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LOGIC OF IMPLICATIONS

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Abstract: A sound and complete logic for implications (or quasi-equations) is presented, extending naturally Birkhoff's equational logic. This is based on a general logic for injectivity, following an idea of G. Roşu.

1. Introduction

The aim of the present paper is an extension of the equational logic of G. Birkhoff to a logic of implications (or quasi-equations) of the form

$$\left(\bigwedge_{i=1}^{n} u_i = v_i\right) \Rightarrow u = v$$

We follow an idea of G. Roşu [5] who presented a logic of injectivity in an abstract category $\mathcal{K}$: recall that an object K is *injective* w.r.t. an epimorphism $e : A \to B$ if and only if for every morphism $f : A \to K$ there exists $g : B \to K$ with $f = g \cdot e$. The logic of injectivity is devoted to characterizing the logical consequences of a set $\mathcal{E}$ of epimorphisms: a logical consequence is an epimorphism e such that an object is e-injective whenever it is f-injective for every $f \in \mathcal{E}$. We formulate deduction rules for injectivity and we prove that the deduction system is sound and complete whenever one restricts to epimorphisms between finitely presentable objects. In our previous work [4] and [6] we investigated similar topics: an abstract characterization of injectivity classes. In [4] we worked with injectivity w.r.t. morphisms (not necessarily epimorphisms) having finitely presentable domains and codomains. In fact, by using one result of that paper we are able to prove the completeness result of our rules in a few lines; see Theorem 2.12 below. The main idea of [4] was to use the calculus of fractions of Gabriel and Zisman [3], and this

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leads us to a deduction system for injectivity which is (very similar, but) slightly different from that of G. Roşu in [5]. Using our injectivity deduction system we obtain, in the last section, a logic of implications which extends the equational logic of G. Birkhoff.

2. Logic of Injectivity

2.1. Remark In the present section a logic of deriving injectivity (=orthogonality) w.r.t. a class of epimorphisms is presented. This is a variation of the logic presented by G. Roşu: the rules are slightly different and the assumptions for proving the completeness of the logic are also slightly different. We use ideas of the classical work [3] of Gabriel and Zisman on the calculus of fractions, as exploited by Hébert, Adámek and Rosický in [4], see also [6]. We start by recalling that concept.

2.2. Definition A class $\mathcal{E}$ of morphisms in a category $\mathcal{A}$ is said to *satisfy the calculus of fractions* provided that

(i) $\mathcal{E}$ contains all identity morphisms,
(ii) $\mathcal{E}$ is closed under composition,
(iii) for every span

$$\begin{tikzcd} & A \arrow[dl, "e\in\mathcal{E}"'] \arrow[dr, "f"] & \\ B & & C \end{tikzcd}$$

there exists a commutative square

$$\begin{tikzcd} & A \arrow[dl, "e\in\mathcal{E}"'] \arrow[dr, "f"] & \\ B \arrow[dr, "f'"'] & & C \arrow[dl, "e'\in\mathcal{E}"] \\ & D & \end{tikzcd}$$

and

(iv) given parallel morphisms $h_1, h_2 : B \to C$ in $\mathcal{A}$ such that $h_1 e = h_2 e$ for some morphism $e \in \mathcal{E}$ then there exists $e' \in \mathcal{E}$ with $e'h_1 = e'h_2$.

2.3. Remark (a) In particular, whenever a category has pushouts, every class $\mathcal{E}$ of epimorphisms containing all identity morphisms and closed under composition and pushout satisfies the calculus of fractions.

(b) If $\mathcal{E}$ is a class of epimorphisms, the condition (iv) can be omitted: from $h_1 e = h_2 e$ we conclude $h_1 = h_2$.

2.4. Notation Given a category $\mathcal{A}$, we denote by $\mathcal{A}_{fp}$ a full subcategory representing up to isomorphism all *finitely presentable* objects, i.e., objects A such that $\hom(A, -) : \mathcal{A} \to \mathbf{Set}$ preserves filtered colimits.

2.5. Proposition(see [4]) *Let $\mathcal{A}$ be a cowellpowered category with colimits, and let $\mathcal{E}$ be a class of epimorphisms of $\mathcal{A}_{fp}$ satisfying the calculus of fractions in $\mathcal{A}_{fp}$. Then every finitely presentable object A in $\mathcal{A}$ has an $\mathcal{E}$-injective reflection $r_A : A \to A^*$ obtained by a filtered colimit of all morphisms $e : A \to X$ of $\mathcal{E}$ with $X \in \mathcal{A}_{fp}$.*

More detailed: let $A \downarrow \mathcal{E}$ be the full subcategory of the comma category $A \downarrow \mathcal{A}_{fp}$ (of all morphisms with domain A and codomain in $\mathcal{A}_{fp}$) formed by members of $\mathcal{E}$. Then the diagram

$$D_A : A \downarrow \mathcal{E} \to \mathcal{A}, e \mapsto X$$

is filtered, and if A^* is the colimit of D_A with the colimit cone $e^* : X \to A^*$ (for $e : A \to X$ in $A \downarrow \mathcal{E}$) then $r_A = id_A^*$ is the reflection of A in the full subcategory of $\mathcal{A}$ of all $\mathcal{E}$-injective objects. This means that A^* is $\mathcal{E}$-injective, and given a morphism $f : A \to B$ such that B is $\mathcal{E}$-injective, then f factorizes through r_A. This was proved in [4] assuming that $\mathcal{A}$ is a finitely accessible category. But finite accessibility was only used to make the diagram D_A essentially small. Since in the present paper $\mathcal{E}$ is a class of epimorphisms, this follows from $\mathcal{A}$ being cowellpowered.

2.6. Notation An epimorphism e is said to be a *logical consequence* of a class $\mathcal{E}$ of epimorphisms if every $\mathcal{E}$-injective object is e-injective. Notation:

$$\mathcal{E} \models e$$

2.7. Remark It was an idea of G. Roşu to present logical rules for axiomatizing the logical consequence $\models$ above. The calculus of fractions above (minus rule (iv) which we do not need since we work with epimorphisms) inspires us to a slightly different selection of rules:

2.8. Injectivity Deduction System *consists of one axiom*

$$\textit{Axiom :} \qquad \frac{}{\mathrm{id}_A}$$

and the following three deduction rules:

$$\textit{Composition:} \frac{e, \ e'}{e' \cdot e} \quad \text{if the codomain of } e \text{ is the domain of } e'$$

$$\textit{Cancellation:} \frac{e' \cdot e}{e}$$

$$\textit{Pushout:} \qquad \frac{e}{e'} \quad \text{for every pushout} \quad \begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "f"'] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "e'"] & \cdot \end{tikzcd}$$

2.9. Notation Let $\mathcal{E}$ be a class of epimorphisms of a category $\mathcal{A}$. We use

$$\mathcal{E} \vdash e$$

to denote the fact that e can be proved from $\mathcal{E}$ by using the Injectivity Deduction System. That is, there exists a list $e_1, e_2, \cdots, e_n$ of morphisms such that $e_n = e$ and, for every $i = 1, 2, \cdots, n$, either e_i is an identity morphism, or e_i is the conclusion of one of the above rules such that the assumptions of that rule are among $e_1, \cdots, e_{i-1}$.

2.10. Lemma *The deduction system 1.8 is sound, i.e.,* $\mathcal{E} \vdash e$ *implies* $\mathcal{E} \models e$.

Proof (1) Every object is injective with respect to id_A, for all A.

(2) Injectivity w.r.t. e and e' implies, obviously, injectivity w.r.t. $e' \cdot e$.

(3) Injectivity w.r.t. $e' \cdot e$ implies, obviously, injectivity w.r.t. e.

(4) Let X be e-injective and let f be an arbitrary morphism from the domain of e' to X

$$\begin{tikzcd}
\cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \arrow[ddr, "g"] & \\
\cdot \arrow[r, "e'"] \arrow[drr, "f"'] & \cdot \arrow[dr, "h"] & \\
 & & X
\end{tikzcd}$$

The e-injectivity of X yields g with $ge = fu$, and the universal property yields h with $f = he'$. □

2.11. Example The above Injectivity Deduction System is not complete in general. Indeed, consider the signature Σ consisting of a countable set of nullary symbols $a_0, a_1, a_2, \cdots$ and let $\mathcal{E}$ be the collection of all epimorphisms

$$e_n : I \to I' \text{ in } Alg\,\Sigma \qquad (n = 1, 2, ...)$$

such that I is the initial algebra and the kernel of e_n has one equivalence class $\{a_0^I, \cdots, a_n^I\}$ and all other equivalence classes are singleton sets.

It is obvious that an algebra A is $\mathcal{E}$-injective iff all the constants in A are equal. Consequently, the trivial quotient

$$e_0 : I \to 1$$

has the property that

$$\mathcal{E} \models e_0.$$

Nevertheless

$$\mathcal{E} \not\vdash e_0.$$

In fact, for every epimorphism

$$e : B \to B' \text{ with } \mathcal{E} \vdash e$$

we see that

(1.1) The number of pairs $i, j \in \omega$ with $a_i^B \neq a_j^B$ but $a_i^{B'} = a_j^{B'}$ is finite.

Indeed, identity morphisms satisfy (1.1), and the collection of all morphisms satisfying (1.1) is closed under composition, cancellation and pushout. Obviously e_0 does not satisfy (1.1).

2.12. Completeness Theorem *Given a cowellpowered category $\mathcal{A}$ with colimits, the Injectivity Deduction System is complete in $\mathcal{A}_{fp}$. That is, for every set $\mathcal{E} \cup \{e_0\}$ of epimorphisms in $\mathcal{A}_{fp}$,*

$$\mathcal{E} \models e_0 \quad \textit{implies } \mathcal{E} \vdash e_0.$$

Proof Let $\overline{\mathcal{E}}$ denote the set of all epimorphims e in $\mathcal{A}_{fp}$ such that $\mathcal{E} \vdash e$ in $\mathcal{A}_{fp}$. Clearly $\overline{\mathcal{E}}$ contains all identity morphisms and is closed under composition and pushout. By 2.3(a), $\overline{\mathcal{E}}$ satisfies the calculus of fractions. By 2.10, $\mathcal{E}$-injectivity implies $\overline{\mathcal{E}}$ -injectivity.

Given a logical consequence $e_0 : A \to B$ of $\mathcal{E}$ in $\mathcal{A}_{fp}$, let $r_A : A \to A^*$ be the $\overline{\mathcal{E}}$-injective reflection of 2.5. Since A^* is e_0-injective, we have a morphism f with $r_A = f \cdot e_0$.

Since B is finitely presentable, $hom(B, -)$ preserves the filtered colimit $A^* = Colim D_A$. Thus, f factorizes as $f = e^* \cdot f'$, for some colimit morphism $e^* : X \to A^*$ of D_A:

$$\begin{tikzcd} A \arrow[r, "e_0"] \arrow[d, "e"'] & B \arrow[d, "f"] \arrow[dl, "f'"'] \\ X \arrow[r, "e^*"'] & A^* \end{tikzcd}$$

The diagram D_A is filtered and the colimit morphism $e^* : X \to A^*$ merges the pair $e, f' \cdot e_0 : A \to X$. Since A is finitely presentable, this implies the existence of an object of $A \downarrow \overline{\mathcal{E}}$ (i.e., a morphism $d : A \to Y$ in $\overline{\mathcal{E}}$) and a morphism $g : X \to Y$ such that $g \cdot e = d$ and g also merges the above pair. Hence $g \cdot f' \cdot e_0 = d$, and then

$$E \vdash d \text{ implies } E \vdash e_0,$$

by Cancellation in 2.8. $\square$

2.13. Remark (i) Our assumptions differ from those used by G. Roşu in [5] for his completeness theorem: he only required the morphism $e_0 : A \to B$ to be finitely presentable as an object of the comma category $A \downarrow \mathcal{A}$, which is strictly weaker than our assumption that e_0 and all morphisms of $\mathcal{E}$ have finitely presentable domains and codomains. However G. Roşu assumed that the domains of his epimorphisms are projective, which is a strong assumption that our intended application (to quasivarieties) does not fulfill. Furthermore, instead of composition in 2.8 G. Roşu uses "union" stating that a pushout of two morphisms derived from $\mathcal{E}$ is derived from $\mathcal{E}$. This follows clearly from Pushout and Composition in 2.8. On the other hand, our formulation of the pushout rule is somewhat more restrictive than that used by G. Roşu : in applications of the above rule Pushout, e is any morphism derived from $\mathcal{E}$, whereas Roşu's rule works with $e \in \mathcal{E}$.

(ii) Every cowellpowered category with colimits has a factorization system (epi, strong mono), see [1]. Theorem 2.12 can be generalized to any cocomplete and E-cowellpowered category with a factorization system (E, M), which is the approach taken in [5].

2.14. Example Here we demonstrate that in Completeness Theorem 2.12 we cannot weaken the assumptions that the domains and codomains of morphisms $\mathcal{E} \cup \{e_0\}$ be finitely presentable to the assumption, used in [5], that these morphisms are finitely presentable, as objects of the arrow category.

In fact, in the category $Alg(\Sigma)$, where Σ consists of nullary symbols $a_n, n \in \mathbb{N}$, we exhibit finitely presentable epimorphisms $\mathcal{E} \cup \{e\}$ with

$$\mathcal{E} \models e \text{ but } \mathcal{E} \nvdash e.$$

A Σ-algebra A is finitely presentable iff

(i) all but finitely many elements of A have the form a_n^A, for some $n \in \mathbb{N}$ and

(ii) there are only finitely many pairs m, n with $a_m^A \neq a_n^A$.

An epimorphism $h : A \to B$ in $Alg(\Sigma)$ is finitely presentable iff there are only finitely many pairs m, n with $a_m^A \neq a_n^A$ and $a_m^B = a_n^B$.

Denote by 1 the terminal Σ-algebra, by $I = \{a_0, a_1, a_2, \cdots\}$ the initial Σ-algebra and by C the algebra $C = \{0, 1\}$ having $a_0^C = 0$ and $a_i{}^C = 1$ for all $i \geq 1$.

Let

$$e_0 : C \to 1$$

be the trivial epimorphism and for every $k \geq 1$ define the quotient

$$e_k : I \to I_k = I/{\sim_k}$$

of I modulo the least congruence $\sim_k$ with a_k congruent to a_{k+1}.

For

$$\mathcal{E} = \{e_0, e_1, e_2, \cdots\}$$

an algebra is $\mathcal{E}$-injective iff all constants of Σ in it are equal. Thus if

$$e : I \to I/\approx$$

denotes the quotient modulo the least congruence with $a_0 \approx a_1$, we have that $\mathcal{E} \models e$.

We will prove that $\mathcal{E} \not\vdash e$ by finding a set $\overline{\mathcal{E}}$ of epimorphisms with

$$\{id_A | A \in Alg(\Sigma)\} \cup \mathcal{E} \subseteq \overline{\mathcal{E}} \quad \text{but} \quad e \notin \overline{\mathcal{E}}$$

which is closed under pushout, composition and left cancellation. This proves that $\overline{\mathcal{E}}$ contains all consequences of $\mathcal{E}$, thus $\mathcal{E} \not\vdash e$.

Let $\overline{\mathcal{E}}$ be the set of all epimorphisms $g : B \to B'$ such that

(1) g is a finitely presentable morphism,
(2) if $g(x) = g(x')$ and $x \neq x'$ then $x = a_i^B$ and $x' = a_j^B$ for some i, j, and
(3) if B is finitely presentable and $a_0^B \neq a_j^B$ for all $j \geq 1$, then $a_0^{B'} \neq a_j^{B'}$ for all $j \geq 1$.

It is clear that $\overline{\mathcal{E}}$ contains $\{id_A | A \in Alg(\Sigma)\} \cup \mathcal{E}$ but it does not contain e (since e does not fulfil (3)).

a. $\overline{\mathcal{E}}$ is closed under pushout. In fact, let

$$\begin{tikzcd} B \arrow[r, "g"] \arrow[d, "h"'] & B' \arrow[d, "l"] \\ D \arrow[r, "f"] & D' \end{tikzcd}$$

be a pushout with $g \in \overline{\mathcal{E}}$. It is easy to see that, since g is finitely presentable, so is f. Thus it remains to verify (2) and (3). By the description of pushouts (in **Set**, hence in $Alg(\Sigma)$) f merges a pair of elements $x \neq x'$ of D iff there exist the following zig-zag of elements $x = x_0, \cdots, x_{n+1} = x'$ of D and elements $y_0, \cdots, y_{2n+1}$ of B

$$\begin{tikzcd}[column sep=small] & y_0 \arrow[dl] & y_1 \arrow[dr] & & y_2 \arrow[dl] & \cdots & y_{2n-1} \arrow[dr] & & y_{2n} \arrow[dl] & y_{2n+1} \arrow[dr] & \\ x & & & x_1 & & \cdots & & x_n & & & x' \end{tikzcd}$$

such that $x_k = h(y_{2k-1}) = h(y_{2k})$ for $k = 1, ..., n$, and $g(y_{2k}) = g(y_{2k+1})$, with $y_{2k} \neq y_{2k+1}$, for all $k = 0, \cdots, n$.

To prove (2), let $f(x) = f(x')$ with $x \neq x'$ and let us choose a zig-zag as above. Since g fulfils (2), the equality $g(y_0) = g(y_1)$ implies $y_0 = a_i^B$ for some i, thus $x = h(y_0) = a_i^D$. Analogously with x'.

To prove (3), we assume that there exists a $k \geq 1$ with $a_0^{D'} = a_k^{D'}$, but $a_0^D \neq a_j^D$, for all $j \geq 1$, then we verify that D is not finitely presentable. We have $a_0^B \neq a_j^B$, for all $j \geq 1$, and we will prove that $a_0^{B'} = a_l^{B'}$ for some $l \geq 1$: it then follows that B is not finitely presentable, since g fulfils (3). Consequently, there exist infinitely many pairs (u, v) with $a_u^B = a_v^B$ but $u \neq v$. For each such pair we have $a_u^D = a_v^D$, thus D is not finitely presentable.

b. $\overline{\mathcal{E}}$ is closed under composition. In fact, given two composable morphisms

$$B \xrightarrow{g} B' \xrightarrow{g'} B''$$

in $\overline{\mathcal{E}}$ it is easy to see that $g' \cdot g$ is finitely presentable. It fulfils (2) because, given $g'(g(x)) = g'(g(x'))$ and $x \neq x'$, then either $g(x) = g(x')$ and we apply (2) to g, or $g(x) \neq g(x')$ and then (2) applied to g' yields $g(x) = a_i^B$ and $g(x') = a_j^B$, and then, again, we apply (2) to g. Finally, $g' \cdot g$ fulfils (3): assume

$$a_0^{B''} = a_k^{B''} \text{ for some } k, \text{ but } \; a_0^B \neq a_j^B \text{ for all } j \geq 1.$$

If $a_0^{B'} = a_l^{B'}$ for some $l \geq 1$, then B is not finitely presentable due to (3) applied to g. If $a_0^{B'} \neq a_l^{B'}$ for all $l \geq 1$ then B' is not finitely presentable, due to (3) applied to g'. The latter implies again that B is not finitely presentable: recall that $g : B \to B'$ is a finitely presentable morphism, thus, B is a finitely presentable object iff B' is one.

c. $\overline{\mathcal{E}}$ is closed under left cancellation. Given

$$B \xrightarrow{g} B' \xrightarrow{g'} B''$$

with $g' \cdot g$ in $\overline{\mathcal{E}}$, then g clearly fulfils (1) and (2). It fulfils (3) because $g' \cdot g$ does.

3. Logic of Implications

3.1. Assumption Σ denotes a finitary, one-sorted signature. We assume a fixed countable set V of variables. A free Σ-algebra on a set $W \subseteq V$ is denoted by $\phi(W)$.

An *equation* is a pair of elements of $\phi(V)$, notation: $u = v$. An *implication* is a formal expression

$$\mathcal{P} \Rightarrow u = v$$

where $\mathcal{P}$ is a finite set of equations.

3.2. Remark (a) If $\mathcal{P} = \{s_1 = t_1, \ldots, s_n = t_n\}$ and $\{x_1, \ldots, x_k\}$ is the set of all variables which appear in the implication, then the implication

$$I \equiv (\mathcal{P} \Rightarrow u = v)$$

is a shorthand for the first-order formula

$$(\forall x_1) \ldots (\forall x_k) \left((\bigwedge_{i=1}^{n} s_i = t_i) \Rightarrow (u = v) \right) .$$

Thus a Σ-algebra A *satisfies* I iff for every interpretation of variables, i.e., every homomorphism $f : \phi(W_I) \to A$, we have:

$$f(s_i) = f(t_i) \text{ for } i = 1, \ldots, n \text{ implies } f(u) = f(v).$$

(b) Below we work with a (non specified) finite set $W \subset V$ of variables. Since we always deal with finitely many implications at a time, some set W like that is always sufficient.

3.3. Notation Given an implication

$$I \equiv (\mathcal{P} \Rightarrow u = v),$$

we denote by $\sim_{\mathcal{P}}$ the congruence on $\phi(W)$ generated by the equations in $\mathcal{P}$ with the corresponding quotient map

$$q_{\mathcal{P}} : \phi(W) \to \phi(W)/ \sim_{\mathcal{P}} \ .$$

And we denote by $\sim_I$ the congruence on $\phi(W)$ generated by the equations in $\mathcal{P} \cup \{u = v\}$ with the corresponding quotient map

$$q_I : \phi(W) \to \phi(W)/ \sim_I \ .$$

We obtain a quotient map

$$e_I : \phi(W)/ \sim_{\mathcal{P}} \longrightarrow \phi(W)/ \sim_I$$

such that the triangle

$$\begin{tikzcd}[column sep=small] & \phi(W) \arrow[dl, "q_{\mathcal{P}}"'] \arrow[dr, "q_I"] & \\ \phi(W)/\sim_{\mathcal{P}} \arrow[rr, "e_I"'] & & \phi(W)/\sim_I \end{tikzcd}$$

commutes.

3.4. Notation A *substitution* is a function assigning to every variable a term or, equivalently, a homomorphism $\sigma : \phi(W) \to \phi(W')$, for W, $W' \subseteq V$. We write u^σ instead of $\sigma(u)$, and for every equation $u = v$ we denote by $(u = v)^\sigma$ the equation $u^\sigma = v^\sigma$; for every implication $I \equiv (\mathcal{P} \Rightarrow u = v)$ we denote by I^σ the implication $\mathcal{P}^\sigma \Rightarrow u^\sigma = v^\sigma$ where $\mathcal{P}^\sigma = \{e^\sigma : e \in \mathcal{P}\}$.

3.5. Remark It has been first observed by B. Banaschewski and H. Herrlich [2] that a Σ-algebra satisfies an implication I iff it is injective w.r.t. the regular epimorphism e_I. And conversely: for every regular epimorphism e in *Alg* Σ whose domain and codomain are finitely presentable, e-injectivity can be expressed by a finite family of implications.

3.6. Deduction System for Implications Our deduction system consists of two axioms

Axiom 1: $$\frac{}{\{u = v\} \Rightarrow u = v}$$

Axiom 2: $$\frac{}{\emptyset \Rightarrow u = u}$$

and the following deduction rules

Symmetry: $$\frac{\mathcal{P} \Rightarrow u = v}{\mathcal{P} \Rightarrow v = u}$$

Transitivity: $$\frac{\mathcal{P} \Rightarrow u = v,\ \mathcal{P} \Rightarrow v = w}{\mathcal{P} \Rightarrow u = w}$$

Congruence: $$\frac{\mathcal{P} \Rightarrow u_1 = v_1,\ \dots,\ \mathcal{P} \Rightarrow u_n = v_n}{\mathcal{P} \Rightarrow f(u_1, \dots, u_n) = f(v_1, \dots, v_n)}$$

for all n-ary symbols f in Σ.

Invariance: $$\frac{\mathcal{P} \Rightarrow u = v}{\mathcal{P}^\sigma \Rightarrow u^\sigma = v^\sigma}$$

for all substitutions σ.

Weakening: $$\frac{\mathcal{P} \Rightarrow u = v}{\mathcal{P} \cup \{u' = v'\} \Rightarrow u = v}$$

Cut: $$\frac{\mathcal{P} \Rightarrow u' = v',\ \mathcal{P} \cup \{u' = v'\} \Rightarrow u = v}{\mathcal{P} \Rightarrow u = v}$$

In all these axioms and rules u, v and w, with additional indices and primes, denote arbitrary terms in $\phi(V)$ and $\mathcal{P}$ denotes an arbitrary finite set of equations.

3.7. Remark This deduction system extends naturally Birkhoff's equational logic (consisting of Axiom 2 and the first four deduction rules with $\mathcal{P} = \emptyset$).

3.8. Notation For a given set E of implications and an implication I, we write

$$E \models I$$

if I is a logical consequence of E, i.e., whenever an algebra satisfies all implications in E, then it satisfies I. And we write

$$E \vdash I$$

if there exists a formal proof of I from E using the Deduction System 3.6.

3.9. Lemma *The following deduction rules follow from 3.6:*

(i) $$\frac{}{\mathcal{P} \Rightarrow u = u}$$

(ii) $$\frac{}{\mathcal{P} \Rightarrow u = v} \qquad \textit{if } u = v \textit{ is a member of } \mathcal{P}.$$

(iii) $$\frac{\mathcal{P} \Rightarrow s_i = t_i (i = 1, \dots, n),\ \mathcal{P} \cup \{s_i = t_i\}_{i=1}^n \Rightarrow u = v}{\mathcal{P} \Rightarrow u = v}$$

Proof (i) Axiom 2 and Weakening yield

$$\frac{}{\{u' = v'\} \Rightarrow u = u}$$

which, using Weakening again, yields

$$\frac{}{\{u' = v',\, u'' = v''\} \Rightarrow u = u}$$

etc.

(ii) Analogous to (i): use Axiom 1 and Weakening.

(iii) If $n = 1$, this is Cut. For $n = 2$ use Weakening and Cut to get

$$\cfrac{\cfrac{\mathcal{P} \Rightarrow s_1 = t_1,\, \mathcal{P} \Rightarrow s_2 = t_2,\, \mathcal{P} \cup \{s_1 = t_1,\, s_2 = t_2\} \Rightarrow u = v}{\mathcal{P} \cup \{s_1 = t_1\} \Rightarrow s_2 = t_2,\, (\mathcal{P} \cup \{s_1 = t_1\}) \cup \{s_2 = t_2\} \Rightarrow u = v}}{\mathcal{P} \cup \{s_1 = t_1\} \Rightarrow u = v}$$

Another application of Cut gets the desired result

$$\frac{\mathcal{P} \cup \{s_1 = t_1\} \Rightarrow u = v,\, \mathcal{P} \Rightarrow s_1 = t_1}{\mathcal{P} \Rightarrow u = v}$$

Analogously for $n \geq 3$. □

3.10. Remark The rule 3.9(iii) has a stronger form, namely,

$$\frac{\mathcal{P} \Rightarrow s_i = t_i (i = 1, \ldots, n),\, \{s_i = t_i\}_{i=1}^n \Rightarrow u = v}{\mathcal{P} \Rightarrow u = v}$$

In fact, from $\{s_i = t_i\}_{i=1}^n \Rightarrow u = v$, we obtain $\mathcal{P} \cup \{s_i = t_i\}_{i=1}^n \Rightarrow u = v$ by applying Weakening successively.

3.11. Lemma *If $\mathcal{P}$ is a finite set of equations in $\phi(W)$ then for every pair $u \sim_{\mathcal{P}} v$ of congruent terms we have a proof of $\mathcal{P} \Rightarrow u = v$. Shortly*

$$\vdash (\mathcal{P} \Rightarrow u = v) \text{ whenever } u \sim_{\mathcal{P}} v.$$

Proof The relation R of all pairs $(u, v) \in \phi(W) \times \phi(W)$ such that $\mathcal{P} \Rightarrow u = v$ is reflexive by 3.9 (i). Using Symmetry, Transitivity and Congruence in 3.6 we conclude that R is a congruence on $\phi(W)$. Since R contains $\mathcal{P}$, due to 3.9(ii), it contains $\sim_{\mathcal{P}}$. Thus, $u \sim_{\mathcal{P}} v$ implies uRv. □

3.12. Lemma *Given homomorphisms*

$$\begin{tikzcd} \phi(W)/\sim_{\mathcal{P}} \arrow[r, "e"] \arrow[d, "f"'] & \phi(W)/\sim_{\mathcal{P}\cup\mathcal{Q}} \\ \phi(W')/\sim_{\mathcal{P}'} & \end{tikzcd}$$

for some finite sets of variables W and W' and some finite sets $\mathcal{P}, \mathcal{Q}$ and $\mathcal{P}'$ of equations, where e is the canonical quotient morphism, there exists a substitution $\sigma : \phi(W) \to \phi(W')$ with

$$\mathcal{P}^\sigma \textit{ included in } \sim_{\mathcal{P}'}$$

such that the canonical quotient homomorphism $e' : \phi(W')/\sim_{\mathcal{P}'} \to \phi(W')/\sim_{\mathcal{P}' \cup \mathcal{P}^\sigma \cup \mathcal{Q}^\sigma}$ forms a pushout of e along f:

$$\begin{tikzcd}
\phi(W)/\sim_{\mathcal{P}} \arrow[r, "e"] \arrow[d, "f"'] & \phi(W)/\sim_{\mathcal{P}\cup\mathcal{Q}} \arrow[d, dashed] \\
\phi(W')/\sim_{\mathcal{P}'} \arrow[r, "e'"'] & \phi(W')/\sim_{\mathcal{P}'\cup\mathcal{P}^\sigma\cup\mathcal{Q}^\sigma}
\end{tikzcd}$$

Proof For the given homomorphism

$$f : \phi(W)/\sim_{\mathcal{P}} \longrightarrow \phi(W')/\sim_{\mathcal{P}'}$$

find a substitution σ such that the square

$$\begin{tikzcd}
\phi(W) \arrow[r, "q_{\mathcal{P}}"] \arrow[d, "\sigma"'] & \phi(W)/\sim_{\mathcal{P}} \arrow[d, "f"] \\
\phi(W') \arrow[r, "q_{\mathcal{P}'}"'] & \phi(W')/\sim_{\mathcal{P}'}
\end{tikzcd}$$

commutes. (To do so, choose a function i splitting $q_{p'}$, i.e., $q_{\mathcal{P}'} \cdot i = \mathrm{id}$. The morphism

$$W \xrightarrow{\eta} \phi(W) \xrightarrow{q_{\mathcal{P}}} \phi(W)/\sim_{\mathcal{P}} \xrightarrow{f} \phi(W')/\sim_{\mathcal{P}'} \xrightarrow{i} \phi(W')$$

has a unique extension to a substitution, i.e., an homomorphism $\sigma : \phi(W) \to \phi(W')$ such that $\sigma \cdot \eta = i \cdot f \cdot q_{\mathcal{P}} \cdot \eta$. Composed with $q_{\mathcal{P}'}$ this yields $(q_{\mathcal{P}'} \cdot \sigma) \cdot \eta = (f \cdot q_{\mathcal{P}}) \cdot \eta$, thus the square above commutes.) Observe that a pushout of the quotient map $e_F : \phi(W)/\sim_{\mathcal{P}} \longrightarrow \phi(W)/\sim_{\mathcal{P}\cup\{u_j=v_j,\, j\in J\}}$ along f is a quotient map $q : \phi(W')/\sim_{\mathcal{P}'} \longrightarrow \phi(W')/\approx$, where $\approx$ is the smallest congruence containing $\sim_{\mathcal{P}'}$ and such that $q \cdot f : \phi(W)/\sim_{\mathcal{P}} \longrightarrow \phi(W')/\approx$ factorizes through e_F. The latter condition implies that given $t, s \in \phi(W)$ with $e_F \cdot q_{\mathcal{P}}(t) = e_F \cdot q_{\mathcal{P}}(s)$ then $q \cdot f \cdot q_{\mathcal{P}}(t) = q \cdot f \cdot q_{\mathcal{P}}(s)$. Now $e_F \cdot q_{\mathcal{P}}$ is the quotient map of $\sim_F$ and $q \cdot f \cdot q_{\mathcal{P}} = q \cdot q_{\mathcal{P}'} \cdot \sigma$ where $q \cdot q_{\mathcal{P}'}$ is the quotient map of $\approx$. Thus, the latter condition states that

$$t \sim_F s \text{ implies } t^\sigma \approx s^\sigma.$$

In other words, $\approx$ is the smallest congruence containing $\mathcal{P}'$ and $\mathcal{P}^\sigma \cup \{u_j^\sigma = v_j^\sigma,\ j \in J\}$. Thus, a pushout of e and f has the form

$$\begin{tikzcd}
\phi(W)/\sim_{\mathcal{P}} \arrow[r, "e"] \arrow[d, "f"'] & \phi(W)/\sim_{\mathcal{P}\cup\mathcal{Q}} \arrow[d, "g"] \\
\phi(W')/\sim_{\mathcal{P}'} \arrow[r, "e'"'] & \phi(W')/\sim_{\mathcal{P}'\cup\mathcal{P}^\sigma\cup\mathcal{Q}^\sigma}
\end{tikzcd}$$

where e' is the canonical quotient morphism.

Moreover $\mathcal{P}^\sigma$ is included in $\sim_{\mathcal{P}'}$ because $q_{\mathcal{P}'} \cdot \sigma = f \cdot q_{\mathcal{P}}$ implies that

$$\text{given } u = v \text{ in } \mathcal{P} \text{ then } u^\sigma \sim_{\mathcal{P}'} v^\sigma. \qquad \square$$

3.13. Remark It is well known that the finitely presentable objects of $Alg\,\Sigma$ are precisely those isomorphic to the quotient algebras $\phi(W)/\sim_{\mathcal{P}}$ where $W \subseteq V$ is a finite set of variables and $\mathcal{P}$ a finite set of equations in $\phi(W)$. And it is easy to verify that, analogously, the epimorphisms between finitely presentable objects are precisely those isomorphic (in the arrow-category) to the canonical quotient maps $e : \phi(W)/\sim_{\mathcal{P}} \to \phi(W)/\sim_{\mathcal{P}\cup\mathcal{Q}}$ (where $\mathcal{P}$ and $\mathcal{Q}$ are finite sets of equations in $\phi(W)$). The set of all these canonical epimorphisms is denoted by $\mathcal{E}_{fp}$. Taking into account that $\mathcal{E}_{fp}$ is closed under composition and left-cancellable, and using Lemma 3.12, it is obvious that the Completeness Theorem 2.12 remains true if we apply it (instead of to all epimorphisms of A_{fp}) just to the set $\mathcal{E}_{fp}$.

3.14. Theorem *The deduction system of 3.6 is sound and complete. That is:*

$$E \models I \quad \textit{iff} \quad E \vdash I$$

for every set E of implications and every implication I.

Proof It is easy to verify the soundness.

To prove that 3.6 is complete, we use the completeness theorem of Section 2 and translate it to the deduction system 3.6. For doing so we are going to work with finite nonempty sets F of implications having the same antecedent,

$$F = \{\mathcal{P} \Rightarrow u_1 = v_1,\ \dots,\ \mathcal{P} \Rightarrow u_n = v_n\}.$$

We denote by $\mathbb{F}$ the set of all such sets F and put

$$\mathcal{P}_F = \mathcal{P} \quad \text{and} \quad \mathcal{Q}_F = \{u_1 = v_1,\ \dots,\ u_n = v_n\}$$

which are finite sets of equations in $\phi(W)$ (for some finite set $W \subseteq V$ of variables).

An implication $I \equiv (\mathcal{P} \Rightarrow u = v)$ is considered as a member of $\mathbb{F}$ by identifying it with the corresponding singleton set I. We write

$$F \vdash G \quad (F, G \in \mathbb{F})$$

if every member of G can be derived from the finite set F by applying the rules of 3.6.

For every $F \in \mathbb{F}$ we form the canonical epimorphism

$$e_F : \phi(W)/\sim_{\mathcal{P}} \longrightarrow \phi(W)/\sim_{\mathcal{P}\cup\mathcal{Q}}$$

(where $\mathcal{P} = \mathcal{P}_F$ and $\mathcal{Q} = \mathcal{Q}_F$, we drop the index F whenever no confusion can arise). This is consistent with Notation 3.3. Let $\mathcal{A}$ be the category of Σ-algebras and let $\mathcal{A}_{fp}$ be the category of finitely presentable algebras of the form $\phi(W)/\sim_{\mathcal{P}}$, where $W \subseteq V$ is a finite set of variables and $\mathcal{P}$ is a finite set of equations (with $\sim_{\mathcal{P}}$ denoting the congruence generated by $\mathcal{P}$). By Theorem 2.12 and Remark 3.13, the Injectivity Deduction System is complete for epimorphisms in $\mathcal{E}_{fp}$, and we will use this completeness to prove the present theorem by verifying the following:

(A) an application of the rules of 2.8 to morphisms $e_{F_1}, \cdots, e_{F_n}$ $(F_i \in \mathbb{F})$ always lead to a conclusion of the form e_F $(F \in \mathbb{F})$ with

$$\bigcup_{i=1}^{n} F_i \vdash F$$

and

(B) if $e_F = e_{F'}$ $(F, F' \in \mathbb{F})$ then $F \vdash F'$.

By proving (A) and (B), the completeness of 3.6 follows: given a set E of implications with a logical consequence I,

$$E \models I,$$

we know from Remark 3.5 that the injectivity w.r.t. e_I is a logical consequence of the injectivity w.r.t. $\hat{E} = \{e_{\hat{I}},\ \hat{I} \in E\}$. By the Completeness Theorem 2.12 we conclude that a formal proof of e_I from $\hat{E}$ exists in the deduction system 2.8, that is, there are implications $I_1, \cdots, I_n \in E$ such that $e_{I_1}, \cdots, e_{I_n} \vdash e_I$ in Injectivity Deduction System. Due to (A), every step in that proof is of the form e_F for some F such that $\{I_1, \cdots, I_n\} \vdash F$.

In particular, the last line, e_I, is equal to some such e_F, which by (B) implies $F \vdash I$. Consequently, we obtain $\{I_1, \cdots, I_n\} \vdash I$, and thus $E \vdash I$, as required.

The statement (B) follows immediately from 3.10 and 3.11.

Proof of (A). Our task is to prove for every rule of 2.8 that if the premises have the form $e_{F_1}, \cdots, e_{F_n}$ then the conclusion has the form e_F where $\cup F_k \vdash F$.

We proceed by inspecting the rules individually.

(1) *Axiom.* Suppose that e_F is an identity morphism. Then the two congruences $\sim_{\mathcal{P}}$ and $\sim_{\mathcal{P}\cup\mathcal{Q}}$ coincide, thus for each $u = v$ in $\mathcal{Q}$ we have

$$u \sim_{\mathcal{P}} v$$

and Lemma 3.11 gives us $\vdash \mathcal{P} \Rightarrow u = v$.

(2) *Composition Rule.* Let e_F and $e_{F'}$ be two morphisms in $\mathcal{E}_{fp}$ which compose, with F and F' members of $\mathbb{F}$:

$$\begin{tikzcd} & \phi(W)/\sim_{\mathcal{P}_F\cup\mathcal{Q}_F} = \phi(W)/\sim_{\mathcal{P}_{F'}} \arrow[dr, "e_{F'}"] & \\ \phi(W)/\sim_{\mathcal{P}_F} \arrow[ur, "e_F"] \arrow[rr, "e"'] & & \phi(W)/\sim_{\mathcal{P}_{F'}\cup\mathcal{Q}_{F'}} \end{tikzcd}$$

Since $\mathcal{P}_F \cup \mathcal{Q}_F$ generates the same congruence as $\mathcal{P}_{F'}$, it follows that $\mathcal{P}_F \cup \mathcal{Q}_F \cup \mathcal{Q}_{F'}$ generates the same congruence as $\mathcal{P}_{F'} \cup \mathcal{Q}_{F'}$, consequently,

$$e = e_{F''}$$

for

$$F'' = F \cup \{\mathcal{P}_F \Rightarrow u = v;\ u = v \text{ in } \mathcal{Q}_{F'}\}.$$

It is our task to prove that

$$F \cup F' \vdash F''.$$

That is, given $u = v$ in $\mathcal{Q}_{F'}$, we are going to derive the implication $\mathcal{P}_F \Rightarrow u = v$ from $F \cup F'$.

Using Lemma 3.11 on any $s = t$ in $\mathcal{P}_{F'}$, we get

$$\vdash \mathcal{P}_F \cup \mathcal{Q}_F \Rightarrow s = t,$$

therefore,

$$F \vdash (\mathcal{P}_F \Rightarrow s = t) \qquad (\text{for all } s = t \text{ in } \mathcal{P}_{F'})$$

see 3.9(iii). Since for $u = v$ in $\mathcal{Q}_{F'}$ we have, trivially,

$$F' \vdash (\mathcal{P}_{F'} \Rightarrow u = v)$$

we conclude (by Weakening) that

$$F \cup F' \vdash (\mathcal{P}_{F'} \Rightarrow u = v) \text{ and } F \cup F' \vdash (\mathcal{P}_F \Rightarrow s = t) \text{ for all } s = t \text{ in } \mathcal{P}_{F'}.$$

It follows from 3.10 that

$$F \cup F' \vdash (\mathcal{P}_F \Rightarrow u = v)$$

as requested.

(3) *Cancellation Rule.* We are given a commutative diagram

$$\begin{tikzcd} \phi(W)/\sim_{\mathcal{P}} \arrow[r, "e_F"] \arrow[d, "e"'] & \phi(W)/\sim_{\mathcal{P}\cup\mathcal{Q}} . \\ \phi(W)/\sim_{\mathcal{P}\cup\mathcal{R}} \arrow[ur, "e'"'] & \end{tikzcd}$$

Then $e = e_{F'}$ for

$$F' = \{\mathcal{P} \Rightarrow u = v;\ u = v \text{ in } \mathcal{R}\}.$$

We now have to prove that $F \vdash F'$, i.e., for every $u = v$ in $\mathcal{R}$ we have to verify

$$F \vdash \mathcal{P} \Rightarrow u = v.$$

Since $e_F = e' \cdot e_{F'}$, we conclude $u \sim_{\mathcal{P}\cup\mathcal{Q}} v$, thus, by Lemma 3.11,

$$\vdash \mathcal{P} \cup \mathcal{Q} \Rightarrow u = v.$$

Therefore, 3.9(iii) yields

$$F \vdash \mathcal{P} \Rightarrow u = v$$

as requested.

(4) *Pushout Rule.* Let

$$\begin{tikzcd} \phi(W)/\sim_{\mathcal{P}} \arrow[r, "e_F"] \arrow[d, "f"'] & \phi(W)/\sim_{\mathcal{P}\cup\mathcal{Q}} \arrow[d, "g"] \\ \phi(W')/\sim_{\mathcal{P}'} \arrow[r, "e'"'] & B \end{tikzcd} \tag{3.1}$$

be a pushout where $F \in \mathbb{F}$ (and we put $\mathcal{P}_F = \mathcal{P}$ and $\mathcal{Q}_F = \mathcal{Q}$), and $e' \in \mathcal{E}_{fp}$. By Lemma 3.12 we have a substitution σ with $\mathcal{P}^\sigma$ included in $\sim_{\mathcal{P}'}$, and

$$e' = e_{F'} \text{ for } F' = \{\mathcal{P}' \Rightarrow u^\sigma = v^\sigma;\ u = v \text{ in } \mathcal{P} \cup \mathcal{Q}\}.$$

Thus, we need to show that $F \vdash F'$. First for every $u = v$ in $\mathcal{P}$ we have

$$\vdash \mathcal{P}' \Rightarrow u^\sigma = v^\sigma,$$

since $u = v$ in $\mathcal{P}$ implies $u^\sigma \sim_{\mathcal{P}'} v^\sigma$, see Lemma 3.11. Secondly, for every $s = t$ in $\mathcal{Q}$ we verify

$$F \vdash \mathcal{P}' \Rightarrow s^\sigma = t^\sigma.$$

In fact, from Invariance in 3.6 we know that

$$F \vdash \mathcal{P}^\sigma \Rightarrow s^\sigma = t^\sigma$$

and since $\vdash \mathcal{P}' \Rightarrow u^\sigma = v^\sigma$ (for all $u^\sigma = v^\sigma$ in $\mathcal{P}^\sigma$), this yields by 3.10 the desired statement $F \vdash \mathcal{P}' \Rightarrow s^\sigma = t^\sigma$. Consequently, $F \vdash F'$. $\square$

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