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SAMPLING THEORY ASSOCIATED TO q-DIFFERENCE EQUATIONS OF THE STURM-LIOUVILLE TYPE

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Abstract: We will show that if the kernel $K(x,t)$ in the representation $f(t) = \int_0^a K(x,t)u(t)d_qt$, with $u \in L_q^2(0,a)$, is a solution of a second order q-Sturm-Liouville boundary problem, then f admits a representation as a sampling formula of the form $f(t) = \sum_{n=0}^{\infty} f(\lambda_n)W(t)/[W'(\lambda_n)\,(t-\lambda_n)]$ where λ_n is the nth eigenvalue of the associated q-Sturm-Liouville boundary problem and $W(t)$ is the Wronskian of two solutions selected in a specified way.

Keywords: sampling theorem, q-difference equations, Sturm-Liouville systems.
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1. Introduction

The celebrated Whittaker-Shannon-Kotel'nikov theorem states that every integral transform of the form

$$f(x) = \int_{-\pi}^{\pi} e^{ixt}u(t)dt$$

where $u \in L^2(-\pi,\pi)$, can be written as the sampling formula

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{\sin \pi(x-n)}{\pi(x-n)} \tag{1}$$

Writing

$$L(x) = \sin \pi z = \pi z \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

then the Whittaker-Shannon-Kotel'nikov theorem is a Lagrange-type interpolation formula of the form

$$f(x) = \sum_{n=-\infty}^{\infty} f(n)\frac{L(x)}{L'(n)(x-n)} \tag{2}$$

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The quest of writing sampling theorems as Lagrange type interpolation formulas has attracted a considerable amount of research over the years. When the kernel e^{ixt} is replaced by some function in the context of Kramer´s lemma, then often there are special function formulae available to perform the task. This is also the case of the functions orthogonal with respect to their own zeros [3]. Another interesting sitution occurs when the kernel of the integral transform arises from a second order differential equation. A major achievement in this direction is due to Zayed, Hinsen and Butzer [20]. They considered the second order Sturm-Liouville eigenvalue

$$-y''(x) + v(x)y(x) = \lambda y(x) \tag{3}$$

where $v(x)$ is defined on the finite interval $[a, b]$, together with the initial conditions

$$\cos\alpha y(a) + \sin\alpha y'(a) = 0$$
$$\cos\beta y(b) + \sin\beta y'(b) = 0$$

and selected a particular solution of the problem satisfying $\varphi(a, \lambda) = \sin\alpha$ and $\varphi'(a, \lambda) = -\cos\alpha$ and another one satisfying $\psi(b, \lambda) = \sin\beta$ and $\psi'(b, \lambda) = -\cos\beta$. Within this setting, their main theorem reads as follows:

Theorem A: *Every function f that can written as an integral transform of the form*

$$f(\lambda) = \int_a^b u(x)\varphi(x, \lambda)dx$$

where $u \in L^2(a, b)$, admits a sampling representation of the form

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{W(\lambda)}{W'(\lambda_n)(\lambda - \lambda_n)}$$

where $W(\lambda)$ is the Wronskian of the functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$.

The construction of expansions in q-Fourier series ([8], [9]) was followed by the derivation of q-sampling theorems ([2], [5], [14]). The most relevant feature present in all of these q-sampling theorems is the sparsity of their sampling nodes, located at the zeros of the q-analogues of the $\sin x$. Recent research about these zeros ([1], [18]) indicates that, for big n, they behave very much like sequences of the form q^{-n}. Therefore, the resulting sampling expansions provide a process of reconstructing signals from samples that become sparse as they move away from the origin. As an instance, every function f within the setting of our main result in [2] contains all its information on the sequence $\{f(q^{-n+\epsilon_n})\}_{n\in\mathbf{N}}$, $0<\epsilon_n<1$, $0<q<1$. That is, whereas the sampling theorems of the Whittaker-Shannon-Kotel'nikov type identify a function with its values over an arithmetic sequence (or close to such a sequence, as in the case of irregular sampling), the q-sampling theorems identify a function with its values over a geometric sequence. The other novel aspect of the q-sampling theorems is the presence of the parameter q itself: For every realization of the parameter we have a different sampling theory, with different sampling nodes. The idea of using q-deformations of classical results has showed before to be a successfull one in physical mathematics as can be testified by the considerable amount of research about the q-harmonic oscilator that followed [7] or the surge of the theory of Quantum Groups and their conections to q-special functions [10].

For a comprehensive introduction to the subject of sampling from a classic point of view we refer to [13]. For an exposition of modern sampling methods and how they were inspired by physical problems in communication, astronomy, and medicine, we suggest the reading of [4] where also are included algorithms for practical reconstruction of signals from samples.

The purpose of the present paper is to construct a q-analogue of Theorem A, building on recent results of Annaby and Mansour [6], who provided a detailed study of the basis properties of solutions of q-Sturm-Liouville systems, inspired by the formal work of Exton [11].

2. Preliminaries on q-calculus

Following the standard notations in [12], consider a number $0<q<1$ and define the q-shifted factorial for n finite and different from zero as

$$(a;q)_n=(1-q)(1-aq)...(1-aq^{n-1})$$

and the zero and infinite cases as

$$(a;q)_0=1$$

$$(a;q)_\infty=\lim_{n\to\infty}(a;q)_n$$

The q-difference operator D_q is defined as

$$D_qf(x)=\frac{f(x)-f(qx)}{x(1-q)}$$

When required, q will be replaced by q^{-1}. The following facts can be verified directly from the definition and will be used often :

$$D_{q^{-1}}f(x) = (D_q f)(xq^{-1})$$

$$D_q^2 f(q^{-1}x) = qD_q\left[D_q f(q^{-1}x)\right] = D_{q^{-1}}D_q f(x)$$

Associated to this operator there is a nonsymmetric formula for the q-differentiation of a product

$$D_q[f(x)g(x)] = f(qx)D_q g(x) + g(x)D_q f(x) \tag{4}$$

The q-integral usually associated to the name of Jackson is defined, in the interval $(0, a)$, as

$$\int_0^a f(x)d_q x = (1-q)\sum_{n=0}^{\infty} f(aq^n)aq^n$$

Let $L_q^2(0, a)$ be the space of all complex-valued functions defined on $(0, a)$ such that

$$\|f\| = \left[\int_0^a |f(x)|^2\, d_q x\right]^{\frac{1}{2}} < \infty$$

The space $L_q^2(0, a)$ is a separable Hilbert space (see [5] for details) with the inner product

$$\langle f, g\rangle = \int_0^a f(x)\overline{g(x)}d_q x$$

Through the remainder of the text we will deal only with functions q-regular at zero, that is, functions such that

$$\lim_{n\to\infty} f(q^n x) = f(0)$$

The class of the functions which are q-regular at zero includes the continuous functions. An example of a function that is not q-regular at zero is given by [17]

$$f(x) = \sin(\alpha \log x) \quad \text{with} \quad \alpha \log q = 2\pi$$

If f and g are both q-regular at zero, there is a rule of q-integration by parts given by

$$\int_0^a g(x)D_q f(x)d_q x = (fg)(a) - (fg)(0) - \int_0^a D_q g(x)f(qx)d_q x. \tag{5}$$

The q appearing in the argument of f in the right hand side integrand is another manifestation of the asymmetry that is everywhere present in q-calculus. As an important special case we have

$$\int_0^a D_q f(x) d_q x = (f)(a) - (f)(0) \tag{6}$$

For these and other formulas we refer to [15].

3. The q-Sturm-Liouville problem

We will consider a q-Sturm-Liouville equation of the form

$$-\frac{1}{q} D_{q^{-1}} D_q y(x) + v(x) y(x) = \lambda y(x) \tag{7}$$

$$0 \leq x \leq a < \infty, \quad \lambda \in \mathbf{C}.$$

with $v(x)$ defined on the interval $[0, a]$, together with the initial conditions

$$a_{11} y(0) + a_{12} D_{q^{-1}} y(0) = 0 \tag{8}$$

$$a_{21} y(a) + a_{22} D_{q^{-1}} y(a) = 0 \tag{9}$$

It was shown in [6] that such a q-Sturm-Liouville problem is formally self adjoint, that is, denoting by ℓy the left hand member of (7), they proved that $\langle \ell y, h \rangle = \langle y, \ell h \rangle$. This self-adjointness property allowed the authors to use the spectral theorem for compact self adjoint operators, after turning the q-difference problem into a q-integral one, by means of the construction of a q-type Green´s function. A key step for the construction of a q-type Green´s function in [6] was the definition of a fundamental set of solutions of (7) by means of the q-analogues of the functions sin and cos defined as

$$\sin(x; q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n^2}}{(q; q)_{2n}} (x(1-q))^{2n} \tag{10}$$

and

$$\cos(x; q) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q; q)_{2n+1}} (x(1-q))^{2n+1} \tag{11}$$

these functions differ slightly from the ones consider in [9]. However, the crucial information about their roots can be obtained by relating them to the third Jackson q-Bessel function

$$J_\nu(x; q) = x^\nu \frac{(q^{\nu+1}; q)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)/2}}{(q^{\nu+1}; q)_n (q; q)_n} x^{2n}$$

Clearly we have the relations

$$\sin(x;q) = \frac{(q^2;q^2)}{(q^3;q^2)} (x(1-q))^{\frac{1}{2}} J_{\frac{1}{2}}(x(1-q);q^2)$$

$$\cos(x;q) = \frac{(q^2;q^2)}{(q;q^2)} (x(1-q))^{\frac{1}{2}} J_{-\frac{1}{2}}(x(1-q)q^{\frac{1}{2}};q^2)$$

combining this relations with the bounds obtained in [1] for the roots of the third q-Bessel function, we obtain the following theorem on the location of the roots of the functions $\sin(x;q)$ and $\cos(x;q)$.

Theorem B *If $q < (1-q^2)^2$ then the nonzero roots of the function $\sin(x;q)$ are of the form*

$$x_n = \frac{q^{-n+\epsilon_n}}{1-q} \tag{12}$$

and if $q^3 < (1-q^2)^2$ then the roots of the function $\cos(x;q)$ are of the form

$$y_n = \frac{q^{-n+1+\epsilon_n}}{1-q} \tag{13}$$

where $0 < \epsilon_n < 1$. The restrictions on q can be removed if n is big enough.

Remark: It can be seen from the rapid growth of their zeros, or directly from the power series expansions (10) and (11), that both $\sin(x;q)$ and $\cos(x;q)$ are functions of order zero, when considered as entire funtions. As a result they are unbounded on the real line (if they were bounded on the real line then we could use a Phragmén-Lindelöf argument to extend the bound to the whole complex plane and this would force the functions to be constant). This unboundedness property constitutes a serious obstacle in obtaining asymptotics for the eigenvalues and eigenfunctions of the q-Sturm-Liouville problem.

We will use the next two results from [6] in our discussion.

Theorem C *Concerning the above definitions, the following propositions hold:*

i) Given $c_1, c_2 \in C$, equation (7) has a unique solution ϕ, q-regular at zero and satisfying

$$\phi(0,\lambda) = c_1, \qquad D_{q^{-1}}\phi(0,\lambda) = c_2, \qquad \lambda \in \mathbf{C}.$$

Moreover, $\phi(x,\lambda)$ is entire in λ for all $x \in [0,a]$, where the Dq^{-1} derivative of a function $f(x)$ at zero is given by

$$D_{q^{-1}}f(x) = \lim_{n\to-\infty} \frac{f(xq^{-n}) - f(0)}{xq^{-n}} = D_q f(0).$$

Theorem D *The eigenvalues of the problem (7)-(9) form an infinite sequence of real numbers which can be ordered in an ascending way. Moreover, the set of all normalized eigenfunctions of (7)-(9) forms an orthonormal basis for $L^2_q(0,a)$.*

Essential in our discussion will be the q-Wronskian of two functions f and g defined as

$$W_q(f,g)(x) = f(x)D_q g(x) - g(x)D_q f(x) \tag{14}$$

It was proved by Meijer and Swarttouw that $\{f,g\}$ forms a complete set of solutions of (7) if and only if their q-Wronskian does not vanish at any point of $[0,a]$. The q-Wronskian of a q-Sturm-Liouville problem will play a fundamental role in the next section.

4. The q-sampling theory

In this section we will establish our main result. The key ingredient will be Kramer´s Lemma, discovered in [16]. It is usually stated with the Lebesgue measure dx but a q-version can be derived without modifying the structure of the proof. Actually, the result can be stated in a very general way, using the inner product in a general Hilbert space. Since it is clear that the q-integral defines a inner product in a Hilbert space, we simply state Kramer´s Lemma in the required form, and appeal to [13] for more information on the subject.

Theorem E *(Kramer´s Lemma). Let $I \subset R$ be a bounded interval, $K(x,t)$ be a kernel belonging to $L^2(I)$ for each fixed t in a suitable subset D of R. Suppose also that, for some sequence of points belonging to D, $\{K(x,\lambda_n)\}$ is an orthogonal basis for $L^2(I)$. Under this conditions, every function f written in the form*

$$f(t) = \int_I g(x)K(x,t)d_qx$$

admits the sampling expansion

$$f(t) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{\int_I \overline{K(x,\lambda_n)}K(x,t)d_qx}{\int_I |K(x,t)|^2\, d_qx}$$

the sampling series converges absolutely, and uniformly on every set $C \subset D$ *for which* $\|K(,t)\|$ *is bounded.*

Before the proof of our main result we need to establish some basic facts about the q-Wronskian of q-Sturm-Liouville problems.

Lemma 1 Let f and g be q-regular at zero. The Wronskian $W_q(f,g)(x)$ of the q-Sturm-Liouville problem (7) does not depend on x.

Proof. Applying the formula (5) in the proper order we obtain

$$\begin{aligned} D_q W_q(f,g)(x) &= D_q\left[f(x)(D_q g)(x) - g(x)(D_q f)(x)\right] \\ &= f(qx)D_q^2 g(x) - g(qx)D_q^2 f(x) \end{aligned}$$

Then

$$D_q W_q(f,g)(q^{-1}x) = f(x)(D_q^2 g)(q^{-1}x) - g(x)(D_q^2 f)(q^{-1}x)$$

$$= f(x)\left[v(x)g(x) - \lambda g(x)\right] - g(x)\left[v(x)f(x) - \lambda f(x)\right] = 0$$

As a result,

$$0 = D_q W_q(f,g)(q^{-1}x) = \frac{W_q(f,g)(q^{-1}x) - W_q(f,g)(x)}{q^{-1}x(1-q)}$$

or, for every $x \neq 0$,

$$W_q(f,g)(x) = W_q(f,g)(q^{-1}x)$$

Iterating gives

$$W_q(f,g)(q^n x) = W_q(f,g)(x)$$

for every $n = 1, 2, \ldots.$ Taking the limit when $n \to \infty$ we conclude that $W_q(f,g)(x) = W_q(f,g)(0)$, since $W_q(f,g)(x)$ is q-regular at zero. $\Box$

From now on, we will invoke Theorem C and choose, from the solutions of (7), a particular solution $\varphi(x,\lambda)$ such that it is an entire function of λ, real valued when λ is real and satisfying

$$\varphi(0,\lambda) = -a_{12}; D_{q^{-1}}\varphi(0,\lambda) = a_{11} \tag{15}$$

and another one satisfying

$$\psi(a,\lambda) = -a_{22}; D_{q^{-1}}\psi(a,\lambda) = a_{21} \tag{16}$$

since the q-wronskian is independent of x, we can evaluate it at $x = a$, and use the above conditions on ψ in order to write

$$W_q(\varphi,\psi)(\lambda) = W(\lambda) = a_{21}\varphi(a,\lambda) + a_{22}D_{q^{-1}}\varphi(a,\lambda) \tag{17}$$

It follows from the initial conditions (9) that $W(\lambda) = 0$ if and only if λ is an eigenvalue of the q-Sturm-Liouville problem. The set up is now complete to state our main result. The proof will go along the same lines of Zayed´s proof in [21].

Theorem 1 *Let $\varphi(x, \lambda)$ and $\psi(x, \lambda)$ be the solutions of (7) selected as above. Then every function f of the form*

$$f(\lambda) = \int_0^a u(x)\varphi(x, \lambda) d_q x, \quad u \in L^2(0, a) \tag{18}$$

can be written as the Lagrange-type sampling expansion

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{W(\lambda)}{W'(\lambda_n)(\lambda - \lambda_n)} \tag{19}$$

where $W(\lambda)$ is the Wronskian of the functions $\varphi(x, \lambda)$ and $\psi(x, \lambda)$.

Proof. Multiply equation (7) by $\varphi(x, \lambda_n)$. Then consider again equation (7), but replace λ by λ_n and multiply this last equation by $\varphi(x, \lambda)$. Subtracting the two results yields

$$(\lambda - \lambda_n)\, \varphi(x, \lambda)\varphi(x, \lambda_n) = D_q^2 \varphi(q^{-1}x, \lambda_n)\varphi(x, \lambda) - D_q^2 \varphi(q^{-1}x, \lambda)\varphi(x, \lambda_n)$$

an application of the rule for the q-differentiation of a product (4) gives, choosing the right order in both q-differentiations,

$$= D_q \left[D_q \varphi(q^{-1}x, \lambda_n)\varphi(x, \lambda) - D_q \varphi(q^{-1}x, \lambda)\varphi(x, \lambda_n) \right]$$

Performing a q-integration by means of (6) gives

$$\begin{aligned}
& (\lambda - \lambda_n) \int_0^a \varphi(x, \lambda)\varphi(x, \lambda_n) d_q x \\
= \;& D_q \varphi(q^{-1}a, \lambda_n)\varphi(a, \lambda) - D_q \varphi(q^{-1}a, \lambda)\varphi(a, \lambda_n) \\
& - D_q \varphi(0, \lambda_n)\varphi(0, \lambda) - D_q \varphi(0, \lambda)\varphi(0, \lambda_n) \\
= \;& \varphi(a, \lambda) D_{q^{-1}}(a, \lambda_n) - \varphi(a, \lambda_n) D_{q^{-1}} \varphi(a, \lambda)
\end{aligned}$$

the justification for the last identity lies on the fact that, by (15) and the initial conditions (8) we have

$$\begin{aligned}
& D_q \varphi(0, \lambda_n)\varphi(0, \lambda) - D_q \varphi(0, \lambda)\varphi(0, \lambda_n) \\
= \;& -D_{q^{-1}} \varphi(0, \lambda_n) a_{12} - a_{11} \varphi(0, \lambda_n) = 0
\end{aligned}$$

Now assume that $a_{21} \neq 0$. Multiply (17) by $\varphi(a, \lambda_n)$ to obtain

$$W(\lambda)\varphi(a, \lambda_n) = a_{21}\varphi(a, \lambda)\varphi(a, \lambda_n) + a_{22} D_{q^{-1}} \varphi(a, \lambda)\varphi(a, \lambda_n).$$

Using the initial condition (8) this identity becomes

$$W(\lambda)\varphi(a,\lambda_n) = -a_{22}D_{q^{-1}}\varphi(a,\lambda)\varphi(a,\lambda) + a_{22}D_{q^{-1}}\varphi(a,\lambda)\varphi(a,\lambda_n)$$
$$= a_{22}\left[\varphi(a,\lambda)D_{q^{-1}}(a,\lambda) - \varphi(a,\lambda_n)D_{q^{-1}}\varphi(a,\lambda)\right]$$

as a result,

$$(\lambda - \lambda_n)\int_0^a \varphi(x,\lambda)\varphi(x,\lambda_n)d_qx = \frac{W(\lambda)\varphi(a,\lambda_n)}{(\lambda-\lambda_n)\,a_{22}}$$

and taking the limit as $\lambda \to \lambda_n$ gives

$$\int_0^a |\varphi(x,\lambda_n)|^2\,d_qx = W'(\lambda_n)\frac{\varphi(a,\lambda_n)}{a_{21}}$$

Now, by Theorem D, $\{\varphi(x,\lambda_n)\}$ forms an orthogonal basis of $L^2(0,a)$. We can therefore apply Kramer´s lemma and write an integral transform of the form (18) as

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{W(\lambda)}{W'(\lambda_n)(\lambda-\lambda_n)} \tag{20}$$

Consider now the case $a_{21} = 0$. Multiply the identity (17) by $D_{q^{-1}}\varphi(a,\lambda_n)$ to obtain

$$W(\lambda)D_{q^{-1}}\varphi(a,\lambda_n) = a_{22}\varphi(a,\lambda)D_{q^{-1}}\varphi(a,\lambda_n)$$

on the other side, by (9) we have $D_{q^{-1}}\varphi(a,\lambda) = 0$, so that

$$(\lambda - \lambda_n)\int_0^a \varphi(x,\lambda)\varphi(x,\lambda_n)d_qx = \varphi(a,\lambda)D_{q^{-1}}\varphi(a,\lambda_n)$$

we conclude that

$$\int_0^a \varphi(x,\lambda)\varphi(x,\lambda_n)d_qx = \frac{W(\lambda)D_{q^{-1}}\varphi(a,\lambda_n)}{a_{22}\,(\lambda-\lambda_n)}$$

taking the limit as $\lambda \to \lambda_n$,

$$\int_0^a |\varphi(x,\lambda_n)|^2\,d_qx = W'(\lambda_n)\frac{D_{q^{-1}}\varphi(a,\lambda_n)}{(\lambda-\lambda_n)\,a_{22}}$$

and as before, the use of Kramer's lemma gives (20). □

Example: Consider the problem

$$-\frac{1}{q}D_{q^{-1}}D_qy(x) = \lambda y(x)$$

and the conditions

$$\begin{aligned} y(0) &= 0 \\ y(1) &= 1 \end{aligned}$$

A fundamental set of solutions is $\{\cos(\sqrt{\lambda}x;q), \sin(\sqrt{\lambda}x;q)\}$. From this we select $\varphi(x,\lambda) = \cos(\sqrt{\lambda}x;q)$ to satisfy the initial conditions. The eigenvalues $\{\lambda_n\}$ of the problem are the zeros of $\sin(\sqrt{\lambda};q)$, and from (12), if $q < (1-q^2)^2$ or if n is big enough, then $\lambda_n = (1-q)^{-2}q^{-2n+\epsilon_n}$. Since the Wronskian is a function of order zero and its zeros are the λ_n's , it follows from Theorem 1 that every function of the form

$$f(t) = \int_0^1 u(x)\cos(\sqrt{\lambda}x;q)d_qx$$

has the representation

$$f(\lambda) = \sum_{n=0}^{\infty} f((1-q)^{-2}q^{-2n+\epsilon_n}) \frac{\sin(\sqrt{\lambda};q)}{[\sin'(\sqrt{x};q)]_{x=\lambda_n}(\lambda-(1-q)^{-2}q^{-2n+\epsilon_n})}$$

The examples 2 and 3 from [5] can be treated in a similar way.

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