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MATRIX REALIZATION OF A PAIR OF TABLEAUX WITH KEY AND SHUFFLING CONDITION

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ABSTRACT: Given a pair of tableaux $(\mathcal{T}, \mathcal{K}(\sigma))$, where $\mathcal{T}$ is a skew-tableau in the alphabet $[t]$ and $\mathcal{K}(\sigma)$ is the *key* associated with $\sigma \in S_t$, with the same evaluation as $\mathcal{T}$, we consider the problem of a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma))$ over a local principal ideal domain [1, 2, 3, 4, 5, 6]. It has been shown that the pair $(\mathcal{T}, \mathcal{K}(\sigma))$ has a matrix realization only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{K}(\sigma)$ [5]. This condition has also been proved sufficient when σ is the identity [1, 2, 4], the reverse permutation in $\mathcal{S}_t$ [2, 3], or any permutation in $\mathcal{S}_3$ [6]. In each of these cases, the plactic class of $\mathcal{K}(\sigma)$ may be described by shuffling together their columns. For $t \geq 4$ this is no longer true for an arbitrary permutation, but shuffling together the columns of a *key* always leads to a congruent word. In [17] A. Lascoux and M. P. Schützenberger have introduced the notions of *frank* word and *key*. It is a simple derivation on Greene's theorem [11] that words congruent with a *key*, and *frank* words are dual of each other as *biwords*. In this paper, we exhibit, for any $\sigma \in \mathcal{S}_t$, a matrix realization for the pair $(\mathcal{T}, \mathcal{K}(\sigma))$, when the word of $\mathcal{T}$ is a shuffle of the columns of $\mathcal{K}(\sigma)$. This construction is based on a biword defined by the columns of the *key* and the places of their letters in the skew-tableau $\mathcal{T}$. The places of these letters are row words which are shuffle components of a *frank* word.

KEYWORDS: Frank words, keys, matrix realization, plactic monoid, shuffle.
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1. Introduction

Given $\sigma \in \mathcal{S}_t$, let $\mathcal{K}(\sigma)$ be an associated *key* [8, 16, 17]. That is, $\mathcal{K}(\sigma)$ is a tableau with columns pairwise comparable for the inclusion order, obtained by taking a sequence of left reordered factors of σ, considered as a word, by decreasing order of length. Given the pair of tableaux $(\mathcal{T}, \mathcal{K}(\sigma))$, where $\mathcal{T}$ is a skew-tableau over the alphabet $[t]$ and $\mathcal{K}(\sigma)$ is the *key* associated with σ with the same evaluation as $\mathcal{T}$, we consider the problem of a matrix realization, over a local principal ideal domain, for the pair $(\mathcal{T}, \mathcal{K}(\sigma))$ [Section 3, Definition 3.1].

When σ is the identity [1, 2, 4], the reverse permutation in $\mathcal{S}_t$ [3], or any permutation in $\mathcal{S}_3$ [6], it has been shown that $(\mathcal{T}, \mathcal{K}(\sigma))$ has a matrix realization if and only if the word of $\mathcal{T}$ is an element of the plactic class of

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$\mathcal{K}(\sigma)$. (Note that in the first two cases, this means that $\mathcal{T}$ is a Littlewood-Richardson and a dual Littlewood-Richardson tableau, respectively). For these permutations, the elements of the plactic class of $\mathcal{K}(\sigma)$ are shuffles of the columns of $\mathcal{K}(\sigma)$ and this property has been used to exhibit a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma))$. For $t \geq 4$, this shuffle property is no longer true for an arbitrary permutation in $\mathcal{S}_t$. For instance, the word 431421 is in the plactic class of the *key* 4321 41, but it is not a shuffle of the columns 4321 and 41. Nevertheless, as we shall see, in subsection 2.2, the plactic class of a *key* $\mathcal{K}(\sigma)$ contains the set of all possible shuffles of their columns. (In [5], a different has been used to prove this result).

In [5], the *only if* condition of the previous results has been generalized for any $\sigma \in S_t$, $t \geq 1$. In this paper, we give an explicit construction of a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma))$, when the word of $\mathcal{T}$ is a shuffle of the columns of $\mathcal{K}(\sigma)$. *Keys* and *frank* words are dual in the sense that they are *different representations* of the same word by biwords. Our construction is based on a biword defined by the columns of the *key* and the places of their letters in the skew-tableau $\mathcal{T}$. The places of these letters are row words which are shuffle components of a *frank* word. Henceforth, for those permutations $\sigma \in \mathcal{S}_t$ for which the plactic class of an associated *key* is described by shuffling together their columns [5], there exists a matrix realization for the pair $(T, K(\sigma))$ if and only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{K}(\sigma)$.

The paper is organized as follows. The next section is divided into three subsections. In subsection 2.1 we introduce some combinatorics of the monoid of tableaux [9, 13]. We define biword as the representation of a word with respect to some skew-tableau. Some properties of these biwords are analyzed. In subsection 2.2, following [17], some combinatorics of *keys* and *frank* words is discussed. The main results are the representation of words congruent with a *key*, and *frank* words by biwords. Using these biwords we show how to generate *frank* words by shuffling rows. These *frank* words are in correspondence with those words which are shuffles of the columns of a *key* and it is shown that they are congruent with that *key*. Finally, in subsection 2.3, based on these biwords we give a graphical interpretation of the words which are shuffles of the columns of a *key*. In section 3, we introduce the concept of a matrix realization of a pair of tableaux $(\mathcal{T}, \mathcal{K}(\sigma))$, with the same evaluation. Our main theorem 3.3 gives an explicit construction of a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma))$, when the word of $\mathcal{T}$ is a shuffle of the columns of $K(\sigma)$. The proofs of the results of section 3 are given in the last section.

[17] ______, *Keys and standard bases*, Invariant theory and tableaux (Minneapolis, MN,1988) IMA, Math. Appl., vol. 19, Springer, New York-Berlin, 1990.
[18] M. Newman, *Integral Matrices*, Academic Press, New York, 1972.
[19] C. Reutenauer, *Free Lie algebras*, Oxford University Press, Oxford, 1993.
[20] C. Schensted, *Longest increasing and decreasing subsequences*, Canadian Journal of Mathematics **13** (1961), 179–191.

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2. Keys, frank words and shuffles

2.1. Young tableaux, words and biwords. Let $\mathcal{S}_t$ be the symmetric group of degree $t \geq 1$, and let $\mathbb{N}$ be the set of positive integers with the usual order " $\leq$ ". Given $k, t \in \mathbb{N}$, $k \leq t$, $[k,t]$ denotes the set $\{k, \ldots, t\}$ in $\mathbb{N}$. When $k = 1$, we put $[t] := [1,t]$. We denote by $[t]^*$ the free monoid in the alphabet $[t]$.

A partition is a sequence of nonnegative integers $a = (a_1, a_2, \ldots)$, all but a finite number of which are nonzero, such that $a_1 \geq a_2 \geq \cdots$ The maximum value of i for which $a_i > 0$ is called the *length* of a, denoted by $l(a)$. If the length of a is zero, we have the null partition $a = (0, 0, \ldots)$. If $a_i = 0$, for $i > k$, we write $a = (a_1, \ldots, a_k)$ as well. Sometimes it is convenient to use the notation

$$a = (a_1^{m_1}, a_2^{m_2}, ..., a_k^{m_k}),$$

where $a_1 > a_2 > \ldots > a_k$ and $a_i^{m_i}$, with $m_i \geq 0$, means that a_i appears m_i times as a part of a. Thus, every partition can be written as $a = (t^{l_t}, \ldots, 2^{l_2}, 1^{l_1})$ for some positive integer t. The *conjugate partition* of a is, therefore, defined as the partition $(\sum_{i=1}^t l_i, \ldots, l_{t-1} + l_t, l_t)$. On the other hand, if $\sigma \in \mathcal{S}_t$ and $m_{\sigma(i)} = \sum_{k=1}^t l_k$, $i = 1, \ldots, t$, we have $(t^{l_t}, \ldots, 2^{l_2}, 1^{l_1}) = \sum_{i=1}^t (1^{m_{\sigma(i)}})$.

Given a word $w = x_1 \cdots x_k$ over the alphabet $[t]$, we denote by $|w|_j$ the multiplicity of the letter $j \in [t]$ in w. Here k is the length of w, denoted by $|w|$. The sequence $(|w|_1, \ldots, |w|_t)$ is called the *evaluation* of w. We have $|w| = |w|_1 + \cdots + |w|_t$. The length and evaluation of the empty word are zero. The word w, with $k \geq 1$, is said a *row* if $x_1 \leq \cdots \leq x_k$, and a column if $x_1 > \cdots > x_k$. If w is a column, w has planar representation according to its name: the letters are displayed in a column by decreasing order from top to bottom. For example, $\begin{matrix} 5 \\ 2 \\ 1 \end{matrix}$ is the planar representation of 521. Let V_t denote the set of all columns in $[t]^*$. Every word in $[t]^*$ has a unique factorization as a product of a minimal number of columns $w = v_1 v_2 \cdots v_r$, with $v_i \in V_t$. We shall call it the *column factorization* of w and denote it occasionally by $v_1 \cdot v_2 \cdot \ldots \cdot v_r$. The *shape* of w is the sequence $(|v_1|, \ldots, |v_r|)$ of the lengths of the column factors v_i of w. For instance, $w = 43 \cdot 32 \cdot 21$ is the column factorization of w, but $w = u_1 u_2 u_3 u_4$, with $u_1 = 43, u_2 = 3, u_3 = 2$, and $u_4 = 21$ is not.

The underlying set of a column defines a bijection $v \to \{v\}$ between the set V_t and the family $2^{[t]}$ of subsets of $[t]$. According to this bijection we often identify a column with its underlying set. This bijection allows to extend to V_t the order $\leq$ on $2^{[t]}$ by letting $u \leq v$ if and only if there is an increasing injection $i : \{u\} \to \{v\}$ such that $a \leq i(a)$, for all $a \in \{u\}$. For instance, $52 \leq 542 \leq 6432$. In particular, if $\{u\} \subseteq \{v\}$ we have $u \leq v$. We define another order $\triangleright$ on $2^{[t]}$, and extend it to V_t, putting $\{u\} \triangleright \{v\}$ if and only if there is an increasing injection $i : \{v\} \to \{u\}$ such that $i(b) \leq b$, for all $b \in \{v\}$ [17]. For instance, $5431 \triangleright 542 \triangleright 3$.

A word $w = v_1 \cdot v_2 \cdot \ldots \cdot v_r$, $v_i \in V_t$, is called a *tableau* if its columns satisfy $v_1 \triangleright v_2 \triangleright \cdots \triangleright v_r$. The shape of a tableau is, therefore, a partition. For instance,

$$5321 \triangleright 41 \triangleright 42 \triangleright 4 = \begin{array}{llll} 5 & & & \\ 3 & & & \\ 2 & 4 & 4 & \\ 1 & 1 & 2 & 4 \end{array}$$

is a tableau of shape $(4, 2, 2, 1) = (4^1, 3^0, 2^2, 1^1)$. The conjugate partition $(4, 3, 1, 1)$ defines the length of the rows of the tableau. A tableau $\mathcal{T} = v_1 \cdot v_2 \cdot \ldots \cdot v_r \in [|\mathcal{T}|]^*$ is said *standard* if there is no repetition of letters.

Knuth's congruence $\equiv$ [12] on words over the alphabet $[t]$ is the congruence in $[t]^*$ generated by the so-called elementary transformations, where x, y and z are letters in $[t]$:

$$xzy \equiv zxy, \quad x \leq y < z, \tag{1}$$

$$yzx \equiv yxz, \quad x < y \leq z. \tag{2}$$

These relations (1),(2), also called *plactic*, are the algebraic version of the plactic congruence in $[t]^*$ [14, 9, 12, 13] obtained by means of Schensted's construction [20], whose main result is summarized in the following theorem.

Theorem 2.1. (*a*) *Each plactic class contains a unique tableau P.*

(*b*) *The words in the plactic class of the tableau P are in bijection with the set of standard tableaux of the same shape as P.*

The set of all tableaux in the alphabet $[t]$ is thus a *section* of the plactic monoid $[t]^*/\equiv$. We denote by $P(w)$ the unique tableau in the plactic class of $w \in [t]^*$, which may be obtained using the Schensted's insertion algorithm on w [20], or by applying *jeu de taquin* to any skew-tableau with word w [9].

Given a word $w \in [t]^*$, let $l(w, k)$ be the maximum of the sum of k disjoint decreasing subwords of w, and let $l'(w, k)$ be the maximum of the sum of

where T_k is a product of diagonal and lower triangular elementary matrices, for $k = 2, \ldots, t$. Finally, use lemma 3.1 to eliminate by right equivalence all diagonal and lower triangular elementary matrices on the right member of (36), starting from right to left. Thus, we find that (36) is equivalent to $D_{[m_1]} \cdots D_{[m_t]}$. □

Remark 4.2. Analyzing the proof above, we find that the sequence A_0, B_1, B_2, B_3, B_4, obtained through the application of algorithm 1 to the shuffle decomposition (28), in example 3.4, is not a matrix realization of $(\mathcal{T}, \mathcal{K}(\sigma))$, since it has produced a situation similar to (34) when computing the invariant partition of $B_1B_2B_3B_4$. This is a consequence of the no satisfaction of proposition 2.13 by the shuffle decomposition.

References

[1] G. Appleby, *A simple approach to matrix realizations for Littlewood-Richardson sequences*, Linear and Multilinear Algebra **291** (1999), 1–14.

[2] O. Azenhas, *Realizações Matriciais de Quadros de Young e Suas Formas Canónicas (in Portuguese)*, Ph.D. thesis, Universidade de Coimbra, 1991.

[3] ______, *Opposite Litlewood-Richardson sequences and their matrix realizations*, Linear Algebra and its Applications **225** (1995), 91–116.

[4] O. Azenhas and E. Marques de Sá, *Matrix realizations of Littlewood-Richardson sequences*, Linear and Multilinear Algebra **27** (1990), 229–242.

[5] O. Azenhas and R. Mamede, *Matrix realizations of pairs of Young tableaux, keys and shuffles*, DMUC preprint **04-40** (2004), available from webpage www.mat.uc.pt/∼cmuc.

[6] ______, *Actions of the symmetric group on sets of skew-tableaux with prescribed matrix realization*, Linear Algebra and its Applications **401** (2005), 221–275.

[7] W. C. Brown, *Matrices over commutative rings*, Monographs and Textbooks in Pure and Applied Mathematics, Marcel Dekker, New York, 1993.

[8] C. Ehresmann, *Sur la topologie de certains espaces homogènes*, Annals of Mathematics, (2) **35** (1934), 396–443.

[9] W. Fulton, *Young Tableaux*, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, Cambridge, 1997.

[10] J. A. Green, *Shuffle Algebras, Lie Algebras and Quantum Groups*, Textos de Matemática, Série B, vol. 9, Departamento de Matemática, Universidade de Coimbra, Coimbra, 1995.

[11] C. Greene, *An extension of schensted's theorem*, Advances in Mathematics **14** (1974), 254–265.

[12] D. E. Knuth, *Permutations, Matrices, and generalized Young Tableaux*, Pacific Journal of Mathematics **34** (1970), 709–727.

[13] A. Lascoux, B. Leclerc, and J-Y Thibon, *The plactic monoid*, in M. Lothaire (ed.), Algebraic Combinatorics on Words, Vol. 90 of Enciclopedia of Mathematics and its Applications, pp. 164-196, Cambridge University Press, Cambridge, 2002.

[14] A. Lascoux and M. P. Schützenberger, *Le monoïde plaxique*, Noncommutative structures in algebra and geometric combinatorics, (Naples, 1978), Quaderni de "La Ricerca Scientifica" **109** (1981).

[15] ______, *The plactic ring*, Lecture Notes (1981).

[16] ______, *Arêtes et tableaux*, Séminaire Lotharingien de Combinatoire **20** (1988), 125–144.

Therefore, any unimodular matrix $T_{ij}(\tau)$, other than the original ones, obtained while applying steps 1. and 2. arise from one (or more) right critical vertex of type I or II of two shuffle components (u_i, u_j), whose critical components u_l, if any, satisfy $\sigma_0^{-1}(u_l^1) < max\{i, j\}$.

We start by analyzing a particular situation. Assume that $T_{ji}(-1)$, $i < j$, is an original or secondary matrix of B_{k+1}, and that $j \leq m_{k+1}, \ldots, m_{k+r}$, for some $r \geq 1$. By lemmas 4.7 and 4.9, this implies the existence of a left critical vertex or right critical vertex of u_j of type I of (u_i, u_j), where u_l is the shuffle component whose right most letter u_l^1 satisfy $\sigma_0^{-1}(u_l^1) = l$, $l = i, j$. The existence of $T_{ij}(\tau)$ on the right of $D_{[m_{k+r}]}$ implies the existence of a right critical vertex of u_i of type I or II of (u_i, u_j). In particular, this means that the vertex in column $k+r$ of u_j is above the correspondent vertex of u_i. Then, proposition 2.13 implies the existence of an integer $q \in [k+1, k+r-1]$ such that $q \in \{u_i\}$ but $q \notin \{u_j\}$. Therefore, $T_{ij}(\tau)$ is transformed into $T_{ij}(\tau p)$ while passing through $D_{[m_q]}$, and situation (34) will not occur.

The general case is treated similarly, for if $T_{ji}(\tau)$, $i < j$, is placed between the diagonal matrices $D_{[m_k]}$ and $D_{[m_{k+1}]}$ with $j \leq m_{k+1}$, there must be a sequence of matrices

$$T_{i_0 i_1}(\tau_0),\, T_{i_1 i_2}(\tau_1), \ldots, T_{i_s i_{s+1}}(\tau_s), \tag{35}$$

$s \geq 1$, giving rise to $T_{ji}(\tau)$ by the application of steps 1 and 2. As we have already seen, the matrices in (35) are coming from critical vertices: the leftmost either from a left critical vertex or a right critical vertex of type I, the rightmost one from a right critical vertex of type I or II, and the remaining from right critical vertices of type I. Thus, if u_l is the shuffle component whose right most letter u_l^1 satisfy $\sigma_0^{-1}(u_l^1) = l$, $l = i, j$, by lemma 3.1 and proposition 2.13, there is $k' \geq k$ such that the vertex of u_j in column k' must be below the correspondent vertex of u_i.

Similarly, the existence of $T_{ij}(\tau')$ on the right of $D_{[m_{k'+r}]}$, $r > 1$, with $j \leq m_k, \ldots, m_{k+r}$, implies that u_i is below u_j in column k'', for some $k'' \geq k' + r$. By proposition 2.13, there must be an integer $q \in [k'+1, k''-1]$ such that $q \in \{u_i\}$ but $q \notin \{u_j\}$. Therefore, $T_{ij}(\tau')$ is transformed into $T_{if}(\tau' p)$ while passing through $D_{[m_q]}$. We may then conclude that a situation like the one described in equation (34) cannot occur, and thus the procedure described above allows us to write

$$B_1 \cdots B_t \sim_L D_{[m_1]} T_2 D_{[m_2]} \cdots T_t D_{[m_t]}, \tag{36}$$

k disjoint nondecreasing subwords of w. These numbers, called *Greene's invariants*, have their name justified by the following theorem. Denote by $(a_1, \ldots, a_s)$ the shape of $P(w)$, and by $(a'_1, \ldots, a'_r)$ its conjugate partition.

Theorem 2.2. (Greene's theorem [11]) *For $k = 1, \ldots, s$, $a_k = l(w,k) - l(w,k-1)$, and for $k = 1, \ldots, r$, $a'_k = l'(w,k) - l'(w,k-1)$.*

Greene's theorem gives an interpretation for the shape of a tableau in terms of the decreasing and nondecreasing subwords of any congruent word.

A *skew-tableau* $\mathcal{T}$ in $[t]^*$ [15] is a tableau on the alphabet $[t] \cup \{\varnothing\}$, where the extra letter $\varnothing$ is such that

$$\varnothing < \varnothing < 1 < 2 < \cdots < t.$$

The word $w(\mathcal{T})$ of the skew-tableau $\mathcal{T}$ is the word in $[t]^*$ obtained by eliminating from $\mathcal{T}$ the extra letter $\varnothing$, and the evaluation of $\mathcal{T}$ is the evaluation of $w(\mathcal{T})$.

Let a be the partition defined by the number of letters $\varnothing$ in each column of $\mathcal{T}$. Then, if c is the shape of $\mathcal{T}$, c/a, called the *skew-shape* of $\mathcal{T}$, denotes the number of letters of $w(\mathcal{T})$ in each column of $\mathcal{T}$. In particular, a tableau in $[t]^*$ is a skew-tableau with $a = 0$. For example, $\mathcal{T} = 43\varnothing\varnothing\, 3\varnothing\varnothing\, 2\varnothing\, 21$ is a skew-tableau of skew-shape $(4,3,2,2)/(2,2,1) = (2,1,1,2)$, and its planar representation is

$$\begin{array}{llll} 4 & & & \\ 3 & 3 & & \\ \varnothing & \varnothing & 2 & 2 \\ \varnothing & \varnothing & \varnothing & 1 \end{array}. \tag{3}$$

We have $w(\mathcal{T}) = 433221$ and the evaluation of $\mathcal{T}$ is $(1,2,2,1)$. Notice that any word $w = v_1 \cdot \ldots \cdot v_r$ in $[t]^*$ may be seen as the word of a skew-tableau in $[t]^*$ with skew-shape $(f_1, \ldots, f_n)$ such that $\sum_{i=1}^k f_i \leq \sum_{i=1}^k |v_i|$, $1 \leq k \leq r$ and $|w| = \sum_{i=1}^n f_i$. For instance, the word $1\,32\,54$ is the word of the skew-tableaux

$$\begin{array}{lll} 1 & 3 & \\ \varnothing & 2 & 5 \\ \varnothing & \varnothing & 4 \end{array} \quad \text{and} \quad \begin{array}{llll} 1 & & & \\ \varnothing & & & \\ \varnothing & \varnothing & 3 & \\ \varnothing & \varnothing & 2 & 5 \\ \varnothing & \varnothing & \varnothing & 4 \end{array}.$$

Given a word $w \in [t]^*$, let $\mathcal{T}$ be a skew-tableau with word w, skew-shape $(f_1, \ldots, f_n)$ and evaluation $(m_1, \cdots, m_t)$. Let

$$\Sigma = \begin{pmatrix} \pi_1 & \cdots & \pi_k \\ x_1 & \cdots & x_k \end{pmatrix} = \begin{pmatrix} 1^{f_1} & \cdots & n^{f_n} \\ w_1 & \cdots & w_n \end{pmatrix} \tag{4}$$

be the biword where the bottom word is $w = x_1 \cdots x_k = w_1 \cdots w_n$, with w_i columns of length f_i if $f_i > 0$, and the top word is $\pi_1\pi_2 \cdots \pi_k = 1^{f_1}2^{f_2} \cdots n^{f_n}$, where π_j is the column index, counting from left to right, of the letter x_j in $\mathcal{T}$, $1 \leq j \leq k$. The billeter $\begin{pmatrix} \pi_j \\ x_j \end{pmatrix}$ means that the letter x_j is placed in the column π_j of $\mathcal{T}$. For each i in $[t]$, let $J_i = y_1^i > \cdots > y_{m_i}^i \in [n]^*$ defined by the indices of the columns of the m_i letters i in $\mathcal{T}$. The columns words $J_1, \ldots, J_t$ are said the *indexing sets* of $\mathcal{T}$ and, as we have just seen, each J_i indicates the indices of the columns where the m_i letters i of w are placed, in the planar representation of $\mathcal{T}$.

The biword Σ (4) is by nonincreasing rearrangement for the antilexicographic order with priority on the first row. (By this we mean that $\pi_i < \pi_j$ or $\pi_i = \pi_j$ and $x_i > x_j$.)

Sorting the billeters in Σ (4), by nonincreasing rearrangement for the lexicographic order with priority on the second row, we obtain another biword

$$\Sigma' = \begin{pmatrix} J_t & \cdots & J_2 & J_1 \\ t^{m_t} & \cdots & 2^{m_2} & 1^{m_1} \end{pmatrix}, \tag{5}$$

where $\begin{pmatrix} J_i \\ i^{m_i} \end{pmatrix}$ is the biword with bottom word i^{m_i} and top word the column $J_i = y_1^i \cdots y_{m_i}^i$.

A skew-tableau determines a unique set of billeters, but not a unique biword. We look at Σ and Σ' as distinguished biwords representing the word w with respect to the skew-tableau $\mathcal{T}$, which are obtained by sorting one of them using the lexicographic order with priority on the second row, or the antilexicographic order with priority on the first row.

Given a sequence of nonegative integers $(m_1, \cdots, m_t)$, its *reverse* sequence is defined by $(m_1, \cdots, m_t)^{rev} = (m_t, \cdots, m_1)$.

Theorem 2.3. (a) *The transformation $\Sigma \leftrightarrow \Sigma'$ establishes a bijective correspondence between the nondecreasing subwords of $J_t \cdots J_1$ and the decreasing subwords of w.*

For $k = 2, \ldots, t$, use corollary 4.8 to write

$$B_k = (\prod_{l=1}^{s_k} T_{i_l j_l}(1)) D_k (\prod_{l=1}^{q_k} T_{j_l i_l}(-1)) D_{[m_k]},$$

where we are assuming that $C_k = \prod_{l=q_k+1}^{s_k} T_{i_l j_l}(1)$ for some integer $s_k \geq q_k$, and $i_l, j_l \in \{i_1, \ldots, i_{q_k}\}$, for $l = q_k + 1, \ldots, s_k$. We will refer to $T_{i_l j_l}(1)$ as original matrices of B_k, if $l = 1, \ldots, q_k$, and as secondary matrices otherwise.

For $k = 2, \ldots, t$, and for each $l = 1, \ldots, s_k$, do the following operations:

(1) use lemma 3.1 to eliminate by left equivalence the upper triangular matrix $T_{i_l j_l}(1)$. Note that this operation will produce some elementary matrices $T_{uv}(x)$ to the left of $D_{[m_{k-1}]}$.
(2) Starting from left to right, use again lemma 3.1 to eliminate by left equivalence all new upper triangular elementary matrices. Again new elementary matrices may appear. Repeat the process until there are no upper triangular elementary matrices between $D_{[m_1]}$ and $D_{[m_{k-1}]}$.

Notice that by lemma 3.1 there is always an elementary matrix E satisfying

$$T_{ji}(\tau)T_{ab}(\tau') = T_{ab}(\tau')T_{ji}(\tau)E, \tag{34}$$

except when τ and τ' are units, $a = i$ and $b = j$. Thus, to ensure that steps 1. and 2. are feasible, we must show that this situation does not occur.

With this propose in mind, we start by noticing that if $T_{ij}(1)$, $i < j$, is an original matrix of B_{k+1}, with $m_k < j$, we have

$$D_{[m_k]}T_{ij}(1) = T_{ij}(p)D_{[m_k]},$$

and, therefore, $T_{ij}(p)$ will not lead to a situation similar to (34). The only case where $T_{ij}(1)$ stays invariant while passing through $D_{[m_k]}$ is when $j \leq m_k$. By lemma 4.5, this occurs if and only if there is a vertex $(a, k) \in u_j$, not positively-linked, and right critical vertex $(x, k) \in u_i$ of type I or II of (u_i, u_j), where u_i and u_j are shuffle components satisfying $\sigma_0^{-1}(u_i^1) = i$ and $\sigma_0^{-1}(u_j^1) = j$, being u_i^1 and u_j^1 the right most letters of u_i and u_j, respectively. By lemma 4.7 and corollary 4.8, we find that the same situation happens when $T_{ij}(1)$ is a secondary matrix of B_{k+1}. In this case, corollary 4.8 implies that for each critical component v_l, $\sigma_0^{-1}(v_l^1) < max\{i, j\}$, where v_l^1 is the right most letter of v_l.

Lemma 4.9. *Suppose $m_k < m_{k+1}$, and let $(x,k),(y,k+1)$ be a pair of linked vertices of $w(\mathcal{T})$, belonging to the shuffle component u. Then, $I + S(x,y,\nu\sigma_k) = T_{ij}(1)$ with $m_k < j \le m_{k+1}$ if and only if there is a vertex $(a,k+1)$, belonging to a shuffle component v, which is a left critical vertex of (u,v), with $\sigma_{k+1}^{-1}(a) = j$.*

Proof: The *if* part.

Without loss of generality, assume $(x,k+1)$ is a left critical vertex of (u,v), with no critical components. Since $\sigma_{k+1}([m_{k+1}]) = J_{k+1}$, there is $j \in [m_{k+1}]$ such that $\sigma_{k+1}(j) = x$. Consider the matrix $S(x,y,\nu\sigma_k)$. Clearly, we may write

$$\sigma_{k+1} = \theta(x\,y)\nu\sigma_k,$$

where $\theta \in \mathcal{S}_n$ satisfy $\theta(x) = x$. Therefore, $\nu\sigma_k(j) = (x\,y)\theta^{-1}\sigma_{k+1}(j) = y$, and thus $I + S(x,y,\nu\sigma_k) = T_{ij}(1)$, for some integer $i \in [m_k]$. Finally, note that since the shuffle component v has no letter k, all vertices in column k must be positively-linked. Thus, by lemma 4.5 we find $m_k < j$.

The *only if* part.

Assume now that $I+S(x,y,\nu\sigma_k) = T_{ij}(1)$, for some integer $m_k < j \le m_{k+1}$. Again, it is clear that we must have

$$\sigma_{k+1} = \rho(x_r\,x_{r-1})\cdots(x_2\,x_1)(x_1\,x)(x\,y)\nu\sigma_k,$$

for some integers $x_r > \cdots > x_1 > x > y$ and a permutation $\rho \in \mathcal{S}_n$ such that $\rho(x_r) = x_r$. Thus, $\sigma_{k+1}(j) = x_r \in J_{k+1}$, since $j \in [m_{k+1}]$. Denote by v the shuffle component containing the vertex $(x_r, k+1)$, and note that since $\rho(x_r) = x_r$, v cannot have the letter k. Thus, $(x_r, k+1)$ is a left critical vertex of (u,v). $\square$

Confront this result with example 4.1, where the column index of $I + S^2_{3,2} = T_{12}(1)$ satisfy $1 = m_1 < 2 = m_2$, and $(3,2)$ is a left critical vertex of $(7654321, 7652)$.

We are now ready to start the proof of our main theorem.

Proof of theorem 3.3.

It was shown in proposition 4.4 that $A_0, B_1, \ldots, B_t$ satisfy conditions I and II of definition 3.1. So it remains to show that $B_1 \cdots B_t$ is equivalent to the diagonal matrix $D_{[m_1]} \cdots D_{[m_t]}$, that is, $B_1, \ldots, B_t$ realizes the key with weight $(m_1, \ldots, m_t)$.

(*b*) *The shapes of $P(w)$ and $P(J_t \cdots J_1)$ are conjugate, and $(|J_t|, \ldots, |J_1|)^{rev}$ is the evaluation of w.*

Proof: (*a*) In fact, if v is a nondecreasing subword of $J_t \cdots J_1$, it is formed by a single letter of each J_i. The correspondent subword v' in the second row of Σ' is necessarily a column, and it is also a subword of w. Reciprocally, a subword in the first row of Σ, corresponding to a decreasing subword of w, is necessarily nondecreasing, and when passing to Σ', it remains a subword in $J_t \cdots J_1$.

(*b*) It is clear that the evaluation of w is the reverse sequence of $(|J_t|, \ldots, |J_1|)$. The result follows from (*a*) and Greene's theorem, theorem 2.2. $\square$

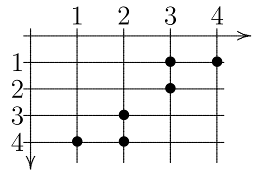
For example, the biwords Σ and Σ' of the tableau (3) are, respectively,

$$\begin{pmatrix} 11\;2\;3\;44 \\ 43\;3\;2\;21 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1\;21\;43\;4 \\ 4\;33\;22\;1 \end{pmatrix}, \tag{6}$$

while the biwords Σ and Σ' of the tableau $\mathcal{T} = 5\varnothing\varnothing\varnothing\,5431\varnothing\,5321\,321\,1$, are, respectively,

$$\begin{pmatrix} 1\;2222\;3333\;444\;5 \\ 5\;5431\;5321\;321\;1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 321\;2\;432\;43\;5432 \\ 555\;4\;333\;22\;1111 \end{pmatrix}. \tag{7}$$

The billeters of the biwords Σ and Σ' with respect to a skew-tableau $\mathcal{T}$, with word w in $[t]^*$, may be represented in a lattice of points of $\mathbb{N}^2$ according to the bijection $\binom{y}{i} \mapsto (y,i) \in \mathbb{N}^2$ such that $y \in J_i$, $1 \le i \le t$. In drawing such a lattice of points, we shall adopt the convention, as with matrices, that the first coordinate, the row index, increases as one goes downwards, and the second coordinate, the column index, increases as one goes from left to right. The points (y,i), in the lattice $\mathbb{N}^2$, are said the *vertices* of the biwords Σ and Σ', or the vertices of w with respect to $\mathcal{T}$. (When there is no danger of confusion, we omit the reference to the skew-tableau.) For example, the vertices of the biwords (6), or of 433221 with respect to (3), above, are represented in the following grid:



. (8)

A partition a and the biword Σ' (5) parameterize completely a skew-tableau, since partition a gives the places for the letters $\varnothing$, and the set J_i the places for the letters i in $\mathcal{T}$, for $i = 1, \ldots, t$.

Given $J \subseteq [t]$, we define the characteristic function of J by $(\chi^J)_i = 1$, if $i \in J$, and $(\chi^J)_i = 0$ otherwise. Given a skew-tableau $\mathcal{T}$, with skew-shape c/a and biword Σ' (5), we may associate the sequence of partitions $(a^0, a^1, \ldots, a^t)$ by setting $a^0 := a$ and $a^i := a_{i-1} + \chi^{J_i}$, $i = 1, \ldots, t$, with $a^t = c$. Clearly, each $a^i = (a^i_1, \ldots, a^i_r)$ is a partition and satisfy

$$a^i_k \leq a^{i+1}_k \leq a^i_k + 1, \tag{9}$$

for $i = 0, 1, \ldots, t-1$, and $k = 1, \ldots, l(c)$. Conversely, any sequence of partitions $(a^0, a^1, \ldots, a^t)$ satisfying (9) gives rise to a skew-tableau $\mathcal{T}$ with biword Σ' defined by the sets $J_i = \{k : a^i_k = a^{i-1}_k + 1\}$, $i = 1, \ldots, t$. For instance, the skew-tableau (3) is defined by the sequence of partitions $\mathcal{T} = (a^0, \ldots, a^4)$, where $a^0 = (2,2,1,0), a^1 = (2,2,1,1), a^2 = (2,2,2,2), a^3 = (3,3,2,2)$ and $a^4 = (4,3,2,2)$.

2.2. Keys, frank words and shuffles. Consider again $w = x_1 \cdots x_k \in [t]^*$, and let I be a subset of $[k]$. We denote by $w|I$ the word $x_{i_1} \cdots x_{i_l}$, if $I = \{i_1 < i_2 < \cdots < i_l\}$. Such a word $w|I$ is called a *subword* of w. Now, let q words $u_1, \ldots, u_q \in [t]^*$ of lengths $k_1, \ldots, k_q$, respectively. Put $k = k_1 + \cdots + k_q$ and let $[k] = \cup_{j=1}^q I_j$, where $(I_1, \ldots, I_q)$ is a q-tuple of pairwise disjoint subsets of $[k]$ with $|I_j| = k_j$, $j = 1, \ldots, q$. Then the word $w|(I_1, \ldots, I_q)$ is defined by $w|I_j = u_j$, for $j = 1, \ldots, q$, [10, 19], and is called a *shuffle* of $u_1, \ldots, u_q$. The words $u_1, \ldots, u_q$ are said the *shuffle components* of $w|(I_1, \ldots, I_q)$. We may have $w|(I_1, \ldots, I_q) = w|(J_1, \ldots, J_q)$ with $(J_1, \ldots, J_q)$ in the conditions above. That is, a word may have different shuffle decompositions with respect to given words. For instance, $w = 543321 \in [5]^*$ is a shuffle of $w|\{1,3,6\} = w|\{1,4,6\} = 531$ and $w|\{2,4,5\} = w|\{2,3,5\} = 432$, and thus we have $w = w|(\{1,3,6\},\{2,4,5\}) = w|(\{1,4,6\},\{2,3,5\})$.

The set of all words obtained by shuffling together the q words $u_1, \ldots, u_q$, is

$$Sh(u_1, \ldots, u_q) =$$
$$= \{w|(I_1, \ldots, I_q) : \cup_{j=1}^q I_j = [k],\ |I_j| = k_j,\ w|I_j = u_j,\ 1 \leq j \leq q\},$$

where $(I_1, \cdots, I_q)$ is a q-tuple of pairwise disjoint subsets of $[k]$.

Given a set $A = \{u_1, \ldots, u_q\} \subseteq [t]^*$, we put $Sh(A) = Sh(u_1, \ldots, u_q)$. If C is another finite set, we have $Sh(A \cup C) = Sh\left[Sh(A), Sh(C)\right]$.

Proof: The *if* part.

Without loss of generality assume that (x_2, k) is a right critical vertex of type I or II of (u_1, u_2), without critical components. Then, $x_1 = s^k_{(y_2, k+\varepsilon_2)}$ and we may write

$$\nu_2 \sigma_k = \theta(x_1\, s^k_{(y_1, k+\varepsilon_1)}) \nu_1 \sigma_k,$$

where $\theta \in \mathcal{S}_n$ satisfy $\theta(x_1) = x_1$. Since $\nu_2\sigma_k(j_1) = \theta(x_1)$, the result follows.

The *only if* part.

Without loss of generality, assume that in the product B_k, there are no matrices $T_{ij_1}(1)$ between $T_{i_1 j_1}(1)$ and $T_{i_2 j_1}(1)$. Then, we may write

$$\nu_2 \sigma_k = \theta(x_1\, s^k_{(y_1, k+\varepsilon_1)}) \nu_1 \sigma_k,$$

for some permutation $\theta \in \mathcal{S}_n$ satisfying $\theta(x_1) = x_1$. By definition, we must have $\nu_2 \sigma_k(j_1) = s^k_{(y_2, k+\varepsilon_2)}$. On the other hand,

$$\nu_2 \sigma_k(j_1) = \theta(x_1\, s^k_{(y_1, k+\varepsilon_1)}) \nu_1 \sigma_k(j_1) = x_1.$$

Therefore, we find that $x_1 = s^k_{(y_2, k+\varepsilon_2)}$, and this equality means that (x_2, k) is a right critical vertex of type I or II of (u_1, u_2). □

This lemma may be illustrated with example 4.1, where the matrices $I + S^5_{2,1} = T_{13}(1)$ and $I + S^5_{3,2} = T_{23}(1)$ have the same column index, and $(3,4)$ is a right critical vertex of type I of $(7654321, 7542)$.

The following corollary is a straightforward application of lemmas 3.1 and 4.7.

Corollary 4.8. *For each $k = 1, \ldots, t-1$, consider the matrix $S_{k+1} = \prod_{l=1}^{q_k} (T_{i_l j_l}(1) T_{j_l i_l}(-1))$ defined in step 2 of algorithm 1. Then, we may write*

$$S_{k+1} = \left(\prod_{l=1}^{q_k} T_{i_l j_l}(1) \right) C_{k+1} D_{k+1} \prod_{l=1}^{q_k} T_{j_l i_l}(-1),$$

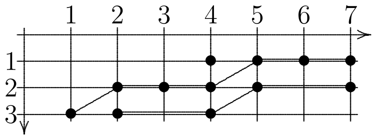
where C_{k+1} [respectively, D_{k+1}] is a product of upper [respectively, lower] triangular elementary matrices $T_{ij}(1)$, with $i, j \in \{i_1, \ldots, i_{q_k}\}$, $i \neq j$.

Moreover, $T_{ij}(1)$ is a factor of $C_{k+1} D_{k+1}$ only if there are shuffle components u_l, with vertices (a_l, k) satisfying $\sigma^{-1}_{k+1}(a_l) = l$, for $l = i, j$, such that (a_i, k) is a right critical vertex of type I or II of (u_i, u_j), with critical components u_{s_l}, whose rightmost letter $u^1_{s_l}$ satisfy $\sigma^{-1}_0(u^1_{s_l}) < max\{i, j\}$, for $l = 1, \ldots, q$.

must have $\sigma_k(j) = a$, where (a,k) is a not positively-linked vertex of a shuffle component $r_{\sigma,q'}$, with $q > q'$. Thus $j = \sigma_k^{-1}(a) \in [L_{q'+1}+1, L_{q'}] \subseteq [m_k] \setminus [|J'_k|]$. $\square$

Corollary 4.6. *Each matrix $I + S_{x,y}^{k+1} = T_{ij}(1)$, defined in step 2 of algorithm 1, is upper triangular with $i \in [m_k]$ and $j > i$. Moreover, $j \in [m_k]$ if and only if there is a vertex (a,k), not positively-linked, belonging to a shuffle component v, and such that (x,k) is a right critical vertex of type I or II of (u,v), where u is the shuffle component containing (x,k).*

Example 4.1. Let $\Pi = \begin{pmatrix} 1112223\ 2211\ 1 \\ 7654321\ 7542\ 4 \end{pmatrix}$ and consider its graphical representation:



(32)

Applying algorithm 1, we must set $\sigma_1 = (13)$, since $s_1^0 = s_{(3,1)}^0 = 3$, $s_2^0 = s_{(3,2)}^0 = 2$ and $s_3^0 = s_{(1,4)}^0 = 1$, and thus $\sigma_i = (3\,2)\sigma_1$, for $i = 2,3,4$ and $\sigma_5 = (3\,2)(2\,1)\sigma_4$. Consider the matrices $S_{3,2}^2 = S(3, s_{(2,2)}^1, \sigma_2)$, $S_{2,1}^5 = S(2, s_{(1,5)}^4, \sigma_4)$, and $S_{3,2}^5 = S(2, s_{(2,5)}^4, (2\,1)\sigma_4)$, produced in step 2 of algorithm 1. Since $s_{(2,2)}^1 = 2$, $s_{(1,5)}^4 = 1$ and $s_{(2,5)}^4 = 2$, we get

$$I + S_{3,2}^2 = T_{12}(1), \quad I + S_{2,1}^5 = T_{13}(1), \quad \text{and } I + S_{3,2}^5 = T_{23}(1). \tag{33}$$

Each matrix in (33) is upper triangular and satisfy lemma 4.5. Notice that in the last two matrices the column index $3 \in [3] = [m_4]$. This is consistent with the previous remark, since the vertex $(1,4)$ is not positively-linked, and $(2,4)$ and $(3,4)$ are right critical vertices of type I of $(7654321, 4)$ and of $(7542, 4)$, respectively.

In the next lemmas, we analyze the relationship between the critical vertices of the fixed biword Π and the matrices defined in step 2 of algorithm 1. This analysis is important in order to prove that $B_1, \ldots, B_t$ realizes $\mathcal{K}(\sigma)$.

Lemma 4.7. *For $q = 1, 2$, let $(x_q, k), (y_q, k+\varepsilon_q)$, be a pair of linked vertices belonging to the shuffle component u_q, with $x_1 < x_2$, and consider the matrices $I + S(x_q, s_{(y_q,k+\varepsilon_q)}^k, \nu_q\sigma_k) = T_{i_q j_q}(1)$. Then, $j_1 = j_2$ if and only if (x_2, k) is a right critical vertex of type I or II of (u_1, u_2).*

A tableau whose columns are pairwise comparable for the inclusion order is called a *key* [17]. That is, a tableau $u_r \cdot \ldots \cdot u_1$ is a key if $\{u_r\} \supseteq \cdots \supseteq \{u_2\} \supseteq \{u_1\}$. By Greene's theorem, a key is the tableau whose shape is the conjugate of the evaluation, by nonincreasing order. Equivalently, the key with evaluation $(m_1, \ldots, m_t)$ is the tableau with this evaluation whose shape is the conjugate of the partition $(m_{\sigma(1)}, \cdots, m_{\sigma(t)})$, for some $\sigma \in \mathcal{S}_t$. Thus given a sequence $(m_1, \ldots, m_t)$ of nonnegative integers, the tableau with this evaluation and shape $\sum_{i=1}^t (1^{m_i})$ is the tableau $(0, (1^{m_1}), (1^{m_1}) + (1^{m_2}), \ldots, \sum_{i=1}^t (1^{m_i}))$. This tableau is the key with evaluation $(m_1, \ldots, m_t)$. For instance,

$$\mathcal{T} = 5321\ 51\ 51\ 5 = \begin{array}{llll} 5 & & & \\ 3 & & & \\ 2 & 5 & 5 & \\ 1 & 1 & 1 & 5 \end{array} \tag{10}$$

is a key. $\mathcal{T}$ is the only key with evaluation $(3,1,1,0,4)$, that is, it is the tableau $(0, (1^3), (1^3)+(1), (1^3)+(1)+(1), (1^3)+(1)+(1)+(1^4))$. The shape $(4,2,2,1)$ of $\mathcal{T}$ is the conjugate of $(4,3,1,1,0) = (4^1, 3^1, 2^0, 1^2)$.

For each pair consisting of a permutation $\sigma \in \mathcal{S}_t$, written as a word $\sigma = a_1 \cdots a_t \in [t]^*$, and a sequence of nonnegative integers $(l_t, \ldots, l_1)$, Ehresmann [8] associated a key, here denoted by $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$, putting

$$\mathcal{K}(\sigma, (l_t, \ldots, l_1)) := (r_{\sigma,t})^{l_t} (r_{\sigma,t-1})^{l_{t-1}} \cdots (r_{\sigma,1})^{l_1},$$

where $r_{\sigma,k}$ is the column with underlined set $\{a_1, \ldots, a_k\}$, $1 \le k \le t$. This key is the tableau with shape $(t^{l_t}, \ldots, 2^{l_2}, 1^{l_1})$ and evaluation $(m_1, \ldots, m_t)$ such that $m_{\sigma(i)} = \sum_{k=i}^t l_k$, $1 \le i \le t$. On the other hand, the key with evaluation $(m_1, \ldots, m_t)$ may be written in the form $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$, for some permutation $\sigma \in \mathcal{S}_t$ and sequence of nonnegative integers $(l_t, \ldots, l_1)$, such that $(t^{l_t}, \ldots, 2^{l_2}, 1^{l_1})$ is the conjugate of the partition $(m_{\sigma(1)}, \ldots, m_{\sigma(t)})$.

For example, the key (10) is the key associated with the permutation $\sigma = 51324 \in \mathcal{S}_5$ and the sequence $(0,1,0,2,1)$,

$$K(\sigma, (0,1,0,2,1)) = (54321)^0 (5321)^1 (531)^0 (51)^2 5^1 = 5321\,51\,51\,5.$$

Notice that the shape of the key is $(4,2,2,1) = (4^1, 3^0, 2^2, 1^1)$, the conjugate of the partition obtained by permuting by σ the entries of the evaluation $(3,1,1,0,4)$.

When there is no danger of confusion, we will drop the "$(l_t, \ldots, l_1)$" in the notation $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$.

In what follows, given the sequence $(l_t,\dots,l_1)$ of nonnegative integers, we set $L_{t+1}:=0$, and $L_k:=L_{k+1}+l_k$, for $k=1,\dots,t$. Then, $m:=(L_1,\dots,L_t)$ is the conjugate partition of $(t^{l_t},\dots,2^{l_2},1^{l_2})$. The action of the symmetric group $\mathcal{S}_t$ on a partition m with length $\le t$, is defined by putting $\sigma m=(m_1,\dots,m_t)$, where $m_{\sigma(i)}=L_i$, $i=1,\dots,t$.

A word $w\in[t]^*$ is said *frank* [17] if its shape is a permutation of the shape of $P(w)$. The following theorem, proved by Lascoux and Schützenberger in [17], shows that the frank words, in a plactic class, are in bijection with the set of permutations of the shape of the tableau in that class.

Theorem 2.4. *Let $\mathcal{T}$ be a tableau with shape m. For each permutation $\sigma\in\mathcal{S}_t$, there exists one and only one word $w\equiv\mathcal{T}$ with shape σm.*

Thus, frank words in a plactic class are uniquely determined by their shapes. Using the transformation $\Sigma\leftrightarrow\Sigma'$, next theorem shows the duality between frank words and keys. We can think on them as *different representations* of the same word by biwords.

Theorem 2.5. *Let $\sigma\in\mathcal{S}_t$. The word $J_t\cdots J_1\in[n]^*$ ($J_i\in V_n$) in Σ' is frank with shape $(\sigma m)^{rev}$ if and only if the word $w:=w_1w_2\cdots w_n\in[t]^*$, ($w_i\in V_t$) in Σ is congruent with the tableau of shape $(t^{l_t},\dots,2^{l_2},1^{l_1})$ and evaluation σm such that $m=(\sum_{k=i}^t l_k)_{1\le i\le t}$, that is, w is congruent with $K(\sigma,(l_t,\dots,l_1))$.*

Proof: It follows from theorem 2.3 that $(\sigma m)^{rev}$ is the shape of $J_t\cdots J_1$ if and only if σm is the evaluation of w and that $(t^{l_t},\dots,1^{l_1})$ is the shape of $P(w)$ if and only if $m=(\sum_{k=i}^t l_k)_{1\le i\le t}$ is the shape of $P(J_t\cdots J_1)$. $\square$

In the conditions of this theorem, given Σ, we say that the first row of the biword Σ' is an *associated frank word* of w. Notice that the frank word in Σ' depends on the biword Σ. Given the word w, Σ depends on the skew tableau that we attach to w. That is, it depends on the way how we decompose the word w into columns.

Given rows $v_r,\dots,v_1\in[n]^*$, and columns $u_r,\dots,u_1\in[t]^*$ such that $|v_i|=|u_i|$, $1\le i\le r$, consider the biword

$$\Pi=\begin{pmatrix} v_r & \cdots & v_2 & v_1\\ u_r & \cdots & u_2 & u_1\end{pmatrix}, \tag{11}$$

such that if $\binom{y}{x}$ and $\binom{y}{z}$ are billeters of Π, then $x\ne z$.

Sorting the billeters of Π by nonincreasing rearrangement, respectively, for the lexicographic order with priority on the second row, and for the

Proof: The proof is by induction on k. For $k=0$, $P_{id}=I$ and $A_0:=diag_p(a)$, and in this case there is nothing to prove. So let k be in $\{1,\dots,t\}$. By induction, A_{k-1} is left equivalent to $diag_p(a+\chi^{J_1}\cdots+\chi^{J_{k-1}})P_{\sigma_{k-1}}$. Therefore, by definition of B_k, A_k is left equivalent to

$$diag_p(a+\chi^{J_1}\cdots+\chi^{J_{k-1}})P_{\sigma_{k-1}}S_kD_{[m_k]}. \tag{30}$$

Recall that $S_k=\prod_{i=1}^{q_k}(I+S^k_{x_i^ky_i^k})(I-S^k_{x_i^ky_i^k})^T$. Therefore, by lemma 3.2 and (30), A_k is left equivalent to

$$diag_p(a+\chi^{J_1}\cdots+\chi^{J_{k-1}})P_{\sigma_k}D_{[m_k]}=diag_p(a+\chi^{J_1}\cdots+\chi^{J_{k-1}})D_{\sigma_k[m_k]}P_{\sigma_k}. \tag{31}$$

By lemma 4.2 (b), we have $\sigma_k[m_k]=J_k$, and the result follows. $\square$

Remark 4.1. In the proof of proposition 4.4 there are no restrictions on the shuffle decomposition that we have considered. Therefore, if $w(\mathcal{T})\in Sh(R^{l_t}_{\sigma,t},\dots,R^{l_1}_{\sigma,1})$, the sequence of matrices $A_0,B_1,\dots,B_t$ given by algorithm 1 satisfy always conditions I and II of definition 3.1 for any biword Π of $\mathcal{T}$. To prove III, we need restrictions on the biword Π, as we have seen in example 3.4. Otherwise, algorithm 1 produces a pair $(\mathcal{T},\mathcal{F})$, where $\mathcal{F}$ is a tableau with the same evaluation as $\mathcal{T}$, but not necessarily the key $\mathcal{K}(\sigma)$.

Lemma 4.5. *Let (x,k) and $(y,k+\varepsilon)$, $\varepsilon\ge 1$, be two linked vertices of a shuffle component $r_{\sigma,q}$, with leftmost vertex (u,k'). Then, the permutation ν and the matrix*

$$S^{k+1}_{x,y}=S(x,s^k_{(y,k+\varepsilon)},\nu\sigma_k)$$

defined in step 2 of algorithm 1 satisfy:

(a) $\nu(x)=x$.

(b) $\nu^{-1}(s^k_{(y,k+\varepsilon)})\notin J'_k=\{a\in J_k : (a,k)$ *is a positively-linked vertex*$\}$.

(c) $I+S^{k+1}_{x,y}=T_{ij}(1)$, *with* $i=\sigma_k^{-1}(x)=\sigma_1^{-1}(s^0_{(u,k')})\in[L_{q+1}+1,L_q]\subseteq[m_k]$, *and* $j\notin[|J'_k|]$.

Proof: (a) and (b) are obvious.

To prove (c), recall that by the definition of the matrix $I+S^{k+1}_{x,y}=(s_{ij})$, we have $s_{ij}\ne 0$ only if $\nu\sigma_k(i)=x$ and $\nu\sigma_k(j)=s^k_{(y,k+\varepsilon)}$. Using ($a$) and corollary 4.3, we obtain

$$i=\sigma_k^{-1}(x)=\sigma_1^{-1}(s^0_{(u,k')})\in[L_{q+1}+1,L_q]\subseteq[m_k].$$

Now, by (b), we find that $\sigma_k(j)=\nu^{-1}(s^k_{(y,k+\varepsilon)})\notin J'_k$. If $\sigma_k(j)\notin J_k$, by lemma 4.2 (b), we must have $j\notin[m_k]$. Assume now that $\sigma_k(j)\in J_k\setminus J'_k$. Then, we

such that $\begin{pmatrix} x \\ j \end{pmatrix}$ and $\begin{pmatrix} n+1 \\ 0 \end{pmatrix}$ are consecutive vertices in Π_0, and in the graphical representation of Π_0, the leftmost vertex of each shuffle component is linked to a distinct vertex $(n+u, 0)$. In particular, the vertices (x, j) and $(n+1, 0)$ are linked. Put $\theta_1 := \nu(n+1 s^0_{(x,j)})$, where ν is defined as in step 2 of algorithm 1. By lemma 4.1, $\theta_j \cdots \theta_2\theta_1(n+1) = x$, that is, $\theta_j \cdots \theta_2(s^0_{(x,j)}) = x$, and the result follows.

(b) Recall that $|J_j| = m_j = L_{\sigma^{-1}(j)} = \sum_{k=\sigma^{-1}(j)}^{t} l_k$. Thus, $[L_{k+1}+1, L_k] \subseteq [m_j]$ for $k \in \{t, \ldots, \sigma^{-1}(j)\}$. We use induction on j to prove that if $x \in J_j$ is such that (x, j) is a vertex of a shuffle component $r_{\sigma,k}$ of $w(\mathcal{T})$, with $k \in \{t, \ldots, \sigma^{-1}(j)\}$, there exists $i_x \in [L_{k+1}+1, L_k]$ such that $\sigma_j(i_x) = x$.

When $j = 1$, each letter in J_1 corresponds to the leftmost vertex of each shuffle component $r_{\sigma,k}$ of $w(\mathcal{T})$, for $k = t, \ldots, \varepsilon$, where $\varepsilon := \sigma^{-1}(1)$. Therefore, we may write

$$\{(x, 1) : x \in J_1\} = \cup_{k=t}^{\varepsilon} \mathrm{X}_k.$$

By the definition of σ_1, we have $\sigma_1([L_{k+1}+1, L_k]) = s(\mathrm{X}_k)$. Since $s(\mathrm{X}_k) = \{x : (x, 1) \in X_k\}$, for $k = t, \ldots, \varepsilon$, the result follows.

Fix now j in $\{2, \ldots, t\}$, and let $y \in J_j$ be the row index of a letter j belonging to a shuffle component $r_{\sigma,k}$ of $w(\mathcal{T})$. Clearly, we must have $k \in \{t, \ldots, \sigma^{-1}(j)\}$. If (y, j) is the leftmost vertex of $r_{\sigma,k}$ then, by (a), we have $\sigma_j(\sigma_1^{-1}(s^0_{(y,j)})) = y$, with $\sigma_1^{-1}(s^0_{(y,j)}) \in [L_{k+1}+1, L_k]$.

Assume now that (y, j) is negatively-linked to a vertex $(x, j-\varepsilon)$ with $x \geq y$, and $\varepsilon \geq 1$. By induction, there exists $i_x \in [L_{k+1}+1, L_k]$ such that $\sigma_{j-\varepsilon}(i_x) = x$, and by lemma 4.1 we find that $\theta_j \cdots \theta_{j-\varepsilon+1}(x) = y$. Thus,

$$\sigma_j(i_x) = \theta_j \cdots \theta_{j-\varepsilon+1}\sigma_{j-\varepsilon}(i_x) = y.$$

By induction, our claim is proved. Thus, for each $x \in J_j$ there is $i_x \in [m_j]$ such that $\sigma_j(i_x) = x$. Since $|J_j| = m_j$, we must have $\sigma_j([m_j]) = J_j$. □

Corollary 4.3. *If (x, j) is a vertex of a shuffle component $r_{\sigma,k}$ with leftmost vertex $(y, j-\varepsilon)$, $\varepsilon \geq 0$, then*

$$\sigma_j^{-1}(x) = \sigma_{j-\varepsilon}^{-1}(y) = \sigma_1^{-1}(s^0_{(y,j-\varepsilon)}) \in [L_{k+1}+1, L_k].$$

Proof: Follows from the proof of the previous lemma. □

Proposition 4.4. *For each $k \in \{0, 1, \ldots, t\}$, the matrix A_k, given by algorithm 1, is left equivalent to $diag_p(a + \chi^{J_0} + \chi^{J_1} \cdots + \chi^{J_k})P_{\sigma_k}$, where $\sigma_0 := id \in \mathcal{S}_n$ and $J_0 := \emptyset$.*

antilexicographic order with priority on the first row, we get the biwords

$$\Sigma' = \begin{pmatrix} J_t & \cdots & J_2 & J_1 \\ t^{m_t} & \cdots & 2^{m_2} & 1^{m_1} \end{pmatrix} \text{ and } \Sigma = \begin{pmatrix} 1\cdots 1 & \cdots & n\cdots n \\ w_1 & \cdots & w_n \end{pmatrix}, \tag{12}$$

where each $J_i \in [n]^*$, $1 \leq i \leq t$, and $w_j \in [t]^*$, $1 \leq j \leq n$, are columns. On the other hand, Σ' is also obtained from Π by shuffling appropriately the biwords $\begin{pmatrix} v_i \\ u_i \end{pmatrix}$. Thus the top word of Σ' is in $Sh(v_r, \ldots, v_1)$, while the bottom word of Σ' is in $Sh(u_r, \ldots, u_1)$. Similarly, Σ is also obtained from Π by shuffling appropriately the biwords $\begin{pmatrix} v_i \\ u_i \end{pmatrix}$. Thus, the top and bottom words of Σ are in $Sh(v_r, \ldots, v_1)$ and $Sh(u_r, \ldots, u_1)$, respectively. Moreover, the evaluation of $J_t \cdots J_1$ is the sequence $(|w_n|, \ldots, |w_1|)$.

As we shall see the biword Π may be used to construct frank words as a shuffle of rows. For this we consider the special biword Π where the second row is a key. Let $v_r, \ldots, v_1$ be rows in $[n]^*$ with $t = |v_r| \geq \cdots \geq |v_1| \geq 1$, such that the columns u_i satisfy:

1. $\{t \cdots 21\} = \{u_r\} \supseteq \cdots \supseteq \{u_2\} \supseteq \{u_1\}$;
2. if $\binom{y}{x}$ and $\binom{y}{z}$ are billeters of Π, then $x \neq z$.

From now on, we shall consider only biwords Π on these conditions.

Theorem 2.6. (a) *The top word $J_t \cdots J_2 J_1$ of Σ' is a frank word in the set $Sh(v_r, \ldots, v_1)$, whose shape is a permutation of the conjugate of $(|v_r|, \ldots, |v_1|)$.*

(b) *The bottom word $w_1 \cdots w_n$ of Σ is in $Sh(u_r, \ldots, u_1)$, and in the plactic class of the key $u_r \cdots u_1$.*

Proof: We already now that $J_t \cdots J_2 J_1 \in Sh(v_r, \ldots, v_1)$. Moreover, by conditions 1 and 2, it is simple to check that $J_t \cdot \ldots \cdot J_1$ is the column factorization of $J_t \cdots J_1$, and its shape is a permutation of the conjugate of $(|v_r|, \ldots, |v_1|)$.

Let $w = w_1 \cdots w_n$ be the word in the second row of Σ. Since $\{u_i\} \supseteq \{u_{i-1}\}$ and $w \in Sh(u_r, \ldots, u_1)$, we must have $l(w, k) = |v_r| + \cdots + |v_k|$, for $k = 1, \ldots, r$. By Greene's theorem, theorem 2.2, we find that the shape of $P(w)$ is the partition $(|v_r|, \ldots, |v_1|)$. Thus, by theorem 2.3, the shape of $P(J_t \cdots J_2 J_1)$ is the conjugate of $(|v_r|, \ldots, |v_1|)$. This means that $J_t \cdots J_1$ is a frank word and, by theorem 2.5, w must be in the plactic class of the key $u_r \cdots u_1$. □

Remark 2.1. Not all frank words can be written in this way. For instance, $v = 21 \cdot 1 \cdot 2 \cdot 21$ has shape $(2,1,1,2)$, and is congruent with the tableau $P(v) = 21 \cdot 21 \cdot 1 \cdot 2$, whose shape is $(2,2,1,1)$. Then, the conjugate of the shape of $P(v)$ is $(4,2)$, but it is not possible to write v as a shuffle of two rows in $[2]^*$ of lengths 4 and 2, respectively.

Let $\sigma \in \mathcal{S}_t$. For $i = 1,\dots,t$, let $R^{l_i}_{\sigma,i}$ be the set of all simultaneous shuffles of l_i columns $r_{\sigma,i}$. For instance, with $\sigma = 2413 \in S_4$, we have $R^2_{\sigma,2} = \{4242, 4422\}$ and $R^2_{\sigma,3} = \{421421, 424121, 424211, 442121, 442211\}$. If $w \in Sh(R^{l_t}_{\sigma,t},\cdots,R^{l_1}_{\sigma,1})$ it is always possible to form a biword Π satisfying condition 1 and 2.

Corollary 2.7. *The set $Sh(R^{l_t}_{\sigma,t},\cdots,R^{l_1}_{\sigma,1})$ is a subset of the plactic class of the key $K(\sigma,(l_t,\dots,l_1))$.*

Proof: If $w \in Sh(R^{l_t}_{\sigma,t},\cdots,R^{l_1}_{\sigma,1})$, define a biword Σ as in (4), and fix a shuffle decomposition of $w = w|(X^{l_t}_t,\cdots,X^1_t,\cdots,X^{l_1}_1,\cdots,X^1_1)$, where $(X^{l_t}_t,\cdots,X^1_t,\cdots,X^{l_1}_1,\cdots,X^1_1)$ is a L_1-tuple of pairwise disjoint subsets of $[|\sum_{i=1}^t il_i|]$, with $w|X^i_j = r_{\sigma,j}$, $1 \le j \le t$, $1 \le i \le l_j$. Denote by v the first row of Σ and let $v|X^i_j = I^i_j$, $1 \le j \le t$, $1 \le i \le l_j$.

Since Σ is a shuffle of the biwords $\begin{pmatrix} I^i_j \\ r_{\sigma,i} \end{pmatrix}$, $1 \le j \le t$, $1 \le i \le l_j$, we consider the biword

$$\Pi = \begin{pmatrix} I^{l_t}_t & \cdots & I^1_t & \cdots & I^{l_1}_1 & \cdots & I^1_1 \\ r_{\sigma,t} & \cdots & r_{\sigma,t} & \cdots & r_{\sigma,1} & \cdots & r_{\sigma,t} \end{pmatrix}, \tag{13}$$

satisfying the conditions 1 and 2 above. Therefore, by the preceding theorem, $w \equiv K(\sigma,(l_t,\dots,l_1))$. □

Corollary 2.8. *Let $w \equiv K(\sigma,(l_t,\dots,l_1))$ and $J_t \cdots J_1$ be an associated frank word. Then, $w \in Sh(R^{l_t}_{\sigma,t},\cdots,R^{l_1}_{\sigma,1})$ if and only if $J_t \cdots J_1 \in Sh(I^1_t,\dots,I^{l_t}_t,\dots,I^1_1,\dots,I^{l_1}_1)$, where I^i_j are rows with $|I^i_j| = j$, for $j = 1,\dots t$, $i = 1,\dots,l_j$, for which it is possible to form a biword Π satisfying conditions 1 and 2.*

Proof: It follows from theorem 2.6 and corollary 2.7. □

Example 2.1. (1) Given the rows $1123, 12, 3$, we may construct the biword

$$\Pi = \begin{pmatrix} 1123 & 23 & 2 \\ 4321 & 43 & 3 \end{pmatrix}.$$

$s^k_{(y,j)} \neq y$ and henceforth, ρ_2 has no transposition $(a\,y)$. Therefore, $\rho_2(y) = y$ and the result follows.

Case 2. Assume now that $s^k_{(y,k+\varepsilon)} \neq y$, and let $(y,k+\varepsilon)(a_1,b_1)\cdots(a_r,b_r)$ be the s-path of $s^k_{(y,k+\varepsilon)}$. Clearly, $s^k_{(y,k+\varepsilon)} = a_r$, the integer r is even, and for each odd integer $i \in [r]$, the vertices (a_i,b_i) and (a_{i+1},b_{i+1}) are linked, with $a_{i-1} = a_i$ and $a_0 = y$.

Let $i_1 \in [r]$ satisfy $a_{i_1} \le a_j$ and $b_{i_1} \le b_j$, for $j \in [r]$. Then, i_1 is odd and the s-path of $s^{b_{i_1}}_{(a_{i_1+i},b_{i_1+1})} = a_r$ is $(a_{i_1+1},b_{i_1+1})\cdots(a_r,b_r)$.

Now, if $i_1 \neq 1$, let $i_2 \in [i_1 - 1]$ satisfy $a_{i_2} \le a_j$ and $b_{i_2} \le b_j$, for $j \in [i_1 - 1]$. Again, we find that i_2 is odd, and the s-path of $s^{b_{i_2}}_{(a_{i_2+1},b_{i_2+1})} = a_{i_1-1} = a_{i_1}$ is $(a_{i_2+1},b_{i_2+1}),\dots,(a_{i_1-1},b_{i_1-1})$.

Continuing this process, we obtain a sequence of integers $i_1 > i_2 > \dots > i_s = 1$, such that for $q = 1,\dots,s$, the integer i_q is odd, satisfy $a_{i_q} \le a_j$ and $b_{i_q} \le b_j$, for $j \in [i_{q-1} - 1]$, and the s-path of $s^{b_{i_q}}_{(a_{i_q+1},b_{i_q+1})} = a_{i_{q-1}-1} = a_{i_{q-1}}$ is $(a_{i_q+1},b_{i_q+1}),\dots,(a_{i_{q-1}-1},b_{i_{q-1}-1})$. Thus, we may write

$$\theta_{k+\varepsilon}\cdots\theta_{k+1} = \rho_s(a_{i_s}\,a_{i_s-1})\rho_{s-1}\cdots\rho_2(a_{i_2}\,a_{i_1})\rho_1(a_{i_1}\,a_r)\rho_0(x\,a_r)\rho'_0,$$

for some $\rho_i, \rho'_0 \in \mathcal{S}_n$, where $\rho'_0(x) = x$. Moreover, from the equality $s^{b_{i_q}}_{(a_{i_q+1},b_{i_q+1})} = a_{i_q-1}$, and using an argument similar to the one used in case 1, we find that $\rho_q(a_{i_q}) = a_{i_q}$, for $q = 1,\dots,s$, $\rho_0(a_r) = a_r$, and the result follows. □

For instance, in example 3.2, we have $\theta_2 = (5\,2)$, $\theta_3 = (4\,3)(3\,2)$, $\theta_4 = (4\,3)(3\,1)$ and $\theta_5 = id$. The vertices $(5,1),(4,3)$ and $(3,3),(1,5)$ are linked, and satisfy $\theta_3\theta_2(5) = 4$ and $\theta_5\theta_4(3) = 1$.

Lemma 4.2. *Let $k \in \{1,\dots,t\}$, and let (x,j) be the leftmost vertex of a shuffle component $r_{\sigma,k}$. Then,*
(a) $\sigma_j(\sigma_1^{-1}(s^0_{(x,j)})) = x$.
(b) $\sigma_j[m_j] = J_j$.

Proof: (*a*) Note that $\sigma_j(\sigma_1^{-1}(s^0_{(x,j)})) = \theta_j\cdots\theta_2(s^0_{(x,j)})$. If there are no vertices in row x to the left of column j, then $s^0_{(x,j)} = x$, $\theta_j\cdots\theta_2(x) = x$, and thus $\sigma_j(\sigma_1^{-1}(s^0_{(x,j)})) = x$.

Suppose there is a vertex in row x to the left of column j. Extend the biword Π to a biword Π_0 by adding extra billeters $\begin{pmatrix} n+u \\ 0 \end{pmatrix}$ for all $u \in [L_1]$,

partition of $B_1B_2B_3B_4$. Using lemma 3.1 $(ii), (v)$ and (vii), we may write

$$\begin{aligned} & B_1B_2B_3B_4 \sim_L \\ \sim_L\ & D_{\{1\}}T_{21}(-1)D_{\{1,2\}}T_{31}(-1)T_{42}(-1)D_{\{1,2\}}T_{12}(1)T_{21}(-1)D_{\{1\}} \\ =\ & D_{\{1\}}T_{21}(-1)T_{12}(1)D_{\{1,2\}}T_{31}(-1)T_{32}(-1)T_{42}(-1)D_{\{1,2\}}T_{21}(-1)D_{\{1\}} \quad (29) \end{aligned}$$

and by 3.1 (vi), we find that (29) $\sim_R$ $D_{\{1\}}T_{21}(-1)T_{12}(1)D_{\{1,2\}}D_{\{1,2\}}D_{\{1\}}$. Since by 3.1 $(viii)$, $T_{21}(-1)T_{12}(1) = T_{22}(-1)T_{12}(1)P_{(1\,2)}$, we obtain

$$\begin{aligned} B_1B_2B_3B_4 &\sim D_{\{1\}}T_{22}(-1)T_{12}(1)P_{(1\,2)}D_{\{1,2\}}D_{\{1,2\}}D_{\{1\}} \\ &\sim_L D_{\{1\}}P_{(1\,2)}D_{\{1,2\}}D_{\{1,2\}}D_{\{1\}} \\ &\sim_R D_{\{1\}}D_{\{1,2\}}D_{\{1,2\}}D_{\{2\}}. \end{aligned}$$

Thus, the tableau realized by the sequence B_1, B_2, B_3, B_4 is

$$\begin{array}{cc} 3 & 4 \\ 2 & 3 \\ 1 & 2 \end{array} \neq \mathcal{K}(\sigma) = \begin{array}{cc} 4 & \\ 3 & \\ 2 & 3 \\ 1 & 2 \end{array}$$

and, therefore, A_0, B_1, B_2, B_3, B_4 is not a matrix realization for the pair $(\mathcal{T}, \mathcal{K}(\sigma))$.

4. Proof of the main theorem

The proof of main theorem 3.3 needs some additional lemmas. In what follows, assume that we have applied algorithm 1 to a biword Π of $\mathcal{T}$.

Lemma 4.1. *Let $k \in [t]$ and let (x, k) and $(y, k+\varepsilon)$, $\varepsilon \geq 1$, be two linked vertices of $w(\mathcal{T})$. Then, $\theta_{k+\varepsilon} \cdots \theta_{k+1}(x) = y$.*

Proof: Case 1. Assume $s^k_{(y,k+\varepsilon)} = y$. This means that there are no vertices in positions (y, j), $k < j < k+\varepsilon$. Thus, we may write

$$\theta_{k+\varepsilon} \cdots \theta_{k+1} = \rho_2(x\ y)\rho_1,$$

for some permutations $\rho_1, \rho_2 \in \mathcal{S}_n$, such that $\rho_1(x) = x$.

We claim that $\rho_2(y) = y$. In fact, for this equality to be false, we should have a vertex (y, j), with $j > k+\varepsilon$, negatively-linked to a vertex (a, i), with $i \in [k, k+\varepsilon-1]$ and $a > y$. Denote by u and v the shuffle components containing the vertices $(y, k+\varepsilon)$ and (y, j), respectively. Since the letter $k+\varepsilon$ is in u but not in v, we find that v must be a subcolumn of u. Thus, we must have $i = k$, and $(y, k+\varepsilon)$ has to be positively-linked. This means that

Then, sorting the billeters of Π, respectively, by nonincreasing rearrangement for the lexicographic order with priority of the second row, and by nonincreasing rearrangement for the antilexicographic order with priority of the first row, we get the biwords

$$\Sigma = \begin{pmatrix} 11\ 222\ 33 \\ 43\ 432\ 31 \end{pmatrix} \text{ and } \Sigma' = \begin{pmatrix} 21\ 321\ 2\ 3 \\ 44\ 333\ 2\ 1 \end{pmatrix}.$$

By the theorem above, the bottom word $w = 43\,432\,31$ of Σ is in the plactic class of the key $4321\,43\,3$, and may be written as $w|(\{1,2,5,7\},\{3,6\},\{4\}) \in Sh(4321, 43, 3)$, while the top word $v = 21\,321\,2\,3$ of Σ' is frank and is written as $v = v|(\{2,5,6,7\},\{1,3\},\{4\}) \in Sh(1123, 23, 2)$. Thus, we may write

$$\Pi = \begin{pmatrix} u|\{1,2,5,7\} & u|\{3,6\} & u|\{4\} \\ w|\{1,2,5,7\} & w|\{3,6\} & w|\{4\} \end{pmatrix} = \begin{pmatrix} v|\{2,5,6,7\} & v|\{1,3\} & u|\{4\} \\ c|\{2,5,6,7\} & c|\{1,3\} & c|\{4\} \end{pmatrix},$$

where u is the top row of Σ and c is the bottom row of Σ'.

(2) In general, the set $Sh(R^{l_t}_{\sigma,t}, \cdots, R^{l_1}_{\sigma,1})$ is not the whole plactic class of $K(\sigma, (l_t, \ldots, l_1))$. Consider the word $w = 431421 \equiv K(\sigma, (1,0,1,0))$, with $\sigma = 1423$. The correspondent frank word is $v = 211221$ which, as we have seen in remark 2.1, cannot be written as a shuffle of two rows of lengths 4 and 2. Thus, w is not in the set $Sh(r_{\sigma,4}, r_{\sigma,2})$.

The frank words with shape $(m_1, \ldots, m_t)$ satisfying $m_1 \geq \cdots \geq m_t$, or $m_1 < \cdots < m_t$, or for $t = 3$, have been characterized as shuffles of rows [4, 3, 6].

Corollary 2.9. [6] *Let σ be the identity or the reverse permutation in $\mathcal{S}_t$, $t \geq 1$, or any permutation in $\mathcal{S}_3$. Then, the plactic class of $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$ is $Sh(R^{l_t}_{\sigma,t}, \cdots, R^{l_1}_{\sigma,1})$.*

The following result, proved in [5], allows us to check whether the plactic class of a key is, or not, described by the shuffles of its columns.

Theorem 2.10. *The plactic class of $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$, $l_i > 0$, $1 \leq i \leq t$, is $Sh(R^{l_t}_{\sigma,t}, \cdots, R^{l_1}_{\sigma,1})$ if and only if for every left factor of the word of σ, by decreasing order, the difference of any two consecutive elements is at most 2.*

2.3. Graphical representation of words in $Sh(R^{l_t}_{\sigma,t},\cdots,R^{l_1}_{\sigma,1})$. Let $w \in Sh(R^{l_t}_{\sigma,t},\cdots,R^{l_1}_{\sigma,1})$, and consider a biword Σ, as in (4). Fix a shuffle decomposition for w, and consider the correspondent biword

$$\Pi = \begin{pmatrix} I^1_t \\ r_{\sigma,t} \end{pmatrix} \cdots \begin{pmatrix} I^{l_t}_t \\ r_{\sigma,t} \end{pmatrix} \cdots \begin{pmatrix} I^1_2 \\ r_{\sigma,2} \end{pmatrix} \cdots \begin{pmatrix} I^{l_2}_2 \\ r_{\sigma,2} \end{pmatrix} \begin{pmatrix} I^1_1 \\ r_{\sigma,1} \end{pmatrix} \cdots \begin{pmatrix} I^{l_1}_1 \\ r_{\sigma,1} \end{pmatrix}, \tag{14}$$

where each I^i_j is a row with $|I^i_j| = j$, for $j = 1, \ldots t$, and $i = 1, \ldots, l_j$, such that if $\binom{y}{x}$ and $\binom{y}{z}$ are billeters of Π, then $x \neq z$. In general, there is more than one biword Π associated with w, with bottom word the key $(r_{\sigma,t})^{l_t} \cdots (r_{\sigma,1})^{l_1}$, each corresponding to a different shuffle decompositions of w in $Sh(R^{l_t}_{\sigma,t},\ldots,R^{l_1}_{\sigma,1})$. Attaching Π to a skew-tableau $\mathcal{T}$, the top word of each factor $\begin{pmatrix} I^i_j \\ r_{\sigma,j} \end{pmatrix}$ gives the indices of the column of $\mathcal{T}$ where the letters of $r_{\sigma,i}$ are placed. Linking, by a straight line, the vertices of consecutive billeters of each factor $\begin{pmatrix} I^i_j \\ r_{\sigma,j} \end{pmatrix}$, we get a graphical representation of each shuffle component of w. We identify each factor with the corresponding shuffle component, and the biword Π with the shuffle decomposition of w. This means that (14) is graphically represented by $L_1 = l_t + \cdots + l_1$ polygonal lines of nonnegative slope, each corresponding to a shuffle component $\begin{pmatrix} I^i_j \\ r_{\sigma,j} \end{pmatrix}$, $1 \leq j \leq t$, $1 \leq i \leq l_j$.

For example, the word in the second row of the biword Σ in (6) is a shuffle of 4321 and 32. We may sort the billeters of (6) in several ways, in order to obtain a biword Π. Take, for instance, the biword

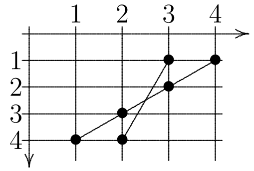
$$\Pi = \begin{pmatrix} 1\,1\,4\,4 & 2\,3 \\ 4\,3\,2\,1 & 3\,2 \end{pmatrix}. \tag{15}$$

Linking, respectively, the vertices of the consecutive billeters of $\begin{pmatrix} 1144 \\ 4321 \end{pmatrix}$ and $\begin{pmatrix} 23 \\ 32 \end{pmatrix}$ by a polygonal line, we get the following graphical representation of $4\,3\,\underline{3}\,\underline{2}\,2\,1 \in Sh(4321, 32)$ (the underlined letters indicate the shuffle

and the permutations $\sigma_2 = (4\,3)\sigma_1$, and $\sigma_4 = \sigma_3 = (4\,2)(3\,1)\sigma_2$. Finally, define $A_0 := diag_p(a)$, $B_1 := P_{\sigma_1}D_{[1]}$, $B_2 := T_{12}(1)T_{21}(-1)D_{[2]}$, $B_3 := T_{14}(1)T_{41}(-1)\ T_{23}(1)T_{32}(-1)D_{[2]}$, and $B_4 := D_{[1]}$.

Applying the same arguments used in example 3.1, we may show that the sequence A_0, B_1, B_2, B_3, B_4 is a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma))$, and it is clearly different from the one obtained in example 3.1.

Example 3.4. Consider again the tableau (3) presented in example 3.1, but now consider the shuffle decomposition of the word of $\mathcal{T}$ displayed in (17):



(28)

Clearly, this representation does not satisfy proposition 2.13, since $(4, 2) \in 32$ is a left critical vertex of $(4321, 32)$, and 32 is above 4321 in column 3. We will show that, in this case, the sequence of matrices obtained by algorithm 1 is not a matrix realization of $(\mathcal{T}, \mathcal{K}(\sigma))$. Following step 1 of algorithm 1, we may define $\sigma_1 = (14)(23)$, since $s^0_1 = s^0_{(4,1)} = 4$ and $s_2 = s^0_{(4,2)} = 3$. By step 2, we define the matrices

$$\begin{aligned}
I + S^2_{4,3} &= I + S(4, 3, \sigma_1) := T_{12}(1),\\
I + S^3_{3,2} &= I + S(3, 2, (4\,3)\sigma_1) := T_{13}(1),\\
I + S^3_{4,1} &= I + S(4, 1, (3\,2)(4\,3)\sigma_1) := T_{24}(1),\\
I + S^4_{2,1} &= I + S(2, 1, (4\,1)(3\,2)(4\,3)\sigma_1) := T_{12}(1),
\end{aligned}$$

and the permutations $\sigma_2 := (4\,3)\sigma_1$, $\sigma_3 := (4\ 1)(3\ 2)\sigma_2$ and $\sigma_4 := (2\ 1)\sigma_3$. Finally, we define the matrices $A_0 := diag_p(a)$, $B_1 := P_{\sigma_1}D_{[1]}$, $B_2 := T_{12}(1)\,T_{21}(-1)D_{[2]}$, $B_3 := T_{13}(1)T_{31}(-1)T_{24}(1)T_{42}(-1)D_{[2]}$, and $B_4 := T_{12}(1)T_{21}(-1)\,D_{[1]}$.

As in example 3.1, using lemma 3.2, we find that A_0, B_1, B_2, B_3, B_4 satisfy conditions I and II of definition 3.1. Let us now compute the invariant

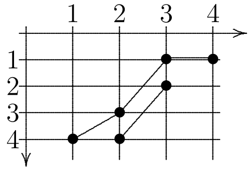
Using lemma 3.2, it is now a simple exercise to prove that $A_0B_1\cdots B_i \sim diag_p(a+\chi^{J_1}+\cdots+\chi^{J_i})$, for $i=1,\ldots,5$. Finally, using lemma 3.1 (ii) and (vii), we find that $B_1B_2B_3B_4B_5$ is left equivalent to

$$D_{[4]}T_{43}(-1)D_{[2]}T_{31}(-1)T_{21}(-1)T_{32}(-1)D_{[3]}T_{52}(-1)T_{32}(-1)T_{53}(-1)D_{[1]}D_{[3]}, \tag{26}$$

and by (vi), (26) $\sim_R$ $D_{[4]}D_{[2]}D_{[3]}D_{[1]}D_{[3]}$. This proves that the sequence $A_0, B_1, B_2, B_3, B_4, B_5$ is a matrix realization of $(\mathcal{T}, \mathcal{K}(\sigma))$.

Given a tableau $\mathcal{T}$ whose word is an element of $Sh(R_{\sigma,t}^{l_t},\ldots,R_{\sigma,1}^{l_1})$, there are, in general, several shuffle decompositions for the word of $\mathcal{T}$, each corresponding to a biword Π. Some of them satisfy proposition 2.13, others do not. For those shuffle decompositions satisfying proposition 2.13, the application of algorithm 1 produces a matrix realization for $(\mathcal{T},\mathcal{K}(\sigma))$. In the next examples, we use the skew-tableau $\mathcal{T}$, described in example 3.1, to show that different shuffle decompositions may give different matrix realizations. Moreover, if we consider a shuffle decomposition that does not satisfy proposition 2.13, then theorem 3.3 is not, necessarily, true.

Example 3.3. Consider the shuffle decomposition of the word of the skew-tableau (3) represented in the following lattice. Note that it satisfies proposition 2.13 but it is different from the one consider in example 3.1.

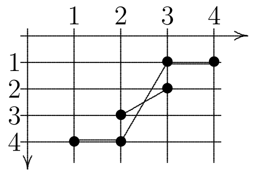


. (27)

Note that $s_1^0 = s_{(4,1)}^0 = 4$ and $s_2^0 = s_{(4,2)}^0 = 3$. Thus, following algorithm 1, we may define $\sigma_1 = (1\,4)(2\,3) \in \mathcal{S}_4$, the matrices

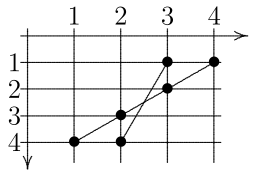
$$\begin{aligned}
I+S_{4,3}^2 &= I+S(4,3,\sigma_1) := T_{12}(1),\\
I+S_{3,1}^3 &= I+S(3,1,(4\,3)\sigma_1) := T_{14}(1),\\
I+S_{4,2}^3 &= I+S(4,2,(3\,1)(4\,3)\sigma_1) := T_{23}(1),\\
I+S_{1,1}^4 &= I+S(1,1,(4\,2)(3\,1)(4\,3)\sigma_1) := I,
\end{aligned}$$

component 32):



. (16)

If, instead, we sort the billeters of (6) in order to get the biword $\Pi' = \begin{pmatrix} 1234\ 14 \\ 4321\ 32 \end{pmatrix}$, we have the following graphical representation of $4\,\underline{3}\,3\,2\,\underline{2}\,1 \in Sh(4321, 32)$:



. (17)

For the rest of this paper, we fix a biword Π, and consider the graphical representation of Π with the consecutive vertices of each shuffle component $\begin{pmatrix} I_j^i \\ r_{\sigma,j} \end{pmatrix}$ linked by a straight line. This means, in particular, that we fix a shuffle decomposition of $w \in Sh(R_{\sigma,t}^{l_t},\ldots,R_{\sigma,1}^{l_1})$. Thus, each shuffle component $r_{\sigma,k}$ of w is identified with a polygonal line, reading it from right to left. Recalling that the shuffle components of w are columns, the rightmost letter of a shuffle component corresponds, in the graphical representation, to the leftmost vertex of its polygonal line. We oftener identify the vertices $\binom{y}{x}$ of a polygonal line with the corresponding letter x. For instance, in (17), the leftmost vertex $(4,1)$ of the polygonal line of 4321 corresponds to the rightmost letter 1 of this column.

Remark 2.2. (a) We stress that since the columns of a key are pairwise comparable for the inclusion order, if a is a letter of the column u, but not of the column v, then $\{v\} \subseteq \{u\}$.

(b) The polygonal lines do not have common vertices since the billeters of Π are all distinct, and the straight lines connecting consecutive vertices of a polygonal line have nonnegative slope.

Given two shuffle components u and v having the letter k, we say that u is *above* [respectively, *below*] v in column k if the vertices $(a,k) \in u$ and $(x,k) \in v$ satisfy $a < x$ [respectively, $a > x$].

Two vertices $(a,b),(x,y)$ are *linked* if they are consecutive vertices of a polygonal line. In this case, if $b < y$, (a,b) is said *positively-linked* to (x,y) (recall that $a \geq x$), and (x,y) is said *negatively-linked* to (a,b). Clearly, if (a,b) is not the rightmost vertex of a polygonal line, there is always a vertex to which (a,b) is positively-linked to. In this case, (a,b) is said positively-linked.

For instance, in the graphical representation (17) of the biword $\begin{pmatrix} 1234\,14 \\ 4321\,32 \end{pmatrix}$, the vertex $(4,1)$ is positively-linked to $(3,2)$, since $1 < 2$ and $\binom{4}{1}$ and $\binom{3}{2}$ are consecutive billeters of the shuffle component $\begin{pmatrix} 1234 \\ 4321 \end{pmatrix}$. The vertex $(3,2)$ is thus negatively-linked to $(4,1)$, and it is also positively-linked to $(2,3)$, since $2 < 3$ and $\binom{3}{2}$, $\binom{2}{3}$ are consecutive billeters of the same shuffle component.

Definition 2.1. Consider the biword Π (14). For each vertex (a,b) of Π, consider the map

$$\begin{array}{rcl} [0,b] & \longrightarrow & [a] \\ n & \longmapsto & s^n_{(a,b)} \end{array}$$

defined as follows:
(I) If there is no vertex of Π in row a to the left of column b, then let $s^n_{(a,b)} := a$.
(II) Otherwise, let (a,b'), $b' < b$, be the rightmost vertex of Π, in row a, to the left of (a,b).
  (a) If $b' \leq n$, put $s^n_{(a,b)} := a$.
  (b) If $b' > n$ and (a,b') is positively-linked to a vertex (x,y) of Π, $x < a$, put $s^n_{(a,b)} := s^n_{(x,y)}$.
  (c) Else, put $s^n_{(a,b)} := s^n_{(a,b')}$.

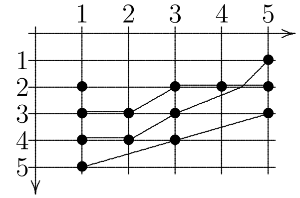
In particular, if the top row of Π (14) has no repeated letters, that is, if the indexing sets of w are pairwise disjoint, the polygonal lines do not have vertices in the same row and we are always in situation (I). Otherwise, the number $s^n_{(a,b)}$, with $n < b$, indicates, according to a certain path, a row $x \leq a$

also satisfy III. So, bearing in mind lemma 3.1, we may write

$$\begin{aligned} B_1B_2B_3B_4 &= P_{\sigma_1}D_{[1]}D_{[2]}T_{23}(1)T_{14}(1)T_{32}(-1)T_{41}(-1)D_{[2]}D_{[1]} \\ &= P_{\sigma_1}T_{23}(p)T_{14}(p^2)D_{[1]}D_{[2]}T_{32}(-1)T_{41}(-1)D_{[2]}D_{[1]} \\ &\sim_L D_{[1]}D_{[2]}T_{32}(-1)T_{41}(-1)D_{[2]}D_{[1]} \\ &\sim_R D_{[1]}D_{[2]}D_{[2]}D_{[1]}. \end{aligned}$$

Therefore, the sequence A_0, B_1, B_2, B_3, B_4 is a matrix realization of the pair $(\mathcal{T}, \mathcal{K}(\sigma))$.

Example 3.2. Let $\Pi = \begin{pmatrix} 22233\ 1344\ 345\ 2 \\ 54321\ 5321\ 531\ 1 \end{pmatrix}$ be the biword associated to skew-tableau $\mathcal{T}$ considered in example 2.4, and represented below. The word is in $Sh(R^1_{\sigma,5}, R^1_{\sigma,4}, R^1_{\sigma,3}, R^0_{\sigma,2}, R^1_{\sigma,1})$, with $\sigma = 13524$.



. (25)

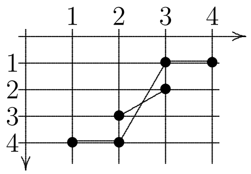
The evaluation of $\mathcal{T}$ is $(4,2,3,1,3)$ and its indexing sets are $J_1 = \{5,4,3,2\}$, $J_2 = \{4,3\}$, $J_3 = \{4,3,2\}$, $J_4 = \{2\}$, and $J_5 = \{3,2,1\}$, with $(L_1,\ldots,L_5) = (4,3,3,2,1)$.

By step 1 of algorithm 1, we must consider $\sigma_1 = (135)(24) \in \mathcal{S}_5$, since the leftmost vertices of the shuffle components of $w(\mathcal{T})$ satisfy $s^1_4 = s^0_{(2,1)} = 2$, $s^0_1 = s^0_{(3,1)} = 3$, $s^0_2 = s^0_{(4,1)} = 4$, and $s^0_3 = s^0_{(5,1)} = 5$, respectively. By step 2 of algorithm 1, define the matrices

$$\begin{array}{ll} I + S^2_{3,3} := I + S^2_{4,4} = I, & I + S^4_{2,2} := I, \\ I + S^2_{5,4} := I + S(5,2,\sigma_1) = T_{34}(1), & I + S^4_{3,1} := I + S(3,1,\sigma_3) = T_{25}, \\ I + S^3_{3,2} := I + S(3,2,\sigma_2) = T_{13}(1), & I + S^4_{4,3} := I + S(4,3,(3\,1)\sigma_3) = T_{35}, \\ I + S^3_{4,3} := I + S(4,3,(3\,2)\sigma_2) = T_{23}, & I + S^5_{2,2} := I, \end{array}$$

and the permutations $\sigma_2 := (5\ 2)\sigma_1$, $\sigma_3 := (4\ 3)(3\ 2)\sigma_2$ and $\sigma_5 := \sigma_4 := (4\ 3)(3\ 1)\sigma_3$. Put $A_0 := diag_p(a)$, $B_1 := P_{\sigma_1}D_{[4]}$, $B_2 := T_{34}(1)T_{43}(-1)D_{[2]}$, $B_3 := T_{13}(1)T_{31}(-1)T_{23}(1)T_{32}(-1)D_{[3]}$, $B_4 := T_{25}(1)T_{52}(-1)T_{35}(1)T_{53}(-1)D_{[1]}$ and $B_5 := D_{[3]}$.

Example 3.1. Consider the biword $\Pi = \begin{pmatrix} 1144\,23 \\ 4321\,32 \end{pmatrix}$ associated to the skew-tableau $\mathcal{T}$ (3). The word 433221 is in $Sh(R^1_{\sigma,4}, R^0_{\sigma,3}, R^1_{\sigma,2}, R^0_{\sigma,1})$, with $\sigma = 2341$. The biword Π is represented in the lattice below and satisfy proposition 2.13, since its representation has neither left nor right critical vertices.



(24)

$\mathcal{T}$ has indexing sets $J_1 = \{4\}$, $J_2 = \{3,4\}$, $J_3 = \{1,2\}$, $J_4 = \{1\}$, and evaluation $(1,2,2,1)$ where $(L_1, L_2, L_3, L_4) = (2,2,1,1)$.

The leftmost vertices of the shuffle components $r_{\sigma,4}$ and $r_{\sigma,2}$ are $(4,1)$ and $(3,2)$, respectively. So, recalling example 2.2, we have $s^0_1 = s^0_{(4,1)} = 4$ and $s^0_2 = s^0_{(3,2)} = 3$, and thus $s(\mathrm{X}_4) = \{4\}$ and $s(\mathrm{X}_2) = \{3\}$. Then, by algorithm 1, step 1, we must consider $\sigma_1 \in \mathcal{S}_4$ satisfying $\sigma_1(1) = 4$ and $\sigma_1(2) = 3$. Take, for instance, $\sigma_1 = (14)(23)$. Next, define

$$\begin{aligned}
I + S^2_{4,4} &:= I + S(4,4,\sigma_1) = I, \text{ and } \sigma_2 := (4\,4)\sigma_1 = \sigma_1;\\
I + S^3_{3,2} &:= I + S(3,2,\sigma_2) := T_{23}(1),\\
I + S^3_{4,1} &:= I + S(4,1,(3\,2)\sigma_2) := T_{14}(1), \text{ and } \sigma_3 := (4\,1)(3\,2)\sigma_2;\\
I + S^4_{1,1} &:= I + S(1,1,\sigma_3) := I, \text{ and } \sigma_4 := (1\,1)\sigma_3 = \sigma_3.
\end{aligned}$$

Finally, define the matrices $A_0 := diag_p(a)$, $B_1 := P_{\sigma_1} D_{[1]}$, $B_2 := D_{[2]}$, $B_3 := T_{23}(1)T_{32}(-1)\; T_{14}(1)T_{41}(-1)D_{[2]}$, and $B_4 := D_{[1]}$.

Clearly, $A_1 = A_0 B_1 \sim diag_p(a + \chi^{J_1})$ and $A_2 = A_0 B_1 B_2 \sim diag_p(a + \chi^{J_1} + \chi^{J_2})$. Since $\sigma_3([2]) = J_3$, by lemma 3.2, we find that

$$\begin{aligned}
A_3 &\sim_L diag_p(a + \chi^{J_1} + \chi^{J_2}) P_{\sigma_2} B_3 \sim_L diag_p(a + \chi^{J_1} + \chi^{J_2}) P_{\sigma_3} D_{[2]}\\
&\sim_R diag_p(a + \chi^{J_1} + \chi^{J_2} + \chi^{J_3}).
\end{aligned}$$

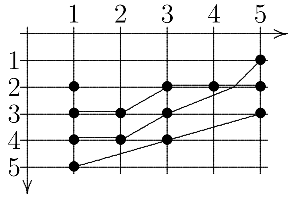
Since $A_4 \sim diag_p(a + \chi^{J_1} + \cdots + \chi^{J_4})$, the sequence $A_0, B_1, \ldots, B_4$ satisfy conditions I and II of definition 3.1. It remains to shows that this sequence

with a vertex (x,y), $n \le y \le b$, such that there are no vertices in the interval $]n, y[$. Next we exemplify the computation of $s^n_{(a,b)}$.

Example 2.2. Consider the graphical representation (16). Since there are no vertices in row 3 to the left of $(3,2)$, we have $s^n_{(3,2)} = 3$, $0 \le n \le 2$. By the same reasoning, $s^n_{(1,3)} = 1$, $0 \le n \le 3$. To compute $s^n_{(4,2)}$, $0 \le n \le 2$, notice that $(4,1)$ is the rightmost vertex, in row 4, to the left of $(4,2)$. Then, by $(II)(c)$, $s^0_{(4,2)} = s^0_{(4,1)} = 4$, since $1 > 0$ and $(4,1)$ is positively-linked with $(4,2)$; by $(II)(a)$, $s^n_{(4,2)} = 4$, for $n = 1,2$. Similarly, $s^n_{(1,4)} = 1$, for $0 \le n \le 4$.

Example 2.3. Consider the graphical representation (17). To compute $s^n_{(4,2)}$, $0 \le n \le 2$, notice that $(4,1)$ is positively-linked to $(3,2)$. Thus, by $(II)(b)$, we put $s^0_{(4,2)} = s^0_{(3,2)} = 3$, since there are no vertices in row 3 to the left of $(3,2)$; by $(II)(a)$, $s^n_{(4,2)} = 4$ for $n = 1,2$. Similarly, we find that $s^n_{(1,4)} = 1$, for $0 \le n \le 4$.

Example 2.4. Consider the biword $\Pi = \begin{pmatrix} 22233\;1344\;345\;2 \\ 54321\;5321\;531\;1 \end{pmatrix}$, obtained from the biwords (7), whose graphical representation is



. (18)

To compute $s^n_{(3,5)}$, $0 \le n \le 5$, note that the vertex $(3,3)$ is the rightmost vertex in row 3 to the left of column 5, and is positively-linked with $(1,5)$. Thus, by $(II)(a)$, $s^n_{(3,5)} = 3$ for $n = 3,4,5$, and by $(II)(b)$, $s^n_{(3,5)} = s^n_{(1,5)} = 1$, for $n = 0,1,2$. Similarly, we find that $s^n_{(4,3)} = 4$ for $n = 3,2$, $s^1_{(4,3)} = s^1_{(3,3)} = s^1_{(2,3)} = 2$, and $s^0_{(4,3)} = s^0_{(3,3)} = s^0_{(2,3)} = s^0_{(2,1)} = 2$.

Definition 2.2. For each vertex $(a,b) \in \Pi$ and each integer $n \in [0,b]$, we define, recursively, the sequence of vertices $S^n_{(a,b)}$, called the *s-path* of $s^n_{(a,b)}$, as follows:

(1) If (a,b) satisfies conditions (I) or $(II)(a)$ of definition 2.1, then $S^n_{(a,b)} := (a,b)$.

(2) Otherwise, let $(a,b') \in \Pi$, $b' < b$, be the rightmost vertex, in row a, to the left of (a,b). If (a,b') is positively-linked to a vertex (x,y), with $x < a$, then $S^n_{(a,b)} := \Big((a,b),(a,b'),S^n_{(x,y)}\Big)$. Else, we put $S^n_{(a,b)} := \Big((a,b),S^n_{(a,b')}\Big)$.

For instance, in the graphical representation of the biword given in (17), the s-path of $s^0_{(3,2)} = 3$ is $(3,2)$, and the s-path of $s^0_{(4,2)} = 3$ is

$$\Big((4,2),(4,1),(3,2)\Big), \tag{19}$$

while $\Big((4,3),(4,2),(3,3),(3,2),(2,3),(2,1)\Big)$ is the s-path of $s^0_{(4,3)} = 2$, in the graphical representation (18).

Lemma 2.11. *Let (x,y) be the leftmost vertex of a polygonal line of Π. Then, $s^0_{(x,y)}$ is computed either using rules (I) or $(II)(b)$.*

Proof: If there are no vertices in row x to the left of column y, we use rule (I) to obtain $s^0_{(x,y)} = x$. Otherwise, let (x,y'), $y' < y$, be the vertex in row x nearest to (x,y). Then (x,y') is a vertex of a shuffle component which also contains the letter y, and we must use rule $(II)(b)$ to obtain

$$s^0_{(x,y)} = s^0_{(x_1,y_1)},$$

where (x_1,y_1) is the vertex negatively-linked to (x,y'), which satisfy $x_1 < x$ and $y_1 \le y$. If there are no vertices in row x_1 to the left of (x_1,y_1), by (I) we have $s^0_{(x,y)} = s^0_{(x_1,y_1)} = x_1$. Otherwise, we use the same reasoning to obtain

$$s^0_{(x,y)} = s^0_{(x_1,y_1)} = s^0_{(x_2,y_2)},$$

where (x_2,y_2) is a vertex of a shuffle component which also contains the letter y, and satisfy $x_2 < x_1$ and $y_2 \le y_1$. Repeating this process, we eventually obtain

$$s^0_{(x,y)} = \cdots = s^0_{(x_r,y_r)} = x_r,$$

where (x_r,y_r) belong to a shuffle component that also contains the letter y, satisfy $x_r < x_{r-1}$ and $y_r \le \cdots \le y_1 \le y$, and such that there are no vertices in row x_r to the left of (x_r,y_r). □

From the proof of the lemma above we find that if $(x,y),(a_1,b_1),\dots,(a_r,b_r)$ is the s-path of $s^0_{(x,y)} \ne x$, with (x,y) the leftmost vertex of a shuffle component of w, then each vertex (a_i,b_i) belongs to a shuffle component that

Step 3. Let $A_0 := diag_p(a)$. Put $B_k := S_k D_{[m_k]}$, for $k = 1,\dots,t$, with $S_1 := P_{\sigma_1}$, and define inductively

$$A_k := A_{k-1}B_k.$$

Remark 3.1. (a) Lemma 2.12 asserts that the permutation $\sigma_1 \in \mathcal{S}_t$, given in step 1 of algorithm 1, is well defined.

(b) The matrix $S^{k+1}_{x^k_j,y^k_j}$, defined in step 2 of algorithm 1, is the null matrix whenever $s^k_{(y^k_j,k_j)} = x^k_j$, that is, if $x^k_j = y^k_j$. Therefore, the sequence $A_0, B_1, \dots, B_t$, obtained through the application of algorithm 1 may be simplified if we use a biword Π of $\mathcal{T}$, whose graphical representation has a reduced number of links between vertices involving distinct rows.

(c) If $u = \sigma_k^{-1}(\nu^k_{j-1})^{-1}(x^k_j)$, and $v = \sigma_k^{-1}(\nu^k_{j-1})^{-1}(s^k_{(y^k_j,k_j)})$, then $I + S^{k+1}_{x^k_j,y^k_j} = T_{uv}(1)$.

(d) For $i = 1,\dots,t$, $|J_i| = m_i = L_{\sigma(i)} > 0$.

(e) $\sigma_1([L_{k+1}{+}1, L_{k+1}{+}l_k]) = s(X_k)$, for $1 \le k \le t$. In particular, $s(X_k) = \emptyset$ iff $l_k = 0$.

(f) $\sigma_{k+1} = \theta_{k+1}\cdots\theta_2\sigma_1$.

Main Theorem 3.3. *Let $\sigma \in \mathcal{S}_t$ and $(l_t,\dots,l_1)$ a sequence of nonnegative integers. Let $\mathcal{T}$ be a skew-tableau of evaluation σm and skew-shape c/a, with $l(c) \le n$, whose word is a shuffle of all columns of the key $\mathcal{K}(\sigma,(l_t,\dots,l_1))$. Then, the sequence $A_0, B_1, \dots, B_t$, given by algorithm 1, is a matrix realization for the pair $(\mathcal{T}, \mathcal{K}(\sigma,(l_t,\dots,l_1))$.*

As already pointed out, it has been proved that $(\mathcal{T},\mathcal{K}(\sigma))$ is admissible only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{K}(\sigma)$ [5]. Thus, when the plactic class of $\mathcal{K}(\sigma)$ is the set of all shuffles of their columns together, we obtain the following characterization for the admissibility of pairs $(\mathcal{T},\mathcal{K}(\sigma))$ with the same evaluation.

Theorem 3.4. *Let $\sigma \in \mathcal{S}_t$ and $(l_t,\dots,l_1)$ a sequence of nonnegative integers such that the plactic class of the key $\mathcal{K}(\sigma,(l_t,\dots,l_1))$ coincides with the set $Sh(R^{l_t}_{\sigma,t},\dots, R^{l_1}_{\sigma,1})$. Let $\mathcal{T}$ be a skew-tableau of evaluation σm. Then, the pair $(\mathcal{T},\mathcal{K}(\sigma,(l_t,\dots,l_1))$ is admissible if and only if the word of $\mathcal{T}$ is in the plactic class of $\mathcal{K}(\sigma,(l_t,\dots,l_1))$.*

Before giving the proof of theorem 3.3, we will compute some examples.

Clearly, $I+S(x,y,\sigma)=T_{uv}(1)$, with $\sigma(u)=x$ and $\sigma(v)=y$. When $x=y$, $S(x,x,\sigma)$ is the null matrix.

Lemma 3.2. *Given $\sigma\in\mathcal{S}_n$ and $x\geq y$ in $[n]$, we have*

$$diag_p(a)P_\sigma(I+S(x,y,\sigma))(I-S(x,y,\sigma)^T)\sim_L diag_p(a)P_{(x\,y)\sigma} \qquad (22)$$

for any partition a of length $\leq n$.

Proof: Let $u,v\in[n]$ such that $\sigma(u)=x$ and $\sigma(v)=y$. Then, $I+S(x,y,\sigma)=T_{uv}(1)$ and $I-S(x,y,\sigma)^T=T_{vu}(-1)$. By lemma 3.1 $(viii)$, the left member of (22) is

$$diag_p(a)P_\sigma T_{uv}(1)T_{vu}(-1)=diag_p(a)P_\sigma T_{vv}(-1)T_{vu}(-1)P_{(uv)}. \qquad (23)$$

Since $x\geq y$, we have

$$(23)\sim_L diag_p(a_{\sigma(1)},\ldots,a_{\sigma(n)})P_{(uv)}\sim_L diag_p(a)P_\sigma P_{(uv)}=diag_p(a)P_{(x\,y)\sigma}.$$

□

Algorithm 1. Let $\sigma\in\mathcal{S}_t$ and $(l_t,\ldots,l_1)$ a sequence of nonnegative integers. Suppose we are given a skew-tableau $\mathcal{T}$ with evaluation σm and skew-shape c/a, with $l(c)\leq n$, such that $w(\mathcal{T})$ is in $Sh(R^{l_t}_{\sigma,t},\ldots,R^{l_1}_{\sigma,1})$. Fix a biword Π of $\mathcal{T}$ satisfying proposition 2.13. Our algorithm is presented as a three-step definition:

Step 1. For each $k=1,\ldots,t$, let $\mathrm{X}_k\subseteq\mathbb{N}^2$ be the set of the leftmost vertices of the l_k shuffle components $r_{\sigma,k}$ of $w(\mathcal{T})$, and define

$$s(\mathrm{X}_k):=\{s^0_{(x,j)}:(x,j)\in\mathrm{X}_k\}=\{s^0_{L_{k+1}+1}<\cdots<s^0_{L_{k+1}+l_k}\}\subseteq[n].$$

Let $\sigma_1\in\mathcal{S}_n$ such that $\sigma_1(i)=s^0_i$, for $i\in[L_1]$.

Step 2. For $k=1,\ldots,t-1$, let $J'_k:=\{x\in J_k:(x,k)\text{ is positively-linked}\}=\{x^k_1<\cdots<x^k_{q_k}\}\subseteq J_k$ and $\nu^k_0:=id\in\mathcal{S}_n$. For each $k=1,\ldots,t-1$ and $j=1,\ldots,q_k$, let (y^k_j,k_j) be the vertex negatively-linked to (x^k_j,k), and define inductively

$$S^{k+1}_{x^k_j y^k_j}:=S(x^k_j,s^k_{(y^k_j,k_j)},\nu^k_{j-1}\sigma_k),\ \text{ and }\ \nu^k_j:=(x^k_j\ s^k_{(y^k_j,k_j)})\nu^k_{j-1}.$$

Define $\theta_{k+1}:=\nu^k_{q_k}$, $\sigma_{k+1}:=\theta_{k+1}\sigma_k$, and $S_{k+1}:=\prod_{i=1}^{q_k}(I+S^{k+1}_{x^k_i y^k_i})(I-S^{k+1}_{x^k_i y^k_i})^T$.

contains the letter y and satisfy $b_i\leq y$, $i=1,\ldots r$. Moreover, the integer r is even, and if i is odd, the vertices $(a_i,b_i),(a_{i+1},b_{i+1})$ are linked, and $a_i=a_{i-1}$, with $a_0:=x$. The s-path (19) of $s^0_{(4,2)}$ obtained from (17), illustrates this, with $r=2$.

Lemma 2.12. *Assume that w has length n. Let $F:=\{(a_i,b_i)\in\Pi|i=1,\ldots,L_1\}$ be the collection of leftmost vertices of the polygonal lines of w. Then, the map $s^0:F\to[n]$ sending (a_i,b_i) into $s^0_{(a_i,b_i)}$ is an injection.*

Proof: This is obvious if the vertices are in pairwise disjoint rows. Otherwise, let (x,y) be the leftmost vertex of a polygonal line of w. By lemma 2.11, $s^0_{(x,y)}$ is computed using only rules (I) and $(II)(b)$. Thus, if we go trough the s-path $((x,y),(x_1,y_1),\ldots,(x_r,y_r))$ of $s^0_{(x,y)}$ backwards, starting in the final vertex (x_r,y_r), we obtain the initial vertex (x,y). Therefore, if the vertices $(x,y),(a,b)$ correspond to the leftmost vertices of two distinct polygonal lines, we must have $s^0_{(x,y)}\neq s^0_{(a,b)}$. □

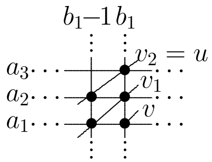
For instance, in example (17), we have $F=\{(4,1),(4,2)\}$ with $s^0_{(4,1)}=4$ and $s^0_{(4,2)}=3$, and in example (18), $F=\{(a,1):a=2,3,4,5\}$ with $s^0_{(a,1)}=a$ for $a=2,3,4,5$.

In a biword Π there are some vertices that play an important role in the proof of our main theorem 3.3. Those vertices, called *critical*, are characterized in the following definitions.

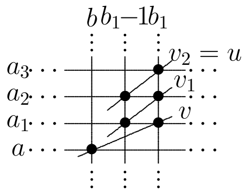
Definition 2.3. Let u,v be two shuffle components of w such that $\{v\}\subseteq\{u\}$. Let $(a_1,b_1)\in v$, either negatively-linked to a vertex (a,b) with $b<b_1-1$, or the leftmost vertex of v. The vertex (a_1,b_1) is said a *left critical vertex* of (u,v) if there are shuffle components v_i and pairs of linked vertices $(a_i,b_1-1),(a_{i+1},b_1)\in v_i$, for $i=1,\ldots,k$, such that $u=v_k$ and $a_{k+1}<\cdots<a_1\leq a$.

The shuffle components $v_1,\ldots,v_{k-1}$ are called the *critical components* of the critical vertex (a_1,b_1).

Schematically, with $k=2$, we either have



or



where $a_{i+1} \leq a_i - 1$, $i = 1, 2, 3$, with $a_0 = a$, $b \leq b_1 - 2$, and columns v_1, v_2, in the right grid, must contain the letter b, since $b \in \{v\} \subseteq \{v_1\}, \{v_2\}$.

For instance, consider once more (17). The vertex $(4, 2)$ is a left critical vertex of $(4321, 32)$, since it corresponds to the leftmost vertex of 32, and there is a pair of linked vertices $(4, 1), (3, 2) \in 4321$. It is the only left critical vertex present in this shuffle decomposition. Notice that if we consider a different biword, the vertex $(4, 2)$ may no longer be a left critical vertex. In fact, the biword Π' represented in (16) has no left critical vertices.

Consider now the graphical representation (18). The vertex $(4, 3)$ is a left critical vertex of $(54321, 531)$ with critical component 5321, since it is negatively-linked to $(5, 1)$, $3 - 1 > 1$, and there are pairs of linked vertices $(4, 2), (3, 3) \in 5321$, and $(3, 2), (2, 3) \in 54321$. Clearly, $(4, 3)$ is also a left critical vertex of $(5321, 531)$. In this biword, there is no other left critical vertex.

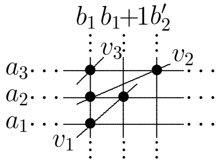
Definition 2.4. Let $v_1, \ldots, v_k$, $k \geq 2$, be shuffle components of w, and let $(a_k, b_1) \in v_k$ and $(a_i, b_1), (a'_i, b'_i) \in v_i$, $i = 1, \ldots, k - 1$, be pairs of linked vertices such that $b_1 < b'_i$ and $s^{b_1}_{(a'_i, b'_i)} = a_{i+1}$, for $i = 1, \ldots, k - 1$. If $b_1 = b'_1 - 1$, the vertex (a_1, b_1) is called a *right critical vertex* of type I of (v_k, v_1). Otherwise, (a_1, b_1) is called a *right critical vertex* of type II of (v_k, v_1).

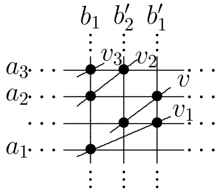
The shuffle components $v_2, \ldots, v_{k-1}$ are called the *critical components* of the critical vertex (a_1, b_1).

Remark 2.3. (a) Notice that in the conditions of the definition above, (a_1, b_1) is also a right critical vertex of type I or II of (v_j, v_1), for $j = 2, \ldots, k$.

(b) According to the definition, the vertex (a_i, b_1) is a right critical vertex of type I or II of (v_k, v_1), for $i = 1, \ldots, k - 1$.

The following diagrams are schematic representations of right critical vertices of types I and II, respectively, with $k = 3$:





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partition of $B_1 \cdots B_r$, $1 \leq r \leq t$, then $(b^1, \ldots, b^t)$ is a tableau with weight $(m_1, \ldots, m_t)$ as well (see [6]).

Definition 3.1. Let $\mathcal{T} = (a^0, a^1, ..., a^t)$ be a skew-tableau and let $\mathcal{F} = (0, b^1, ..., b^t)$ be a tableau, both of evaluation $(m_1, \ldots, m_t)$. We say that a sequence of $n \times n$ nonsingular matrices $A_0, B_1, ..., B_t$ is a matrix realization of the pair $(\mathcal{T}, \mathcal{F})$ (or realizes $(\mathcal{T}, \mathcal{F})$) if:

I. For each $r \in \{1, ..., t\}$, the matrix B_r has invariant partition $(1^{m_r}, 0^{n-m_r})$.

II. For each $r \in \{0, 1, ..., t\}$, the matrix $A_r := A_0 B_1 ... B_r$ has invariant partition a^r.

III. For each $r \in \{1, ..., t\}$, the matrix $B_1 ... B_r$ has invariant partition b^r.

$(\mathcal{T}, \mathcal{F})$ is called an *admissible pair* of tableaux.

Actually, I and II say that $A_0, B_1, \ldots, B_t$ realizes $\mathcal{T}$, and III says that $B_1 \ldots B_t$ realizes $\mathcal{F}$.

Recall that the key $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$ is the only tableau of evaluation $(m_1, \ldots, m_t)$ and shape $\sum_{i=1}^{t}(1^{m_i})$. Actually,

$$\mathcal{K}(\sigma, (l_t, \ldots, l_1)) = (0, (1^{m_1}), \sum_{i=1}^{2}(1^{m_i}), \ldots, \sum_{i=1}^{t}(1^{m_i})).$$

Thus, when $\mathcal{F} = \mathcal{K}(\sigma, (l_t, \ldots, l_1))$, in order to verify property III, it is sufficient to show that $B_1 \cdots B_t$ has invariant partition $(1^{m_1}) + \cdots + (1^{m_t})$. For the purpose of this paper, we shall consider only pairs $(\mathcal{T}, \mathcal{K}(\sigma, (l_t, \ldots, l_1))$ such that the word of $\mathcal{T}$ is an element of $Sh(R^{l_t}_{\sigma,t}, \ldots, R^{l_1}_{\sigma,1})$.

It has been shown that $(\mathcal{T}, \mathcal{K}(\sigma, (l_t, \ldots, l_1))$ is an admissible pair only if the word of $\mathcal{T}$ is an element of the plactic class of $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$ [5]. Thus, the following problem arises: *given a pair $(\mathcal{T}, \mathcal{K}(\sigma, (l_t, \ldots, l_1)))$, such that the word of $\mathcal{T}$ is a shuffle of all columns of the key $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$, do there exist a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma, (l_t, \ldots, l_1))$?*

The following algorithm and theorem 3.3 give an answer to this problem. For this, we need the following definition.

Definition 3.2. Given $\sigma \in \mathcal{S}_n$ and $x \geq y$ in $[n]$, we define the $n \times n$ matrix $S(x, y, \sigma) = [s_{ij}]$, with

$$s_{ij} = \begin{cases} 1 & \text{, if } \sigma(i) = x \text{ and } \sigma(j) = y \neq x \\ 0 & \text{, otherwise.} \end{cases}$$

is a sequence of nonnegative integers, we have $\sigma m = P_\sigma [m_1 \cdots m_t]^T$. It is a simple exercise to prove that

$$P_\sigma diag_p(a) = diag_p(\sigma a) P_\sigma, \quad P_\sigma^T diag_p(a) P_\sigma = diag_p(\sigma^{-1} a),$$
$$\text{and} \quad P_\sigma D_J = D_{\sigma(J)} P_\sigma.$$

We denote by E_{ij} the $n \times n$ matrix having 1 in position (i,j) and $0's$ elsewhere, and define the elementary unimodular matrices $T_{ij}(x)$ as follows:

$$T_{ij}(x) = I + xE_{ij}, \quad \text{where } i \neq j \text{ and } x \in \mathcal{R}_p;$$
$$T_{ii}(v) = I + (v-1)E_{ii}, \quad \text{where } v \text{ is a unit of } \mathcal{R}_p.$$

It is obvious that $E_{ij}E_{rs} = \delta_{jr}E_{is}$. Therefore, if $i \neq j$ and $r \neq s$, we find that $T_{ij}(x)T_{rs}(y) = I + xE_{ij} + yE_{rs} + xy\delta_{jr}E_{is}$. In the lemma below we state some basic properties of the these elementary matrices $T_{ij}(x)$, which will be used in the sequel.

Lemma 3.1. *Let $i, j, r, s, m \in [n]$, and $x, y, v \in \mathcal{R}_p$, such that v is a unit. Then,*

(i) $T_{ij}(x)T_{rs}(y) = T_{rs}(y)T_{ij}(x)$, *whenever $i \neq s$ and $j \neq r$.*
(ii) $T_{ij}(x)T_{js}(y) = T_{js}(y)T_{ij}(x)T_{is}(xy)$, *if $i \neq s$.*
(iii) $T_{ii}(v)T_{rs}(x) = T_{rs}(ux)T_{ii}(v)$, *for some unit u.*
(iv) $T_{ji}(y)T_{ij}(xp) = T_{ij}(u_1xp)T_{ii}(u_2)T_{jj}(u_3)T_{ji}(u_4y)$, *for some units u_i, $i = 1, \ldots, 4$.*
(v) $T_{ij}(x)D_{[m]} = D_{[m]}T_{ij}(x)$, *if $i, j > m$.*
(vi) $T_{ij}(x)D_{[m]} = D_{[m]}T_{ij}(xp)$, *if $i > m \geq j \geq 1$.*
(vii) $T_{ij}(xp)D_{[m]} = D_{[m]}T_{ij}(x)$, *if $j > m \geq i \geq 1$.*
(viii) $T_{ji}(-1)T_{ij}(1) = T_{jj}(-1)T_{ij}(1)P_{(i\,j)}$.

Proof: Straightforward. □

Let $\sigma \in \mathcal{S}_t$, $t \geq 1$, and $(l_t, \ldots, l_1)$ a sequence of nonnegative integers with $l_t > 0$. Consider the sequence $(m_1, \ldots, m_t)$, where $m_{\sigma(i)} = L_i$, $1 \leq i \leq t$. In what follows, $\mathcal{T}$ will denote a skew-tableau of evaluation $(m_1, ..., m_t)$ and skew-shape c/a, with $l(c) \leq n$. Following [2, 4, 6], we introduce the definition of a matrix realization of a pair of tableaux $(\mathcal{T}, \mathcal{F})$, with $\mathcal{F}$ a tableau of evaluation $(m_1, ..., m_t)$ and shape b^t.

Let A_0 be a matrix with invariant partition a^0, and for $r = 1, \ldots, t$, let B_r be a matrix with invariant partition $(1^{m_r}, 0^{n-m_r})$. If a^r is the invariant partition of $A_0B_1 \cdots B_r$, $1 \leq r \leq t$, then it is clear that $(a^0, a^1, \ldots, a^t)$ is a skew-tableau with weight $(m_1, \ldots, m_t)$. Similarly, if b^r is the invariant

Notice that $a_{i+1} \leq a_i - 1$, $i = 1, 2$; in the left grid, v_1 must have letter b_2', and in the right grid, v and v_2 must have the letter b_1 and b_1', respectively. Note also that (a_2, b_1) is a right critical vertex of type I or II of (v_3, v_2).

For example, in (17), the vertex $(2,3) \in 4321$ is a right critical vertex of type I of $(4321, 32)$, with no critical components, since it is positively-linked to $(1,4)$, satisfy $s^3_{(1,4)} = 1$, and there is a vertex $(1,3) \in 32$. This is the only right critical vertex of this shuffle decomposition.

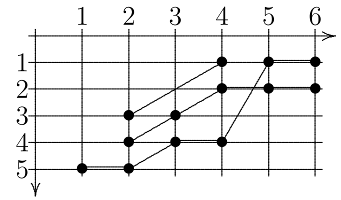
Consider now example (18). Notice that $(4,3)$ is a right critical vertex of type II of $(5321, 531)$, since it is positively-linked to $(3,5)$, with $3 \neq 5-1$, satisfy $s^3_{(3,5)} = 3$, and there is a vertex $(3,3) \in 5321$. This example shows that a vertex may be simultaneously a left and a right critical vertex. The vertex $(5,1)$ is a right critical vertex of type II of $(1, 531)$, since it is positively-linked to $(4,3)$, with $1 \neq 3-1$, satisfy $s^1_{(4,3)} = 2$, and there is a vertex $(2,1) \in 1$. Finally, the vertex $(4,2)$ is a right critical vertex of type I of $(54321, 5321)$, since it is positively-linked to $(3,3)$, with $2 = 3-1$, satisfy $s^2_{(3,3)} = 3$, and there is a vertex $(3,2) \in 54321$.

Given a word w in $Sh(R^{l_t}_{\sigma,t}, \ldots, R^{l_1}_{\sigma,1})$, we have seen that, in general, it admits several shuffle decompositions. But as we shall see in the next section, example 3.4, not all shuffles decompositions allow us to obtain a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma))$. In what follows, given a shuffle decomposition of w, we will adjust the links between the vertices of the biword Π, and, therefore, the shuffle decomposition itself, forming a new biword Π' in order to achieve the matrix realization.

So, consider the graphical representation of the biword Π, that we have associated with w, and suppose there is a pair of shuffle components u, v for which there are integers $k < k'$ and a vertex (a, k) satisfying:

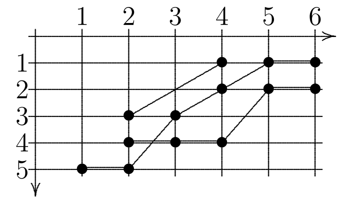
(i) $[k, k'] \subseteq \{u\} \cap \{v\}$;
(ii) (a, k) is a left critical vertex or a right critical vertex of type I of (u, v), with no critical components;
(iii) v is above [respectively, below] u in column k' if $(a, k) \in v$ [respectively, $(a, k) \in u$].

In this case, it is easy to check that we may re-link the vertices of u, v, between columns k and k', in such a way that the vertex (a, k) is no longer a critical vertex, forming a new biword Π'. For instance, consider the graphical representation of $\begin{pmatrix} 114455\ 22234\ 13 \\ 654321\ 65432\ 42 \end{pmatrix}$:



(20)

Notice that (i) both polygonal lines $u = \binom{114455}{654321}$ and $v = \binom{22234}{65432}$ have the integers $2, 3, \ldots, 6$; (ii) $(5, 2) \in u$ is a right critical vertex of type I of (u, v) with no critical components, and (iii) u is above v in column 5. We may re-link the vertices of u and v between columns 2 and 6, obtaining another biword with the following graphical representation:



(21)

The application of this procedure to a graphical representation of the biword Π implies, for the new biword, that whenever a vertex $(a, k) \in v$ [respectively, $(a, k) \in u$] is a left critical vertex or a right critical vertex of type I of (u, v), with no critical components, and the letters $k, \ldots, k + r$ are both in u and v, then the vertices of the shuffle component v must be below [respectively, above] the vertices of u in columns $k, \ldots, k + r$. For instance, in (21), $(4, 4) \in v$ is a right critical vertex of type I of (u, v) and v is below u in columns 5 and 6.

For the rest of this paper, we fix a biword Π, previously adjusted by the procedure described above. The graphical representation of this shuffle decomposition has, therefore, the following easily deduced property.

Proposition 2.13. *Let u, v be two shuffle components of w with $[k, k+r] \subseteq \{u\} \cap \{v\}$, for some $k, k + r \in [t]$, with $r \geq 1$, and let (a, k) be either a left critical vertex, or a right critical vertex of type I of (u, v), with critical components u_l satisfying $[k, k+r] \subseteq \{u_l\}$, for $l = 1, \ldots q$. Then, if $(a, k) \in v$ [respectively, $(a, k) \in u$], the vertices of v are below [respectively, above] the vertices of u in columns $k, k + 1, \ldots, k + r$.*

Proof: We consider the case $(a, k) \in v$, the other case is similar. The proof is by induction on the number $q \geq 0$ of critical components of (a, k). When there are no critical components, there is nothing to prove since we have assumed that the procedure described above have been applied. So, let $(a, k) \in v$ be a left or right critical vertex of type I of (u, v) with $q > 0$ critical components, $u_1, \ldots, u_q$, and assume the result for $q - 1$ critical components.

By definition, (a, k) is also a left or right critical vertex of type I of (v, u_q), with critical components $u_1, \ldots, u_{q-1}$. By induction, the vertices of v are below the vertices of u_q in columns $k, k + 1, \ldots, k + r$. Finally, recalling remark 2.3, there is a right critical vertex $(a', k) \in u_q$ of type I of (u, u_q). Therefore, the vertices of u_q must be below the vertices of u in columns $k, k + 1, \ldots, k + r$. $\square$

3. An algorithm and statement of results

Let $\mathcal{R}_p$ be a local principal ideal domain with maximal ideal (p). In this paper, all matrices are $n \times n$ and nonsingular with entries over $\mathcal{R}_p$; A^T denotes the transpose of matrix A. Let $\mathcal{U}_n$ be the group of $n \times n$ unimodular matrices over $\mathcal{R}_p$. Given $n \times n$ matrices A and B, we say that B is *left equivalent* to A (written $B \sim_L A$) if $B = UA$ for some $U \in \mathcal{U}_n$; B is *right equivalent* to A (written $B \sim_R A$) if $B = AV$ for some $V \in \mathcal{U}_n$; and B is *equivalent* to A (written $B \sim A$) if $B = UAV$ for some $U, V \in \mathcal{U}_n$. The relations $\sim_L$, $\sim_R$ and $\sim$ are equivalence relations in the set of all $n \times n$ nonsingular matrices over $\mathcal{R}_p$.

Let A be a $n \times n$ nonsingular matrix. By the Smith normal form theorem [7, 18], there exist nonnegative integers $a_1, ..., a_n$ with $a_1 \geq ... \geq a_n \geq 0$ such that A is equivalent to

$$diag(p^{a_1}, ..., p^{a_n}).$$

The sequence $a = (a_1, ..., a_n)$, of the exponents of the p-powers in the Smith normal form of A, is a partition of length $\leq n$, uniquely determined by the matrix A. We call a the *invariant partition* of A. More generally, if we are given a sequence of nonnegative integers $e_1, ..., e_n$, the following notation for p-powered diagonal matrices will be used:

$$diag_p(e_1, ..., e_n) := diag(p^{e_1}, ..., p^{e_n}).$$

Given a subset $J \subseteq [n]$, put $D_J := diag_p(\chi^J)$. If $\sigma \in \mathcal{S}_n$, we denote by P_σ the permutation matrix having $\delta_{i\sigma(j)}$ in position (i, j). Then, if $m = (m_1, \ldots, m_t)$