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ON THE EIGENVALUES OF SOME TRIDIAGONAL MATRICES

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ABSTRACT: A solution is given for a problem on eigenvalues of some symmetric tridiagonal matrices suggested by William Trench. The method used is generalizable to other problems.

KEYWORDS: Tridiagonal matrices, eigenvalues, recurrence relations, Chebyshev polynomials.
AMS SUBJECT CLASSIFICATION (2000): 15A18, 65F15, 15A09, 15A47, 65F10.

1. Inverse of a tridiagonal matrix

Let us consider the n-by-n nonsingular tridiagonal matrix T

$$T = \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & c_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix} .$$

In [4], Usmani gave an elegant and concise formula for the inverse of the tridiagonal matrix T:

$$(T^{-1})_{ij} = \begin{cases} (-1)^{i+j} b_i \dots b_{j-1} \theta_{i-1} \phi_{j+1}/\theta_n, & \text{if } i \leq j \\ (-1)^{i+j} c_j \dots c_{i-1} \theta_{j-1} \phi_{i+1}/\theta_n, & \text{if } i > j \, , \end{cases} \tag{1.1}$$

where θ_i's verify the recurrence relation

$$\theta_i = a_i \theta_{i-1} - b_{i-1} c_{i-1} \theta_{i-2} \, , \quad \text{for } i = 2, \dots, n \, ,$$

with initial conditions $\theta_0 = 1$ and $\theta_1 = a_1$, and ϕ_i's verify the recurrence relation

$$\phi_i = a_i \phi_{i+1} - b_i c_i \phi_{i+2} \, , \quad \text{for } i = n-1, \dots, 1 \, ,$$

with initial conditions $\phi_{n+1} = 1$ and $\phi_n = a_n$. Observe that $\theta_n = \det T$.

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In [6], W.F. Trench proposed and solved the problem of finding eigenvalues and eigenvectors of the classes of symmetric matrices:

$$A = [\min\{i, j\}]_{i,j=1,\ldots,n}$$

and

$$B = [\min\{2i - 1, 2j - 1\}]_{i,j=1,\ldots,n} \ .$$

A. Kovačec has presented a different proof of this problem [2]. These two matrices are in fact particular cases of a more general matrix

$$C = [\min\{ai - b, aj - b\}]_{i,j=1,\ldots,n} \ ,$$

with $a > 0$ and $a \neq b$. It is very interesting that, under the above conditions, C is always invertible and its inverse is a tridiagonal matrix.

Proposition 1.1. *The tridiagonal matrix of order n*

$$T_n = \begin{bmatrix} 1 + \frac{a}{a-b} & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

is the inverse of $\frac{1}{a}C$.

Proof: Notice that θ_i's verify the recurrence relation $\theta_i = 2\theta_{i-1} - \theta_{i-2}$, for $i = 2, \ldots, n-1$, and $\theta_n = \theta_{n-1} - \theta_{n-2}$, with initial conditions $\theta_0 = 1$ and $\theta_1 = \frac{2a-b}{a-b}$. Then $\theta_i = \frac{(i+1)a-b}{a-b}$, for $i = 1, \ldots, n-1$, and $\theta_n = \frac{a}{a-b}$. The ϕ_i's verify the recurrence relation $\phi_i = 2\phi_{i+1} - \phi_{i+2}$, for $i = n-1, \ldots, 2$, with initial conditions $\phi_{n+1} = 1$ and $\phi_n = 1$. Therefore, since $\phi_i = 1$, for $i = n+1, \ldots, 2$, and $\phi_1 = \frac{a}{a-b}$.

Consequently, the inverse of T_n is the symmetric matrix such that

$$(T_n^{-1})_{ij} = (-1)^{i+j}(-1)^{j-i}\frac{\frac{ai-b}{a-b}}{\frac{a}{a-b}} = \frac{ai-b}{a}$$

for $i \leq j$. ∎

2. Eigenpairs of a particular tridiagonal matrix

According to the initial section the problem of finding the eigenvalues of C is equivalent to describing the spectra of a tridiagonal matrix. Here we give a general procedure to locate the eigenvalues of the matrix T_n from Proposition 1.1.

Let us consider the set of polynomials $\{Q_k(x)\}$ defined by the recurrence relation given by $Q_0(x) = 1$ and $Q_1(x) = (ax+1)Q_0(x)$,

$$Q_k(x) = (ax+2)Q_{k-1}(x) - Q_{k-2}(x) \ , \quad \text{for} \quad k = 2, \ldots, n-1 \ ,$$

and

$$Q_n(x) = \left(ax + \frac{2a-b}{a-b} \right) Q_{n-1}(x) - Q_{n-2}(x) \ .$$

Note that each polynomial $Q_k(x)$, for $k = 0, \ldots, n$, is of degree k. The last recurrence relation has the following matricial form:

$$x \begin{bmatrix} Q_{n-1}(x) \\ Q_{n-2}(x) \\ \vdots \\ Q_1(x) \\ Q_0(x) \end{bmatrix} = -\frac{1}{a} \begin{bmatrix} \frac{2a-b}{a-b} & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \begin{bmatrix} Q_{n-1}(x) \\ Q_{n-2}(x) \\ \vdots \\ Q_1(x) \\ Q_0(x) \end{bmatrix} + Q_n(x) \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} .$$

Since $Q_k(x) = U_k\left(\frac{ax}{2}+2\right) - U_{k-1}\left(\frac{ax}{2}+2\right)$, for $k = 0, \ldots, n-1$, and

$$\begin{aligned} Q_n(x) &= \left(ax + 1 + \frac{a}{a-b} \right) \left(U_{n-1}\left(\frac{ax}{2}+1\right) - U_{n-2}\left(\frac{ax}{2}+1\right) \right) - \\ &\qquad - \left(U_{n-2}\left(\frac{ax}{2}+2\right) - U_{n-3}\left(\frac{ax}{2}+1\right) \right) \\ &= U_n\left(\frac{ax}{2}+1\right) - U_{n-1}\left(\frac{ax}{2}+1\right) - \\ &\qquad - \left(1 - \frac{a}{a-b} \right) \left(U_{n-1}\left(\frac{ax}{2}+1\right) - U_{n-2}\left(\frac{ax}{2}+1\right) \right) , \end{aligned}$$

where $U_k(x)$, for $k = 0, \ldots, n$, are the Chebyshev polynomials of second kind of degree k, the zeros of $Q_n(x)$ are exactly the eigenvalues of $-\frac{1}{a}C$, i.e., the (real) values which satisfy the equality

$$p_n(x) := \frac{U_n\left(\frac{ax}{2}+1\right) - U_{n-1}\left(\frac{ax}{2}+1\right)}{U_{n-1}\left(\frac{ax}{2}+1\right) - U_{n-2}\left(\frac{ax}{2}+1\right)} = 1 - \frac{a}{a-b} \ . \tag{2.1}$$

In general, (2.1) means that the eigenvalues of $-\frac{1}{a}C$ are the intersections of the graph of $p_n(x)$ with the line $y = 1 - \frac{a}{a-b}$.

As a first consequence consider the case when $a = 1$ and $b = 0$. The eigenvalues of $-A$ are the solutions of the equation $U_n\left(\frac{x}{2}+1\right) - U_{n-1}\left(\frac{x}{2}+1\right) = 0$, which are, for $k = 0, \ldots, n-1$,

$$\lambda_k = 2\cos\left(\frac{2k+1}{2n+1}\,\pi\right) - 2 \ .$$

The value of an eigenvector associated to λ_k follows immediately:

$$\left[\ Q_{n-1}(\lambda_k) \quad \cdots \quad Q_1(\lambda_k) \quad Q_0(\lambda_k)\ \right]^t \ .$$

Hence we proved the following:

Theorem 2.1 ([2, 6]). *The matrix A of order n, $n \geq 3$, has the eigenpairs (λ_k, v_k) given by*

$$\lambda_k = \frac{1}{2}\left(1 - \cos\left(r_k\right)\right)^{-1} \quad \text{and} \quad v_k = \left[\sin(jr_k)\right]^t_{j=1,\ldots,n} \ ,$$

where

$$r_k = \frac{2k+1}{2n+1}\,\pi \ ,$$

for $k = 0, \ldots, n-1$.

If $a = 2$ and $b = 1$, then the eigenvalues of $-\frac{1}{2}B$ are solutions of the equation $U_n\left(x+1\right) - U_{n-2}\left(x+1\right) = 0$, which are, for $k = 0, \ldots, n-1$,

$$\cos\left(\frac{2k+1}{2n}\,\pi\right) - 1 \ .$$

3. Location of eigenvalues

Since $p_n(x)$ defined in (2.1) is strictly increasing, even if it is impossible to evaluate exactly the eigenvalues of C, one can locate them. For example, if $b < 0$, then $1 - \frac{a}{a-b} > 0$, and each eigenvalue λ_k is located between the zeros of $U_n\left(\frac{ax}{2}+1\right) - U_{n-1}\left(\frac{ax}{2}+1\right)$ and the zeros of $U_{n-1}\left(\frac{ax}{2}+1\right) - U_{n-2}\left(\frac{ax}{2}+1\right)$, i.e., lies in the intervals

$$\left]\frac{2}{a}\left(\cos\left(\frac{2k+1}{2n+1}\,\pi\right) - 1\right), \frac{2}{a}\left(\cos\left(\frac{2k-1}{2n-1}\,\pi\right) - 1\right)\right[\ ,$$

for $k = n-1, \ldots, 1$, and λ_0 is on the right side of $\frac{2}{a}\left(\cos\left(\frac{1}{2n+1}\,\pi\right) - 1\right)$. If $1 - \frac{a}{a-b} < 0$, i.e., $b > 0$, one can make an analogous consideration.

Let us consider the matrix T_6 from Proposition 1.1 with $a = -b = 2$, i.e.,

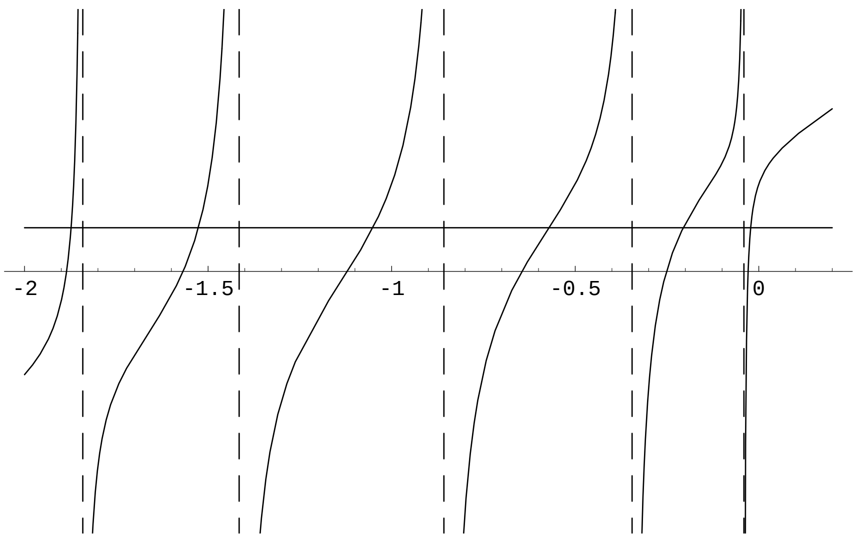
$$\begin{bmatrix} \frac{3}{2} & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} .$$

The eigenvalues of this matrix are located in the intervals

$$\begin{array}{c}]-0.2514892, -0.0405070[\\ ]-0.6453951, -0.3451392[\\ ]-1.1205366, -0.8576851[\\ ]-1.5680647, -1.4154150[\\ ]-1.8854560, -1.8412535[\ . \end{array}$$

and one is greater than -0.0290581. In fact, they are approximately

$$\begin{array}{rcl} \lambda_0 & = & -0.0220986 \\ \lambda_1 & = & -0.2058355 \\ \lambda_2 & = & -0.5715577 \\ \lambda_3 & = & -1.0510977 \\ \lambda_4 & = & -1.5262645 \\ \lambda_5 & = & -1.8731458 \ . \end{array}$$



The intersection of the graphs $y = p_6(x)$ and $y = \frac{1}{2}$.

FIGURE 1

4. A matrix of maximums

In the second section we have considered the matrix $[\min\{i,j\}]_{i,j}$. What happens if instead of the minimum we have the maximum? We note that the inverse of C must be tridiagonal because the upper and the lower triangular parts of C have rank 1 form.

Theorem 4.1. *For a positive integer n, consider the tridiagonal matrix of order n*

$$M = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1+\frac{1}{n} \end{bmatrix} . \tag{4.1}$$

Then M is invertible and the inverse is

$$M^{-1} = [\max\{i,j\}]_{i,j=1,\ldots,n} \ .$$

Proof: From (1.1) we have $\theta_i = -2\theta_{i-1} - \theta_{i-2}$, for $i = 2,\ldots,n-1$ and $\theta_n = \left(-1+\frac{1}{n}\right)\theta_{n-1} - \theta_{n-2}$, with initial conditions $\theta_0 = 1$ and $\theta_1 = -1$. Then $\theta_i = (-1)^i$, for $i = 0,\ldots,n-1$, and $\theta_n = (-1)^{n-1}\frac{1}{n} = \det M$. The ϕ_i's verify the recurrence relation $\phi_i = -2\phi_{i+1} - \phi_{i+2}$, for $i = n-1,\ldots,2$, with initial conditions $\phi_{n+1} = 1$ and $\phi_n = -1+\frac{1}{n}$, and $\phi_1 = -\phi_2 - \phi_3$. Then $\phi_i = (-1)^{n-i+1}(i-1)\frac{1}{n}$, for $i = 2,\ldots,n+1$. Finally, the inverse of M is the symmetric matrix such that

$$(M^{-1})_{ij} = (-1)^{i+j}\frac{(-1)^{i-1}(-1)^{n-j}\frac{j}{n}}{(-1)^{n-1}\frac{1}{n}} = j \quad \text{for } i \leq j \ ,$$

i.e.,

$$M^{-1} = [\max\{i,j\}]_{i,j=1,\ldots,n} \ .$$

■

Let us consider again the recurrence relation of $Q_k(x)$ already defined, with $a = 1$ and $b = n+1$. This recurrence relation is equivalent to

$$x\begin{bmatrix} Q_0(x) \\ Q_1(x) \\ \vdots \\ Q_{n-2}(x) \\ Q_{n-1}(x) \end{bmatrix} = \begin{bmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -1+\frac{1}{n} \end{bmatrix}\begin{bmatrix} Q_0(x) \\ Q_1(x) \\ \vdots \\ Q_{n-2}(x) \\ Q_{n-1}(x) \end{bmatrix} + Q_n(x)\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} .$$

Therefore one can located the eigenvalues of the matrix M using the arguments of the last section. Note that

$$\left[\begin{array}{cccc} Q_0(\lambda_k) & Q_1(\lambda_k) & \cdots & Q_{n-1}(\lambda_k) \end{array}\right]^t$$

is an eigenvector of M associated to the eigenvalue λ_k.

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