] \nleq [y]$ then $x \nleq y$ and so there exists a clopen decreasing subset U_1 of X which contains y but not x. Then the set $U = \{[a] : a \in U_1\}$ is a clopen subset of $X/\sim$, because $r_X^{-1}(U) = U_1$, and it is decreasing: if $[x] \leq [y] \in U$ then $x \leq y \in U_1$ which implies that $x \in U_1$ and so that $[x] \in U$. Thus $I(X)$ belongs to PSp.

Furthermore, given a morphism $g : X \to Y$ with $Y \in \mathcal{P}sp$ the unique function g' such that $g' \cdot r_X = g$ is continuous and order preserving. Thus the regular epimorphism r_X is the reflection of X in $\mathcal{P}sp$. ∎

Proposition 2.7. *Let $f : X \to Y$ be a morphism in $\mathcal{P}sp$.*

(i) *f is a regular epimorphism in $\mathcal{P}sp$ if and only if it is a regular epimorphism in $\mathcal{P}Preord$;*

(ii) *f is a universal regular epimorphism in $\mathcal{P}sp$ if and only if $R_Y = f \times f(R_X)$.*

Proof: (i) In categories with a system of factorization $(RegularEpi, Mono)$ the embedding of each regular epireflective subcategory preserves and reflects regular epimorphisms.

(ii) In the proof of 2.5 (ii), $A' \in \mathcal{P}sp$ and the proof that the condition is necessary follows in a completely analogous way. ■

Finally, being a regular epireflective subcategory of a category with a factorization system $(RegularEpi, Mono)$, $\mathcal{P}sp$ also admits a $(RegularEpi, Mono)$-factorization system.

3. Descent in $\mathcal{P}Preord$

By Proposition 2.5(ii), a morphism $p : E \to B$ is a descent morphism in $\mathcal{P}Preord$ if and only if for each $b \leq b'$ in B there exist $e \leq e'$ in E such that $p(e) = b$ and $p(e') = b'$, and these are the descent morphisms in $\mathcal{P}Preord$ as well as in the full subcategory $\mathcal{F}inPreord$ of finite preordered sets.

Proposition 3.1. *(3.4, [3]) For a morphism $p : E \to B$ in $\mathcal{P}reord$ (or in $\mathcal{F}inPreord$) the following are equivalent:*
(i) *p is an effective descent morphism;*
ii *for every $b_0 \leq b_1 \leq b_2$ in B there exists $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$, for i=0,1,2.*

Proposition 3.2. *(3.9, [4]) Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ closed under pullback. If a morphism p in $\mathcal{D}$ is an effective descent morphism in $\mathcal{C}$ then the following are equivalent:*

(i) *p is an effective descent morphism in $\mathcal{D}$;*
(ii) *for every pullback*

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"] & B \end{tikzcd} \tag{2}$$

in $\mathcal{C}$, $A \in \mathcal{D}$ whenever $E \times_B A \in \mathcal{D}$.

Corollary 3.3. *Let $\mathcal{D}$ be a full subcategory of $\mathcal{C}$ closed under pullbacks. If the descent morphisms in $\mathcal{D}$ are effective descent morphisms in $\mathcal{C}$ then the following are equivalent:*

(i) *p is an effective descent morphisms in $\mathcal{D}$;*
(ii) *p is a descent morphism in $\mathcal{D}$ and, for every pullback (2) in $\mathcal{C}$, $A \in \mathcal{D}$ whenever $E \times_B A \in \mathcal{D}$.*

Proposition 3.4. *In $\mathcal{S}toneRel$ the effective descent morphisms are the regular epimorphisms.*

Proof: The universal epimorphisms in $\mathcal{S}toneRel$ are the morphisms whose underlying morphisms in $\mathcal{S}tone$ and in $\mathcal{R}el$ are universal regular epimorphisms. Indeed, Proposition 2.2 is still true if instead of $\mathcal{C}Preord$ we consider $\mathcal{C}Rel$, the category of spaces of $\mathcal{C}$ equipped with an arbitrary binary relation. In the this case only the left adjoint to the forgetful functor $U : \mathcal{C}Rel \to \mathcal{C}$, has a different definition: $F(C) = (C, \emptyset)$ for each $C \in \mathcal{C}$.

In $\mathcal{S}tone$ the regular epimorphisms are universal and they are exactly the surjective maps. In $\mathcal{R}el$ a regular epimorphism is a morphism $f : X \to Y$ such that $f(X) = Y$ and $R_Y = f \times f(R_X)$ and so it is also a universal regular epimorphism. Consequently, every regular epimorphism in $\mathcal{S}toneRel$ is a universal regular epimorphism. Now the conclusion follows by applying Theorem 1.4. ∎

Proposition 3.5. *For a morphism $p : E \to B$ in $\mathcal{P}Preord$ the following are equivalent:*

(i) *p is an effective descent morphism;*
(ii) *p is a descent morphism and, for every pullback (2) in $\mathcal{S}toneRel$, $A \in \mathcal{P}Preord$ whenever $E \times_B A \in \mathcal{P}Preord$.*

Proof: We can apply Corollary 3.3 to $\mathcal{D} = \mathcal{P}Preord$ and $\mathcal{C} = StoneRel$. Indeed, $\mathcal{P}Preord$ is a full subcategory of $\mathcal{S}toneRel$ closed under pullbacks and every descent morphism in $\mathcal{P}Preord$ is an effective descent morphism in $\mathcal{S}toneRel$. ∎

We can also apply 3.3 to $\mathcal{D} = StonePreord$ and $\mathcal{C} = StoneRel$. In this case this case one can give an explicit description of the effective descent morphisms in $\mathcal{S}tonePreord$.

Proposition 3.6. *For a morphism $p : E \to B$ in $\mathcal{S}tonePreord$ the following are equivalent:*

(i) *p is an effective descent morphism;*
(ii) *p is a descent morphism and, for every pullback (2) in $\mathcal{S}toneRel$, $A \in \mathcal{S}tonePreord$ whenever $E \times_B A \in \mathcal{S}tonePreord$*
(iii) *for every $b_0 \leq b_1 \leq b_2$ in B there exists $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$, for i=0,1,2.*

Proof: By 3.4 in [3], that we recall in 3.1, *(ii)* $\Leftrightarrow$ *(iii)* tell us that the effective descent morphisms in $\mathcal{S}tonePreord$ are exactly those morphisms whose underlying morphisms in $\mathcal{P}reord$ are the effective descent morphisms in this category. The proof given there still holds if we replace sets by Stone spaces as we sketch now.

If p satisfies (ii) and $b_0 \leq b_1 \leq b_2$ in B let A be the set $\{a_0, a_1, a_2\}$ equipped with the binary relation $R_A = \{(a_0, a_1), (a_1, a_2)\} \cup \Delta_A$ and the discrete topology and α be defined by $\alpha(a_i) = b_i$ for i=0, 1, 2. For the pullback (2) in $\mathcal{S}toneRel$, since $E \times_B A \notin \mathcal{S}tonePreord$, $R_{E\times_B A}$ is not transitive. Hence there exist $(x_0, x_1), (x_1, x_2) \in R_{E\times_B A}$ but not (x_1, x_2). Since π_2 is a regular epimorphism in $\mathcal{S}toneRel$ we can conclude that x_i belongs to $p^{-1}(b_i) \times \{a_i\}$ and then that $e_i = \pi_1(x_i)$ form a chain $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$ for i = 0, 1, 2.

Conversely, it is easy to show that condition (iii) implies that R_A is transitive when the relation $R_{E\times_B A}$ is transitive. ■

Proposition 3.7. *A morphism $p : E \to B$ in $\mathcal{P}Preord$ is an effective descent morphism if:*

(i) *for every $b_0 \leq b_1 \leq b_2$ in B there exists $e_0 \leq e_1 \leq e_2$ in E such that $p(e_i) = b_i$, for i=0,1,2.*

(ii) *for every pullback (2) in $\mathcal{S}tonePreord$, if $E \times_B A$ belongs to $\mathcal{P}Preord$ then also A belongs to $\mathcal{P}Preord$.*

Proof: Since $\mathcal{D} = \mathcal{P}Preord$ is a full subcategory of $\mathcal{C} = \mathcal{S}tonePreord$ closed under pullbacks, we apply 3.2 and the characterization above to conclude that these are sufficient conditions for a descent morphism in $\mathcal{P}Preord$ to be an effective descent morphism. ■

Condition (ii) above is necessary for a descent morphism in $\mathcal{P}Preord$ to be an effective descent morphism.

Proposition 3.8. *If p is an effective descent morphism in $\mathcal{P}Preord$ then for every pullback (2) in $\mathcal{S}tonePreord$, if $E \times_B A$ belongs to $\mathcal{P}Preord$ then also A belongs to $\mathcal{P}Preord$.*

Proof: Given a pullback (2) in $\mathcal{S}tonePreord$ with $E \times_B A \in \mathcal{P}Preord$ then we have that

- $(E \times_B A, \pi_1, \pi_{13}) \in (\mathcal{P}Preord \downarrow E)^{\mathbb{T}}$;
- π_2 is a coequalizer of (π_{23}, π_{13}) in $\mathcal{S}tonePreord$, because p is a descent morphism;

- $q = r_A \cdot \pi_2$ is the coequalizer of (π_{23}, π_{13}) in $\mathcal{P}Preord$.

By 1.4, q is a universal regular epimorphism in $\mathcal{P}Preord$. Then for $a \leq a'$ in $I(A)$ there exist $(e, a) \leq (e', a') \in E \times_B A$ and so $a \leq a'$ in A. Hence r_A is an isomorphism and so that A belongs to $\mathcal{P}Preord$. ■

We can apply 3.2 to $\mathcal{D} = \mathcal{F}inPPreord$, the subcategory of finite spaces in $\mathcal{P}Preord$, which is isomorphic $\mathcal{F}inPreord$, and $\mathcal{C} = \mathcal{P}Preord$ to conclude that a morphism $p \in \mathcal{D}$ which is an effective descent morphism in $\mathcal{P}Preord$ is also an effective descent morphism in $\mathcal{D}$. Then, the class of descent morphisms in $\mathcal{P}Preord$ strictly contains the one of the effective descent morphisms in this category since we also have strict inclusion of the corresponding classes in $\mathcal{F}inPreord$.

4. Descent in $\mathcal{P}sp$

Let $H : \mathcal{X} \to \mathcal{C}$ be the inclusion and I the reflection of $\mathcal{C}$ in $\mathcal{X}$. Reflections that preserve pullbacks of all pairs with codomain in $\mathcal{X}$ are said to have *stable units* in [2].

Lemma 4.1. *Let $\mathcal{C}$, $\mathcal{C}'$, $\mathcal{X}$ and $\mathcal{X}'$ be categories with pullbacks. Given reflections $H \dashv I : \mathcal{C} \to \mathcal{X}$ and $I' \dashv H' : \mathcal{X}' \to \mathcal{C}'$ and pullback preserving functors U and V for which the diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "I"] \arrow[d, "V"'] & \mathcal{X} \arrow[l, shift left, "H"] \arrow[d, "U"] \\ \mathcal{C}' \arrow[r, shift left, "I'"] & \mathcal{X}' \arrow[l, shift left, "H'"] \end{tikzcd}$$

commutes, if $H' \dashv I'$ has stable units and U reflects isomorphisms then $H \dashv I$ also has stable units.

Proposition 4.2. *The reflection $H \dashv I : \mathcal{P}Preord \to \mathcal{P}sp$ has stable units.*

Proof: Consider the following commutative diagram

$$\begin{tikzcd} \mathcal{P}Preord \arrow[r, shift left, "I"] \arrow[d, "V"'] & \mathcal{P}Sp \arrow[l, shift left, "H"] \arrow[d, "U"] \\ \mathcal{P}reord \arrow[r, shift left, "I'"] & \mathcal{O}rd \arrow[l, shift left, "H'"] \end{tikzcd}$$

where the bottom is the reflection of $\mathcal{P}reord$ in $\mathcal{O}rd$, the category of (partially) ordered sets and U, V are the obvious forgetful functors. In [7], J. Xarez proved that the reflection of $\mathcal{P}reord$ in $\mathcal{O}rd$ has stable units. Since U and V preserves pullbacks and U reflects isomorphisms we conclude that the reflection of $\mathcal{P}Preord$ in $\mathcal{P}sp$ also has stable units. ■

Theorem 4.3. *A morphism $p : E \to B$ in $\mathcal{P}sp$ is an effective descent morphism in this category if and only if it is an effective descent morphism in $\mathcal{P}Preord$*

Proof: Let $p : E \to B \in \mathcal{P}sp$ be an effective descent morphism in $\mathcal{P}Preord$. Since $\mathcal{P}sp$ is a full subcategory of $\mathcal{P}Preord$ closed under pullbacks, by 3.2, for a pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"] & B \end{tikzcd}$$

in $\mathcal{P}Preord$ with $E \times_B A \in \mathcal{P}sp$ we have to prove that $A \in \mathcal{P}Preord$. Since the reflection $I : \mathcal{P}Preord \to \mathcal{P}sp$ has stable units and $B \in \mathcal{P}sp$ then I preserve the pullback. Thus, in the diagram

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "r_{E\times_B A}"'] \arrow[dr, phantom, "(1)"] & A \arrow[d, "r_A"] \\ I(E \times_B A) \arrow[r, "I(\pi_2)"] \arrow[d, "I(\pi_1)"'] \arrow[dr, phantom, "(2)"] & I(A) \arrow[d, "I(\alpha)"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

the outer rectangle (1) + (2) as well as the square (2) are pullbacks. Consequently, (1) is a pullback and so, for the pullback functor $p^* : \mathcal{P}Preord \downarrow B \to \mathcal{P}Preord \downarrow E$, $p^*(r_A) = r_{E\times_B A}$ which is an isomorphism because $E \times_B A$ belongs to $\mathcal{P}sp$. But $p^* : \mathcal{P}Preord \downarrow B \to \mathcal{P}Preord \downarrow E$, being monadic, reflects isomorphisms. Therefore, r_A is an isomorphism and so A belongs to $\mathcal{P}sp$.

Conversely, if the morphism p is an effective descent morphism in $\mathcal{P}sp$ it is a descent morphism also in $\mathcal{P}Preord$. Indeed, let (C, γ, ξ) be a $\mathbb{T}$-algebra for the monad induced in $\mathcal{P}Preord \downarrow E$ by the adjunction $p! \dashv p^* : \mathcal{P}Preord \downarrow B \to \mathcal{P}Preord \downarrow E$.

Since I preserves pullbacks of morphisms with codomain in $\mathcal{P}sp$ it is easy to see that $(I(C), I(\gamma), I(\xi))$ is a $\mathbb{T}$-algebra for the monad induced in $\mathcal{P}sp \downarrow E$ by the adjunction $p! \dashv p^* : \mathcal{P}sp \downarrow B \to \mathcal{P}sp \downarrow E$.

In the diagram

$$\begin{tikzcd} C \arrow[r, "q"] \arrow[d, "r_C"'] \arrow[dr, phantom, "(1)"] & Q \arrow[d, "r_Q"] \\ I(C) \arrow[r, "I(q)"] \arrow[d, "I(\gamma)"'] \arrow[dr, phantom, "(2)"] & I(Q) \arrow[d, "I(\delta)"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

the square (2) is a pullback: since p is an effective descent morphism in $\mathcal{P}sp$ and the forgetful functor from $\mathcal{P}sp$ to $\mathcal{S}et$ preserves pullbacks and coequalizers the conclusion follows from 1.4.

We can apply 1.4 to $\mathcal{C} = \mathcal{P}Preord$ and prove that the outer rectangle (1) + (2) is a pullback, which is equivalent to prove that (1) is a pullback.

Let $h : C \to I(C) \times_{I(Q)} Q$ be the morphism defined by $h(c) = ([c], q(c))$. There exists an isomorphism $t : E \times_B Q \to I(C) \times_{I(Q)} Q$ such that $h = t \cdot \alpha_{(C,\gamma,\xi)}$. Then, by 1.3, we conclude that h is a bijective map and so an homeomorphism.

Furthermore it is an order isomorphism: if $h(c) \leq h(c')$ then $[c] \leq [c']$ and so $c \leq c'$ in C. Thus h is an isomorphism in $\mathcal{P}Preord$ and so p is an effective descent morphism in $\mathcal{P}Preord$. ■

References

[1] J. Bénabou and J. Roubaud, Monades et descente, *Comptes Rendus Acad. Sc. Paris* **270** A (1970), 96-98.

[2] C. Cassidy, M. Hébert and G. M. Kelly, Reflective subcategories, localization and factorization systems, *J. Austral. Math. Soc. (Ser. A)* **38**(1985), 287-329.

[3] G. Janelidze and M. Sobral, Finite preorders and topological descent I, *J. Pure Appl. Algebra* **175** (2002), 187-205.

[4] G. Janelidze, M. Sobral and W. Tholen, Beyond Barr Exactness: Effective Descent Morphisms, in *Categorical Foundations. Special Topics in Order, Topology, Algebra and Sheaf Theory*, Cambridge University Press, 2004.

[5] G. Janelidze and W. Tholen, Facets of descent I, *Applied Cat. Struct.* **2** (1994), 245-281.
[6] M. Sobral and W. Tholen, Effective descent morphisms and effective equivalence relations, *Conference Proceedings of the Canadian Mathematical Society* **13**(1992), 421-433.
[7] J. Xarez, The monotone-light factorization for categories via preordered and ordered sets, PhD Thesis, Universidade de Aveiro (Portugal), 2003.

Margarida Dias
Departamento de Matemática, Universidade dos Açores, 9501-801 Ponta Delgada, Portugal
E-mail address: mdias@notes.uac.pt

Manuela Sobral
Departamento de Matemática, Universidade de Coimbra, 3001-454 Coimbra, Portugal
E-mail address: sobral@mat.uc.pt