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TORSION THEORIES AND RADICALS IN NORMAL CATEGORIES

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ABSTRACT: We introduce a relativized notion of (semi)normalcy for categories that come equipped with a proper stable factorization system, and we use radicals and normal closure operators in order to study torsion theories in such categories. Our results generalize and complement recent studies in the realm of semi-abelian and, in part, homological categories. In particular, we characterize both, torsion-free and torsion classes, in terms of their closure under extensions. We pay particular attention to the homological and, for our purposes more importantly, normal categories of topological algebra, such as the category of topological groups. But our applications go far beyond the realm of these types of categories, as they include, for example, the normal, but non-homological category of pointed topological spaces, which is in fact a rich supplier for radicals of topological groups.

1. Introduction

Ever since the concept was defined in [22], *semi-abelian categories* have been investigated intensively by various authors. Their main building block, Bourn protomodularity [8], in conjunction with Barr exactness [3], and the presence of a zero object and of finite limits and colimits provide all the tools necessary for pursuing many themes of general algebra and of homology theory of not necessarily commutative structures. The monograph [5] gives a comprehensive account of these developments.

It has been observed in [6] and [9] that the category TopGrp of topological groups satisfies all conditions of a semi-abelian category, except for Barr-exactness, i.e., equivalence relations are not necessarily effective. But TopGrp is still regular, that is: it has a pullback-stable $(\mathcal{R}eg\mathcal{E}pi, \mathcal{M}ono)$-factorization system. The term *homological* was used in [5, 6] for finitely complete, regular and protomodular categories with a zero object. Hence, TopGrp is the prototype of a homological category, which still allows for the establishment of the essential lemmata of homology theory, but fails to be semi-abelian.

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Let us examine TopGrp's failure to be semi-abelian a bit more closely. Recall that a category C is semi-abelian if, and only if, (see [22]):

(1) C is finitely complete and has a zero object;
(2) C has a pullback-stable $(\mathcal{R}egEpi, \mathcal{M}ono)$-factorization system;
(3) for every commutative diagram

$$\begin{tikzcd} {} \arrow[r] \arrow[d] \arrow[dr, phantom, "\boxed{1}"] & {} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\boxed{2}"] & {} \arrow[d] \\ {} \arrow[r] & {} \arrow[r] & {} \end{tikzcd}$$

in C with a regular epimorphism p, if $\boxed{1}$ and $\boxed{1|2}$ are pullback diagrams, $\boxed{2}$ is also one;
(4) equivalence relations in C are effective, i.e., are kernel pairs;
(5) C has finite coproducts.

A semi-abelian category has, in fact, all finite colimits. Conditions (1)-(3) say precisely that C is homological. In a homological category, $\mathcal{R}egEpi = \mathcal{N}ormEpi =$ the class of normal epimorphisms, i.e., of cokernels, and $\mathcal{M}ono = 0\text{-}\mathcal{K}er =$ the class of morphisms with zero kernel (see [8, 5]). Now, in the presence of the other conditions, (4) may be rephrased in old-fashioned terms by:

(4′) images of *normal monomorphisms* (=kernels) under normal epimorphisms are normal monomorphisms.

Here "image" is to be understood with respect to the $(\mathcal{R}egEpi, \mathcal{M}ono) = (\mathcal{N}ormEpi, 0\text{-}\mathcal{K}er)$-factorization system of C. But in TopGrp the natural image (as a subobject of the codomain) is *not* formed via the $(\mathcal{R}egEpi, \mathcal{M}ono)$- but the $(\mathcal{E}pi, \mathcal{R}egMono)$-factorization system. (Consider, for example, the normal epimorphism $q : \mathbb{R} \twoheadrightarrow \mathbb{R}/\mathbb{Z}$ and the cyclic subgroup H of $\mathbb{R}$ generated by $\sqrt{2}$; $q(H)$ is, as a subgroup of $\mathbb{R}/\mathbb{Z}$, dense, whereas the quotient topology of H would make it discrete.) Hence, the only condition that prevents TopGrp (and even $\mathsf{TopAbGrp}$, the category of topological abelian groups) from being semi-abelian could be rescued if we would interpret "image" naturally, i.e., switch to the "correct" factorization system. The example $\mathsf{HausGrp}$ of Hausdorff topological groups shows that, in general, the "correct" factorization system (namely: (surjective, embeddings)) may be given neither as the $(\mathcal{R}egEpi, \mathcal{M}ono)$- nor as the $(\mathcal{E}pi, \mathcal{R}egMono)$-system. This is why we are proposing relativized notions of homologicity and of semi-abelianess in this paper, for categories that come equipped with a proper and stable $(\mathcal{E}, \mathcal{M})$-factorization system, which we call $(\mathcal{E}, \mathcal{M})$-*seminormal* and $(\mathcal{E}, \mathcal{M})$-*normal*, respectively. (We are aware of the fact that the term *normal* for categories has been used in the older literature, particularly in [28, 29], but not with any lasting impact. Hence, we hope that our terminology does not lead to any confusion.) These relativized notions reach categories such as Set_* and Top_* (pointed sets and pointed topological spaces), which are very far from being homological. Nevertheless, our axioms are strong enough to establish the key characterization theorems for torsion and torsion-free classes, in terms of their closure property under extensions. In fact, most of the results in this paper remain true for categories that satisfy only a fraction of the conditions for $(\mathcal{E}, \mathcal{M})$-(semi)normalcy. We therefore refer always directly to the conditions used at each stage, rather than working with the blanket assumption of $(\mathcal{E}, \mathcal{M})$-normalcy.

The paper rests on two (well-known, in principle) correspondences, between torsion theories and radicals, and between radicals and closure operators, the latter of which is adapted from [18], and the composition of which has been the subject of a recent paper by Bourn and Gran [9]. We generalize and extend their results from the context of homological (or semi-abelian categories) to that one of seminormal (or normal) categories, and beyond, by adapting to the present context many results on (pre)radicals presented in [18]. This adaptation follows the lead of [9] where the notion of closure operator for arbitrary subobjects (as used in [17, 18]) is restricted to one for normal subobjects. We believe that the factorization of the correspondence between torsion theories and closure operators through radicals greatly clarifies matters, and it also connects the results better to existing work, especially to that of Barr [4] and Lambek [23]. (The Lambek paper contains many references to the literature of the time, which was predominantly concerned with torsion theories for R-modules. Later papers present general categorical approaches, for example [30, 27, 11].)

Consequently, after specifying our setting in Section 2, we start off by presenting (pre)radicals in Section 3. Their easy relation with torsion theories follows in Section 4, while the more involved relation with closure operators is presented in Section 5. We give a summary of results in Section 6 and present examples in Section 7.

2. The setting

Throughout this paper *we consider a pointed, finitely complete category* C*, which has cokernels of kernels.* We also assume that C *comes with a fixed (orthogonal) factorization system* $(\mathcal{E},\mathcal{M})$*, which is proper and stable.* Hence, $\mathcal{E}\subseteq\mathcal{E}pi$ and $\mathcal{M}\subseteq\mathcal{M}ono$ are pullback-stable classes of morphisms such that $\mathsf{C}=\mathcal{M}\cdot\mathcal{E}$, and the so called $(\mathcal{E},\mathcal{M})$-diagonalization property holds. As a consequence, for every morphism $f : X\to Y$ one has the adjunction

$$\begin{tikzcd} \mathcal{M}/X \arrow[r, shift left, "f(-)"] \arrow[r, phantom, "\perp"] & \mathcal{M}/Y \arrow[l, shift left, "f^{-1}(-)"] \end{tikzcd}$$

between inverse image and image under f of $\mathcal{M}$-subobjects, given by pullback and $(\mathcal{E},\mathcal{M})$-factorization, respectively. Note that, because of pullback stability, one has $f\in\mathcal{E}$ if, and only if,

$$f(f^{-1}(N)) = N \text{ for all } N\in\mathcal{M}/Y.$$

We also note that, because the system is proper, every regular monomorphism represents an $\mathcal{M}$-subobject, and every regular epimorphism lies in $\mathcal{E}$; in particular, every *normal monomorphism* (=kernel of some morphism) is in $\mathcal{M}$, and every *normal epimorphism* (=cokernel of some morphism) is in $\mathcal{E}$.

Denoting by 0-$\mathcal{K}er$ the class of morphisms with trivial kernel (which contains all monomorphisms), first we note that C automatically has a second factorization system given by $(\mathcal{N}ormEpi, 0\text{-}\mathcal{K}er)$ which, however, generally fails to be proper or stable, but which coincides with $(\mathcal{R}egEpi,\mathcal{M}ono)$ when C is homological.

Proposition 2.1. *Every morphism factors into a normal epimorphism followed by a morphism with a trivial kernel, and this constitutes an orthogonal factorization system.*

Proof: For every f, in the factorization $f = (X\xrightarrow{p} X/K\xrightarrow{h} Y)$ with $K=\ker f$, the morphism $K \overset{k}{\rightarrowtriangle} X \xrightarrow{p} X/K$ factors through $M=\ker h$, by a morphism $p' : K\to M$, which is epic since, with $p\in\mathcal{E}$, also $p'\in\mathcal{E}$. Since $pk=0$, we have $M=0$.

To show the diagonalization property, let $tp = ms$ with a normal epimorphism p and $\ker m = 0$, and let m' be the pullback of m along t. Then

$\ker m' = \ker m = 0$, and for the arrow v induced by the pullback property one has $vk = 0$, with $k : \ker p \to X$.

$$\begin{tikzcd}
X \arrow[rr, "p"] \arrow[dr, "v"] \arrow[dd, "s"'] & & Y \arrow[dd, "t"] \\
& P \arrow[ur, "m'"] \arrow[dl, "t'"] & \\
U \arrow[rr, "m"'] & & V
\end{tikzcd}$$

The induced arrow $Y \to P$, together with $t' : P \to U$, gives the desired fill-in morphism. ■

The additional conditions used in this paper, which will give the notion of $(\mathcal{E}, \mathcal{M})$-(semi)normalcy, are essentially about the interaction of the two factorization systems of C, $(\mathcal{E}, \mathcal{M})$ and $(\mathcal{N}ormEpi, 0\text{-}\mathcal{K}er)$, and are best motivated by the interaction of the quotient and subspace topology in TopGrp (where quotient map means normal epimorphism, and where we would choose for $\mathcal{M}$ the class of embeddings). To this end we first consider a commutative square

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[d, "q"] \\
Z \arrow[r, "g"] & W
\end{tikzcd} \tag{$*$}$$

in C and show:

Proposition 2.2. *Let $f \in \mathcal{M}$ and $p \in \mathcal{E}$. Then* $\ker p = \ker q$ *if, and only if,* $\ker g = 0$ *and* $\ker q \leq X$ *(so that* $\ker q \rightarrowtail Y$ *factors through f). These equivalent conditions hold true when* (*) *is a pullback diagram, and they imply that* (*) *is a pushout diagram in case q is a normal epimorphism.*

Proof: One always has $\ker p \leq \ker q$. If (*) is a pullback one obtains h with $fh = k = (\ \ker q \rightarrowtail Y\ )$ and $ph = 0$. The latter identity actually follows from the former if $\ker g = 0$, since $g(ph) = qk = 0$. Now $ph = 0$ makes h factor through $\ker p$, whence $\ker p = \ker q$. Conversely, $\ker p = \ker q$ trivially implies $\ker q \leq X$, and

$$\ker g = p(p^{-1}(\ker g)) = p(f^{-1}(\ker q)) = p(\ker p) = 0.$$

Furthermore, any morphisms a, b with $ap = bf$ satisfy

$$b(\ker q) = a(p(\ker p)) = a(0),$$

so that b factors uniquely as $b = cq$ when q is a normal epimorphism. Since $p \in \mathcal{E}$ is epic, $cg = a$ follows. ■

Definition 2.3. (1) A commutative diagram (*) is a *basic image square* (*bis*) if $p \in \mathcal{E}$, $f \in \mathcal{M}$, $q \in \mathcal{N}ormEpi$, and $\ker p = \ker q$.

(2) C is $(\mathcal{E}, \mathcal{M})$-*seminormal* when, for every bis (*),

$$g \in \mathcal{M} \ \text{ if, and only if, } p \in \mathcal{N}ormEpi,$$

and when, in either case, the bis is a pullback diagram.

Remark 2.4. There are, implicitly, three conditions that make up $(\mathcal{E}, \mathcal{M})$-seminormalcy, which we may state independently from each other, as follows.

(CT) Every bis (*) with $g \in \mathcal{M}$ is a pullback diagram.

Equivalently: for every normal epimorphism $q : Y \to W$ and for every $\mathcal{M}$-subobject X of Y with $\ker q \leq X$, one has $q^{-1}(q(X)) = X$. Since we already know that $q \in \mathcal{E}$ satisfies $q(q^{-1}(Z)) = Z$ for all $\mathcal{M}$-subobjects Z of W, condition (CT) in fact means that the *Correspondence Theorem* of algebra holds true, i.e., $\mathcal{M}$-subobjects of Y above the kernel of q correspond bijectively to $\mathcal{M}$-subobjects of W:

$$\ker q \backslash (\mathcal{M}/Y) \underset{q^{-1}(-)}{\overset{q(-)}{\rightleftarrows}} \mathcal{M}/W \quad (\sim)$$

Here is an equivalent formulation of (CT) that we shall use frequently:

(CT′) For every normal epimorphism $q : Y \to W$ and all $\mathcal{M}$-subobjects X_1, X_2 of Y with $\ker q \leq X_2$ and $q(X_1) \leq q(X_2)$, one has $X_1 \leq X_2$.

The next ingredient to $(\mathcal{E}, \mathcal{M})$-seminormalcy is:

(PN) For every bis (*) that is a pullback, $g \in \mathcal{M}$ implies $p \in \mathcal{N}ormEpi$.

(Of course, in the presence of (CT), there is no need to mention the pullback provision.) Since $\mathcal{E}$ and $\mathcal{M}$ are stable under pullback, and in light of Proposition 2.2, (PN) just means *p*ullback stability of *n*ormal epimorphisms along $\mathcal{M}$-morphisms, that is:

(PN′) For every pullback diagram (*) with $g \in \mathcal{M}$ and $q \in \mathcal{N}ormEpi$, also $p \in \mathcal{N}ormEpi$.

The third ingredient to $(\mathcal{E}, \mathcal{M})$-seminormalcy is:

(QN) For every bis (*), $p \in \mathcal{N}ormEpi$ implies $g \in \mathcal{M}$.

Since a bis is a pushout diagram (by Prop. 2.2), (QN) requests $\mathcal{M}$ to be stable under certain pushouts along normal epimorphisms, briefly referred to as stability under *n*ormal *q*uotients:

(QN′) For every $\mathcal{M}$-morphism $f : X \to Y$ and every normal subobject $N \rightarrowtail Y$ with $N \leq X$, the induced morphism $g : X/N \to Y/N$ lies also in $\mathcal{M}$.

We shall now consider two further conditions which, in conjunction with (CT), (PN) and (QN), will make C $(\mathcal{E}, \mathcal{M})$-normal.

(IN) For every commutative square (*) with $p, q \in \mathcal{E}$ and $f, g \in \mathcal{M}$, if f is a normal monomorphism, g is also one.

Hence, (IN) stipulates that $\mathcal{M}$-*i*mages of *n*ormal monomorphisms along an $\mathcal{E}$-morphism be normal monomorphisms:

(IN′) For every $\mathcal{E}$-morphism $q : Y \to W$ and every normal subobject $N \rightarrowtail Y$, the $\mathcal{M}$-subobject $q(N) \to W$ is also normal.

The last condition is just a mild existence condition for particular colimits:

(JN) Every two normal subobjects have a join, that is: for any two normal subobjects $N \rightarrowtail X$ and $K \rightarrowtail X$, there is a least normal subobject $N \vee K \rightarrowtail X$ containing N and K.

Since we may switch back and forth between kernels and cokernels, (JN) means equivalently:

(JN′) The pushout of any two normal epimorphisms (with common domain) exists.

Definition 2.5. C is $(\mathcal{E}, \mathcal{M})$-*normal* if C is $(\mathcal{E}, \mathcal{M})$-seminormal and (IN) and (JN) hold; equivalently, if C satisfies (CT), (PN), (QN), (IN) and (JN).

Examples 2.6. (1) Every homological category C is $(\mathcal{R}egEpi, \mathcal{M}ono)$-seminormal. In fact, (PN) is just a particular instance of $\mathcal{R}egEpi$'s pullback stability, and (QN) is (vacuously) satisfied, since $\mathcal{M}ono = 0\text{-}\mathcal{K}er$ (by Prop. 2.2). To prove (CT) in a homological category, one can just apply property

(3) of the Introduction to the diagram

$$\begin{tikzcd} \ker q \arrow[r] \arrow[d] & X \arrow[r, "f"] \arrow[d, "p"] & Y \arrow[d, "q"] \\ 0 \arrow[r] & Z \arrow[r, "g"] & W \end{tikzcd}$$

In other words, (CT) is just a very special instance of the protomodularity condition of a homological category.

(2) A homological category C with binary coproducts is $(\mathcal{R}egEpi, \mathcal{M}ono)$-normal if, and only if, it is semi-abelian. In fact, (IN) coincides with (4′) of the Introduction in this case.

(3) The category TopGrp (and, more generally, the category of models in Top for any semi-abelian theory, see [6]) is homological and $(\mathcal{E}, \mathcal{M})$-normal, with $(\mathcal{E}, \mathcal{M})$=(surjections, embeddings), but (generally) not semi-abelian. Similarly for $\mathsf{HausGrp}$, the category of Hausdorff groups, etc.

(4) The categories Set_* and Top_* of pointed sets and topological spaces, respectively, are $(\mathcal{E}, \mathcal{M})$-normal, with $(\mathcal{E}, \mathcal{M})$= (surjections, embeddings), but certainly not homological (since $\mathcal{M}ono \neq 0\text{-}\mathcal{K}er$).

3. Normal preradicals

Definition 3.1. A *(normal) preradical* of the category C (as in Section 2) is a normal subfunctor $\mathbf{r} \rightarrowtriangle \mathrm{Id}_{\mathsf{C}}$ of the identity functor of C, i.e., for all X we have a normal monomorphism $\mathbf{r}X \rightarrowtriangle X$, so that every morphism $f : X \to Y$ restricts to $\mathbf{r}X \to \mathbf{r}Y$. It is

- *idempotent* if $\mathbf{r}(\mathbf{r}X) = \mathbf{r}X$ for all objects X;
- a *radical*, if $\mathbf{r}(X/\mathbf{r}(X)) = 0$ for all objects X;
- $\mathcal{M}$-*hereditary* if $f^{-1}(\mathbf{r}Y) = \mathbf{r}X$ for every $\mathcal{M}$-morphism $f : X \to Y$;
- $\mathcal{E}$-*cohereditary* if $f(\mathbf{r}X) = \mathbf{r}Y$ for every $\mathcal{E}$-morphism $f : X \to Y$.

Let us note immediately that $\mathcal{M}$-*heredity* implies idempotency (consider $f : \mathbf{r}X \rightarrowtriangle X$), and $\mathcal{E}$-*coheredity* forces $\mathbf{r}$ to be a radical (consider $f : X \to X/\mathbf{r}X$). Both the least preradical $\mathbf{0}$ (given by $0 \to X$) and the largest preradical $\mathbf{1}$ (given by $X \to X$) satisfy these additional properties.

Remark 3.2. With every preradical $\mathbf{r}$ of C one associates the full subcategory $\mathsf{F}_{\mathbf{r}}$ of $\mathbf{r}$-*torsion-free* objects X, which must satisfy $\mathbf{r}X = 0$. Denoting by

$$\varrho_X : X \to X/\mathbf{r}X = RX$$

the canonical projection, one obtains an endofunctor R, pointed by a natural transformation $\varrho : \mathrm{Id}_{\mathsf{C}} \to R$, which is pointwise a normal epimorphism and has $\mathsf{F}_{\mathbf{r}}$ as its fixed subcategory:

$$\mathsf{F}_{\mathbf{r}} = \mathrm{Fix}(R, \varrho) = \{X : \varrho_X \text{ iso }\}.$$

Conversely, any endofunctor S, pointed by a pointwise normal epimorphism $\sigma : \mathrm{Id}_{\mathsf{C}} \to S$, gives the preradical $\ker \sigma$. We thus have an adjunction

$$\{\text{ preradicals }\} \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} \{\text{ normal-pointed endofunctors }\}.$$

Under this adjunction, radicals correspond bijectively to those (S, σ) with $S\sigma$ iso, i.e., to full and replete normal-epireflective subcategories,

$$\{\text{ radicals }\} \underset{\longleftarrow}{\overset{\longrightarrow}{\sim}} \{\text{ normal-epireflective subcategories }\}^{\mathrm{op}}.$$

For a preradical $\mathbf{r}$ of C, $\mathsf{T}_{\mathbf{r}}$ denotes the full subcategory of all $\mathbf{r}$-*torsion* objects X, defined by the condition $\mathbf{r}X = X$. Preradicals are nothing but normal copointed endofunctors, and $\mathsf{T}_{\mathbf{r}}$ is just the fixed subcategory of that copointed endofunctor of C. By restriction of the principal adjunction one obtains the bijective correspondence

$$\{\text{ idempotent preradicals }\} \underset{\longleftarrow}{\overset{\longrightarrow}{\sim}} \{\text{ normal-monocoreflective subcategories }\}.$$

Proposition 3.3. *Let* $\mathbf{r}$ *be a preradical of* C. *Then:*

(1) $\mathbf{r}$ *is* $\mathcal{M}$*-hereditary if, and only if,* $\mathbf{r}$ *is idempotent and* $\mathsf{T}_{\mathbf{r}}$ *closed under* $\mathcal{M}$*-subobjects in* C.

(2) $\mathbf{r}$ *is* $\mathcal{E}$*-cohereditary if, and only if,* $\mathbf{r}$ *is a radical and* $\mathsf{F}_{\mathbf{r}}$ *is closed under* $\mathcal{E}$*-images in* C*, provided that* C *satisfies that* (CT) *and* (IN).

Proof: (1) Let $f : X \to Y$ be in $\mathcal{M}$. If $\mathbf{r}Y = Y$, then $f^{-1}(\mathbf{r}Y) = X$, and $\mathbf{r}X = X$ follows when $\mathbf{r}$ is hereditary. Conversely, assuming closure of $\mathsf{T}_{\mathbf{r}}$ under $\mathcal{M}$-subobjects, since $\mathbf{r}Y \in \mathsf{T}_{\mathbf{r}}$ by idempotency of $\mathbf{r}$, we conclude $f^{-1}(\mathbf{r}Y) \in \mathsf{T}_{\mathbf{r}}$. Trivially, $f^{-1}(\mathbf{r}Y) \leq X$, hence $f^{-1}(\mathbf{r}Y) = \mathbf{r}(f^{-1}(\mathbf{r}Y)) \leq \mathbf{r}X$, and $\mathbf{r}X \leq f^{-1}(\mathbf{r}Y)$ is always true.

(2) Let $f : X \to Y$ be in $\mathcal{E}$. If $\mathbf{r}X = 0$, then $f(\mathbf{r}X) = 0$, and $\mathbf{r}Y = 0$ follows when $\mathbf{r}$ is cohereditary. Conversely, we now assume $\mathsf{F}_{\mathbf{r}}$ to be closed under $\mathcal{E}$-images. From (IN) we have that $f(\mathbf{r}X)$ is normal in Y, and we can

form the commutative diagram

$$\begin{tikzcd} \mathbf{r}X \arrow[r] \arrow[d] & X \arrow[r, "\varrho_X"] \arrow[d, "f"] & X/\mathbf{r}X \arrow[d, "\overline{f}"] \\ f(\mathbf{r}X) \arrow[r] & Y \arrow[r, "q"] & Y/f(\mathbf{r}X) \end{tikzcd}$$

Since $\overline{f}\varrho_X = qf \in \mathcal{E}$, also $\overline{f} \in \mathcal{E}$, so that from $X/\mathbf{r}X \in \mathsf{F}_{\mathbf{r}}$ (for the radical $\mathbf{r}$) one can derive $Z := Y/f(\mathbf{r}X) \in \mathsf{F}_{\mathbf{r}}$, by hypothesis. Since trivially $f(\mathbf{r}X) \leq \mathbf{r}Y$, in the presence of (CT) we can deduce equality of these two $\mathcal{M}$-subobjects from $q(\mathbf{r}Y) = 0$; in fact, $\mathbf{r}Y \rightarrowtail Y \xrightarrow{q} Z$ factors through $\mathbf{r}Z = 0$. ■

Remarks 3.4. (1) Of course, for any epimorphism $f : X \to Y$ in C and any preradical $\mathbf{r}$, when $X \in \mathsf{T}_{\mathbf{r}}$ also $Y \in \mathsf{T}_{\mathbf{r}}$. Likewise, if f is a monomorphism, $Y \in \mathsf{F}_{\mathbf{r}}$ implies $X \in \mathsf{F}_{\mathbf{r}}$. These assertions are immediate consequences of the functoriality of $\mathbf{r}$, as an inspection of the naturality diagram

$$\begin{tikzcd} \mathbf{r}X \arrow[r] \arrow[d] & \mathbf{r}Y \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

reveals.

(2) For an idempotent radical $\mathbf{r}$, both $\mathsf{T}_{\mathbf{r}}$ and $\mathsf{F}_{\mathbf{r}}$ are *closed under extensions*, that is: if in the short exact sequence

$$0 \longrightarrow Y \xrightarrow{k} X \xrightarrow{p} Z \longrightarrow 0$$

both Y and Z lie in $\mathsf{T}_{\mathbf{r}}$ ($\mathsf{F}_{\mathbf{r}}$), X also lies in $\mathsf{T}_{\mathbf{r}}$ ($\mathsf{F}_{\mathbf{r}}$). More precisely, $\mathsf{T}_{\mathbf{r}}$ ($\mathsf{F}_{\mathbf{r}}$) is closed under extensions if $\mathbf{r}$ is a radical (idempotent preradical, respectively). In fact, for a radical $\mathbf{r}$ consider the commutative diagram

$$\begin{tikzcd} \mathbf{r}Y \arrow[r] \arrow[d] & \mathbf{r}X \arrow[r] \arrow[d] & \mathbf{r}Z \arrow[d] \\ Y \arrow[r, "k"] \arrow[d, "\varrho_Y"] & X \arrow[r, "p"] \arrow[d, "\varrho_X"] & Z \arrow[d, "\varrho_Z"] \\ RY \arrow[r] & RX \arrow[r] & RZ \end{tikzcd}$$

If $Y, Z \in \mathsf{T}_{\mathbf{r}}$, so that $\mathbf{r}Y = Y$, $\mathbf{r}Z = Z$, ϱ_X must factor through p by a (normal) epimorphism $Z \to RX$, so that with Z also $RX \in \mathsf{T}_{\mathbf{r}}$, by (1). But

$RX \in \mathsf{F}_{\mathbf{r}}$ since $\mathbf{r}$ is a radical, so that $RX \in \mathsf{T}_{\mathbf{r}} \cap \mathsf{F}_{\mathbf{r}}$ must be 0, which means $X \in \mathsf{T}_{\mathbf{r}}$. Symmetrically, if $\mathbf{r}$ is an idempotent preradical and $Y, Z \in \mathsf{F}_{\mathbf{r}}$, so that $\mathbf{r}Y = 0 = \mathbf{r}Z$, $\mathbf{r}X \to X$ must factor through k by a monomorphism $\mathbf{r}X \to Y$, so that with Y also $\mathbf{r}X \in \mathsf{F}_{\mathbf{r}}$, by (1). But $\mathbf{r}X \in \mathsf{T}_{\mathbf{r}}$, since $\mathbf{r}$ is idempotent, so that $\mathbf{r}X \in \mathsf{T}_{\mathbf{r}} \cap \mathsf{F}_{\mathbf{r}}$, must be 0, which means $X \in \mathsf{F}_{\mathbf{r}}$.

(3) $\mathcal{M}$-heredity of a preradical $\mathbf{r}$ means, by definition, that every $\mathcal{M}$-morphism $f : X \to Y$ yields a pullback diagram

$$\begin{tikzcd} \mathbf{r}X \arrow[r] \arrow[d] & \mathbf{r}Y \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

in C. Next we show that $\mathcal{E}$-coheredity fully deserves the dual name also under this perspective.

Corollary 3.5. *A preradical* $\mathbf{r}$ *is* $\mathcal{E}$*-cohereditary if, and only if, for every* $\mathcal{E}$*-morphism* $f : X \to Y$*,*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varrho_X"'] & Y \arrow[d, "\varrho_Y"] \\ X/\mathbf{r}X \arrow[r, "Rf"] & Y/\mathbf{r}Y \end{tikzcd}$$

is a pushout diagram in C.

Proof: Assume $\mathcal{E}$-coheredity and consider $g : X/\mathbf{r}X \to Z$, $h : Y \to Z$ with $g\varrho_X = hf$. $(\mathcal{E}, \mathcal{M})$-factoring and exploiting the $(\mathcal{E}, \mathcal{M})$-diagonalization property we see that we may assume $g, h \in \mathcal{E}$, without loss of generality. Then $X/\mathbf{r}X \in \mathsf{F}_{\mathbf{r}}$ implies $Z \in \mathsf{F}_{\mathbf{r}}$ with 3.3(2), so that h factors through the $\mathsf{F}_{\mathbf{r}}$-reflection ϱ_Y by a morphism $t : Y/\mathbf{r}Y \to Z$, which also satisfies $tRf = g$ since ϱ_X is epic. Conversely, exploiting the pushout diagram for $f = \varrho_X$ we obtain ϱ_Y iso with $Y = X/\mathbf{r}X$, hence $\mathbf{r}Y = 0$. For general $f \in \mathcal{E}$ we just need to show that $X \in \mathsf{F}_{\mathbf{r}}$ implies $Y \in \mathsf{F}_{\mathbf{r}}$; but that is trivially true since ϱ_X iso implies ϱ_Y iso. ■

Relativizing the notion given in [21], let us call a full replete subcategory B of C to be *$\mathcal{E}$-Birkhoff* when B is normal-epireflective and closed under $\mathcal{E}$-images in C. Dually, B is called *$\mathcal{M}$-co-Birkhoff* if B is normal-monocoreflective and closed under $\mathcal{M}$-subobjects in C.

Restricting the correspondences of 3.2 one obtains from 3.3:

Theorem 3.6. (1) *$\mathcal{M}$-hereditary preradicals of C correspond bijectively to $\mathcal{M}$-co-Birkhoff subcategories of C.*
(2) *If C satisfies* (CT) *and* (IN), *$\mathcal{E}$-cohereditary (pre)radicals of C correspond bijectively to $\mathcal{E}$-Birkhoff subcategories of C.*

A useful observation is:

Lemma 3.7. *For any preradical $\mathbf{r}$ and every morphism $f : X \to Y$ one has $f^{-1}(\mathbf{r}Y) = \mathbf{r}X$ if, and only if,* $\ker(Rf) = 0$, *with Rf as in* 3.5.

Proof: First assume $f^{-1}(\mathbf{r}Y) = \mathbf{r}X$, i.e., that the naturality diagram 3.4 is a pullback, and form the pullback diagram

$$\begin{array}{ccc} \varrho_X^{-1}(\ker(Rf)) & \xrightarrow{k'} & X \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle \varrho_X} \\ \ker(Rf) & \xrightarrow{k} & X/\mathbf{r}X \end{array}$$

Since $\varrho_Y f k' = (Rf)ke = 0$, fk' must factor through $\mathbf{r}Y$, and then k' must factor through $\mathbf{r}X$. This yields $\varrho_X k' = ke = 0$, which forces $k = 0$ since e is epic.

Conversely, assume $\ker(Rf) = 0$ and consider morphisms g, h with $fg = mh$, with $m = (\mathbf{r}Y \rightarrowtail Y)$. Hence g factors through $\mathbf{r}X$, which is a factorization also for h since m is monic. ■

Corollary 3.8. *A radical $\mathbf{r}$ is $\mathcal{M}$-hereditary if, and only if, the reflector R of $\mathsf{F}_{\mathbf{r}}$ takes every $\mathcal{M}$-morphism to a morphism with trivial kernel.*

We note that the corollary remains valid if $\mathbf{r}$ is just a preradical, since the endofunctor R is still available in that case.

Examples 3.9. (1) The only preradicals in Set_* (see 2.6(4)) are $\mathbf{0}$ and $\mathbf{1}$; likewise in the category of vector spaces over a field.

(2) The category Top_* (see 2.6(4)) has a very large supply of preradicals. Here is a first general scheme for obtaining radicals, that actually works in the abstract category C of our setting, provided that the needed limits exist. Let B be a class of objects in C, and for $X \in \mathsf{C}$ let $\mathbf{q}_{\mathsf{B}}X$ be the intersection of the kernels of all morphisms $X \to B$, $B \in \mathsf{B}$. Of course, when B is reflective, $\mathbf{q}_{\mathsf{B}}X = \ker \varrho_X$, as described in 3.2. Then $\mathbf{q}_{\mathsf{B}}$ is a radical, which we call the B-*radical* in C, and just B-radical when $\mathsf{B} = \{B\}$. In Top_* now, taking for B the Sierpinski dyad S with closed base point, then $\mathbf{q}_S(X, x) = \overline{\{x\}}$ is the

closure of x in X. If S^* denotes the Sierpinski dyad with open base point, $\mathbf{q}_{S^*}(X,x)$ is the intersection of all open neighbourhoods of x in X. Both, $\mathbf{q}_S$ and $\mathbf{q}_{S^*}$ are ($\mathcal{M}$-)hereditary (with $\mathcal{M} = \{\text{embeddings}\}$) and therefore idempotent. Applying the same procedure to the 2-point discrete space D we obtain for $\mathbf{q}_D(X,x)$ the quasi-component of x in X, a non-idempotent radical. Likewise, for the real line $\mathbb{R}$ pointed by 0, the radical $\mathbf{q}_\mathbb{R}$ fails to be idempotent. ($\mathbf{q}_\mathbb{R}(X,x)$ contains all points in X that cannot be separated from x by a continuous $\mathbb{R}$-valued function; non-idempotency is therefore witnessed by an infinite pointed regular T_1-space (X,x) such that $\mathbf{q}_\mathbb{R}(X,x) \neq \{x\}$ is finite.)

(3) Let A be is a class of objects of Top_*, and for X in Top_* let $\mathbf{p}_\mathsf{A}$ be the union of the images of all maps $A \to X$, $A \in \mathsf{A}$. When A is normal-monocoreflective, only one such morphism suffices. Then $\mathbf{p}_\mathsf{A}$ is an idempotent preradical of Top_*. Certainly, one can replace A by its normal-monocoreflective hull. In the particular case when the class A is singleton $\{A\}$, we just put $\mathbf{p}_A$. It is easy to see that $\mathbf{p}_S = \mathbf{q}_{S^*}$, and $\mathbf{p}_{S^*} = \mathbf{q}_S$, while $\mathbf{p}_\mathbb{R}$ coincides with the arc-component of the base point (so it differs from $\mathbf{q}_\mathbb{R}$). (In general, $\mathbf{p}_\mathsf{A} = \mathbf{q}_\mathsf{B}$ may occur only when A and B have at most the zero objects in common.)

(4) If A is a connectedness in Top and if B is the corresponding disconnectedness (in the sense of [1]), then $\mathbf{p}_\mathsf{A} = \mathbf{q}_\mathsf{B}$. In particular, we denote by $\mathbf{p}$ the idempotent radical of Top_* obtained by the connected component of the base point (see also [11]).

(5) Every preradical $\mathbf{r}$ of the semi-abelian category Grp of groups may be naturally lifted to a preradical of the $(\mathcal{E},\mathcal{M})$-normal category TopGrp (see 2.6), by just regarding $\mathbf{r}G$ as a subspace of the topological space G, for a given topological group G. Surprisingly, essentially every preradical $\mathbf{r}$ of Top_* may also be lifted to TopGrp, that is: for a topological group G one can expect $\mathbf{r}G$ to be not just a subspace of G but a subgroup. In fact, this is true for any of the examples mentioned in (2), (3) and (4). (A more precise discussion of this phenomenon follows in 5.11 below.) Specifically, the connected component $\mathbf{p}G$ of the neutral element of the topological group G defines an idempotent radical of TopGrp. Likewise, $\mathbf{q}G = \mathbf{q}_D(G, e_G)$ defines a radical $\mathbf{q}$ of TopGrp that, also in this category, fails to be idempotent: for every ordinal α one can find a topological group G_α such that the transfinite iterations $\mathbf{q}G_\alpha$, $\mathbf{q}(\mathbf{q}G_\alpha)$, etc., form a chain of subgroups of G_α of length α ([13]).

It is interesting to note that, when G is a compact abelian Hausdorff group, the topologically defined subgroup $\mathbf{p}G$ coincides with the algebraically defined maximally divisible subgroup $\mathbf{d}G$ (cf. [12, Example 4.1]).

(6) Let us now apply the principle described in (2) to subclasses B of TopGrp directly. For example (see [14]), for Z the class of zero-dimensional groups (so that they have a base of clopen neighbourhoods), we obtain the radical $\mathbf{z} = \mathbf{q}_{\mathsf{Z}}$ which, like $\mathbf{q} = \mathbf{q}_D$, is not idempotent (for the same reason as $\mathbf{q}$). Another non-idempotent radical is obtained by considering the class D of discrete groups. Then $\mathbf{o}G = \mathbf{q}_{\mathsf{D}}G$ is the intersection of all open normal subgroups of G. There is a chain of inclusions

$$\mathbf{p}G \subseteq \mathbf{q}G \subseteq \mathbf{z}G \subseteq \mathbf{o}G,$$

each of which may be proper. The inequality $\mathbf{p} \neq \mathbf{q}$ follows from the fact that $\mathbf{p}$ is idempotent while $\mathbf{q}$ is not. The highly non-trivial fact that $\mathbf{q} \neq \mathbf{z}$ was established by Megrelishvili [25]. Answering a question of Arhangel′skij, he gave an example of a totally disconnected group G (so, $\mathbf{q}G = \{e\}$) that admits no coarser Hausdorff zero-dimensional group topology (so, $\mathbf{z}G \neq \{e\}$). Finally, the properness of the last inclusion is witnessed by the subgroup $G = \mathbb{Q}/\mathbb{Z}$ of all torsion elements of the circle group (it is obviously zero-dimensional, but has no proper clopen subgroup, so $\mathbf{z}G = \{e\}$, while $\mathbf{o}G = G$).

(7) Here is an important classical example which, again, arises by the scheme of (6). The class $\mathsf{CompGrp}$ of compact Hausdorff groups is a reflective subcategory of TopGrp. For every topological group G the reflection $\varrho_G : G \to bG$ has dense-image and is the *Bohr compactification* of G. The $\mathsf{CompGrp}$-radical is known as *von Neumann's kernel* and usually denoted by $\mathbf{n}$. According to von Neumann [26], the groups of $\mathsf{F}_{\mathbf{n}}$ are called *maximally almost periodic* (briefly, MAP) while the groups of $\mathsf{T}_{\mathbf{n}}$ are called *minimally almost periodic* (briefly, MinAP). The radical $\mathbf{n}$ is neither idempotent ([24]), nor cohereditary (there exists a MAP group G with a non-trivial Hausdorff quotient G/N that is MinAP [2]). More about this radical can be found in [16].

4. Torsion theories

Definition 4.1. [9] A *torsion theory* of C is a pair (T, F) of full replete subcategories of C such that:

(1) for all $Y \in \mathsf{T}$ and $Z \in \mathsf{F}$, every morphism $f : Y \to Z$ is zero;

(2) for every object $X \in \mathsf{C}$ there exists a short exact sequence

$$0 \longrightarrow Y \overset{k}{\rightarrowtail} X \overset{p}{\twoheadrightarrow} Z \longrightarrow 0$$

(so that $k = \ker p$ and $p = \operatorname{coker} k$) with $Y \in \mathsf{T}$ and $Z \in \mathsf{F}$.

T is the *torsion* part of the theory, and F is its *torsion-free* part. Any full replete subcategory of C is called *torsion* (*torsion-free*) if it is the torsion (torsion-free, respectively) part of a torsion theory.

One proves easily:

Theorem 4.2. (T, F) *is a torsion theory of* C *if, and only if, there exists a (uniquely determined) idempotent radical* $\mathbf{r}$ *of* C *with* $\mathsf{T} = \mathsf{T}_{\mathbf{r}}$ *and* $\mathsf{F} = \mathsf{F}_{\mathbf{r}}$.

Proof: For an idempotent radical $\mathbf{r}$ and any object X, we have the short exact sequence

$$0 \longrightarrow \mathbf{r}X \rightarrowtail X \twoheadrightarrow X/\mathbf{r}X = RX \longrightarrow 0$$

with $\mathbf{r}X \in \mathsf{T}_{\mathbf{r}}$ and $RX \in \mathsf{F}_{\mathbf{r}}$. Furthermore, because of the commutativity of

$$\begin{tikzcd} \mathbf{r}Y \arrow[r] \arrow[d] & \mathbf{r}Z \arrow[d] \\ Y \arrow[r, "h"'] & Z \end{tikzcd}$$

any morphism h must be 0 when $Y \in \mathsf{T}_{\mathbf{r}}$ and $Z \in \mathsf{F}_{\mathbf{r}}$, i.e., when $\mathbf{r}Y = Y$ and $\mathbf{r}Z = 0$.

Conversely, given a torsion theory (T, F), let us first point out that Y, Z of 4.1(2) depend functorially on X. Indeed, given any morphism $f : X \to X'$, consider

$$\begin{tikzcd} 0 \arrow[r] & Y \arrow[r, tail] & X \arrow[r, two heads] \arrow[d, "f"'] & Z \arrow[r] & 0 \\ 0 \arrow[r] & Y' \arrow[r, tail] & X' \arrow[r, two heads] & Z' \arrow[r] & 0 \end{tikzcd}$$

with both rows short exact, $Y' \in \mathsf{T}$ and $Z' \in \mathsf{F}$. Then, since any arrow $Y \to Z'$ is 0 we have the fill-in arrows $Y \to Y'$ and $Z \to Z'$. In particular, given X we can put $\mathbf{r}X = Y$ and $RX = X/\mathbf{r}X \cong Z$ to obtain a preradical $\mathbf{r}$. Clearly, $\mathsf{T} = \mathsf{T}_{\mathbf{r}}$ since, for $X \in \mathsf{T}$, because $RX \in \mathsf{F}$, the morphism ϱ_X is 0, so that $\mathbf{r}X = \ker 0 = X$; conversely, when $X = \mathbf{r}X$ one has $X \in \mathsf{T}$ since $\mathbf{r}X \in \mathsf{T}$. Analogously, $\mathsf{F} = \mathsf{F}_{\mathbf{r}}$. These identities tell us also that $\mathbf{r}$ is an idempotent radical. Finally, as the coreflector of T, $\mathbf{r}$ is uniquely determined by T. ∎

Remark 4.3. For every preradical $\mathbf{r}$ of C, the pair $(\mathsf{T}_{\mathbf{r}}, \mathsf{F}_{\mathbf{r}})$ satisfies (1) of 4.1, but need not be a torsion theory. However, if it is, then $\mathbf{r}$ is necessarily an idempotent radical, by the uniqueness part of Theorem 4.2.

With the initial considerations of Section 3 one obtains immediately:

Corollary 4.4. *In a torsion theory, the torsion part and the torsion-free part determine each other uniquely. A full replete subcategory is torsion if, and only if, it is normal-monocoreflective such that the coreflector is a radical; it is torsion-free if, and only if, it is normal-epireflective such that the (pre)radical given by the kernels of its reflections is idempotent.*

Corollary 4.5. *The following conditions are equivalent for a pair* (T, F) *of full replete subcategories of* C *satisfying condition* 4.1(2)*:*

(i) (T, F) *is a torsion theory;*
(ii) $\mathsf{T} \cap \mathsf{F} = \{0\}$*,* T *is closed under* $\mathcal{E}$*-images, and* F *is closed under* $\mathcal{M}$*-subobjects.*
(iii) $\mathsf{T} \cap \mathsf{F} = \{0\}$*, and for all morphisms* $f : X \to Y$ *one has*

$$(f \textit{ normal epi } \& \ X \in \mathsf{T} \Rightarrow Y \in \mathsf{T}), \quad (\ker f = 0 \ \& \ Y \in \mathsf{F} \Rightarrow X \in \mathsf{F}).$$

Proof: Assuming (i), we have $(\mathsf{T}, \mathsf{F}) = (\mathsf{T}_{\mathbf{r}}, \mathsf{F}_{\mathbf{r}})$ for an idempotent radical $\mathbf{r}$. Clearly, $\mathsf{T}_{\mathbf{r}} \cap \mathsf{F}_{\mathbf{r}} = \{0\}$, and (ii) follows from 3.4(1). Now only the last assertion of (iii) needs proof; but the diagram

$$\begin{tikzcd} \mathbf{r}X \arrow[r] \arrow[d] & \mathbf{r}Y \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

shows that, when $\mathbf{r}Y = 0$, $\mathbf{r}X$ factors through $\ker f$, hence $\mathbf{r}X = 0$ when $\ker f = 0$.

Conversely, assuming (ii) or (iii), in order to show that any $f : X \to Y$ with $X \in \mathsf{T}$ and $Y \in \mathsf{F}$ is 0, factor f $(\mathcal{E}, \mathcal{M})$ or $(\mathcal{N}ormEpi, 0\text{-}\mathcal{K}er)$ (see 2.1), respectively. Then the factoring object must be 0 by hypothesis, in each of the two cases. ∎

We now prove that closure under extensions (see 3.4(2)) is characteristic for both torsion-free and torsion subcategories.

Theorem 4.6. (1) *Let* C *satisfy* (QN) *and* (IN). *Then a normal-epireflective subcategory* F *of* C *is torsion-free if, and only if, it is closed under extensions and, for its induced radical* $\mathbf{r}$, $\mathbf{rr}X$ *is normal in* X, *for all objects* X.

(2) *Let* C *satisfy* (PN). *Then a normal-monocoreflective subcategory* T *of* C *is torsion if, and only if, it is closed under extensions.*

Proof: (1) It suffices to show that a radical $\mathbf{r}$ is idempotent when $\mathsf{F}_{\mathbf{r}}$ is closed under extensions and $\mathbf{rr}X$ is normal in X, for all $X \in \mathsf{C}$. The left square of

$$\begin{tikzcd} \mathbf{r}X \arrow[r, "f", tail] \arrow[d, "p"', two heads] & X \arrow[r, "1"] \arrow[d, "q", two heads] & X \arrow[d, "\varrho_X", two heads] \\ R\mathbf{r}X = \mathbf{r}X/\mathbf{rr}X \arrow[r, "g"] & X/\mathbf{rr}X \arrow[r, "h"] & X/\mathbf{r}X = RX \end{tikzcd}$$

is a bis for all objects X; in fact, $\ker p = \mathbf{rr}X = \ker q$. By (QN), with $f \in \mathcal{M}$ we have $g \in \mathcal{M}$, and by (IN) we know that g is even a normal monomorphism. Since $hq = \varrho_X$ is a normal epimorphism, h is also one, with kernel g:

$$\ker h = h^{-1}(0) = q(q^{-1}(h^{-1}(0))) = q(\varrho_X^{-1}(0)) = q(\mathbf{r}X) = g.$$

Consequently, with $R\mathbf{r}X$ and $RX \in \mathsf{F}_{\mathbf{r}}$, also $X/\mathbf{rr}X \in \mathsf{F}_{\mathbf{r}}$, that is: $\mathbf{r}(X/\mathbf{rr}X) = 0$. The commutative diagram

$$\begin{tikzcd} \mathbf{r}X \arrow[r] \arrow[d, tail] & \mathbf{r}(X/\mathbf{rr}X) = 0 \arrow[d, tail] \\ X \arrow[r, "q"] & X/\mathbf{rr}X \end{tikzcd}$$

now shows $\mathbf{r}X \leq \ker q = \mathbf{rr}X$, as desired.

(2) It suffices to show that an idempotent preradical $\mathbf{r}$ is a radical when $\mathsf{T}_{\mathbf{r}}$ is closed under extensions. With $RX = X/\mathbf{r}X$ and $\varrho_X : X \to RX$ the projection, we form the pullback

$$\begin{tikzcd} \varrho_X^{-1}(\mathbf{r}RX) \arrow[r, tail] \arrow[d, "p"'] & X \arrow[d, "\varrho_X"] \\ \mathbf{r}RX \arrow[r, tail] & RX \end{tikzcd}$$

(In the terminology of 5.3 below, $\varrho_X^{-1}(\mathbf{r}RX) = \max_X^{\mathbf{r}}(\mathbf{r}X)$.) By (PN), p is a normal epimorphism, and by Prop. 2.2,

$$\ker p = \ker \varrho_X = \mathbf{r}X.$$

Hence, we may apply the hypothesis to the short exact sequence

$$0 \longrightarrow \mathbf{r}X \rightarrowtail \varrho_X^{-1}(\mathbf{r}RX) \twoheadrightarrow \mathbf{r}RX \longrightarrow 0$$

and obtain $\varrho_X^{-1}(\mathbf{r}RX) \in \mathsf{T}_{\mathbf{r}}$. Consequently, we have the commutative diagram

$$\begin{tikzcd}
\varrho_X^{-1}(\mathbf{r}RX) \arrow[r, tail] \arrow[d, equal] & \mathbf{r}X \arrow[d] \\
\varrho_X^{-1}(\mathbf{r}RX) \arrow[r, tail] & X
\end{tikzcd}$$

that is: $\varrho_X^{-1}(\mathbf{r}RX) \leq \mathbf{r}X$, and trivially $\mathbf{r}X \leq \varrho_X^{-1}(\mathbf{r}RX)$ (from the pullback property). This fact easily implies $RX \in \mathsf{F}_{\mathbf{r}}$ (as desired), as we shall show explicitly, and more generally, in 5.3 below: since $\mathbf{r}X$ is $\max^{\mathbf{r}}$-closed, $RX = X/\mathbf{r}X \in \mathsf{F}_{\mathbf{r}}$. ■

Remark 4.7. (1) The additional condition in 4.6(1) that $\mathbf{rr}X$ be normal in X (i.e., that the composite

$$\mathbf{rr}X \rightarrowtail \mathbf{r}X \rightarrowtail X$$

of normal monomorphisms be normal again) is, of course, satisfied in categories of modules where each subobject is normal. But it also holds in Grp (since, generally, $\mathbf{rr}X$ is a fully invariant subobject of X), and therefore in TopGrp. But we do not know whether the condition is redundant in a more general context, in all semi-abelian categories, in all categories of models in Top for a semi-abelian theory?

(2) One may wonder why no extra condition is needed in 4.6(2): simply because normal epimorphisms are, unlike normal monomorphisms, always closed under composition in C. The proof refers to this property implicitly since $\varrho_X^{-1}(\mathbf{r}RX)$ is the kernel of the composite morphism

$$X \twoheadrightarrow RX \twoheadrightarrow RRX,$$

which makes the intrinsic duality of the two proofs more apparent. Also note that, instead of the conjunction of (QN) and (IN), in 4.6(1) it would suffice to require the precise categorical dual of (PN′), namely that normal monomorphisms be stable under pushout along $\mathcal{E}$-morphisms (or just along normal epimorphisms).

(3) Without any additional hypotheses on C one can easily show that a normal-epireflective subcategory F of C is torsion-free if, and only if, it is

closed under extensions, and the full image in C of its induced radical functor $\mathbf{r}$ is closed under (normal) epimorphisms. For C homological, this criterion was proved in [9].

We say that a full reflective subcategory B of C *has stable reflections* if, for all objects Y and all $f : X \to RY$ with $X \in \mathsf{B}$, the morphism e in the $(\mathcal{N}ormEpi, 0\text{-}\mathcal{K}er)$-factorization $m \cdot e$ of the pullback of the B-reflection $\varrho_Y : Y \to RY$ is also a B-reflection (of $X \times_{RX} Y$). Under condition (CT), this property turns out to be characteristic for torsion-free subcategories.

Theorem 4.8. *Let* C *satisfy condition* (CT). *Then a normal-epireflective subcategory is torsion-free if, and only if, it has stable reflections.*

Proof: Let us first assume that F is torsion-free, hence $\mathsf{F} = \mathsf{F}_{\mathbf{r}}$ for an idempotent radical $\mathbf{r}$. We consider a pullback diagram

$$\begin{tikzcd} P \arrow[r, "f'"] \arrow[d, "p"] & Y \arrow[d, "\varrho_Y"] \\ X \arrow[r, "f"] & RY \end{tikzcd}$$

with $X \in \mathsf{F}$. Since $(\mathbf{r}P \to P \xrightarrow{p} X)$ factors through $\mathbf{r}X = 0$, there is a morphism $v : RP \to X$ with $v\varrho_P = p$. By the pullback property, there is a unique k making the diagram

$$\begin{tikzcd} \mathbf{r}Y \arrow[r, "k"] \arrow[d] & P \arrow[r, "f'"] \arrow[d, "p"] & Y \arrow[d, "\varrho_Y"] \\ 0 \arrow[r] & X \arrow[r, "f"] & RY \end{tikzcd}$$

commutative, with $f'k = (\mathbf{r}Y \rightarrowtail Y)$. Since the right and the whole rectangle are pullbacks, so is the left, that is: $\mathbf{r}Y$ is the kernel of p. The morphism v gives $\mathbf{r}P \leq \mathbf{r}Y$, and in order to show equality, by (CT) it suffices to show $\varrho_Y(\mathbf{r}P) = 0$; but that follows trivially from the commutativity of

$$\begin{tikzcd} \mathbf{r}Y \arrow[r, "k"] \arrow[d, "\varrho_{\mathbf{r}Y}"'] & P \arrow[d, "\varrho_P"] \\ 0 = R\mathbf{r}Y \arrow[r, "Rk"] & RP \end{tikzcd}$$

(the identity $0 = R\mathbf{r}Y$ is due to the idempotency of $\mathbf{r}$). Since $k = \ker p = \mathbf{r}Y = \mathbf{r}P$, ϱ_P coincides with the normal-epi component in the $(\mathcal{N}ormEpi, 0\text{-}\mathcal{K}er)$-factorization of p. This proves that F has stable reflections.

Conversely, let us consider a normal-epireflective subcategory F with stable reflection ϱ; we may assume $\mathsf{F} = \mathsf{F}_{\mathbf{r}}$ for a radical $\mathbf{r}$ and must show idempotency of $\mathbf{r}$. For every object X, since $\mathbf{r}X = \ker \varrho_X$, we have the pullback diagram

$$\begin{tikzcd} \mathbf{r}X \arrow[r] \arrow[d] & X \arrow[d, "\varrho_X"] \\ 0 \arrow[r] & RX \end{tikzcd}$$

Since $0 \in \mathsf{F}_{\mathbf{r}}$ and $\mathbf{r}X \to 0$ is a normal epimorphism, the hypothesis of stability gives $R\mathbf{r}X = 0$, so that $\mathbf{r}(\mathbf{r}X) = \ker(\mathbf{r}X \to R\mathbf{r}X) = \ker(\mathbf{r}X \to 0) = \mathbf{r}X$. ∎

Following [9] we say that a full replete subcategory B of C is a *fibred reflection* if for all $f : X \to RY$ with $X \in \mathsf{B}$ and $\varrho_Y : Y \to RY$ the B-reflection of some C-object Y, the pullback of ϱ_Y along f is also a B-reflection. Clearly, this property is stronger than having stable reflections, but if $\mathcal{N}ormEpi$ is pullback-stable (in particular, in homological categories), these properties coincide. As an immediate corollary of the above theorem we obtain the following fact, proved in [9, Theorem 4.11] for homological categories.

Corollary 4.9. *Let* C *satisfy condition* (CT)*, and let normal epimorphisms be stable under pullback in* C*. Then a normal-epireflective subcategory is torsion-free if, and only if, it is a fibred reflection.*

The following example shows that the above corollary fails to be true without the assumption of pullback-stability, and it justifies our definition of having stable reflections as used in Theorem 4.8. Consider the idempotent radical $\mathbf{p}$ of Top_* defined in Example 3.9(4). Then the torsion-free category $\mathsf{F}_{\mathbf{c}}$ fails to be a fibred reflection. For a counter-example take $Y = ([0,1], 0)$ and $X = (\mathbb{Q}, 0)$, then $\mathbf{c}Y = Y$, so $RY = 0$ and $f : X \to 0$. Consequently, the pullback of $\varrho_Y : Y \to 0$ is nothing else but the projection $X \times Y \to X$. Obviously, it is not a reflection map.

Definition 4.10. A torsion theory (T, F) is $\mathcal{M}$-*hereditary* if T is closed under $\mathcal{M}$-subobjects, and it is $\mathcal{E}$-*cohereditary* if F is closed under $\mathcal{E}$-images.

¿From 3.3 and 4.2 we have:

Corollary 4.11. *A torsion theory* (T, F) *in* C *is* $\mathcal{M}$*-hereditary (*$\mathcal{E}$*-cohereditary) if, and only if,* T *is an* $\mathcal{M}$*-co-Birkhoff (*F *is an* $\mathcal{E}$*-Birkhoff) subcategory of* C*, and (if* C *satisfies* (CT) *and* (IN)*) it is therefore equivalently described by an* $\mathcal{M}$*-hereditary (idempotent* $\mathcal{E}$*-cohereditary) radical of* C*.*

5. Normal closure operators

Definition 5.1. A *normal closure operator* $c = (c_X)_{X \in \mathsf{C}}$ of C assigns to every normal subobject $N \rightarrowtriangle X$ a normal subobject $c_X(N)$ of X such that, for every object X,

- c_X is *extensive*: $N \le c_X(N)$ for all N;
- c_X is *monotone*: $N \le K \Rightarrow c_X(N) \le c_X(K)$;
- the *continuity condition* is satisfied: $c_X(f^{-1}(L)) \le f^{-1}(c_Y(L))$ for all $f : X \to Y$ and normal subobjects L of Y.

The normal subobject N of X is *c-closed* in X if $c_X(N) = N$, and *c-dense* in X if $c_X(N) = X$. One calls c

- *idempotent*, if $c_X(N)$ is c-closed in X for all $N \rightarrowtriangle X$;
- *weakly hereditary*, if N is c-dense in $c_X(N)$ for all $N \rightarrowtriangle X$.

Remarks 5.2. (1) A *closure operator* c of C can be defined just like a normal closure operator, except that c_X acts on all $\mathcal{M}$-subobjects of X (see [17, 18]). However, such a closure operator very often maps normal subobjects to normal subobjects, i.e., yields a normal closure operator. For example this property was observed in [18, Exercise 4V] for every closure operator of the category of groups; in fact, it holds for every closure operator of a semi-abelian variety of universal algebras (cp. [6, 7]).

If, in our category C, every $\mathcal{M}$-subobject $M \to X$ has a *normal closure* $\nu_X(M)$ in X, i.e., a least normal subobject containing M, then every normal closure operator c is the restriction to normal subobjects of a closure operator $d \ge \nu$ which maps normal subobjects to normal subobjects: simply put $d := c\nu$, i.e., $d_X(M) := c_X(\nu_X(M))$. The existence of a normal closure of every $\mathcal{M}$-subobject is guaranteed if every $\mathcal{M}$-subobject has a cokernel; then $\nu_X(M)$ is simply the kernel of $\mathrm{coker}(M \to X)$. In particular, the normal closure exists when C is finitely cocomplete.

(2) Every normal closure operator c of C induces a normal preradical of C, namely

$$\mathrm{rad}^c X := c_X(0),$$

defining in fact a functor of preordered classes:

$$\mathrm{rad} : \{\text{normal closure operators}\} \longrightarrow \{\text{(normal) preradicals}\}.$$

Given a preradical $\mathbf{r}$, there is a largest normal closure operator c with $\mathrm{rad}^c = \mathbf{r}$, namely $c = \max^{\mathbf{r}}$, defined by

$$\max{}^{\mathbf{r}}_X(N) = p^{-1}(\mathbf{r}(X/N)) \text{ for all } \; N \rightarrowtail X \; ;$$

here $p : X \to X/N$ is the projection (see [17, 18]).

If (JN) is satisfied in C (see Section 2), then for every preradical $\mathbf{r}$ there is also a least normal closure operator c with $\mathrm{rad}^c = \mathbf{r}$, namely $c = \min^{\mathbf{r}}$, defined by

$$\min{}^{\mathbf{r}}_X(N) = \mathbf{r}X \vee N \;\text{ for all }\; N \rightarrowtail X \; .$$

One calls $\max^{\mathbf{r}}$ and $\min^{\mathbf{r}}$ the *maximal* and *minimal* (normal) closure operator induced by $\mathbf{r}$, respectively, and easily verifies that for any normal closure operator c,

$$(c \leq \max{}^{\mathbf{r}} \Leftrightarrow \mathrm{rad}^c \leq \mathbf{r}) \;\text{ and }\; (\min{}^{\mathbf{r}} \leq c \Leftrightarrow \mathbf{r} \leq \mathrm{rad}^c).$$

In other words, the functor rad has both adjoints:

$$\begin{tikzcd}[column sep=huge] \{\text{normal closure operators}\} \arrow[r, "\mathrm{rad}" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \{\text{ (normal) preradicals}\} \arrow[l, shift right=4, "\min"'] \arrow[l, shift left=4, "\max"'] \end{tikzcd}$$

Obviously, in the presence of (JN), $N \rightarrowtail X$ is $\min^{\mathbf{r}}$-closed if, and only if, $\mathbf{r}X \leq N$. Furthermore, $\min^{\mathbf{r}}$ is idempotent, and it is weakly hereditary if and only if $\mathbf{r}$ is idempotent. The corresponding statements for $\max^{\mathbf{r}}$ are more involved:

Proposition 5.3. *Let* $\mathbf{r}$ *be a preradical of* C.

(1) *A normal subobject* $N \rightarrowtail X$ *is* $\max^{\mathbf{r}}$*-closed* ($\max^{\mathbf{r}}$*-dense) if and only if* X/N *is* $\mathbf{r}$*-torsion-free* ($\mathbf{r}$*-torsion, respectively).*
(2) *For every normal subobject* $N \rightarrowtail X$ *one has the isomorphism*

$$(X/N)/\mathbf{r}(X/N) \cong X/\max{}^{\mathbf{r}}_X(N)$$

and, under condition (PN), $\mathbf{r}(X/N) \cong \max^{\mathbf{r}}_X(N)/N$.
(3) $\max^{\mathbf{r}}$ *is idempotent if, and only if,* $\mathbf{r}$ *is a radical, and, under condition* (PN), $\max^{\mathbf{r}}$ *is weakly hereditary if, and only if,* $\mathbf{r}$ *is idempotent.*

Proof: In the diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d] & \max^{\mathbf{r}}_X(N) \arrow[r] \arrow[d] & X \arrow[d, -{Triangle[open]}] \\
0 \arrow[r] & \mathbf{r}(X/N) \arrow[r] & X/N
\end{tikzcd}$$

the whole and the right rectangle are pullbacks (by definition), hence also the left rectangle is one. Consequently, $N = \max^{\mathbf{r}}_X(N)$ if and only if $\mathbf{r}(X/N) = 0$. Moreover, the right pullback diagram shows immediately that $\max^{\mathbf{r}}_X(N) = X$ holds if and only if $\mathbf{r}(X/N) = X/N$.

(2) Extending the above diagram to the right we obtain

$$\begin{tikzcd}
\max^{\mathbf{r}}_X(N) \arrow[r] \arrow[d] & X \arrow[r, -{Triangle[open]}] \arrow[d, -{Triangle[open]}] & X/\max^{\mathbf{r}}_X(N) \arrow[d, -{Triangle[open]}] \\
\mathbf{r}(X/N) \arrow[r] & X/N \arrow[r, -{Triangle[open]}] & (X/N)/\mathbf{r}(X/N)
\end{tikzcd}$$

Since $N \le \max^{\mathbf{r}}_X(N)$ there is a diagonal morphism $X/N \to X/\max^{\mathbf{r}}_X(N)$ for the right rectangle, which is easily seen to induce an inverse for the right vertical arrow.

In order to show the second isomorphism, we complete the defining pullback diagram for $Y = \max^{\mathbf{r}}_X(N)$, as follows:

$$\begin{tikzcd}
Y \arrow[rr] \arrow[dr, -{Triangle[open]}] \arrow[dd, -{Triangle[open]}] & & X \arrow[dd] \\
& Y/N \arrow[dl, -{Triangle[open]}, "t"] \arrow[dr, "s"] & \\
\mathbf{r}(X/N) \arrow[rr] & & X/N
\end{tikzcd}$$

Condition (PN) guarantees that $Y \to \mathbf{r}(X/N)$ is a normal epimorphism, hence, $t : Y/N \to \mathbf{r}(X/N)$ is also a normal epimorphism. In order to show that t is actually an isomorphism, it suffices to prove $\ker t = 0$. But that is obvious since, thanks to 2.1, $Y \to Y/N \to X/N$ is the $(\mathcal{N}ormEpi, 0\text{-}\mathcal{K}er)$-factorization of $Y \rightarrowtail X \to X/N$, hence $\ker s = 0$ and, consequently, $\ker t = 0$.

(3) When $\mathbf{r}$ is a radical, with (2) we have

$$\mathbf{r}(X/\max{}^{\mathbf{r}}_X(N)) \cong \mathbf{r}((X/N)/\mathbf{r}(X/N)) = 0,$$

which means that $\max^{\mathbf{r}}_X(N)$ is $\max^{\mathbf{r}}$-closed in X, for every $N \rightarrowtail X$. Hence $\max^{\mathbf{r}}$ must be idempotent. Conversely, assuming this property, we have that $\mathbf{r}X = \max^{\mathbf{r}}_X(0)$ is $\max^{\mathbf{r}}$-closed, which means $\mathbf{r}(X/\mathbf{r}X) = 0$ by (1). Hence, $\mathbf{r}$ must be a radical.

When $\max^{\mathbf{r}}$ is weakly hereditary, 0 is $\max^{\mathbf{r}}$-dense in $\max^{\mathbf{r}}_X(0) = \mathbf{r}X$. By (1), $\mathbf{r}X \cong \mathbf{r}X/0$ is $\mathbf{r}$-torsion, hence $\mathbf{rr}X = \mathbf{r}X$. Conversely, if r is idempotent, we must show that the normal subobject N of X is $\max^{\mathbf{r}}$-dense in $Y = \max^{\mathbf{r}}(N)$, that is: $Y/N \in \mathsf{F}_{\mathbf{r}}$. But we have $Y/N \cong \mathbf{r}(X/N)$ by (2), so that $\mathbf{r}(Y/N) = Y/N$ follows from the idempotency of $\mathbf{r}$. ■

Remarks 5.4. (1) We note that in 5.3 only a weak form of (PN) is involved, namely that normal epimorphisms be stable under pullback along normal monomorphisms.

(2) For a normal closure operator c of C, a morphism $f : X \to Y$ is called *c-open* if $f^{-1}(c_Y(N)) = c_X(f^{-1}(N))$ for all normal subobjects $N \rightarrowtail Y$ (see [18]). In the presence of (IN) we call f *c-closed* if $f(c_X(K)) = c_Y(f(K))$ for every normal subobject $K \rightarrowtail X$.

Proposition 5.5. *Let* $\mathbf{r} = \mathrm{rad}^c$ *be the induced preradical of a normal closure operator* c *of* C. *Then:*

(1) *A morphism* $f : X \to Y$ *with trivial kernel can be c-open only if* $f^{-1}(\mathbf{r}Y) = \mathbf{r}X$. *Hence, c-openness of* $\mathcal{M}$*-subobjects implies* $\mathcal{M}$*-heredity of* $\mathbf{r}$.
(2) *c-openness of all normal epimorphisms implies* $c = \max^{\mathbf{r}}$.
(3) *In the presence of* (IN), *a morphism* $f : X \to Y$ *can be c-closed only if* $f(\mathbf{r}X) = \mathbf{r}Y$. *Hence, c-closedness of all* $\mathcal{E}$*-morphisms implies* $\mathcal{E}$*-coheredity of* $\mathbf{r}$.

Proof: (1) $f^{-1}(\mathbf{r}Y) = f^{-1}(c_Y(0)) = c_X(f^{-1}(0)) = c_X(0) = \mathbf{r}X$.
(2) For $N \rightarrowtail X$ and $f : X \to X/N$ the projection, one has

$$c_X(N) = c_X(f^{-1}(0)) = f^{-1}(c_{X/N}(0)) = f^{-1}(\mathbf{r}(X/N)) = \max^{\mathbf{r}}_X(N).$$

(3) $f(\mathbf{r}X) = f(c_X(0)) = c_Y(f(0)) = c_Y(0) = \mathbf{r}Y$. ■

Establishing converse statements takes more effort:

Theorem 5.6. *Let* c *be a normal closure operator of* C *with* $\mathbf{r} = \mathrm{rad}^c$. *Then*

(1) $c = \max^{\mathbf{r}}$ *if and only if all normal epimorphisms are c-open.*

(2) *All morphisms of* C *are* $\max^{\mathbf{r}}$*-open if and only if* $f^{-1}(\mathbf{r}Y) = \mathbf{r}X$ *for all morphisms* f *with trivial kernel.*

Proof: For every morphism $f : X \to Y$ and $N \rightarrowtail X$ we have the commutative diagram

$$\begin{tikzcd}
 & \mathbf{r}(X/f^{-1}(N)) \arrow[rr] \arrow[dd] & & \mathbf{r}(Y/N) \arrow[dd] \\
\max^{\mathbf{r}}_X(f^{-1}(N)) \arrow[ur, "p'"] \arrow[rr] \arrow[dd] & & \max^{\mathbf{r}}_Y(N) \arrow[ur, "q'"] \arrow[dd] & \\
 & X/f^{-1}(N) \arrow[rr, "\overline{f}"] & & Y/N \\
X \arrow[ur, "p"] \arrow[rr, "f"] & & Y \arrow[ur, "q"] &
\end{tikzcd}$$

$\max^{\mathbf{r}}$-openness of f means, by definition, that the front face is a pullback, for every $N \rightarrowtail Y$. But since the two side faces are pullbacks, the front face is a pullback whenever the back face is one.

Now, for the morphism $\overline{f}$, let us first note that $\ker\overline{f} = p(p^{-1}(\ker\overline{f})) = p(f^{-1}(N)) = 0$. To complete the proof of (1), after 5.5(2), we can let f be a normal epimorphism. Then also $\overline{f}$ is one, hence, in fact an isomorphism, making the back face of the above cube a trivial pullback. To complete the proof of (2), after 5.5(1), we just point out that, since $\ker\overline{f} = 0$, the back face of the cube becomes a pullback by hypothesis. ■

Normal closure operators satisfying the condition of 5.6(1) were called homological in [9], but from the perspective of preradicals, maximal seems more appropriate. Hence, c is maximal precisely when $c = \max^{\mathrm{rad}^c}$.

Corollary 5.7. *Let* $\mathbf{r}$ *be a preradical, and let* C *satisfy* (JN). *Then* $\min^{\mathbf{r}} = \max^{\mathbf{r}}$ *(so that there is exactly one normal closure operator inducing* $\mathbf{r}$*) if and only if* $f^{-1}(N \vee \mathbf{r}Y) = f^{-1}(N) \vee \mathbf{r}X$ *for all normal epimorphisms* $f : X \to Y$ *and for all* $N \rightarrowtail X$ *.*

Proof: The condition just means that all normal epimorphisms are $\min^{\mathbf{r}}$-open, and the assertion follows from 5.6(1). ■

Corollary 5.8. *Assume that* C *satisfies* (JN) *and that, for every* $f : X \to Y$ *in* $\mathcal{M}$ *and all* $N_1, N_2 \rightarrowtail Y$ *,* $f^{-1}(N_1 \vee N_2) = f^{-1}(N_1) \vee f^{-1}(N_2)$*. Then all* $f \in \mathcal{M}$ *are* $\min^{\mathbf{r}}$*-open if and only if* $\mathbf{r}$ *is* $\mathcal{M}$*-hereditary.*

Proof: Apply 5.5(1), with $c = \min^{\mathbf{r}}$. ■

Unfortunately, the assumption that $f^{-1}(-)$ preserve the join of normal subobjects is quite restrictive. Again, it seems more natural to consider $\max^{\mathbf{r}}$ rather than $\min^{\mathbf{r}}$. In order to do so, one calls a normal closure operator c *$\mathcal{M}$-hereditary* if, for every $f : X \to Y$ in $\mathcal{M}$ and every normal subobject $N \rightarrowtail Y$ with $N \leq X$ (so that $(f^{-1}(N) = N)$, one has $c_X(N) = f^{-1}(c_Y(N))$. Obviously, $\mathcal{M}$-heredity makes c weakly hereditary (for $K \rightarrowtail X$, consider $f : c_X(K) \to X$).

Theorem 5.9. *Of the following statements for a preradical* $\mathbf{r}$*, each one implies the next. Conditions* (i)$-$(iv) *are equivalent if* C *satisfies* (QN)*, and all are equivalent if every morphism with trivial kernel lies in* $\mathcal{M}$*.*

(i) *Every morphism in* $\mathcal{M}$ *is* $\max^{\mathbf{r}}$*-open.*
(ii) $\max^{\mathbf{r}}$ *is* $\mathcal{M}$*-hereditary.*
(iii) *There is an* $\mathcal{M}$*-hereditary normal closure operator* c *with* $\mathrm{rad}^c = \mathbf{r}$*.*
(iv) $\mathbf{r}$ *is* $\mathcal{M}$*-hereditary.*

Proof: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is trivial, and so is (iii) $\Rightarrow$ (iv):

$$\mathbf{r}X = c_X(0) = f^{-1}(c_Y(0)) = f^{-1}(\mathbf{r}Y).$$

For (iv) $\Rightarrow$ (ii) consider the cube of the proof of 5.6 again. Under (QN), $f \in \mathcal{M}$ implies $\overline{f} \in \mathcal{M}$ when $N \leq X$, so that the back face of that cube is a pullback diagram by hypothesis, and so is the front face.

(ii) $\Rightarrow$ (i) for $\mathcal{M} = 0$-$\mathcal{K}er$ was already stated and proved in 5.5(2). ■

Theorem 5.9 can be "dualized", as follows:

Theorem 5.10. *For a preradical* $\mathbf{r}$*, the following statements are equivalent when* C *satisfies* (IN) *and* (JN)*:*

(i) $\mathbf{r}$ *is* $\mathcal{E}$*-cohereditary;*
(ii) *every* $\mathcal{E}$*-morphism is* $\min^{\mathbf{r}}$*-closed;*
(iii) *every* $\mathcal{E}$*-morphism is* c*-closed, for some normal closure operator with* $\mathrm{rad}^c = \mathbf{r}$*.*

When, moreover, C *satisfies* (CT)*, these conditions imply* $\min^{\mathbf{r}} = \max^{\mathbf{r}}$*, so that all* $\mathcal{E}$*-morphisms are* c*-closed* (*for the only normal closure operator* c *with* $\mathrm{rad}^c = \mathbf{r}$)*; furthermore, normal epimorphisms are also* c*-open and satisfy the condition given in* 5.7*.*

Proof: (i) $\Rightarrow$ (ii) Using (IN) and (JN), for every $f : X \to Y$ in $\mathcal{E}$ and $K \rightarrowtriangle X$ we have

$$f(\min^{\mathbf{r}}_X(K)) = f(K \vee \mathbf{r}X) = f(K) \vee f(\mathbf{r}X) = f(K) \vee \mathbf{r}Y = \min{}^{\mathbf{r}}_Y(f(K)).$$

(ii) $\Rightarrow$ (iii) is trivial, and for (iii) $\Rightarrow$ (i) use 5.5(3).

In order to show that (ii) implies $\min^{\mathbf{r}}_X(K) = \max^{\mathbf{r}}_X(K)$ for all $K \rightarrowtriangle X$, we note that, with f denoting the projection $X \to X/K$, one has

$$f(\min^{\mathbf{r}}_X(K)) = \min^{\mathbf{r}}_{X/N}(f(K)) = \min^{\mathbf{r}}_{X/N}(0) = \mathbf{r}(X/N)$$

and $f(\max{}^{\mathbf{r}}_X(N)) \leq \mathbf{r}(X/N)$. Since $K = \ker f \leq \min^{\mathbf{r}}_X K \leq \max^{\mathbf{r}}_X K$ in the presence of (CT), this shows $\min^{\mathbf{r}}_X(K) = \max^{\mathbf{r}}_X(K)$.

The remaining claims follow from 5.6(1) and 5.7. ■

We return to the issue of lifting preradicals from Top_* to TopGrp (see 3.9(5)). For a preradical $\mathbf{r}$ of Top_*, we can think of $\mathbf{r}(X, x)$ as of the closure of the point x in the space X. In fact, writing

$$c_X(x) = \mathbf{r}(X, x),$$

one defines a fully additive closure operator $c = c^{\mathbf{r}}$ of Top (see 5.2(1)), with full additivity referring to the property $c_X(\bigcup\limits_i M_i) = \bigcup\limits_i c_X(M_i)$ for all families of subsets $M_i \subseteq X$. In this way preradicals of Top_* correspond bijectively to fully additive closure operators of Top.

When can we "lift" the preradical $\mathbf{r}$ from Top_* to TopGrp? That is: when is $\mathbf{r}G = \mathbf{r}(G, e_G)$ a (normal) subgroup of G, for every topological group G?

Proposition 5.11. *For a preradical $\mathbf{r}$ of Top_* and a topological group G, the following conditions are equivalent:*

(i) $\mathbf{r}G$ *is a normal subgroup of* G*;*
(ii) $\mathbf{r}G \cdot \mathbf{r}G \subseteq \mathbf{r}G$*;*
(iii) $c^{\mathbf{r}}_G(c^{\mathbf{r}}_G(\{e_G\})) = c^{\mathbf{r}}_G(\{e_G\})$*;*
(iv) $c^{\mathbf{r}}_G(c^{\mathbf{r}}_G(H)) = c^{\mathbf{r}}_G(H)$*, for every subgroup* H *of* G*.*

If these equivalent conditions are satisfied for all topological groups G, $\mathbf{r}$ can be considered as a preradical of TopGrp and $c^{\mathbf{r}}$ as a normal closure operator of TopGrp, which coincides with the minimal closure operator induced by $\mathbf{r}$.

Proof: Let us note first that, by definition of $c^{\mathbf{r}}$, for all $H \leq G$

$$c^{\mathbf{r}}_G(H) = \bigcup_{a \in H} c^{\mathbf{r}}_G(\{a\}).$$

Considering the Top_*-maps

$$(G,e) \xrightarrow{(-)\cdot a} (G,a) \xrightarrow{(-)\cdot a^{-1}} (G,e)$$

we also see that $c^{\mathbf{r}}_G(\{a\}) = \mathbf{r}G \cdot a$ for all $a \in G$. Hence, $c^{\mathbf{r}}_G(H) = \mathbf{r}G \cdot H = H \cdot \mathbf{r}G$, a formula that shows (i) $\Rightarrow$ (iv) immediately. (iv) $\Rightarrow$ (iii) is trivial. (iii) $\Rightarrow$ (ii):

$$\mathbf{r}G \cdot \mathbf{r}G = c^{\mathbf{r}}_G(\mathbf{r}G) = c^{\mathbf{r}}_G(c^{\mathbf{r}}_G(\{e_G\})) = c^{\mathbf{r}}_G(\{e_G\}) = \mathbf{r}G.$$

(ii) $\Rightarrow$ (i) Since $\mathbf{r}G$ is always invariant under inversion and inner automorphisms, closure under multiplication makes $\mathbf{r}G$ a normal subgroup of G. ∎

Problem 5.12. *Is there a preradical* $\mathbf{r}$ *of* Top_* *and a topological group* G *such that* $\mathbf{r}G$ *is not a subgroup of* G*?*

Remarks 5.13. (1) Prop. 5.11 shows that the answer to 5.12 is negative if we would restrict the search to preradicals $\mathbf{r}$ with $c^{\mathbf{r}}$ idempotent in Top, or to preradicals $\mathbf{r}$ with $\mathbf{r}(G \times G) = \mathbf{r}G \times \mathbf{r}G$ for every topological group G, since the latter condition implies 5.11(ii). (In fact, idempotency of $c^{\mathbf{r}}$ implies preservation of finite products in Top by $c^{\mathbf{r}}$, and therefore by $\mathbf{r}$; see [18, Prop. 4.11].) One easily checks that $c^{\mathbf{r}}$ is idempotent when $\mathbf{r}$ is a radical of Top_*. Consequently, all radicals of Top_* are liftable to TopGrp.

(2) If there is an idempotent preradical $\mathbf{r}$ of Top_* as a witness to a positive answer to 5.12 (so that there is a group G, such that $\mathbf{r}G$ fails to be a subgroup of G), then there is even such witness of the form $\mathbf{s} = \mathbf{p}_D$ for some $D \in \mathsf{Top}_*$ (see 3.9(3)). In fact, if $\mathbf{r}G$ fails to be a subgroup of G, consider the space $D = \mathbf{r}G$ and let $\mathbf{s} = \mathbf{p}_D$. Then, trivially, $D = \mathbf{s}D \hookrightarrow \mathbf{s}G$, and since $D \in \mathsf{T}_{\mathbf{r}}$ (by the idempotency of $\mathbf{r}$), also $\mathbf{s}G \hookrightarrow \mathbf{r}G$. Hence, $\mathbf{s}G = \mathbf{r}G$, which still fails to be a subgroup of G.

6. Summary

6.1 (1) The assignments

$$\begin{array}{ccccccc} \mathsf{F} & \longmapsto & \ker(\text{reflection}), & & \mathbf{r} & \longmapsto & \max{}^{\mathbf{r}} \\ \mathsf{F}_{\mathbf{r}} & \longleftarrow\!\!\!\shortmid & \mathbf{r}, & & \mathrm{rad}^c & \longleftarrow\!\!\!\shortmid & c \end{array}$$

define bijections

$$\{\text{normal epirefl. subcateg.}\} \longleftrightarrow \{\text{radicals}\} \longleftrightarrow \{\text{idpt. maximal nco's}\}$$

(see 3.2 and 5.3); here "nco" stands for "normal closure operator", "subcategory" means "full replete subcategory". In what follows we will also use the abbreviations idpt. = idempotent, wh. = weakly hereditary, hered. = hereditary and cohered. = cohereditary.

(2) Under condition (PN), the bijection (1) restricts to bijections

$$\{\text{torsion-free subcateg.}\} \longleftrightarrow \{\text{idpt. radicals}\} \longleftrightarrow \{\text{wh. idpt. max. nco's}\}$$

(see 4.2 and 5.3). In addition, if (CT) holds, torsion-free subcategories are described as normal-epireflective subcategories with stable reflections (see 4.8), and under conditions (QN) and (IN) there is a criterion involving closure under extensions (see 4.6(1)).

(3) Under conditions (CT), (IN) and (JN), the bijection (1) restricts also to bijections

$$\{\mathcal{E}\text{-Birkhoff subcat.}\} \longleftrightarrow \{\mathcal{E}\text{-cohered.rad.}\} \longleftrightarrow \{\text{idpt.nco's with closed } \mathcal{E}\text{'s}\}$$

(see 3.2 and 5.10).

(4) Under conditions (CT), (IN), (JN) and (PN) (hence, in every $(\mathcal{E}, \mathcal{M})$-normal category) one obtains from (2), (3) the bijections

$$\{\mathcal{E}\text{-cohered.tors.th.}\} \longleftrightarrow \{\mathcal{E}\text{-cohered.idpt.rad.}\} \longleftrightarrow \{\text{wh.idpt.nco's w.closed } \mathcal{E}\}$$

6.2 (1) The assignments

$$\begin{array}{llll} \mathsf{T} \longmapsto \text{coreflection}, & & \mathbf{r} \longmapsto \max{}^{\mathbf{r}} \\ \mathsf{T}_{\mathbf{r}} \longleftarrow\!\shortmid \ \mathbf{r}, & & \mathrm{rad}^{c} \longleftarrow\!\shortmid \ c \end{array}$$

define bijections

$$\{\text{normal monocorefl.subcat.}\} \longleftrightarrow \{\text{idpt.prerad.}\} \longleftrightarrow \{\text{wh.maximal nco's}\}$$

(see 3.2 and 5.3).

(2) The bijection (1) restricts to bijections (under condition (PN))

$$\{\text{torsion subcat.}\} \longleftrightarrow \{\text{idpt. rad.}\} \longleftrightarrow \{\text{wh. idpt. maximal nco's}\}$$

(see 4.2 and 5.3). In addition, if (PN) holds, torsion subcategories are characterized as the normal-monocoreflective subcategories closed under extensions (see 4.6(1)).

(3) Under conditions (PN) and (QN), the bijection (1) restricts also to bijections

$$\{\mathcal{M}\text{-co-Birkhoff subcat.}\} \longleftrightarrow \{\mathcal{M}\text{-hered.prerad.}\} \longleftrightarrow \{\mathcal{M}\text{-hered.max.nco's}\}$$

(see 3.2 and 5.9).

(4) Under conditions (PN) and (QN) (hence, in every $(\mathcal{E}, \mathcal{M})$-seminormal category) one obtains from (2), (3) the bijections

$$\{\mathcal{M}\text{-hered.torsion th.}\} \longleftrightarrow \{\mathcal{M}\text{-hered.rad.}\} \longleftrightarrow \{\mathcal{M}\text{-hered.idpt.max.nco's}\}$$

When every morphism with trivial kernel lies in $\mathcal{M}$ (as in every homological category with its $(\mathcal{R}egEpi, \mathcal{M}ono)$-factorization system), $\mathcal{M}$-hereditary idempotent maximal nco's are simply idempotent nco's for which every morphism is open (see 5.6).

7. Examples

Remark 7.1. In generalization of 3.2, a preradical $\mathbf{r}$ of the category C (as in Section 2) may be obtained from any pointed endofunctor $\sigma : \mathrm{Id}_{\mathsf{C}} \to S$ of C, as $\mathbf{r}X = \ker \sigma_X$. In the particular case $S = \mathrm{Id}_{\mathsf{C}}$, $\mathbf{r}$ is actually $\mathcal{M}$-hereditary:

$$f^{-1}(\mathbf{r}XY) = f^{-1}(\sigma_Y^{-1}(0)) = \sigma_X^{-1}(f^{-1}(0)) = \sigma_X^{-1}(0) = \mathbf{r}X,$$

for all $f : X \to Y$ in $\mathcal{M}$. Dually, any copointed endofunctor $\tau : T \to \mathrm{Id}_{\mathsf{C}}$ gives a preradical R of C via $RX = \ker(\operatorname{coker} \tau_X)$, granted the existence of cokernels. In the particular case $T = \mathrm{Id}_{\mathsf{C}}$, if the induced morphisms $X \to RX$ lie in $\mathcal{E}$, R is $\mathcal{E}$-cohereditary

$$f(RX) = f(\tau_X(X)) = \tau_Y(f(X)) = \tau_Y(Y) = RY.$$

In an abelian category C (with $(\mathcal{E}, \mathcal{M}) = (\mathcal{N}ormEpi, \mathcal{N}ormMono)$), any natural transformation $\sigma : \mathrm{Id}_{\mathsf{C}} \to \mathrm{Id}_{\mathsf{C}}$ gives both, $\mathbf{r}$ and R as above, with $RX = X/\mathbf{r}X$:

$$\begin{tikzcd} & & RX \arrow[dr] & & \\ \mathbf{r}X = \ker \sigma_X \arrow[r] & X \arrow[ur, "\varrho_X"] \arrow[rr, "\sigma_X"] & & X \arrow[r] & \operatorname{coker} \sigma_X \end{tikzcd}$$

Moreover, if C has a generator G, so that the hom-set $\mathsf{C}(G, X)$ is jointly epic, the morphism σ_X is determined by σ_G, since the diagrams

$$\begin{tikzcd} G \arrow[r, "\sigma_G"] \arrow[d, "x"'] & G \arrow[d, "x"] \\ X \arrow[r, "\sigma_X"] & X \end{tikzcd}$$

commute for all $x \in \mathsf{C}(G, X)$.

Example 7.2. In the category AbGrp of abelian groups, with its generator $\mathbb{Z}$, a natural transformation $\sigma : \mathrm{Id} \to \mathrm{Id}$ is determined by $\sigma_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$, i.e., by $m = \sigma_{\mathbb{Z}}(1) \in \mathbb{Z}$: $\sigma_A(x) = mx$ for all $x \in \mathsf{A} \in \mathsf{AbGrp}$. One obtains with 7.1

$$A[m] := \mathbf{r}A = \{x \in A \ : \ mx = 0\},$$
$$mA := RA = \{mx \ : \ x \in A\} \cong A/\mathbf{r}A.$$

$\mathbf{r}$ is a hereditary preradical which, however, fails to be a radical, unless $m = 0$ or $m = \pm 1$, so that $\mathbf{r} = \mathbf{0}$ or $\mathbf{r} = \mathbf{1}$, producing only trivial torsion theories. Likewise, R is a cohereditary radical that is idempotent only for trivial m. In fact, AbGrp *has no non-trivial cohereditary torsion theories* (see 7.3 below).

By contrast, *there are precisely* $2^{\aleph_0}$ *hereditary radicals in* AbGrp. We give a brief indication of the proof (see [20]). For every prime p let $\mathbf{t}_p(A)$ denote the p-primary component of the subgroup of all torsion elements of an abelian group A. It is easy to see that $\mathbf{t}_p$ is a hereditary radical of AbGrp. For every set P of prime numbers, the supremum $\mathbf{t}_P$ of all $\mathbf{t}_p$'s when p runs over P is again a hereditary radical. There are no other hereditary radicals of AbGrp beyond these. Indeed, for any hereditary radical $\mathbf{r}$ of AbGrp, one can completely determine $\mathbf{r}$ by its values on divisible groups (as every abelian group A is a subgroup of some divisible group). Since every divisible group is a direct sum of copies of $\mathbb{Q}$ and the Prüfer groups, the values of $\mathbf{r}$ are determined by $\mathbf{r}\mathbb{Q}$ and $\mathbf{r}(\mathbb{Z}(p^\infty))$. Since $\mathbf{r}A$ is a fully invariant subgroup of $\mathbb{Q}$ (by functoriality), it is either $\mathbb{Q}$ or 0. In the former case, since every Prüfer group $\mathbb{Z}(p^\infty)$ is a quotient of $\mathbb{Q}$, also $\mathbf{r}(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)$, so that $\mathbf{r}$ coincides with the trivial radical $\mathbf{1}$. In case $\mathbf{r}\mathbb{Q} = 0$, we are left with determining the values of $\mathbf{r}(\mathbb{Z}(p^\infty))$. By the radical condition for $\mathbf{r}$, this subgroup can only be either 0 or $\mathbb{Z}(p^\infty)$. With $P = \{p \ : \ \mathbf{r}(\mathbb{Z}(p^\infty)) = \mathbb{Z}(p^\infty)\}$, one obtains $\mathbf{r} = \mathbf{t}_P$.

Example 7.3. For a commutative unital ring S, every ideal $\mathfrak{a}$ of S gives (in generalization of the radical $m(\ )$ of 7.2) a cohereditary radical $\mathbf{r}_{\mathfrak{a}}$ of the category Mod_S of S-modules, namely: $\mathbf{r}_{\mathfrak{a}}M = \mathfrak{a}M = \{ax \ : \ a \in \mathfrak{a},\, x \in M\}$.

Any cohereditary radical $\mathbf{r}$ of Mod_S arises in this way: $\mathbf{r} = \mathbf{r}_{\mathfrak{a}}$, with $\mathfrak{a} = \mathbf{r}S$. Indeed, for a free module $M = \bigoplus_{\alpha} S$ we certainly have $\mathbf{r}M = \mathfrak{a}M$, and coheredity of $\mathbf{r}$ gives the same for all (quotients of free) modules.

$\mathbf{r}_{\mathfrak{a}}$ gives rise to a cohereditary torsion theory precisely when $\mathbf{r}_{\mathfrak{a}}$ is idempotent, that is: when the ideal $\mathfrak{a} = \mathfrak{a}^2$ is idempotent. The existence of such ideals of S depends, of course, on the structure of S. If $\mathfrak{a} = Sa$ is principal, we mention two antipodal cases:

(1) If S is an integral domain, there are *no* non-trivial idempotent ideals.
(2) If S is regular von Neumann, *every* principal ideal is idempotent, each of which gives a cohereditary torsion theory. Unless S is a field, Mod_S will therefore have plenty of non-trivial cohereditary torsion theories. Examples include: any Boolean ring (that is not a field); any product of more than one field; the ring $C(X)$ of continuous real-valued functions on a non-trivial P-space X (so that countable intersections of open sets are still open in X).

There is also a complete description of hereditary torsion theories in Mod_S, based on the existence of an injective cogenerator of this category. We must refer to [20] for details.

Example 7.4. There is a bijective correspondence between torsion theories of Top_* (see 2.6(4)) and pairs of classes (A, B) in Top that form a connectedness/disconnectedness in the sense of [1] (see 3.9(4)). For each such pair one can take as the torsion class T all pointed spaces in A, and as the torsion-free class all pointed spaces such that the A-component of the basepoint is trivial. Top_* has no non-trivial $\mathcal{E}$-cohereditary radicals. Indeed, since every pointed space can be regarded as the surjective image of a discrete space (with the same underlying set), any $\mathcal{E}$-cohereditary radical is determined by its values on the discrete pointed spaces, a subcategory isomorphic to Set_*. But Set_* has only trivial preradicals (see 3.9(1)).

There are precisely 4 non-trivial $\mathcal{M}$-hereditary preradicals of Top_*, namely (in the notation of 3.9(2),(4)) $\mathbf{q}_S$, $\mathbf{q}_{S^*}$, $\mathbf{p}_T$ and $\mathbf{p}_I = \mathbf{p}_{\{I\}}$, where $T = \{0 \leq 1 \leq 2\}$ has the order topology, with basepoint 1, and where I is a 2-point indiscrete space. Indeed, as observed before Prop. 5.11, preradicals of Top_* correspond precisely to fully additive closure operators of Top, and $\mathcal{M}$-heredity gets transferred by this correspondence either way. It was shown in [19, Theorem 5.1] that $k^{\oplus}$ (= the fully additive core of the usual Kuratowski closure operator k of Top), k^* (= the "inverse" Kuratowski closure operator), $k^\oplus \vee k^*$, and the least non-discrete closure operator μ of Top are the only non-trivial fully additive hereditary closure operators of Top. Their induced preradicals are precisely the ones listed earlier.

$\mathbf{q}_S$, $\mathbf{q}_{S^*}$ and $\mathbf{p}_I$ are idempotent radicals and give the torsion theories (spaces with dense base point, spaces with closed base point), (spaces X for which X is the only neighbourhood of the base point, spaces where the base point x does not belong to the closure of any point distinct from x), (indiscrete spaces, spaces in which the base point does not belong to any indiscrete subspace with more than one point), but $\mathbf{p}_T$ fails to be a radical.

Example 7.5. In 3.9(5) we already mentioned the idempotent radical $\mathbf{p}$ of TopGrp, lifted from Top_*. It gives the torsion theory (connected groups, hereditarily disconnected groups), but it is, of course, neither $\mathcal{M}$-hereditary nor $\mathcal{E}$-cohereditary. The radicals $\mathbf{q}_S$, $\mathbf{q}_{S^*}$ and $\mathbf{p}_I$ of Top_*, when lifted to TopGrp, give the same hereditary torsion theory: (indiscrete groups, Hausdorff groups).

A large collection of well-studied hereditary preradicals of the category $\mathsf{TopAbGrp}$ of topological abelian groups arise from a natural generalization of the concept of m-torsion for discrete groups (see 7.2). For any sequence $m = (m_i)$ of integers, call an element x of a topological abelian group A m-torsion if $m_i x$ converges to 0 in A. Now the subgroup $\mathbf{t}_m A$ of all m-torsion elements in A defines a hereditary preradical of $\mathsf{TopAbGrp}$. Of particular importance is the case $m_i = p^i$ for a prime number p, giving the notion of p-torsion as studied already in the 1940s (see [10, 31]). In this case $\mathbf{t}_m A$ is referred to as the topological p-Sylow subgroup of A, which plays an important role in the structure theory of topological (abelian) groups (see [12, 15, 16]). For a general sequence m, $\mathbf{t}_m$ fails to be a radical, unless it is eventually constant 0.

Example 7.6. Let CRng denote the pointed category of commutative, but not necessarily unital rings, considered as a semi-abelian category. Although many classical examples of radicals, like the Jacobson radical, fail to be functorial and therefore do not fit the setting of this paper, there are important examples of torsion theories. For example, for a ring S, let $\mathbf{t}S$ be the set of nilpotent elements x of S (so that $x^n = 0$ for some $n > 0$). Then $\mathbf{t}$ defines an idempotent radical that induces the torsion theory whose torsion-free part contains precisely the rings for which $x^n = 0$ only if $x = 0$. $\mathbf{t}$ is hereditary but fails to be cohereditary.

Here is a far-reaching generalization of the above example, where we replace the monomials x^n $(n > 0)$ by any set P of polynomials in m indeterminates over the integers: $P \subseteq \mathbb{Z}[x_1, \ldots, x_m]$. The full subcategory F of CRng containing precisely the rings S such that, for every $p \in P$, the implication

$$p(a_1, \ldots, a_m) = 0 \quad \Rightarrow \quad a_1 = \cdots = a_m = 0$$

holds in S, is closed under products and subobjects, hence, it is normal-epireflective in CRng. Since it also closed under extensions, the only obstacle that may prevent F from being torsion is that, with $\mathbf{r}$ denoting its induced radical, $\mathbf{rr}S$ may fail to be an ideal of S (see 4.6(1)). However, if we trade our ambient category for a subvariety of CRng, in which being an ideal is a transitive property, such as the category of Boolean rings, then F is a torsion-free class, even though it may be hard to characterize its torsion part and describe the radical in question.

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