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BOUNDARY REGULARITY AT $\{t = 0\}$ FOR A SINGULAR FREE BOUNDARY PROBLEM

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ABSTRACT: In this note it is shown that the weak solutions of the Stefan problem for the singular p-Laplacian are continuous up to $\{t = 0\}$. The result is a follow-up to a recent paper of the authors concerning the interior regularity.

KEYWORDS: Singular PDE, boundary regularity, intrinsic scaling, Stefan problem.
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1. The problem and the regularity result

In a recent paper (cf. [5]), the authors obtained interior continuity results for the weak solutions of the singular parabolic PDE

$$\partial_t \eta - \Delta_p \theta = 0 \ , \quad \eta \in \gamma(\theta) \ ; \qquad 1 < p < 2 \ , \tag{1}$$

where γ is a maximal monotone graph and $\Delta_p u = \operatorname{div} |\nabla u|^{p-2} \nabla u$ is the p-Laplacian. When γ has a single jump at the origin, this equation generalizes to a nonlinear setting the modelling of the classical Stefan problem that corresponds to the case $p = 2$ and describes a phase transition at constant temperature for a substance obeying Fourier's law. Equation (1) is singular both in space and time since $1 < p < 2$ and, roughly speaking, $\gamma'(0) = \infty$.

In this note it is shown that, for continuous initial data, the continuity result holds up to $\{t = 0\}$ so that, in a way, the solution inherits the continuity properties of the boundary data. We consider a regularized approximated problem and show that the sequence of approximate solutions is equicontinuous up to $\{t = 0\}$. Due to the singularities in the equation we need to use intrinsic scaling to uniformly reduce the oscillation of the approximate solutions in a sequence of shrinking cylinders laying at the bottom of the space-time domain. For a modern account of intrinsic scaling and related matters, we suggest the reading of the recent survey [4].

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To fix ideas, assume that an incompressible material (say pure water) occupies a bounded domain $\Omega \subset \mathbb{R}^N$, with two phases, a solid phase corresponding to the region $\{\theta < 0\}$ and a liquid phase corresponding to the region $\{\theta > 0\}$, separated by an interface $\Phi = \{\theta = 0\}$, the free boundary. We denote $\Omega_T = \Omega \times (0,T)$ and $\Sigma = \partial\Omega \times (0,T)$, for some $T > 0$. The problem in its strong formulation reads

$$
(\mathbf{P}) \left\{ \begin{array}{rcll}
\partial_t \, \theta & = & \Delta_p \theta & \text{in } \; \Omega_T \setminus \Phi = \{\theta < 0\} \cup \{\theta > 0\} \\[2ex]
\left[|\nabla \theta|^{p-2} \nabla \theta \right]_-^+ \cdot \mathbf{n} & = & \lambda \, \mathbf{w} \cdot \mathbf{n} & \text{on } \; \Phi = \{\theta = 0\} \\[2ex]
\theta & = & 0 & \text{on } \; \Sigma \\[2ex]
\theta(0) & = & \theta_0 & \text{in } \; \Omega \times \{0\}
\end{array} \right.
$$

where $\mathbf{n}$ is the unit normal to Φ, pointing to the solid region, $\mathbf{w}$ the velocity of the free boundary and $\lambda = [e]_-^+ > 0$ the latent heat of phase transition (e is the internal energy), with $[\,.\,]_-^+$ denoting the jump across Φ.

As usual, a weak formulation, in which all explicit references to the free boundary are absent, is obtained considering the maximal monotone graph H associated with the Heaviside function, and introducing a new unknown function, the enthalpy η, such that

$$\eta \in \gamma(\theta) := \theta + \lambda H(\theta) \ .$$

A formal integration by parts against appropriate test functions and the replacement of the initial condition for θ by a more adequate initial condition for η, leads to an integral relation that we adopt as definition of weak solution.

Definition 1.1. *We say that (η, θ) is a* weak solution *of problem* $(\mathbf{P})$*, if*

$$\theta \in L^p\Big(0,T; W_0^{1,p}(\Omega)\Big) \cap L^\infty(\Omega_T) \quad ;$$

$$\eta \in L^\infty(\Omega_T) \quad \text{and} \quad \eta \in \gamma(\theta) \ , \text{ a.e. in } \ \Omega_T \ ;$$

$$-\int_{\Omega_T} \eta \, \partial_t \xi + \int_{\Omega_T} |\nabla \theta|^{p-2} \nabla \theta \cdot \nabla \xi = \int_\Omega \eta_0 \, \xi(0) \ \ , \ \ \forall \xi \in \mathcal{T}(\Omega_T) \ .$$

The space of test functions we are considering is

$$\mathcal{T}(\Omega_T) := \left\{ \xi \in L^p\Big(0,T;W_0^{1,p}(\Omega)\Big) \ : \ \partial_t\xi \in L^2(\Omega_T) \ , \ \xi(T)=0 \right\} ,$$

and we assume that

$$\eta_0 \in \gamma(\theta_0) \ , \quad \text{and} \quad \exists M>0 \ : \ |\theta_0(x)| \le M \ , \ \text{a.e. } x\in\Omega \ . \tag{2}$$

Let $0<\epsilon\ll 1$ and consider the bilipschitzian function

$$\gamma_\epsilon(s) = s + \lambda H_\epsilon(s) \ ,$$

where H_ϵ is a $\mathcal{C}^\infty$-approximation of the Heaviside function. Taking also functions $\theta_{0\epsilon}\in W^{1,p}(\Omega)$ such that

$$\theta_{0\epsilon} \to \theta_0 \ , \ \gamma_\epsilon(\theta_{0\epsilon}) \to \eta_0 \quad \text{in} \quad L^p(\Omega) \quad \text{and} \quad |\theta_{0\epsilon}| \le M \ , \quad \text{a.e. in } \Omega$$

we define a sequence of approximated problem as follows

(P_ϵ): For each $0<\epsilon\ll 1$, find a function

$$\theta_\epsilon \in H^1\big(0,T;L^2(\Omega)\big) \cap L^\infty\big(0,T;W_0^{1,p}(\Omega)\big) \cap L^\infty(\Omega_T)$$

such that

$$-\int_{\Omega_T} \gamma_\epsilon(\theta_\epsilon)\,\partial_t\xi + \int_{\Omega_T} |\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon\cdot\nabla\xi = \int_\Omega \gamma_\epsilon(\theta_{0\epsilon})\,\xi(0) \ , \quad \forall \xi\in\mathcal{T}(\Omega_T) \ . \tag{3}$$

In the presence of the regularity required, equation (3) can be shown to be equivalent to the two conditions: $\theta_\epsilon(0)=\theta_{0\epsilon}$ and, for a.e. $t\in(0,T)$,

$$\int_{\Omega\times\{t\}} \partial_t[\gamma_\epsilon(\theta_\epsilon)]\,\varphi + \int_{\Omega\times\{t\}} |\nabla\theta_\epsilon|^{p-2}\nabla\theta_\epsilon\cdot\nabla\varphi = 0 \ , \quad \forall\varphi\in W_0^{1,p}(\Omega) \ . \tag{4}$$

It was shown in [7] that this approximated problem has a unique solution and enough *a priori* estimates were derived to pass to the limit and obtain a solution of the original problem. In particular, the sequence of approximate solutions was shown to be equibounded.

We show here that there exists a uniform, i.e. independent of ϵ, modulus of continuity for θ_ϵ up to $\{t=0\}$ and this will allow us to obtain a continuous solution up to $\{t=0\}$ for the original problem as a consequence of Ascoli's theorem. We need to assume, in addition to (2), that

$$\theta_0 \in C(\Omega) \quad \text{and} \quad (\theta_{0\epsilon})_\epsilon \text{ is equicontinuous.} \tag{5}$$

This means that, over a compact $K\subset\Omega$, each $\theta_{0\epsilon}$ and θ_0 have the same modulus of continuity.

We will prove the following regularity result.

Theorem 1.2. *The sequence $(\theta_\epsilon)_\epsilon$ is equicontinuous up to $\{t = 0\}$. Then the weak solution of problem (P) is continuous up to $\{t = 0\}$. Moreover, for any compact $K \subset \Omega$, there exists a non-decreasing continuous function $\omega_K : \mathbb{R}^+ \to \mathbb{R}^+$, $\omega_K(0) = 0$, depending only upon the data and the modulus of continuity of θ_0, such that*

$$|\theta(x_1, t_1) - \theta(x_2, t_2)| \leq \omega_K \left(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{p}} \right) ,$$

for every pair of points $(x_1, t_1), (x_2, t_2) \in K' \times [0, T]$, and every compact $K' \subset K$.

In face of the recent results of [5], we clearly only need to prove the continuity at $t = 0$.

2. Energy and logarithmic estimates near $\{t = 0\}$

The building blocks of regularity theory leading to continuity results are energy and logarithmic estimates. These are the fundamental tools to proof Proposition 3.1 and will be derived next.

The crucial observation here is that, when deriving estimates for (1) in cylinders laying at the bottom of Ω_T, the term involving $(\theta_\epsilon - k)_\pm$ with power one, is absent, unlike in the interior case, which strongly simplifies the analysis. This is due to the choice of an independent of time cutoff function, which suffices for our purposes, and an appropriate selection of levels k, according to the initial data.

Given a point $x_0 \in \mathbb{R}^N$, $K_\rho(x_0)$ denotes the N-dimensional cube with centre at x_0 and wedge 2ρ:

$$K_\rho(x_0) := \left\{ x \in \mathbb{R}^N : \max_{1 \leq i \leq N} |x_i - x_{0i}| < \rho \right\} .$$

Fix $(x_0, t_0) \in \Omega_T$ and consider the cylinder

$$(x_0, t_0) + Q(\tau, \rho) = K_\rho(x_0) \times (t_0 - \tau, t_0)$$

where τ is such that $t_0 - \tau = 0$ so the cylinder lies at the bottom of Ω_T. Consider a piecewise smooth cutoff function ξ, independent of $t \in (0, t_0)$, satisfying

$$0 \leq \xi \leq 1 , \quad |\nabla \xi| < \infty \quad \text{and} \quad \xi(x) = 0 , \; x \notin K_\rho(x_0) . \tag{6}$$

In the weak formulation (4), take $\varphi = \pm\,(\theta_\epsilon - k)_\pm\ \xi^p \in W_0^{1,p}\left(K_\rho(x_0)\right)$ and then integrate in time over $(0,t)$, for $t \in (0,t_0)$. Since

$$\pm\partial_t\left(\gamma_\epsilon(\theta_\epsilon)\right)(\theta_\epsilon - k)_\pm = \pm\gamma_\epsilon'(\theta_\epsilon)\partial_t\theta_\epsilon(\theta_\epsilon - k)_\pm = \partial_t\left(\int_0^{(\theta_\epsilon - k)_\pm} \gamma_\epsilon'(k \pm s)\ s\ ds\right) ,$$

recalling the t-independence of ξ and the definition of γ_ϵ, we obtain the following bound from below for the term involving the time derivative

$$\frac{1}{2}\int_{K_\rho(x_0)\times\{t\}} (\theta_\epsilon - k)_\pm^2\ \xi^p - 2(M+\lambda)\int_{K_\rho(x_0)\times\{0\}} (\theta_\epsilon - k)_\pm\ \xi^p\ . \tag{7}$$

Observe that, if we choose

$$k \geq \sup_{x\in K_\rho(x_0)} \theta_{0\epsilon}(x) \tag{8}$$

when working with $(\theta_\epsilon - k)_+$, and

$$k \leq \inf_{x\in K_\rho(x_0)} \theta_{0\epsilon}(x) \tag{9}$$

for $(\theta_\epsilon - k)_-$, the second term of (7) vanishes. On the other hand, the term concerning the space derivatives is estimated above by

$$\frac{1}{2}\int_0^t\int_{K_\rho(x_0)} |\nabla(\theta_\epsilon - k)_\pm|^p\ \xi^p - C(p)\int_0^t\int_{K_\rho(x_0)} (\theta_\epsilon - k)_\pm^p\ |\nabla\xi|^p$$

using Young's inequality with $\varepsilon = (2(p-1))^{\frac{p-1}{p}}$. We thus obtain

Proposition 2.1. *There exists a constant C, that can be determined a priori in terms of the data and independently of ϵ, such that for every $(x_0,t_0) \in \Omega_T$, for every cylinder $(x_0,t_0) + Q(\tau,\rho)$ such that $t_0 - \tau = 0$, and for every level k verifying* (8) *or* (9),

$$\sup_{0<t<t_0}\int_{K_\rho(x_0)\times\{t\}} (\theta_\epsilon - k)_\pm^2\ \xi^p + \int_0^{t_0}\int_{K_\rho(x_0)} |\nabla(\theta_\epsilon - k)_\pm|^p\ \xi^p$$
$$\leq \int_0^{t_0}\int_{K_\rho(x_0)} (\theta_\epsilon - k)_-^p\ |\nabla\xi|^p\ . \tag{10}$$

Now consider the logarithmic function

$$\Psi^\pm = \Psi\left(H_k^\pm, (\theta_\epsilon - k)_\pm, c\right) = \left(\ln\left(\frac{H_k^\pm}{H_k^\pm + c - (\theta_\epsilon - k)_\pm}\right)\right)_+ , \quad 0 < c < H_k^\pm$$

where

$$H_k^{\pm} = \operatorname*{ess\,sup}_{(x_0,t_0)+Q(\tau,\rho)} (\theta_\epsilon - k)_\pm \ .$$

In the weak formulation (4) take

$$\varphi = \left[\left(\Psi^{\pm}\right)^2\right]' \xi^p = 2\Psi^{\pm} \ \left(\Psi^{\pm}\right)' \ \xi^p \ ,$$

where ξ is defined as in (6). Observing that

$$\begin{cases} \Psi^+(x,0) = 0 \quad \text{for} \quad k \geq \sup\limits_{x\in K_\rho(x_0)} \theta_{0\epsilon}(x) \\ \\ \Psi^-(x,0) = 0 \quad \text{for} \quad k \leq \inf\limits_{x\in K_\rho(x_0)} \theta_{0\epsilon}(x) \end{cases} ,$$

and using Young's inequality with $\varepsilon = (2(p-1))^{\frac{p-1}{p}}$ we arrive at

Proposition 2.2. *There exists a constant C, determined a priori only in terms of the data and independently of ϵ, such that for every $(x_0,t_0) \in \Omega_T$, for every cylinder $(x_0,t_0)+Q(\tau,\rho)$ such that $t_0-\tau = 0$ and for every level k verifying (8) or (9),*

$$\sup_{0<t<t_0} \int_{K_\rho(x_0)\times\{t\}} \left(\Psi^{\pm}\right)^2 \xi^p \leq \int_0^{t_0}\int_{K_\rho(x_0)} \Psi^{\pm} \ \left|\left(\Psi^{\pm}\right)'\right|^{2-p} \ |\nabla\xi|^p \ . \qquad (11)$$

3. Reduction of the oscillation in rescaled cylinders

Fix $(x_0,0) \in \Omega\times\{0\}$, and take $R>0$ such that $K_{2R}(x_0) \subset \Omega$. By translation, we may assume that $x_0 = 0$. Introduce the cylinder

$$Q\left(R^p, 2R\right) := K_{2R}\times(0,R^p)$$

and define

$$\mu^+ = \operatorname*{ess\,sup}_{Q(R^p,2R)} \theta_\epsilon \ ; \quad \mu^- = \operatorname*{ess\,inf}_{Q(R^p,2R)} \theta_\epsilon \ ; \quad \omega = \operatorname*{ess\,osc}_{Q(R^p,2R)} \theta_\epsilon = \mu^+ - \mu^- \ .$$

Construct the cylinder

$$Q\left(a_0R^p, R\right) = K_R\times(0,a_0R^p) \ , \quad a_0 = \left(\frac{\omega}{2^m}\right)^{2-p} \ ,$$

where $m>1$ is to be chosen. Without loss of generality, we may assume that $\frac{\omega}{2^m} \leq 1$ so that the following relations hold:

$$Q\left(a_0R^p, R\right) \subset Q\left(R^p, 2R\right) \quad \text{and} \quad \operatorname*{ess\,osc}_{Q(a_0R^p,R)} \theta_\epsilon \leq \omega \ .$$

The proof of Theorem 1.2 is a well-known consequence of the following iterative argument.

Proposition 3.1. *There exist constants $\sigma \in (0,1)$, and $C, m > 1$, that can be determined a priori only in terms of the data, such that constructing the sequences*

$$\begin{cases} \omega_0 = \omega \\ \omega_{n+1} = \sigma\, \omega_n \end{cases} \qquad \text{and} \qquad \begin{cases} R_0 = R \\ R_{n+1} = \frac{R}{C^n} \end{cases}$$

and the family of boxes

$$Q_n = (a_n R_n^p, R_n) \ , \qquad a_n = \left(\frac{\omega_n}{2^m}\right)^{2-p} \ ,$$

we have

$$Q_{n+1} \subset Q_n \quad \text{and} \quad \operatorname*{\mathit{ess\ osc}}_{Q_n} \ \theta_\epsilon \le \max\left\{\omega_n, 2 \operatorname*{\mathit{ess\ osc}}_{K_{R_n}} \ \theta_{0\epsilon}\right\} \ , \tag{12}$$

for all $n = 0, 1, 2, \ldots$.

To prove Proposition 3.1, assume first that both inequalities

$$\mu^+ - \frac{\omega}{4} \le \mu_0^+ := \operatorname*{ess\,sup}_{K_R} \ \theta_{0\epsilon} \qquad \text{and} \qquad \mu^- + \frac{\omega}{4} \ge \mu_0^- := \operatorname*{ess\,inf}_{K_R} \ \theta_{0\epsilon} \tag{13}$$

hold. Subtracting the second inequality from the first one we get

$$\frac{\omega}{2} \le \mu_0^+ - \mu_0^- = \operatorname*{ess\,osc}_{K_R} \ \theta_{0\epsilon} \ .$$

and the proposition is trivially proved.

Without loss of generality, assume that the second inequality in (13) fails. Then the levels $k = \mu^- + \frac{\omega}{2^s}$, for $s \ge 2$, verify $k \le \mu_0^-$ and, consequently, the energy and logarithmic estimates (10) and (11), respectively, hold for $(\theta_\epsilon - k)_-$. The next result has a double scope: it determines the parameter m that defines the height of the constructed initial cylinder and defines a level such that the subset of $Q\left(a_0 R^p, \frac{R}{2}\right)$ where θ_ϵ is below that level is small.

Lemma 3.2. *For all $\nu \in (0,1)$, there exists $m > 3$, depending only on the data, such that*

$$\left|(x,t) \in Q\left(a_0 R^p, \frac{R}{2}\right) \ : \ \theta_\epsilon(x,t) < \mu^- + \frac{\omega}{2^m}\right| < \nu \left|Q\left(a_0 R^p, \frac{R}{2}\right)\right| \ .$$

Proof. Consider estimate (11) written for $(\theta_\epsilon - k)_-$, with $k = \mu^- + \frac{\omega}{4}$, and for a cutoff function $0 \leq \xi \leq 1$, defined in K_R, and verifying

$$\xi \equiv 1 \quad \text{in } K_{\frac{R}{2}} \ ; \qquad \xi \equiv 0 \quad \text{on } |x| = R \ ; \qquad |\nabla \xi| \leq \frac{2}{R} \ .$$

Take $m > 3$ sufficiently large so that $0 < c = \frac{\omega}{2^m} < H_k^-$. The logarithmic function Ψ^- is well-defined and, since $H_k^- \leq \frac{\omega}{4}$, the following inequalities hold

$$\Psi^- \leq (m-2) \ \ln 2 \qquad \text{and} \qquad \left| \left(\Psi^-\right)' \right|^{2-p} \leq \left(\frac{\omega}{2^m} \right)^{p-2} \ .$$

Then, from (11), we get for all $t \in (0, a_0 R^p)$, the estimate

$$\int_{K_R \times \{t\}} \left(\psi^-\right)^2 \ \xi^p \leq C \ (m-2) \ \left| K_{\frac{R}{2}} \right| \ .$$

Next, integrate over the smaller set

$$\left\{ x \in K_{\frac{R}{2}} : \theta_\epsilon(x,t) < \mu^- + \frac{\omega}{2^m} \right\} \ , \qquad \forall t \in (0, a_0 R^p)$$

where $\xi = 1$ and $\Psi^- \geq (m-3) \ \ln 2$, since $H_k^- \leq \frac{\omega}{4}$. Consequently, for all $t \in (0, a_0 R^p)$,

$$\left| x \in K_{\frac{R}{2}} \ : \ \theta_\epsilon(x,t) < \mu^- + \frac{\omega}{2^m} \right| \leq C \ \frac{m-2}{(m-3)^2} \ \left| K_{\frac{R}{2}} \right| \ .$$

The proof is complete if we choose m so large that $C \ \frac{m-2}{(m-3)^2} < \nu$.

$\square$

The next lemma provides a uniform lower bound for θ_ϵ within a smaller cylinder, through a specific choice of the value ν that appears in Lemma 3.2.

Lemma 3.3. *There exists $\nu_0 \in (0,1)$, depending only on the data, such that if*

$$\left| Q\left(a_0 R^p, \frac{R}{2}\right) \ : \ \theta_\epsilon(x,t) \leq \mu^- + \frac{\omega}{2^m} \right| \leq \nu_0 \left| Q\left(a_0 R^p, \frac{R}{2}\right) \right|$$

then

$$\theta_\epsilon(x,t) \geq \mu^- + \frac{\omega}{2^{m+1}} \ , \quad \text{a.e. } (x,t) \in Q\left(a_0 R^p, \frac{R}{4}\right) \ .$$

Proof. Consider the decreasing sequences of real numbers

$$R_n = \frac{R}{4} + \frac{R}{2^{n+2}} \ ; \qquad k_n = \mu^- + \frac{\omega}{2^{m+1}} + \frac{\omega}{2^{m+1+n}} \ , \quad n = 0, 1, \ldots$$

and, in the energy estimates (10), take $\varphi = -(\theta_\epsilon - k_n)_-\, \xi_n^p$, where $0 \le \xi_n \le 1$ are smooth cutoff functions, defined in K_{R_n}, and verifying

$$\xi \equiv 1 \quad \text{in } K_{R_{n+1}}\ ; \qquad \xi \equiv 0 \quad \text{on } |x| = R_n\ ; \qquad |\nabla \xi_n| \le \frac{2^{n+3}}{R}\ .$$

Introduce the level

$$\bar{k}_n = \frac{k_n + k_{n+1}}{2}\ .$$

Since

$$\begin{aligned}
\int_{K_{R_n}\times\{t\}} (\theta_\epsilon - k_n)_-^2\, \xi_n^p &= \int_{K_{R_n}\times\{t\}} (\theta_\epsilon - k_n)_-^p\, (\theta_\epsilon - k_n)_-^{2-p}\, \xi_n^p \\
&\ge (k_n - \bar{k}_n)^{2-p} \int_{K_{R_n}\times\{t\}} (\theta_\epsilon - \bar{k}_n)_-^p\, \xi_n^p \\
&= a_0\, 2^{-(n+3)p} \int_{K_{R_n}\times\{t\}} (\theta_\epsilon - \bar{k}_n)_-^p\, \xi_n^p
\end{aligned}$$

and $(\theta_\epsilon - k_n)_-^p \le \left(\frac{\omega}{2^m}\right)^p$, the referred energy estimates take the form

$$\sup_{0<t<a_0R^p} \int_{K_{R_n}\times\{t\}} (\theta_\epsilon - \bar{k}_n)_-^p\, \xi_n^p + \frac{1}{a_0}\, 2^{-(n+3)p} \iint_{Q(a_0R^p, R_n)} \left|\nabla(\theta_\epsilon - \bar{k}_n)_-\right|^p\, \xi_n^p$$
$$\le C(p) \left(\frac{\omega}{2^m}\right)^p \frac{2^{2pn}}{R^p}\, \frac{1}{a_0} \iint_{Q(a_0R^p, R_n)} \chi_{[(\theta_\epsilon - k_n)_- > 0]}\ .$$

Introducing the change of variable $z = \frac{t}{a_0}$, defining the new functions

$$\bar{\theta}_\epsilon(x,z) = \theta_\epsilon(x, a_0 z)\ ; \quad \bar{\xi}_n(x,z) = \xi_n(x, a_0 z)\ ,$$

and denoting $V^p = L^\infty(L^p) \cap L^p(W^{1,p})$, we arrive at

$$\left\|(\bar{\theta}_\epsilon - \bar{k}_n)_-\right\|_{V^p(Q(R^p, R_{n+1}))}^p \le C(p)\, \frac{2^{2pn}}{R^p} \left(\frac{\omega}{2^m}\right)^p A_n\ ,$$

where

$$A_n := \int_0^{R^p} |A_n(z)|\ dz\ , \qquad A_n(z) := \left\{x \in K_{R_n} : (\bar{\theta}_\epsilon - k_n)_- > 0\right\}\ .$$

Since

$$\begin{aligned}
\left(\frac{\omega}{2^m}\right)^p 2^{-(n+3)p}\, A_{n+1} &\le \iint_{Q(R^p, R_{n+1})} (\bar{\theta}_\epsilon - \bar{k}_n)_-^p \\
&\le C\, A_n^{1+\frac{p}{N+p}}\ \left|\left|(\bar{\theta}_\epsilon - \bar{k}_n)_-\right|\right|_{V^p(Q(R^p, R_{n+1}))}^p\ ,
\end{aligned}$$

using Corollary 3.1 of [3, page 9], we conclude

$$A_{n+1} \le C \, \frac{2^{3pn}}{R^p} \, A_n^{1+\frac{p}{N+p}}$$

and, consequently,

$$Y_{n+1} \le C \, 2^{3pn} \, A_n^{1+\frac{p}{N+p}} \ , \qquad \text{for} \quad Y_n := \frac{A_n}{\left| Q\left(R^p, R_n\right)\right|} \ .$$

If $Y_0 \le C^{-\frac{N+p}{p}} \, 2^{-\frac{3(N+p)^2}{p}}$ then, by Lemma 4.1 of [3, page 12], $Y_n \to 0$ when $n \to \infty$ which completes the proof. Observe that, by the hypothesis,

$$Y_0 = \frac{\left|(x,z) \in Q(R^p, R) : \bar{\theta}_\epsilon(x,z) < \mu^- + \frac{\omega}{2^m}\right|}{|Q(R^p, R)|} \le \nu_0$$

so we just have to take

$$\nu_0 \equiv C^{-\frac{N+p}{p}} \, 2^{-\frac{3(N+p)^2}{p}} \ .$$

□

Now we can finally conclude the first iteration step in the proof of Proposition 3.1. Indeed, taking $\nu = \nu_0$ from Lemma 3.3, and determining the corresponding value m with the help of Lemma 3.2, we arrive at

$$\theta_\epsilon(x,t) \ge \mu^- + \frac{\omega}{2^{m+1}} \ , \quad \text{a.e.}(x,t) \in Q\left(a_0 R^p, \frac{R}{4}\right) \ ,$$

and then we conclude that

$$\operatorname*{ess\,osc}_{Q\left(a_0 R^p, \frac{R}{4}\right)} \ \theta_\epsilon \le \left(1 - \frac{1}{2^{m+1}}\right) \omega = \sigma \, \omega \ .$$

Taking $C = 4$ in Proposition 3.1, we get $Q_1 \subset Q\left(a_0 R^p, \frac{R}{4}\right)$, and then

$$\operatorname*{ess\,osc}_{Q_1} \ \theta_\epsilon \le \operatorname*{ess\,osc}_{Q\left(a_0 R^p, \frac{R}{4}\right)} \ \theta_\epsilon \le \sigma \, \omega = \omega_1 \ .$$

We can now repeat the whole process starting from Q_1.

Remark 3.4. *Observe that we don't get a reduction on the t-direction since the cutoff functions ξ are independent of t.*

Remark 3.5. *The regularity result can be further extended; one can obtain continuity up to the lateral boundary Σ using a reasoning similar to the one presented in* [2] *and* [8].

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