Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 05–29

REDUCTION OF LAGRANGIAN MECHANICS ON LIE ALGEBROIDS

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Abstract: We prove that if a surjective submersion which is a homomorphism of Lie algebroids is given, then there exists another homomorphism between the corresponding prolonged Lie algebroids and a relation between the dynamics on these Lie algebroid prolongations is established. We also propose a geometric reduction method for dynamics on Lie algebroids defined by a Lagrangian and the method is applied to regular Lagrangian systems with nonholonomic constraints.

Keywords: Reduction, Lie algebroids, Lagrangian mechanics.
AMS Subject Classification (2000): 37J15, 37J60, 70H33, 53D17.

1. Introduction

The Lie algebroids were introduced by Pradines [21] as infinitesimal objects for differential groupoids, and since then they are receiving an increasing interest from mathematics and theoretical physicists. A Lie algebroid can be seen as a generalization of both a Lie algebra and a tangent bundle, these being the simplest (no trivial) examples of Lie algebroids. Another relevant example of Lie algebroid with equal importance to mathematics and physics is the gauge algebroid TP/G associated to a principal bundle $P(M, G)$, where in the classical field theory M is the space-time manifold and G is the gauge group. For the basic proprieties and literature on the subject we refer to the book by Cannas [1] and the survey paper and book by Mackenzie [13, 14].

The aim of this paper is to study the reduction of the dynamics on Lie algebroids defined through a Lagrangian function, which can be carried out by using the prolongation of a Lie algebroid over a map, introduced by Higgins and Mackenzie [8]. The study of Lagrangian mechanics on Lie algebroids was first proposed by Weinstein [23] (see also [12]), and then Martínez [16], adapting the definition of prolongation of a Lie algebroid over a map, developed a formalism for Lagrangian mechanics on Lie algebroids using the

Received November 25, 2005.
The authors acknowledge the financial support from PRODEP/5.3/2003, POCI/MAT/58452/2004, CMUC/FCT and project BFM-2003-02532.

generalization of the fundamental ingredients of geometric Lagrangian mechanics: the vertical endomorphism, the Liouville vector field and the Cartan forms. Afterward several papers on related subjects were developed, see e.g. [11] and references therein.

The theory of reduction has many applications and has been shown to be extremely useful for a deep understanding of many physical theories including, among others, systems with symmetry, Poisson structures, stability theory and integrable systems. The reduction of the dynamics has been previously considered in many papers (see [15] and the references therein) but is not a well known subject in Lie algebroids dynamics. This happens because the meaning of Lie algebroid reduction has not been clearly stated; this issue was clarified in a previous paper [4]. With the study of reduction of the dynamics on Lie algebroids defined by a Lagrangian we generalize a previous work by Rodríguez-Olmos [20], where the author reduced the dynamics of Lie algebroids with symmetry, that is, a Lie algebroid where a Lie group acts and whose action is defined by a Lie algebroid representation of the Lie group.

The paper is organized as follows. In the first three sections, we recall the definition of prolongation of a Lie algebroid A (see [16]) and how the dynamics on the prolongation of A defined by a Lagrangian function can be found (see [11, 16]). We prove in section 4 that given a surjective submersion $\Pi : A \to \widehat{A}$ that is a homomorphism of Lie algebroids, there exists a surjective map between their prolonged Lie algebroids $\overline{\Pi}|_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\widehat{A}$ that is a homomorphism of Lie algebroids too, *i.e.* we can defined a Lie algebroid reduction between the corresponding prolonged Lie algebroids. The particular case of Lie algebroids with symmetry is analyzed with an special attention to the gauge algebroid. In section 5 we show how the dynamics can be reduced, establishing a relation between the dynamics in the Lie algebroid prolongation $\mathcal{T}A$ and the dynamics in the reduced Lie algebroid prolongation $\mathcal{T}\widehat{A}$. Finally, in the last section the Chetaev formulation for nonholonomic systems in Lie algebroids is given and a reduction procedure for Lagrangian systems on Lie algebroids with nonholonomic constraints is explained. We show that the dynamics of a system with nonholonomic constraints can be reduced if the system has a regular and $\overline{\Pi}$-invariant Lagrangian $L = l \circ \overline{\Pi}$ with $l \in C^\infty(\widehat{A})$.

2. Basic concepts of Lie algebroids

Recall that a *Lie algebroid* is a vector bundle $p : A \to M$ over a manifold M together with a vector bundle morphism $\rho : A \to TM$ over the identity map on M (called the anchor) and a Lie bracket $[\cdot,\cdot]_A$ on the $C^\infty(M)$-modulo $\Gamma(A)$ of sections for p satisfying

$$[v, fw]_A = f[v, w]_A + (\rho(v)f)w$$

for every pair of sections v and w and any smooth function f on M. We denote the Lie algebroid by $(A, \rho, [\cdot,\cdot]_A)$ or simply by A whenever it is clear which Lie algebroid structure we refer to. Note that the anchor is a $C^\infty(M)$-linear map of the space $\Gamma(A)$ into the space $\mathfrak{X}(M)$ of vector fields on M, and one can easily prove, using the above condition and the Jacobi identity of the Lie bracket $[\cdot,\cdot]_A$, that the anchor is a Lie algebra homomorphism [9]. For a detailed lecture on the subject see e.g. [1, 14].

Let $(q^1, \ldots, q^n)$ be local coordinates in a chart on an open set $U \subset M$, and let $\{e_\alpha \mid \alpha = 1, \ldots, r\}$ be a basis of local sections of the bundle $p|_{U_A} : U_A = p^{-1}(U) \to M$. Each local section V_U is written $V_U = \xi^\alpha\, e_\alpha$. The local coordinates of $a \in U_A$ are $(q^1, \ldots, q^n, \xi^1, \ldots, \xi^r)$. The local expressions for the Lie product and the anchor map are, respectively,

$$[e_\alpha, e_\beta]_{\mathcal{A}} = c_{\alpha\beta}{}^{\gamma}\, e_\gamma\,, \qquad \rho(e_\alpha) = \rho^i{}_\alpha\, \frac{\partial}{\partial q^i}\,, \tag{2.1}$$

where $c_{\alpha\beta}{}^{\gamma}$ and $\rho^i{}_\alpha$ are the so called *structure functions* of the Lie algebroid relative to $\{e_\alpha\}$. As ρ is an homomorphism of Lie algebras this function satisfies:

$$\rho^j{}_\alpha\, \frac{\partial \rho^i{}_\beta}{\partial q^j} - \rho^j{}_\beta\, \frac{\partial \rho^i{}_\alpha}{\partial q^j} = \rho^i{}_\gamma\, c_{\alpha\beta}{}^{\gamma}\,, \qquad i = 1, \ldots, n\,. \tag{2.2}$$

Moreover the compatibility condition, and the Jacobi identity implies:

$$\sum_{\mathrm{cycl}(\alpha,\beta,\gamma)} \left[\rho^i{}_\gamma\, \frac{\partial c_{\beta\alpha}{}^{\mu}}{\partial q^i} + c_{\alpha\beta}{}^{\nu}\, c_{\nu\gamma}{}^{\mu}\right] = 0\ , \qquad \mu = 1, \ldots, r\,. \tag{2.3}$$

The equations (2.2) and (2.3) are known as the *compatibility equations of the structure functions.*

A Lie algebroid (A, p, M) is endowed with a differential operator d_A that is a nilpotent ($d_A^2 = 0$) derivative of degree one in the exterior graded algebra of A-forms, $\Omega^\bullet(A)$; d_A is the *exterior derivative* of the Lie algebroid. We say that a vector bundle morphism $(\Pi, \pi) : (A, p, M) \to (A', p', M')$ between two

Lie algebroids is a *homomorphism of Lie algebroids* if $d_A \circ \Pi^* = \Pi^* \circ d_{A'}$ (see [22]). This definition is equivalent to the definition of homomorphism of Lie algebroids introduced in [8]: a vector bundle morphism $(\Phi, \phi) : (A, p, M) \to (A', p', M')$ is a homomorphism of the Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ in the Lie algebroid $(A', \rho', [\cdot,\cdot]_{A'})$ when $T\phi \circ \rho = \rho' \circ \Phi$ and, for any pair $v, w \in \Gamma(A)$ with Φ-decompositions $\Phi \circ v = \sum_i f_i\,(v'_i \circ \phi)$ and $\Phi \circ w = \sum_j g_j\,(w'_j \circ \phi)$, the following condition is satisfied:

$$\Phi \circ [v, w]_A = \sum_{i,j} f_i\, g_j\,([v'_i, w'_j]_{A'} \circ \phi) + \sum_j (\rho(v) g_j)\,(w'_j \circ \phi) - \sum_i (\rho(w) f_i)\,(v'_i \circ \phi). \tag{2.4}$$

Given a surjective morphism of vector bundles $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ between two Lie algebroids $(A, \rho, [\cdot,\cdot]_A)$ and $(\widehat{A}, \widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$, respectively, we say that $\widehat{A}$ is a *reduced Lie algebroid* of A if Π is a homomorphism of Lie algebroids (see [4]).

3. Lagrangian mechanics on Lie algebroids

In this section, we recall the definition of prolongation of a Lie algebroid A and the fundamental elements of the Lagrangian mechanics on this prolonged Lie algebroid (see [11] and [16]).

Let $p : A \to M$ be a vector bundle over M with Lie algebroid structure $(\rho, [\cdot,\cdot]_A)$. The *prolongation of the Lie algebroid* A is a vector bundle $\mathcal{T}A$ over A, where $\mathcal{T}A$ is the total space of the pullback of the vector bundle $Tp : TA \to TM$ by the anchor map $\rho : A \to TM$; such total space is (see [16])

$$\mathcal{T}A = \{(b, v) \in A \times TA \mid \rho(b) = Tp(v)\}\,.$$

The projection $p_{\mathcal{T}A} : \mathcal{T}A \to A$ is defined by $p_{\mathcal{T}A}(b, v) = a$, where $p_{TA}(v) = a \in A$ with $p_{TA} : TA \to A$ the canonical projection of the tangent bundle TA over the base A. Note that the pullback of the bundle $Tp : TA \to TM$ by the anchor map $\rho : A \to TM$ coincides with the induced or inverse-image Lie algebroid of A over $p : A \to M$ in the terminology of Higgins and Mackenzie [8] (see also [11]).

A element (b, v) of $\mathcal{T}A$ will be denoted by (a, b, v), where $a \in A$ is the point where v is tangent to A. With this notation

$$\mathcal{T}A = \{(a, b, v) \in A \times A \times TA \mid p(a) = p(b),\ \rho(b) = T_a p(v) \text{ with } v \in T_a A\}\,.$$

So, we have in a natural way the following projections:

$$p_{\mathcal{T}A}(a,b,v)=a, \quad p_2(a,b,v)=b, \quad \rho_{\mathcal{T}A}(a,b,v)=v,$$

which we represent in the commutative diagram

$$\begin{tikzcd} \mathcal{T}A \arrow[r, "\rho_{\mathcal{T}A}"] \arrow[d, "p_2"'] & TA \arrow[d, "Tp"] \arrow[dl, "p_{\mathcal{T}A}"] \\ A \arrow[r, "\rho"'] & TM \end{tikzcd}$$

The structure of vector space of each fibre $\mathcal{T}_aA$ of $p_{\mathcal{T}A}:\mathcal{T}A\to A$ is

$$(a,b_1,v_1)+(a,b_2,v_2)=(a,b_1+b_2,v_1+v_2), \qquad \lambda(a,b_1,v_1)=(a,\lambda b_1,\lambda v_1),$$

for each $(a,b_i,v_i)\in\mathcal{T}_aA$ and $\lambda\in\mathbb{R}$, with $i=1,2$. If r is the rank of the vector bundle A and n is the dimension of M, the dimension of each fibre of $\mathcal{T}A$ is $2r$ because the tangent map $Tp:TA\to TM$ is surjective, and so $\dim(\operatorname{Im}\rho_m+\operatorname{Im}T_ap)=\dim T_mM=n$, where $p(a)=m$ (see [11]).

The vector bundle $p_{\mathcal{T}A}:\mathcal{T}A\to A$ can be endowed with a Lie algebroid structure, where the anchor is the map $\rho_{\mathcal{T}A}:\mathcal{T}A\to TA$ and the Lie bracket in the space of sections is defined using the bracket of a certain type of sections, the so-called *projectable sections*, that generate all space of sections of $\mathcal{T}A$. A section $V\in\Gamma(\mathcal{T}A)$ is said to be *projectable* if there exists a $\sigma\in\Gamma(A)$ such that $p_2\circ V=\sigma\circ p$ or, equivalently, if $V(a)=(a,\sigma(p(a)),X(a))$ for all $a\in A$, where $\sigma\in\Gamma(A)$ and $X\in\mathfrak{X}(A)$ are such that $Tp\circ X=\rho(\sigma)\circ p$. Given two projectable sections V and V' of $\mathcal{T}A$ the bracket of these two sections is defined by:

$$[V,V']_{\mathcal{T}A}(a)=(a,[\sigma,\sigma']_A(p(a)),[X,X'](a)).$$

A section V of $\mathcal{T}A$ is said to be *vertical* if $p_2\circ V=0$. Obviously, all vertical sections of $\mathcal{T}A$ are projectable and the bracket of vertical sections is also vertical.

For example, if A is the tangent bundle to a manifold Q, $A=TQ$, endowed with its usual Lie algebroid structure, the prolongation of the Lie algebroid A is the tangent bundle $T(TQ)$ to TQ endowed with its usual structure of Lie algebroid over TQ (see [16]).

As pointed out in [11], with the above structure of Lie algebroid in $\mathcal{T}A$, $p_2 : \mathcal{T}A \to A$ is a homomorphism of Lie algebroids over $p : A \to M$.

$$\begin{tikzcd} \mathcal{T}A \arrow[r, "p_2"] \arrow[d, "p_{\mathcal{T}A}"'] & A \arrow[d, "p"] \\ A \arrow[r, "p"] & M \end{tikzcd}$$

In fact, we have $\rho \circ p_2(a,b,v) = \rho(b) = Tp(v) = Tp \circ \rho_{\mathcal{T}A}(a,b,v)$ for all $(a,b,v) \in \mathcal{T}A$; moreover, for every projectable sections V and V' of $\mathcal{T}A$, with $p_2 \circ V = \sigma \circ p$ and $p_2 \circ V' = \sigma' \circ p$, we have $p_2 \circ [V,V']_{\mathcal{T}A} = [\sigma,\sigma']_A \circ p$. Since p_2 is a surjective morphism we can conclude that p_2 is a homomorphism of Lie algebroids [8]. As a consequence, A is a reduced Lie algebroid of $\mathcal{T}A$ (see [4]).

The prolongation of the Lie algebroid $p : A \to M$ plays a relevant role in the definition of Lagrangian mechanics on Lie algebroids [16]. First, if $a, b \in A$, we call *vertical lift* of b on a to the element of $\mathcal{T}_aA$ given by

$$b^V(a) = (a, 0, b_a^V),$$

where $b_a^V f = d/dt\,[f(a+tb)]|_{t=0}$ for all $f \in C^\infty(A)$. Thus, given a section $\sigma \in \Gamma(A)$ the vertical lift of σ is a section σ^V of $\mathcal{T}A$ given by

$$\sigma^V(a) = (\sigma(p(a)))^V(a), \quad a \in A.$$

This allows to define the *vertical endomorphism* S in $\mathcal{T}A$ as follows: if $(a,b,v) \in \mathcal{T}_aA$, then

$$S(a,b,v) = (a, 0, b_a^V).$$

The *Liouville section* Δ is the vertical section of $\mathcal{T}A$ given by

$$\Delta(a) = (a, 0, a_a^V) = a^V(a),\ a \in A.$$

From the definition of vertical endomorphism we verify that S transforms any sections of $\mathcal{T}A$ in a vertical section and that $\operatorname{Im} S = \operatorname{Ker} S$, therefore, $S^2 = 0$. We call *second order differential equation* (SODE) to a section D of $\mathcal{T}A$ such that $S(D) = \Delta$ or, equivalently, $p_2 \circ D = \mathrm{id}_A$.

Let $L \in C^\infty(A)$ be the Lagrangian of a system on the Lie algebroid A and $d_{\mathcal{T}A}$ denote the exterior derivative of the Lie algebroid $\mathcal{T}A$. The *Cartan forms*, θ_L and ω_L, are defined, respectively, by

$$\theta_L = d_{\mathcal{T}A}L \circ S \quad \text{and} \quad \omega_L = -d_{\mathcal{T}A}\theta_L.$$

If $(q^1, ..., q^n, \mathbf{v}^1, ..., \mathbf{v}^s)$ is a system of local coordinates of $p : A \to M$ associated with the choice of a basis of local sections $\{e_\alpha \mid \alpha = 1, ..., s\}$ for which the structure functions of the Lie algebroid are $[e_\alpha, e_\beta]_A = c_{\alpha\beta}{}^\gamma\, e_\gamma$, $\rho(e_\alpha) = \rho^i{}_\alpha\, \partial/\partial q^i$, we can consider the following basis of sections of $\mathcal{T}A$,

$$\mathcal{X}_\alpha(a) = (a, e_\alpha(p(a)), \rho^i{}_\alpha \frac{\partial}{\partial q^i}|_a) \quad\text{and}\quad \mathcal{V}_\alpha(a) = (a, 0, \frac{\partial}{\partial \mathbf{v}^\alpha}|_a);$$

denote by $\mathcal{X}^\alpha$ and $\mathcal{V}^\alpha$ the sections of $(\mathcal{T}A)^*$ corresponding to its dual basis. The vertical endomorphism and Liouville section in these local coordinates are given by,

$$S = \mathcal{X}^\alpha \otimes \mathcal{V}_\alpha, \quad \Delta = \mathbf{v}^\alpha\, \mathcal{V}_\alpha,$$

and the Cartan forms are written:

$$\begin{aligned}
\theta_L &= \frac{\partial L}{\partial \mathbf{v}^\alpha} \mathcal{X}^\alpha \\
\omega_L &= \frac{1}{2}\left(c_{\alpha\beta}{}^\gamma \frac{\partial L}{\partial \mathbf{v}^\gamma} - \rho^i{}_\alpha \frac{\partial^2 L}{\partial q^i \partial \mathbf{v}^\beta} + \rho^j{}_\beta \frac{\partial^2 L}{\partial q^j \partial \mathbf{v}^\alpha} \right) \mathcal{X}^\alpha \wedge \mathcal{X}^\beta \\
&\quad + \left(\frac{\partial^2 L}{\partial \mathbf{v}^\beta \partial \mathbf{v}^\alpha} \right) \mathcal{X}^\alpha \wedge \mathcal{V}^\beta
\end{aligned}$$

The *energy of the system* is given by $E_L = \rho_{\mathcal{T}A}(\Delta)L - L$ or, in local coordinates, by

$$E_L = \mathbf{v}^\alpha \frac{\partial L}{\partial \mathbf{v}^\alpha} - L$$

and its differential is given by

$$d_{\mathcal{T}A} E_L = \left(\mathbf{v}^\alpha \rho^i{}_\beta \frac{\partial^2 L}{\partial q^i \partial \mathbf{v}^\alpha} - \rho^i{}_\beta \frac{\partial L}{\partial q^i} \right) \mathcal{X}^\beta + \mathbf{v}^\alpha \frac{\partial^2 L}{\partial \mathbf{v}^\beta \partial \mathbf{v}^\alpha} \mathcal{V}^\beta.$$

The Lagrangian L is said to be (hyper-) regular if the Legendre transformation $\mathcal{F}L : A \to A^*$, defined by

$$\mathcal{F}L(a)(b) := \frac{d}{dt} L(a + tb)\Big|_{t=0},$$

is a (global) diffeomorphism; note that the Legendre transformation is regular if the matrix $\left(\partial^2 L/\partial \mathbf{v}^\alpha \partial \mathbf{v}^\beta\right)$ is invertible. When the Lagrangian is regular the Cartan 2-form ω_L is a symplectic form. Therefore, the *dynamical equation*

$$i(V_L)\omega_L = d_{\mathcal{T}A} E_L \tag{3.1}$$

has a unique solution $V_L \in \Gamma(\mathcal{T}A)$ that is a SODE, *i.e.* $S(V_L) = \Delta$.

Let us suppose that the Lagrangian L is regular and that $V_L \in \Gamma(\mathcal{T}A)$, given by $V_L = a^\alpha \mathcal{X}_\alpha + b^\alpha \mathcal{V}_\alpha$, is the solution of the dynamics. Then, the integral curves of $\rho_{\mathcal{T}A}(V_L) = a^\alpha \rho^i{}_\alpha \, \partial/\partial q^i + b^\alpha \, \partial/\partial \mathbf{v}^\alpha$ satisfy

$$\begin{cases} \dot{q}^i = a^\alpha \rho^i{}_\alpha = \mathbf{v}^\alpha \rho^i{}_\alpha \\ \dot{\mathbf{v}}^\alpha = b^\alpha = W^{\alpha\beta} \left[\mathbf{v}^\epsilon c_{\epsilon\beta}{}^\gamma \dfrac{\partial L}{\partial \mathbf{v}^\gamma} - \mathbf{v}^\epsilon \rho^i{}_\epsilon \dfrac{\partial^2 L}{\partial q^i \partial \mathbf{v}^\beta} + \rho^i{}_\beta \dfrac{\partial L}{\partial q^i} \right], \end{cases}$$

where $(W^{\alpha\beta})$ represents the inverse matrix of $\left(\partial^2 L / \partial \mathbf{v}^\alpha \partial \mathbf{v}^\beta\right)$. When V_L is a SODE, the dynamics equation (3.1) is equivalent to

$$\pounds_{V_L} \theta_L = d_{\mathcal{T}A} L$$

where $\pounds_{V_L} = i(V_L) \circ d_{\mathcal{T}A} + d_{\mathcal{T}A} \circ i(V_L)$ denotes the 'Lie derivative' with respect to the section V_L of $\mathcal{T}A$. Then, the *Euler-Lagrange equations* are given by

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}^\alpha}\right) = \rho^i{}_\alpha \frac{\partial L}{\partial q^i} + \mathbf{v}^\beta c_{\beta\alpha}{}^\gamma \frac{\partial L}{\partial \mathbf{v}^\gamma}$$

with $\dot{q}^i = \mathbf{v}^\alpha \rho^i{}_\alpha$; the above equations are the Lagrange equations obtained by Weinstein [23].

4. Prolongation of a reduced Lie algebroid

Let $\widehat{A}$ be a Lie algebroid which is a reduction of the Lie algebroid A. In this section, we will show that there exists a homomorphism of Lie algebroids between the prolongation of the Lie algebroids A and $\widehat{A}$, in such a way that $\mathcal{T}\widehat{A}$ is a reduced Lie algebroid of $\mathcal{T}A$. This construction is a generalization of the work developed in [20].

Let A and $\widehat{A}$ be vector bundles endowed with the Lie algebroid structures $(\rho, [\cdot,\cdot]_A)$ and $(\widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$, respectively. Suppose that $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ is a surjective submersion of vector bundles that is a homomorphism of Lie algebroids.

Let us consider the surjective morphism of vector bundles $\overline{\Pi} = (\Pi, \Pi, T\Pi)$ over Π. We will show that the restriction of $\overline{\Pi}$ to $\mathcal{T}A$ is a surjective map with values in $\mathcal{T}\widehat{A}$ and a homomorphism of Lie algebroids too. First, we will show that $\overline{\Pi}|_{\mathcal{T}A}(\mathcal{T}A)$ is contained in $\mathcal{T}\widehat{A}$. Let $(a, b, v) \in \mathcal{T}A$, then $\overline{\Pi}(a, b, v) = (\Pi(a), \Pi(b), T\Pi(v))$. Since $p(a) = p(b)$, we have that $\pi(p(a)) = \pi(p(b))$ and so $\widehat{p}(\Pi(a)) = \widehat{p}(\Pi(b))$. On the other hand, $\widehat{\rho}(\Pi(b)) = T\pi(\rho(b)) = T\pi(Tp(v)) = T\widehat{p}(T\Pi(v))$; therefore, $\overline{\Pi}(a, b, v) \in \mathcal{T}\widehat{A}$.

Before proving that $\overline{\Pi}|_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\widehat{A}$ is a homomorphism of Lie algebroids over $\Pi : A \to \widehat{A}$, we need two auxiliary results.

$$\begin{tikzcd}
 & \mathcal{T}A \arrow[r, "\overline{\Pi}|_{\mathcal{T}A}"] \arrow[dl, "p_{\mathcal{T}A}"'] \arrow[d, "\rho_{\mathcal{T}A}"'] & \mathcal{T}\widehat{A} \arrow[d, "\widehat{\rho}_{\mathcal{T}\widehat{A}}"'] \arrow[dr, "\widehat{p}_{\mathcal{T}\widehat{A}}"] & \\
A \arrow[dr, "\mathrm{id}_A"'] & TA \arrow[l, "\tau_A"] \arrow[r, "T\Pi"] \arrow[d, "\tau_A"'] & T\widehat{A} \arrow[r, "\tau_{\widehat{A}}"'] \arrow[d, "\tau_{\widehat{A}}"'] & \widehat{A} \arrow[dl, "\mathrm{id}_{\widehat{A}}"] \\
 & A \arrow[r, "\Pi"] & \widehat{A} &
\end{tikzcd}$$

Lemma 4.1. *Every section V of $p_{\mathcal{T}A} : \mathcal{T}A \to A$ is written in the form $V = \sum_{i\in I} f_i V_i + Z$, where $Z \in \Gamma(\operatorname{Ker}\overline{\Pi})$ and, for each $i \in I$, $f_i \in C^\infty(A)$ and $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$ is a projectable section of $\mathcal{T}A$, with $\Pi \circ \sigma_i = \sigma_i' \circ \pi$ for $\sigma_i' \in \Gamma(\widehat{A})$ and $T\Pi(X_i) = X_i' \circ \Pi$ for $X_i' \in \mathfrak{X}(\widehat{A})$.*

Proof. The $\overline{\Pi}$-projection of a section V of $\mathcal{T}A$ is given by

$$\overline{\Pi} \circ V = \sum_i f_i (V_i' \circ \Pi),$$

with $f_i \in C^\infty(A)$ and V_i' a projectable section of $\mathcal{T}\widehat{A}$. Each section V_i' is given by $V_i'(a') = (a', \sigma_i'(\widehat{p}(a)), X_i'(a'))$ for all $a' \in \widehat{A}$. Then,

$$\begin{aligned}
\overline{\Pi} \circ V(a) &= \sum_i f_i(a)(\Pi(a), \sigma_i' \circ \widehat{p} \circ \Pi(a), X_i' \circ \Pi(a)) \\
&= \sum_i f_i(a)(\Pi(a), \sigma_i' \circ \pi \circ p(a), X_i' \circ \Pi(a))
\end{aligned}$$

because $\widehat{p} \circ \Pi = \pi \circ p$. Once Π and $T\Pi$ are surjective, there exist $\sigma_i \in \Gamma(A)$ and $X_i \in \mathfrak{X}(A)$, such that, $\Pi \circ \sigma = \sigma_i' \circ \pi$ and $T\Pi \circ X = X_i' \circ \Pi$. Thus,

$$\begin{aligned}
\overline{\Pi} \circ V(a) &= \sum_i f_i(a)(\Pi(a), \Pi \circ \sigma_i \circ p(a), T\Pi \circ X_i(a)) \\
&= \sum_i f_i(a)(\overline{\Pi} \circ V_i)(a) = \overline{\Pi} \circ \left(\sum_i f_i V_i \right)(a),
\end{aligned}$$

with $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$, where $\Pi \circ \sigma_i = \sigma_i' \circ \pi$ and $T\Pi \circ X_i = X_i' \circ \Pi$. Therefore, $V = \sum_i f_i V_i + Z$ for $Z \in \operatorname{Ker}\overline{\Pi}$. $\square$

Lemma 4.2. *Let be Z a section of* $\operatorname{Ker}\overline{\Pi}$ *and V_i projectable sections of $\mathcal{T}A$ fulfilling the conditions of the above lemma. Then, the following conditions are satisfied:*

(1) $\rho_{\mathcal{T}A}(Z) \in \operatorname{Ker} T\Pi$*;*
(2) $[V_i, Z]_{\mathcal{T}A} \in \operatorname{Ker}\overline{\Pi}$*;*
(3) $\Gamma(\operatorname{Ker}\overline{\Pi})$ *is a Lie subalgebra of* $\Gamma(\mathcal{T}A)$.

Proof. If $Z \in \Gamma(\operatorname{Ker}\overline{\Pi})$ then Z is of the form $Z(a) = (a, \sigma(p(a)), X(a))$ for all $a \in A$, where $X \in \mathfrak{X}^V(A)$ is a Π-vertical vector field on A and $\Pi \circ \sigma = 0 \circ \pi$. Since $\rho_{\mathcal{T}A}(Z)(a) = \rho_{\mathcal{T}A}(a, \sigma(p(a)), X(a)) = X(a)$, then (1) holds. The bracket between the sections V_i and Z is given by

$$[V_i, Z]_{\mathcal{T}A}(a) \;=\; (a, [\sigma_i, \sigma]_A(p(a)), [X_i, X](a)).$$

Since

$$T\Pi \circ [X_i, X] = [X_i', 0] \circ \Pi = 0 \circ \Pi$$

and

$$\Pi \circ [\sigma_i, \sigma]_A = [\sigma_i', 0]_{\widehat{A}} \circ \pi = 0 \circ \pi,$$

we have that $\overline{\Pi} \circ [V_i, Z]_{\mathcal{T}A} = (\Pi(a), 0_{\pi(p(a))}, 0_{\Pi(a)})$, that is, condition (2) holds. Now, we suppose that $Z, Z' \in \Gamma(\operatorname{Ker}\overline{\Pi})$. Then, $Z(a) = (a, \sigma(p(a)), X(a))$ and $Z'(a) = (a, \sigma'(p(a)), X'(a))$, with $X, X' \in \mathfrak{X}^V(A)$, $\Pi \circ \sigma = 0 \circ \pi$ and $\Pi \circ \sigma' = 0 \circ \pi$. Thus,

$$[Z, Z']_{\mathcal{T}A}(a) = (a, [\sigma, \sigma']_A(p(a)), [X, X'](a)).$$

Since

$$T\Pi \circ [X, X'] = 0 \circ \Pi \;\text{ and }\; \Pi \circ [\sigma, \sigma']_A = 0 \circ \pi,$$

we can conclude that $\overline{\Pi} \circ [Z, Z']_{\mathcal{T}A} = (\Pi(a), 0_{\pi(p(a))}, 0_{\Pi(a)})$. $\square$

Now, we prove the main result of this section.

Proposition 4.3. *The map $\overline{\Pi}|_{\mathcal{T}A} : \mathcal{T}A \to \mathcal{T}\widehat{A}$ is a homomorphism of Lie algebroids over $\Pi : A \to \widehat{A}$. Therefore, $\mathcal{T}\widehat{A}$ is a reduced Lie algebroid of $\mathcal{T}A$.*

Proof. We first remark that $T\Pi \circ \rho_{\mathcal{T}A} = \rho_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}$. In fact, if $(a, b, v) \in \mathcal{T}A$, then

$$T\Pi \circ \rho_{\mathcal{T}A}(a, b, v) = T\Pi(v) = \rho_{\mathcal{T}\widehat{A}}(\Pi(a), \Pi(b), T\Pi(v)) = \rho_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}(a, b, v).$$

Now, let us prove that (2.4) holds. Let V and V' be two sections of $\mathcal{T}A$. Then, according to Lemma 4.1, there exist functions $f_i, g_j \in C^\infty(A)$ and projectable sections $V_i, W_j \in \Gamma(\mathcal{T}A)$ such that $V = \sum_i f_i V_i + Z_1$ and $W = \sum_j g_j W_j + Z_2$, where $Z_1, Z_2 \in \Gamma(\operatorname{Ker}\overline{\Pi})$, $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$ and $W_j(a) = (a, \varsigma_j(p(a)), Y_j(a))$, with: $\Pi \circ \sigma_i = \sigma_i' \circ \pi$, $\Pi \circ \varsigma_j = \varsigma_j' \circ \pi$, $T\Pi(X_i) = X_i' \circ \Pi$ and $T\Pi(Y_j) = Y_j' \circ \Pi$. Thus, the $\overline{\Pi}$-projections of V and W are given by

$$\overline{\Pi} \circ V = \sum_i f_i (V_i' \circ \Pi) \quad \text{and} \quad \overline{\Pi} \circ W = \sum_j g_j (W_j' \circ \Pi),$$

with $V_i'(a') = (a', \sigma_i'(\widehat{p}(a')), X_i'(a'))$ and $W_j'(a') = (a', \varsigma_j'(\widehat{p}(a')), Y_j'(a'))$ projectable sections of $\mathcal{T}\widehat{A}$. The bracket $[\cdot,\cdot]_{\mathcal{T}A}$ of the sections V and W is given by

$$\begin{aligned}
[V, W]_{\mathcal{T}A} &= \sum_{i,j} f_i g_j [V_i, W_j]_{\mathcal{T}A} + \sum_j (\rho_{\mathcal{T}A}(v) g_j) W_j - \sum_i (\rho_{\mathcal{T}A}(w) f_i) V_i \\
&\quad + \sum_j g_j [Z_1, W_j]_{\mathcal{T}A} + \sum_i f_i [V_i, Z_2]_{\mathcal{T}A} + [Z_1, Z_2]_{\mathcal{T}A}\,,
\end{aligned}$$

that is,

$$\begin{aligned}
[V, W]_{\mathcal{T}A}(a) &= \sum_{i,j} (f_i g_j)(a)\,(a, [\sigma_i, \varsigma_j]_A(p(a)), [X_i, Y_j](a)) \\
&\quad + \sum_j (\rho_{\mathcal{T}A}(v) g_j)(a)\,(a, \varsigma_j(p(a)), Y_j(a)) \\
&\quad - \sum_i (\rho_{\mathcal{T}A}(w) f_i)(a)\,(a, \sigma_i(p(a)), X_i(a)) \\
&\quad + \sum_j g_j [Z_1, W_j]_{\mathcal{T}A} + \sum_i f_i [V_i, Z_2]_{\mathcal{T}A} + [Z_1, Z_2]_{\mathcal{T}A},
\end{aligned}$$

for all $a \in A$. Thus, by Lemma 4.2, we have

$$\begin{aligned}
\overline{\Pi} \circ [V, W]_{\mathcal{T}A}(a) &= \sum_{i,j} (f_i g_j)(a)\,(\Pi(a), (\Pi \circ [\sigma_i, \varsigma_j]_A)(p(a)), (T\Pi \circ [X_i, Y_j])(a)) \\
&\quad + \sum_j (\rho_{\mathcal{T}A}(v) g_j)(a)\,(\Pi(a), (\Pi \circ \varsigma_j)(p(a)), (T\Pi \circ Y_j)(a)) \\
&\quad - \sum_i (\rho_{\mathcal{T}A}(w) f_i)(a)\,(\Pi(a), (\Pi \circ \sigma_i)(p(a)), (T\Pi \circ X_i)(a))\,.
\end{aligned}$$

Since the sections σ_i and ς_j are Π-projectable and the vectors field X_i and Y_j are $T\Pi$-projectable, we may write

$$\begin{aligned}\overline{\Pi}\circ[V,W]_{\mathcal{T}A}(a) &= \sum_{i,j}(f_ig_j)(a)\,\big(\Pi(a),([\sigma_i',\varsigma_j']_A\circ\pi)(p(a)),([X_i',Y_j']\circ\Pi)(a)\big)\\ &\quad+\sum_j(\rho_{\mathcal{T}A}(v)g_j)(a)\,\big(\Pi(a),(\varsigma_j'\circ\pi)(p(a)),(Y_j'\circ\Pi)(a)\big)\\ &\quad-\sum_i(\rho_{\mathcal{T}A}(w)f_i)(a)\,(\Pi(a),(\sigma_i'\circ\pi)(p(a)),(X_i'\circ\Pi)(a))\ ,\end{aligned}$$

that is,

$$\begin{aligned}\overline{\Pi}\circ[V,W]_{\mathcal{T}A}(a) &= \sum_{i,j}(f_ig_j)(a)\,\big(\Pi(a),([\sigma_i',\varsigma_j']_A\circ\widehat{p})(\Pi(a)),[X_i',Y_j'](\Pi(a))\big)\\ &\quad+\sum_j(\rho_{\mathcal{T}A}(v)g_j)(a)\,\big(\Pi(a),(\varsigma_j'\circ\widehat{p})(\Pi(a)),Y_j'(\Pi(a))\big)\\ &\quad-\sum_i(\rho_{\mathcal{T}A}(w)f_i)(a)\,(\Pi(a),(\sigma_i'\circ\widehat{p})(\Pi(a)),X_i'(\Pi(a)))\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\overline{\Pi}\circ[V,W]_{\mathcal{T}A} &= \sum_{i,j}f_ig_j([V_i',W_j']_{\mathcal{T}\widehat{A}}\circ\Pi)+\sum_j(\rho_{\mathcal{T}A}(V)g_j)(W_j'\circ\Pi)\\ &\quad-\sum_i(\rho_{\mathcal{T}A}(W)f_i)(V_i'\circ\Pi).\quad\square\end{aligned}$$

Note that, since $\overline{\Pi}$ is surjective we could have shown the condition of homomorphism (2.4) only on projectable sections of $\mathcal{T}A$ of the form $V_i(a)=(a,\sigma_i(p(a)),X_i(a))$, with $\Pi\circ\sigma_i=\sigma_i'\circ\pi$ for $\sigma_i'\in\Gamma(\widehat{A})$ and $T\Pi(X_i)=X_i'\circ\Pi$ for $X_i'\in\mathfrak{X}(\widehat{A})$. This proposition is a particular case of a more general result obtain in [17].

The statement of Proposition 4.3 means that $d_{\mathcal{T}A}\circ(\overline{\Pi}|_{\mathcal{T}A})^*=(\overline{\Pi}|_{\mathcal{T}A})^*\circ d_{\mathcal{T}\widehat{A}}$, where $d_{\mathcal{T}A}$ and $d_{\mathcal{T}\widehat{A}}$ are the exterior derivatives of the Lie algebroids $\mathcal{T}A$ and $\mathcal{T}\widehat{A}$, respectively. Moreover, one can easily prove that the Lie algebroid structure on $\mathcal{T}\widehat{A}$ is the unique structure for which $\overline{\Pi}|_{\mathcal{T}A}:\mathcal{T}A\to\mathcal{T}\widehat{A}$ is a homomorphism of Lie algebroids over $\Pi:A\to\widehat{A}$.

By hypothesis, Π is a homomorphism of Lie algebroids, *i.e.* $d_A\circ\Pi^*=\Pi^*\circ d_{\widehat{A}}$. We also have that $p_2:\mathcal{T}A\to A$ is a homomorphism of Lie algebroids so, $d_{\mathcal{T}A} \circ p_2^* = p_2^* \circ d_A$. Then, $d_{\mathcal{T}A} \circ (\Pi \circ p_2)^* = (\Pi \circ p_2)^* \circ d_{\widehat{A}}$, that is, $\widehat{A}$ is a reduced Lie algebroid of $\mathcal{T}A$.

4.1. Lie algebroids with symmetry. Let Φ be a representation of the Lie group G in the Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ in the sense of [4]. Suppose that Φ and its contragradient representation define free and proper actions of G on the fibre bundles (A, p, M) and (A^*, τ, M), respectively. In these conditions $\overline{\Phi} = (\Phi, \Phi, T\Phi)$ defines a Lie algebroid representation of G in the Lie algebroid prolongation $\mathcal{T}A$ of A [20], that is

Proposition 4.4. *The morphism $\overline{\Phi} : G \to \mathrm{Aut}(\mathcal{T}A)$ is a Lie algebroid representation of the Lie group G on the Lie algebroid $\mathcal{T}A$.*

Proof. First of all we need to prove that $\overline{\Phi}_g := \overline{\Phi}(g)$ preserves $\mathcal{T}A$, *i.e.* $\overline{\Phi}_g(\mathcal{T}A) \subset \mathcal{T}A$. Let $(a, b, v) \in \mathcal{T}_aA$, then $\overline{\Phi}_g(a, b, v) = (\Phi_g(a), \Phi_g(b), T\Phi_g(v))$, with $\Phi_g := \Phi(g)$. Since $p(a) = p(b)$, $\phi_g(p(a)) = \phi_g(p(b))$, where ϕ_g is the base map of Φ_g; then $p \circ \Phi_g(a) = p \circ \Phi_g(b)$. We have $Tp(T\Phi_g(v)) = T(p \circ \Phi_g)(v) = T\phi_g \circ Tp(v) = T\phi_g \circ \rho(b)$, since Φ_g is a Lie algebroid representation, and then $Tp(T\Phi_g(v)) = \rho(\Phi_g(b))$. Therefore $\overline{\Phi}_g(a, b, v) \in \mathcal{T}_{\Phi_g(a)}A$.

Now, in order to show that $\overline{\Phi}_g$ is an automorphism of Lie algebroids, we need to prove that $T\Phi_g \circ \rho_{\mathcal{T}A} = \rho_{\mathcal{T}A} \circ \overline{\Phi}_g$ and $\overline{\mathcal{R}}_g([V_1, V_2]_{\mathcal{T}A}) = [\overline{\mathcal{R}}_g(V_1), \overline{\mathcal{R}}_g(V_2)]_{\mathcal{T}A}$ for all $V_1, V_2 \in \Gamma(\mathcal{T}A)$, where $\overline{\mathcal{R}}_g(V) = \overline{\Phi}_g \circ V \circ \Phi_{g^{-1}}$ for all $V \in \Gamma(\mathcal{T}A)$. For the first condition, we have, for all $(a, b, v) \in \mathcal{T}_aA$,

$$\rho_{\mathcal{T}A} \circ \overline{\Phi}_g(a, b, v) = \rho_{\mathcal{T}A}(\Phi_g(a), \Phi_g(b), T\Phi_g(v)) = T\Phi_g(v) = T\Phi_g \circ \rho_{\mathcal{T}A}(a, b, v)$$

Suppose now that $V_1, V_2 \in \Gamma(\mathcal{T}A)$ are projectable sections, that is $V_i(a) = (a, \sigma_i(p(a)), X_i(a))$, $i = 1, 2$. Then $\overline{\mathcal{R}}_g(V_i)(a) = (a, \mathcal{R}(\sigma)(p(a))), X_i'(a))$ where $\mathcal{R}(\sigma) = \Phi_g \circ \sigma \circ \phi_g^{-1}$ and $X_i' = T\Phi_g \circ X_i \circ \Phi_{g^{-1}}$ for $i = 1, 2$. Since

$$\overline{\mathcal{R}}_g([V_1, V_2]_{\mathcal{T}A})(a) = (a, \Phi_g([\sigma_1, \sigma_2]_A)(\phi_{g^{-1}}(p(a))), T\Phi_g([X_1, X_2])(\Phi_{g^{-1}}(a))),$$

$a \in A$, and Φ_g and $T\Phi_g$ are homomorphisms of Lie algebroids, we conclude

$$\begin{aligned} \overline{\mathcal{R}}_g([V_1, V_2]_{\mathcal{T}A})(a) &= (a, [\mathcal{R}_g(\sigma_1), \mathcal{R}_g(\sigma_2)]_A)(p(a))), [X_1', X_2'](a)) \\ &= [\overline{\mathcal{R}}_g(V_1), \overline{\mathcal{R}}_g(V_2)]_{\mathcal{T}A}(a). \ \square \end{aligned}$$

Let Π be the canonical projection of A onto A/G. We know (see [4]) that A/G is endowed with a vector bundle structure in such a way that $(\Pi, \pi) : (A, p, M) \to (A/G, \widehat{p}, \widehat{M})$ is a surjective submersion of vector bundles. Moreover, A/G is endowed with a Lie algebroid structure such that

Π is a homomorphism of Lie algebroids. Thus, by Proposition 4.3, $\overline{\Pi}|_{\mathcal{T}A}$ is a homomorphism of Lie algebroids over Π and $\mathcal{T}(A/G)$ is a reduced Lie algebroid of $\mathcal{T}A$. One can easily prove that $\overline{\Pi}$ is $\overline{\Phi}$-invariant, that is $\overline{\Pi} \circ \overline{\Phi}_g = \overline{\Pi}$ for all $g \in G$; we just have to note that Π is Φ-invariant and so $T\Pi \circ T\Phi_g = T(\Pi \circ \Phi_g) = T\Pi$ for all $g \in G$. Moreover,

Proposition 4.5. *For each $a \in A$,*

(i) *the map $\overline{\Pi}_a : \mathcal{T}_aA \to \mathcal{T}_{[a]}(A/G)$ is a isomorphism;*
(ii) *the fibre $\mathcal{T}_{[a]}(A/G)$ is isomorphic to the fibre $(\mathcal{T}A/G)_{[a]}$.*

Proof. (i) Let $(a, b, v), (a, b', v') \in \mathcal{T}_aA$ such that $\overline{\Pi}_a(a, b, v) = \overline{\Pi}_a(a, b', v')$, that is, $(\Pi(a), \Pi(b), T\Pi(v)) = (\Pi(a), \Pi(b'), T\Pi(v'))$. So $\Pi(b) = \Pi(b')$ and $T\Pi(v) = T\Pi(v')$. Since Π is the canonical projection defined by the Lie algebroid representation Φ of the Lie group G on A, there exists $g \in G$ such that $b = \Phi_g(b')$. Then, $p(b) = p \circ \Phi_g(b')$, that is, $p(a) = \phi_g \circ p(b') = \phi_g(p(a))$ where ϕ_g is the base map of Φ_g that defines a free and proper action of the Lie group G on the vector bundle $p : A \to M$. Therefore, $g = e$ and so $b = b'$.

Now, we prove that $v = v'$. We have $v - v' \in \operatorname{Ker} T_a\Pi$. In these conditions, $\operatorname{Ker} T_a\Pi$ is generated by the fundamental vector fields in $a \in A$, defined by the free and proper action associated to the representation Φ of G on A. So, $v = v' + \sum_i \lambda_i X^i_A(a)$, where X^i represents the elements of a basis of the Lie algebra $\mathfrak{g}$ of G. Since $b = b'$, we know that $T_ap(v) = T_ap(v')$, then

$$T_ap(v') + \sum_i \lambda_i T_ap(X^i_A(a)) = T_ap(v'),$$

that is,

$$\sum_i \lambda_i T_ap(X^i_A(a)) = 0.$$

But $\sum_i \lambda_i T_ap(X^i_A(a)) = 0$ is equivalent to $\sum_i \lambda_i X^i_M(m) = 0$, where $p(a) = m$ and X^i_M is the fundamental vector field in M associated to the element X^i of $\mathfrak{g}$. Since the fundamental vector fields associated to elements of a basis of $\mathfrak{g}$ are independent, then, all λ_i are null. Therefore, $v = v'$.

(ii) The Lie algebroid representation $\overline{\Phi} = (\Phi, \Phi, T\Phi)$ defines a free and proper action of the Lie group G on the Lie algebroid $\mathcal{T}A$. Therefore, the canonical projection $\underline{\Pi} : \mathcal{T}A \to \mathcal{T}A/G$ is in each point of A an isomorphism. So, $\mathcal{T}_aA \simeq (\mathcal{T}A/G)_{[a]}$ and therefore by (i) $\mathcal{T}_{[a]}(A/G) \simeq (\mathcal{T}A/G)_{[a]}$, for each $a \in A$. $\square$

For example, consider a principal fibre bundle $P(M, G)$ and the gauge algebroid associated $(TP/G, p, M)$ (see [14]). The canonical projection $\Pi : TP \to TP/G$ determines a homomorphism of Lie algebroids $\overline{\Pi}|_{\mathcal{T}(TP)} : \mathcal{T}(TP) \to \mathcal{T}(TP/G)$ over Π. Let ϕ be the (right) action of the group G on P. Then, $\Phi(g) := T\phi_g$ defines a Lie algebroid representation of the group G in TP. Thus, $\overline{\Phi} = (\Phi, \Phi, T\Phi)$ is a Lie algebroid representation of the group G in $\mathcal{T}(TP) = T(TP)$. By the above proposition $\mathcal{T}(TP)/G \cong \mathcal{T}(TP/G)$, that is, $T(TP)/G \cong \mathcal{T}(TP/G)$. In the recent paper of M. de Léon et *al.* [11], the above results for gauge algebroids are proved by a different approach.

5. Reduction of a Lagrangian dynamics

Let (A, p, M) and $(\widehat{A}, \widehat{p}, \widehat{M})$ be vector bundles endowed with the Lie algebroid structures $(\rho, [\cdot,\cdot]_A)$ and $(\widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$, respectively, and suppose that $(\Pi, \pi) : (A, p, M) \to (\widehat{A}, \widehat{p}, \widehat{M})$ is a surjective submersion of vector bundles that is a homomorphism of Lie algebroids.

A function $F \in C^\infty(A)$ is said to be Π-*invariant* if there exists a function $f \in C^\infty(\widehat{A})$ such that $F = f \circ \Pi$ for ; this is equivalent to say that its differential dF belongs to the annihilator of $\operatorname{Ker} T\Pi$.

Next, given a dynamical system in A defined by a regular and Π-invariant Lagrangian function $L \in C^\infty(A)$, we will show how the dynamics is reduced. The following results generalize those obtained in [20] for the reduction of dynamics on Lie algebroids with symmetry.

Lemma 5.1. *The morphism $\overline{\Pi}$ intertwines the vertical endomorphism S in $\mathcal{T}A$ with the vertical endomorphism S' in $\mathcal{T}\widehat{A}$, and the Liouville section Δ of $\mathcal{T}A$ is $\overline{\Pi}$-related with the Liouville section Δ' of $\mathcal{T}\widehat{A}$,* i.e.

$$\overline{\Pi} \circ S = S' \circ \overline{\Pi} \quad \textit{and} \quad \overline{\Pi} \circ \Delta = \Delta' \circ \Pi.$$

Proof. Let $a \in A$, then

$$\overline{\Pi}(b^V(a)) = (\Pi(a), 0, T\Pi(b^V_a)).$$

Since

$$T\Pi(b^V_a)f = \frac{d}{dt}(f \circ \Pi)(a + tb)\,|_{t=0} = \frac{d}{dt} f(\,\Pi(a) + t\,\Pi(b)\,)\,|_{t=0}$$

for all $f \in C^\infty(A)$, we have

$$T\Pi(b^V_a) = (\Pi(b))^V_{\Pi(a)}.$$

Therefore,

$$\overline{\Pi}(b^V(a)) = (\Pi(b))^V(\Pi(a)). \tag{5.1}$$

From the above equality, we show that $\overline{\Pi} \circ S = S' \circ \overline{\Pi}$. In fact, for each $(a, b, v) \in \mathcal{T}_a A$, we have that

$$S' \circ \overline{\Pi}(a, b, v) \;=\; S'(\Pi(a), \Pi(b), T_a\Pi(v)) = (\Pi(b))^V(\Pi(a)).$$

By (5.1) we conclude that $S' \circ \overline{\Pi}(a, b, v) = \overline{\Pi}(b^V(a)) = \overline{\Pi} \circ S(a, b, v)$. Now, let us prove that $\overline{\Pi} \circ \Delta = \Delta' \circ \Pi$. We have $\overline{\Pi} \circ \Delta(a) = \overline{\Pi}(a^V(a))$, for all $a \in A$. From (5.1), we may write

$$\overline{\Pi} \circ \Delta(a) = (\Pi(a))^V(\Pi(a))$$

and therefore $\overline{\Pi} \circ \Delta(a) = \Delta'(\Pi(a)) = \Delta' \circ \Pi(a)$ for all $a \in A$. □

The main result of this section is:

Theorem 5.2. *Let us suppose that the Lagrangian $L \in C^\infty(A)$ of a dynamical system in the Lie algebroid A is regular and Π-invariant, that is, there exists $l \in C^\infty(\widehat{A})$ such that $L = l \circ \Pi$. Then, the following conditions are satisfied:*

(i) *if E_L and E'_l are the energies of the dynamics on the Lie algebroids A and $\widehat{A}$, respectively, then $E'_l \circ \overline{\Pi} = E_L$;*
(ii) *if θ_L and θ'_l are the Cartan 1-forms defined by L and l on the Lie algebroids A and $\widehat{A}$, respectively, then $\overline{\Pi}^* \theta'_l = \theta_L$. As a consequence, we have that $\overline{\Pi}^* \omega'_l = \omega_L$;*
(iii) *the induced Lagrangian l is regular;*
(iv) *if V_L and V'_l are the solutions of the dynamics on the Lie algebroids A and on $\widehat{A}$, respectively, then $\overline{\Pi} \circ V_L = V'_l \circ \Pi$.*

Therefore, the dynamics in A induced by a regular and Π-invariant Lagrangian $L = l \circ \Pi$ reduces to the Lagrangian dynamics in $\widehat{A}$ given by l.

Proof. (i) We have $E'_l := \rho_{\mathcal{T}\widehat{A}}(\Delta')l - l$. Then,

$$E'_l \circ \Pi(a) = \rho_{\mathcal{T}\widehat{A}}(\Delta')l \circ \Pi(a) - l \circ \Pi(a) = \rho_{\mathcal{T}\widehat{A}}(\Delta'(\Pi(a)))l - L(a).$$

for all $a \in A$. By Lemma 5.1 we have

$$E'_l \circ \Pi(a) = \rho_{\mathcal{T}\widehat{A}}(\overline{\Pi}(\Delta(a)))l - L(a)$$

and by $\rho_{\mathcal{T}\widehat{A}} \circ \overline{\Pi} = T\Pi \circ \rho_{\mathcal{T}A}$ we obtain

$$E'_l \circ \Pi(a) = T\Pi \circ \rho_{\mathcal{T}A}(\Delta(a))l - L(a) = \rho_{\mathcal{T}A}(\Delta(a))(l \circ \Pi) - L(a)$$

for all $a \in A$. Therefore, $E'_l \circ \Pi = E_L$.

(ii) Let us prove that $\overline{\Pi}^* \theta'_l = \theta_L$. If $V \in \mathcal{T}_a A$, then

$$\overline{\Pi}^* \theta'_l(V) = \theta'_l(\overline{\Pi}(V)) = d_{\mathcal{T}\widehat{A}} l \circ S'(\overline{\Pi}(V)).$$

and using the results of the Lemma 5.1 we may write

$$\overline{\Pi}^* \theta'_l(V) = d_{\mathcal{T}A} l \circ \overline{\Pi} \circ S(V) = \overline{\Pi}^* \circ d_{\mathcal{T}A} l \circ S(V).$$

But $\overline{\Pi}^* \circ d_{\mathcal{T}A} = d_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}^*$, and then

$$\overline{\Pi}^* \theta'_l(V) = d_{\mathcal{T}\widehat{A}} \circ \overline{\Pi}^* l \circ S(V) = d_{\mathcal{T}\widehat{A}} L \circ S(V) = \theta_L(V),$$

that is, $\overline{\Pi}^* \theta'_l = \theta_L$. Thus, by definition of the Cartan 2-form we deduce $\overline{\Pi}^* \omega'_l = -\overline{\Pi}^*(d_{\mathcal{T}\widehat{A}} \theta'_l)$. The exterior derivative commutes with the morphism $\overline{\Pi}^*$, so

$$\overline{\Pi}^* \omega'_l \;=\; -d_{\mathcal{T}A}(\overline{\Pi}^* \theta'_l) = -d_{\mathcal{T}A} \circ \theta_L = \omega_L.$$

(iii) If $L \in C^\infty(A)$ is regular then ω_L is symplectic. By the above condition we have $\overline{\Pi}^* \omega'_l = \omega_L$, then, ω'_l is also symplectic because $\overline{\Pi}$ is a surjective morphism. So the reduced Lagrangian $l \in C^\infty(\widehat{A})$ is regular.

(iv) We have

$$\overline{\Pi}^* \omega'_l(V_L, X) \;=\; \omega'_l(\overline{\Pi}(V_L), \overline{\Pi}(X)) = i(\overline{\Pi}(V_L)) \omega'_l(\overline{\Pi}(X)), \tag{5.2}$$

for all $X \in \Gamma(\mathcal{T}A)$. On the other hand,

$$\overline{\Pi}^* \omega'_l(V_L, X) = \omega_L(V_L, X) = i(V_L) \omega_L(X) = d_{\mathcal{T}A} E_L(X). \tag{5.3}$$

Since $E_L = E'_l \circ \overline{\Pi} = \overline{\Pi}^* E'_l$, then by (5.2) and (5.3) we have

$$i(\overline{\Pi}(V_L)) \omega'_l(\overline{\Pi}(X)) = (d_{\mathcal{T}A} \circ \overline{\Pi}^*) E'_l(X) = (\overline{\Pi}^* \circ d_{\mathcal{T}\widehat{A}}) E'_l(X) = d_{\mathcal{T}\widehat{A}} E'_l(\overline{\Pi}(X))\,,$$

for all $X \in \Gamma(\mathcal{T}A)$. So $\overline{\Pi}(V_L)$ is a global solution of the dynamics in $\mathcal{T}\widehat{A}$. Once L is regular so is l, then the Cartan 2-form ω'_l is symplectic and so the dynamical equation has just one solution. Therefore, $\overline{\Pi}(V_L) = V'_l$, that is, $\overline{\Pi} \circ V_L = V'_l \circ \Pi$. $\square$

In general, the regularity of l does not imply the regularity of L. However, in the case of Lie algebroids with symmetry, since $\overline{\Pi}$ is an isomorphism in each fibre, we have that a projectable Lagrangian $L = l \circ \Pi$ in A is regular iff the reduced Lagrangian l in A/G is regular. Moreover, if $\mathcal{F}l : \widehat{A} \to \widehat{A}^*$ denotes the Legendre transformation associated with the Lagrangian $l \in C^\infty(\widehat{A})$ we have

$$\mathcal{F}L(a)(b) = \frac{d}{dt}L(a+tb)|_{t=0} = \frac{d}{dt}l(\Pi(a)+t\Pi(b))|_{t=0} = \mathcal{F}l(\Pi(a))(\Pi(b))$$

for all $a, b \in A$, then $\mathcal{F}L = \Pi^* \circ \mathcal{F}l \circ \Pi$.

We can weaken the conditions of the Theorem 5.2, by considering a Π-invariant (possibly degenerated) Lagrangian $L \in C^\infty(A)$ that admits a global dynamics, *i.e.* there exists a globally defined section V of $\mathcal{T}A$ satisfying the equation $i(V)\omega_L = d_{\mathcal{T}A}E_L$. With these hypotheses we have

$$i(\overline{\Pi}(V))\omega'_l = d_{\mathcal{T}\widehat{A}}E'_l,$$

that is, the reduced dynamics admits a global solution given by $\overline{\Pi}(V)$. If the solution of the initial dynamics V is a SODE, then the solution of the reduced dynamics is a SODE too, because

$$S'(\overline{\Pi}(V)) = \overline{\Pi}(S(V)) = \overline{\Pi}(\Delta) = \Delta'.$$

Of course, if the Cartan 2-form ω'_l is symplectic then $\overline{\Pi}(V)$ is always a SODE. In this case, $\operatorname{Im} S' = \operatorname{Ker} S'$ is a Lagrangian subspace with respect to ω'_l because $\omega'_l(S'(X), S'(Y)) = 0$ for all $X, Y \in \mathcal{T}\widehat{A}$, that is, $\dim \operatorname{Im} S' = \frac{1}{2}\dim \mathcal{T}\widehat{A}$.

5.1. Examples of dynamical reduction.

1. **Reduction of degenerated Lagrangian systems.** In standard classical dynamics, let us consider a Lagrangian $L \in C^\infty(TQ)$ satisfying the following conditions (see [2]):
   - (A1) the Cartan 2-form ω_L is presymplectic, *i.e.* it is a constant rank closed form;
   - (A2) the Lagrangian L admits a global dynamics;
   - (A3) the foliation defined by ω_L is regular, *i.e.* the quotient space $TQ/\operatorname{Ker}\widehat{\omega}_L$ has a differentiable manifold structure and the projection $\Pi : TQ \to TQ/\operatorname{Ker}\widehat{\omega}_L$ is a surjective submersion, where $\widehat{\omega}_L(X) = i(X)\omega_L = \omega_L(X, \cdot)$ for all $X \in \mathfrak{X}(TQ)$.

   Under these conditions, let $p : TQ \to Q$ be the canonical projection of TQ and let us suppose that:
   - (A4) the distribution $D = Tp(\operatorname{Ker}\widehat{\omega}_L)$ defines a regular foliation of Q, *i.e.* the space of the leaves $\widehat{Q} = Q/D$ admits a structure of differentiable manifold for which the canonical projection $\pi : Q \to \widehat{Q}$ is a surjective submersion.

We can prove that there exists a unique vector bundle structure in the quotient manifold $\widehat{TQ} = TQ/\mathrm{Ker}\,\widehat{\omega}_L$ such that $\widehat{p} \circ \Pi = \pi \circ p$, where $\widehat{p} : \widehat{TQ} \to \widehat{Q}$ is given by $\widehat{p}([X]) = \pi(p(X))$ for each $X \in TQ$ such that $\Pi(X) = [X]$.

We know that the tangent bundle TQ is a Lie algebroid over Q whose anchor is the identity map on TQ and whose Lie algebra structure in the set of sections is given by the usual bracket of vector fields in Q. If the surjective submersion of vector bundles (Π, π) : $(TQ, p, Q) \to (\widehat{TQ}, \widehat{p}, \widehat{Q})$ satisfies the conditions of the reduction theorem stated in [4], then the bundle $\widehat{TQ}$ is endowed with a (reduced) Lie algebroid structure, such that, (Π, π) is a homomorphism of Lie algebroids. From what we have prove so far, if L is Π-invariant, the dynamics in $T(TQ) = \mathcal{T}(TQ)$ reduces to the dynamics in $\mathcal{T}\widehat{TQ}$. In other words, the dynamics solution in $T(TQ)$, given by a vector field V in TQ, projects into a section of $\mathcal{T}\widehat{TQ}$ that satisfies the dynamics equation in $\mathcal{T}\widehat{TQ}$. In these conditions, we can conclude that there exists a unique symplectic form $\widetilde{\omega}$ in $\mathcal{T}\widehat{TQ}$ such that $\omega_L = \overline{\Pi}^*\widetilde{\omega}$ and, therefore, $\omega'_l = \widetilde{\omega}$ is a symplectic form. So the solution of the reduced dynamics is a SODE, *i.e.* $S'(\overline{\Pi}(V)) = \Delta$.

2. **Reduction of a principal fibre bundle.** Let $P(M, G)$ be a principal fibre bundle. We saw in [4] that the gauge algebroid TP/G is a reduced Lie algebroid of the tangent bundle TP endowed with its usual Lie algebroid structure, where the canonical projection (surjective submersion) $\Pi : TP \to TP/G$ is a homomorphism of Lie algebroids. Given a Π-invariant Lagrangian $L = l \circ \Pi \in C^\infty(TP)$ with a global dynamics (solution) V on TP, we have that $\overline{\Pi}(V)$ is a global dynamics (solution) on TP/G, *i.e.* $i(\overline{\Pi}(V))\omega'_l = d_{\mathcal{T}(TP/G)}E'_l$. If $V' = \overline{\Pi}(V)$ is a SODE, then the reduced dynamics equation is equivalent to $\pounds_{V'}\theta'_l = d_{\mathcal{T}(TP/G)}l$, with $\theta'_l = d_{\mathcal{T}(TP/G)}l \circ S'$. In this case we have $T(TP)/G \cong \mathcal{T}(TP/G)$.

6. Reduction of Nonholonomic systems on Lie algebroids

Let (A, p, M) and $(\widehat{A}, \widehat{p}, \widehat{M})$ be two vector bundles endowed with Lie algebroid structures $(\rho, [\cdot,\cdot]_A)$ and $(\widehat{\rho}, [\cdot,\cdot]_{\widehat{A}})$, respectively, and suppose that the vector bundle morphism $(\Pi,\pi):(A,p,M)\to(\widehat{A},\widehat{p},\widehat{M})$ is a surjective submersion and a homomorphism of Lie algebroids. We have proved before that Π induces a homomorphism of Lie algebroids between the prolongation of the Lie algebroids A and $\widehat{A}$,

$$\overline{\Pi}|_{\mathcal{T}A}=(\Pi,\Pi,T\Pi)|_{\mathcal{T}A}:\mathcal{T}A\to\mathcal{T}\widehat{A}.$$

Let $L\in C^\infty(A)$ be a regular Lagrangian of a dynamical system on A, which is Π-invariant (*i.e.* $L=l\circ\Pi$ for some $l\in C^\infty(\widehat{A})$). As we have proved in section 5, the reduced dynamics on $\widehat{A}$ admits a global solution $X'_l=\overline{\Pi}(X_L)$,

$$i(\overline{\Pi}(X_L))\omega'_l=d_{\mathcal{T}\widehat{A}}E'_l,$$

where $\overline{\Pi}^*\omega'_l=\omega_L$ and $\overline{\Pi}^*E'_l=E_L$.

6.1. Nonholonomic systems on a Lie algebroid. The first time nonholonomic systems in the framework of Lie algebroids is dealt with was in [5].

Let us consider a system on the Lie algebroid A with nonholonomic constraints given by a vector subbundle B of A, where the submanifold B is defined by the vanishing of a set of independent linear functions $\{\phi_a=\phi_{a\beta}\mathbf{v}^\beta\mid a=1,...,k\}$. In a parallel way to the usual formalism in classical mechanics on the tangent bundle (see e.g. [3]), the constrained system equations of motion can be written in a global form

$$\left\{\begin{array}{l} i(V)\omega_L-d_{\mathcal{T}A}E_L\in S^*((\mathcal{T}B)^0)\\ V|_B\in\mathcal{T}B,\end{array}\right. \tag{6.1}$$

where

$$\mathcal{T}B=\{(b,c,v)\in B\times B\times TB\mid p(b)=p(c),\ \rho(c)=Tp(v)\text{ with }v\in T_bB\}\,^*$$

is a subbundle of $p_{\mathcal{T}A}|_B:\mathcal{T}_BA\to B$ and $(\mathcal{T}B)^0\subset(\mathcal{T}A)^*$ denotes the annihilator of $\mathcal{T}B$. The above formulation of the nonholonomic system in the Lie algebroid A is called *Chetaev formulation.*

Note that V is a SODE of $\mathcal{T}A$ since 1-forms in $S^*((\mathcal{T}B)^0)$ are semibasic, that is, vanishing on vertical sections of $\mathcal{T}A$: $S^*(d_{\mathcal{T}A}\phi_a)=\partial\phi_a/\partial\mathbf{v}^\beta\,\mathcal{X}^\beta$. The semibasic forms $S^*(d_{\mathcal{T}A}\phi_a)$ are called *reactions forces* on the Lie algebroid A.

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*In order to simplify the text, we write p (resp. Π and ρ) instead of $p|_B$ (resp. $\Pi|_B$ and $\rho|_B$) and we write $\widehat{p}$ (resp. $\widehat{\rho}$) instead of $\widehat{p}|_{\widehat{B}}$ (resp. $\widehat{\rho}|_{\widehat{B}}$).

Definition 6.1. A nonholonomic constraint ϕ on a Lie algebroid is called ideal if the Liouville section of $\mathcal{T}A$ in B belongs to $\mathcal{T}B$ where B is the subbundle of A defined by $\phi = 0$, that is, $\pounds_\Delta \phi = d_{\mathcal{T}A}\phi(\Delta) = 0$.

With this definition of ideal constraint, we can show:

Proposition 6.2. *When the constrains ϕ_α are ideal and V is a solution of (6.1), the energy of the system is conserved, that is, $\pounds_V E_L = 0$.*

Proof. We have

$$\pounds_V E_L = \rho_{\mathcal{T}A}(V)E_L = \lambda^a S^*(d_{\mathcal{T}A}\phi_a)(V) = \lambda^a d_{\mathcal{T}A}\phi_a(S(V))$$

and, since V is a SODE, then $S(V) = \Delta$ and

$$\pounds_V E_L = \lambda^a d_{\mathcal{T}A}\phi_\alpha(\Delta) = \lambda^a \pounds_\Delta \phi_a.$$

But as the constraints were assumed to be ideal, the right hand term is zero. □

The solution of the nonholonomic system (6.1) is given by

$$V = V_L + \lambda^a Z_a,$$

where V_L is a solution of the initial dynamics (without constraints), Z_a is a vertical section of $\mathcal{T}A$ given by $i(Z_a)\omega_L = S^*(d_{\mathcal{T}A}\phi_a)$ and λ^a is a function on A determined in such a way that $V|_B \in \mathcal{T}B$, *i.e.* $\pounds_V \phi_a = 0$ or $\rho_{\mathcal{T}A}(V)\phi_a = 0$ for all $a = 1, ..., k$.

If $(q^1, ..., q^n, \mathbf{v}^1, ..., \mathbf{v}^s)$ is a system of local coordinates of $p : A \to M$ associated with the choice of a basis of local sections $\{e_\alpha \mid \alpha = 1, .., s\}$, the Euler-Lagrangian equations of the constrained system (6.1) are:

$$\begin{cases} \dot{q}^i &= \rho^i{}_\alpha \mathbf{v}^\alpha \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}^\alpha}\right) &= \rho^i{}_\alpha \frac{\partial L}{\partial q^i} + \mathbf{v}^\beta c_{\beta\alpha}{}^\gamma \frac{\partial L}{\partial \mathbf{v}^\gamma} + \lambda^a \frac{\partial \phi_a}{\partial \mathbf{v}^\alpha} \end{cases}$$

with $\lambda^a \in C^\infty(A)$ for all $a = 1, ..., k$. We can use a new set of local coordinates adapted to the constraints, that is, let us consider a new set of local coordinates in the Lie algebroid A, $\{(q^i, \mathbf{w}^\alpha) \mid i = 1, ..., n, \ \alpha = 1, ..., s\}$, associated with the basis of sections $\{f_\alpha \mid \alpha = 1, ..., s\}$ of A that satisfy:

$$\mathbf{w}^\alpha = \widehat{\Phi_\alpha}(q, \mathbf{v}) = \Phi_{\alpha\beta}(q)\mathbf{v}^\beta, \qquad \mathbf{v}^\alpha = \widehat{\Psi_\alpha}(q, \mathbf{w}) = \Psi_{\alpha\beta}(q)\mathbf{w}^\beta, \quad (6.2)$$

for all $\alpha = 1, ..., s$, where $\widehat{\Phi_\alpha}$ and $\widehat{\Psi_\alpha}$ are linear functions in A associated to the A-1-forms Φ_α and Ψ_α, respectively, that satisfies $\Psi_{\alpha\beta}\Phi_{\beta\gamma} = \delta_{\alpha\gamma}$ and are defined by

$$\Phi = \begin{pmatrix} I_{s-k} & 0_{(s-k)\times k} \\ A_{21} & A_{22} \end{pmatrix}, \qquad \Psi = \begin{pmatrix} I_{s-k} & 0_{(s-k)\times k} \\ B_{21} & B_{22} \end{pmatrix},$$

where the matrix $A = (A_{21}\ A_{22})$ is given by $A_{a\beta} = \phi_{a\beta}$ for all $a = 1, ..., k$ and $\beta = 1, ..., s$ and the matrix $B = (B_{21}\ B_{22})$ is given by $B_{21} = -A_{22}^{-1}A_{21}$ and $B_{22} = A_{22}^{-1}$. We have the following relations between the local sections:

$$f_\alpha = \Psi_{\beta\alpha} e_\beta\,, \quad e_\alpha = \Phi_{\beta\alpha} f_\beta\,.$$

In these new coordinates, the Euler-Lagrange equations of the nonholonomic system on the Lie algebroid A are given by:

$$\left\{ \begin{array}{rcl} \dot{q}^i & = & \rho^i{}_\beta \Psi_{\beta\bar{a}} \mathbf{w}^{\bar{a}} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{w}^{\bar{a}}}\right) & = & \rho^i{}_\beta \Psi_{\beta\bar{a}} \frac{\partial L}{\partial q^i} + \mathbf{w}^{\bar{b}} \gamma^\beta_{\bar{b}\bar{a}} \frac{\partial L}{\partial \mathbf{w}^\beta} \end{array} \right.$$

for all $\bar{a} = 1, ..., s-k$, where $[f_{\bar{b}}, f_{\bar{a}}]_A = \gamma^\beta_{\bar{b}\bar{a}} f_\beta$. These are, precisely, the equations (5) obtained by Mestdag *et al.* in [18], because:

$$\begin{array}{rcl} \mathbf{w}^{\bar{b}} \gamma^\beta_{\bar{b}\bar{a}} \dfrac{\partial L}{\partial \mathbf{w}^\beta} & = & \mathbf{w}^{\bar{b}} \gamma^\beta_{\bar{b}\bar{a}} \Psi_{\eta\beta} \dfrac{\partial L}{\partial \mathbf{v}^\eta} \\ & = & \mathbf{w}^{\bar{b}} \left\langle e^\eta, [f_{\bar{b}}, f_{\bar{a}}]_A \right\rangle \dfrac{\partial L}{\partial \mathbf{v}^\eta} \\ & = & \mathbf{w}^{\bar{b}} (\Psi_{\beta\bar{b}} \Psi_{\alpha\bar{a}}\, c_{\beta\alpha}{}^\eta + \rho^i{}_\beta\, \Psi_{\beta\bar{b}} \dfrac{\partial \Psi_{\eta\bar{a}}}{\partial q^i} - \rho^i{}_\alpha \Psi_{\alpha\bar{a}} \dfrac{\partial \Psi_{\eta\bar{b}}}{\partial q^i}) \dfrac{\partial L}{\partial \mathbf{v}^\eta}. \end{array}$$

Now, suppose that the subbundle B of A is a Lie subalgebroid of A, that is, there exists an injective morphism $\iota : B \to A$ such that ι is a homomorphism of Lie algebroids; we will represent the Lie algebroid structure of B by $(\lambda, [\cdot,\cdot]_B)$. In these conditions we have that $\bar{\iota} = (\iota, \iota, T\iota)|_{\mathcal{T}B} : \mathcal{T}B \to \mathcal{T}A$ is a homomorphism of Lie algebroids. In fact, $\bar{\iota} \circ V|_B = V \circ \iota$ for all $V \in \Gamma(\mathcal{T}A)$ such that $V|_B \in \Gamma(\mathcal{T}B)$ and $[V_1|_B, V_2|_B]_B \in \Gamma(\mathcal{T}B)$ for all sections $V_1, V_2 \in \mathcal{T}A$ such that $V_1|_B, V_2|_B \in \Gamma(\mathcal{T}B)$. Then, we can prove that

$$\bar{\iota}^*(S^*(d_{TA}\phi_a)) = 0 \tag{6.3}$$

Indeed

$$\bar{\iota}^*(S^*(d_{TA}\phi_a)) = S_B^*(\iota^* d_{\mathcal{T}A}\phi_a),$$

where S_B is the vertical endomorphism in $\mathcal{T}B$; then, because ι is a homomorphism of Lie algebroids we have

$$\bar{\iota}^*(S^*(d_{\mathcal{T}A}\phi_a)) = S_B^*(d_{\mathcal{T}B}(\bar{\iota}^*\phi_a)),$$

and then as $\overline{\iota}^*\phi_a = 0$ we have proved that $\overline{\iota}^*(S^*(d_{\mathcal{T}A}\phi_a)) = 0$.

Consider the set $\mathcal{T}^A B = \{(b, c, v) \in B \times A \times TB \mid p(b) = p(c),\ \rho(c) = Tp(v) \text{ with } v \in T_b B\}$. The bundle $\tau : \mathcal{T}^A B \to B$, with $\tau(b, c, v) = b$ for all $(b, c, v) \in \mathcal{T}^A_b B$, is endowed with a Lie algebroid structure, induced by the Lie algebroid structure in A, whose anchor is defined by $\varrho(b, c, v) = v \in T_c B$ for all $(b, c, v) \in \mathcal{T}^A_b B$ and the Lie bracket of projectable sections of $\mathcal{T}^A B$, that is, of sections of the form $V(b) = (b, \sigma(p(b)), X((p(b))))$, where $\sigma \in \Gamma(A)$ and $X \in \mathfrak{X}(B)$ are such that $Tp \circ X = \rho \circ \sigma$, is defined in the usual way (see section 3 in [18] and also [11]). Note that

$$\iota = I^A \circ I \tag{6.4}$$

where $I : \mathcal{T}B \to \mathcal{T}^A B$ is defined by $I(b, c, v) = (b, \iota(c), v)$ and $I^A : \mathcal{T}^A B \to \mathcal{T}A$ is defined by $I^A(b, c, v) = (\iota(b), c, T\iota(v))$; both I^A and I are Lie algebroids homomorphisms. In these conditions, we can give a geometric proof of the relation (14) obtained by Mestdag *et.al* in [18]:

$$i(V|_B)\delta\widetilde{\theta}_L = -\delta\widetilde{E}_L, \tag{6.5}$$

where $\widetilde{\theta}_L = (I^A)^*\theta_L$, $\widetilde{E}_L = (I^A)^* E_L$, $\delta = I^* \circ d_{\mathcal{T}^A B} = d_{\mathcal{T}B} \circ I^*$ and where V is the SODE solution to the constrained system (6.1). Indeed, by the system (6.1) we can write

$$\overline{\iota}^*(\, i(V)\omega_L - d_{\mathcal{T}A} E_L \,) \overset{(6.3)}{=} 0,$$

that is,

$$\overline{\iota}^*(i(V)\omega_L) = \overline{\iota}^*(d_{\mathcal{T}A} E_L) = \delta\widetilde{E}_L,$$

which is equivalent to

$$i(V|_B)(\overline{\iota}^*\omega_L) = \delta\widetilde{E}_L.$$

Since $\overline{\iota}^*\omega_L = -\overline{\iota}^*(d_{\mathcal{T}A}\theta_L) \overset{(6.4)}{=} -I^* \circ d_{\mathcal{T}^A B}((I^A)^*\theta_L) = -\delta\widetilde{\theta}_L$, we obtain

$$i(V|_B)\delta\widetilde{\theta}_L = -\delta\widetilde{E}_L.$$

From the relation (6.5) we can write

$$i(V|_B) d_{\mathcal{T}B}(\overline{\iota}^*\theta_L) = -d_{\mathcal{T}B}(\overline{\iota}^* E_L),$$

which is equivalent to

$$i(V|_B) d_{\mathcal{T}B}\theta_{\overline{L}} = -d_{\mathcal{T}B} E_{\overline{L}},$$

with $\theta_{\overline{L}} = \overline{\iota}^*\theta_L$ and $E_{\overline{L}} = \overline{\iota}^*E_L$, where $\overline{L} = L \circ \iota : B \to \mathbb{R}$ is a differentiable function on B. Therefore,

$$i(V|_B)\omega_{\overline{L}} = d_{\mathcal{T}B}E_{\overline{L}}, \tag{6.6}$$

with $\omega_{\overline{L}} = -d_{\mathcal{T}B}\theta_{\overline{L}}$; in general this 2-form is degenerate.

6.2. Reduction of nonholonomic systems. Let $\widehat{B}$ be a subbundle of $\widehat{A}$ given by $\operatorname{Im}\Pi|_B$, $\Pi|_B(B) = \widehat{B}$. Next, we will prove that the constrained dynamics on A reduces into a dynamics on $\widehat{A}$ whose solution is a section on $\mathcal{T}\widehat{B}$.

First of all, we will show that $\overline{\Pi}(V|_B)$ belongs to the subbundle $\mathcal{T}\widehat{B}$ of $\mathcal{T}_{\widehat{B}}\widehat{A}$, with total space

$$\mathcal{T}\widehat{B} = \left\{(b', c', v') \in \widehat{B} \times \widehat{B} \times T\widehat{B} \mid \widehat{p}(b') = \widehat{p}(c'), \widehat{\rho}(c') = T\widehat{p}(v') \text{ with } v \in T_{a'}\widehat{B}\right\},$$

where V is the solution of the constrained dynamics on A that satisfies the system (6.1). Let $V(b) = (b, c, v) \in \mathcal{T}_bB$, then $\overline{\Pi}(V) = (\Pi(b), \Pi(c), T\Pi(v))$. Thus, $\widehat{p}(\Pi(b)) = \pi(p(b)) = \pi(p(c)) = \widehat{p}(\Pi(c))$; on the other hand, we have

$$\begin{aligned} T\widehat{p}(T\Pi(v)) &= T(\widehat{p} \circ \Pi)(v) = T(\pi \circ p)(v) \\ &= T\pi(Tp(v)) = T\pi(\rho(b)) \\ &= (\widehat{\rho} \circ \Pi)(b) = \widehat{\rho}(\Pi(b)). \end{aligned}$$

Therefore, $\overline{\Pi}(V|_B) \in \mathcal{T}\widehat{B}$. Moreover, $V' = \overline{\Pi}(V)$ is a SODE since

$$S'(V') = S'(\overline{\Pi}(V)) = \overline{\Pi}(S(V)) = \overline{\Pi}(\Delta) = \Delta'.$$

In the following result we prove that $i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l$ is equal to the reaction force of the reduced system in $\widehat{A}$.

Lemma 6.3. $i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l \in S'^*((\mathcal{T}\widehat{B})^0)$.

Proof. We know that

$$i(V)\omega_L - d_{\mathcal{T}A}E_L \in S^*((\mathcal{T}B)^0).$$

So,

$$i(V)(\overline{\Pi}^*\omega'_l) - d_{\mathcal{T}A}(\overline{\Pi}^*E'_l) \in S^*((\mathcal{T}B)^0),$$

that is,

$$\overline{\Pi}^*[i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l] \in S^*((\mathcal{T}B)^0).$$

But V' is a SODE then $i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l$ is a semibasic 1-form. Therefore, there exists $\Phi \in (\mathcal{T}\widehat{A})^*$ such that

$$i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l = S'^*(\Phi).$$

Since $\overline{\Pi}^*(S'^*(\Phi)) \in S^*((\mathcal{T}B)^0)$ and $\overline{\Pi} \circ S = S' \circ \overline{\Pi}$, then, $\overline{\Pi}^*(\Phi) \in (\mathcal{T}B)^0$. Therefore, $\Phi \in (\mathcal{T}\widehat{B})^0$. □

Thus, we have proved:

Theorem 6.4. *The constrained dynamics on A reduces into a dynamics on $\widehat{A}$ whose solution satisfies*

$$\begin{cases} i(V')\omega'_l - d_{\mathcal{T}\widehat{A}}E'_l \in S'^*((\mathcal{T}\widehat{B})^0) \\ V'|_{\widehat{B}} \in \mathcal{T}\widehat{B}\ . \end{cases}$$

Since $V = V_L + \lambda^\alpha Z_\alpha$ then

$$V' = \overline{\Pi}(V) = \overline{\Pi}(V_L) + \overline{\Pi}(\lambda^\alpha Z_\alpha),$$

where $V'_l = \overline{\Pi}(V_L)$ is the SODE solution of the reduced dynamics without constraints on $\mathcal{T}\widehat{A}$ and $\overline{\Pi}(\lambda^\alpha Z_\alpha)$ is a vertical section of $\mathcal{T}\widehat{A}$ satisfying $i(\overline{\Pi}(\lambda^\alpha Z_\alpha))\omega'_l \in S'^*((\mathcal{T}^{\widehat{A}}\widehat{B})^0)$, that is, $i(\overline{\Pi}(\lambda^\alpha Z_\alpha))\omega'_l = \Phi'$ is a semibasic 1-form such that $\overline{\Pi}^*(\Phi') = \lambda^\alpha S^*(d_{\mathcal{T}A}\phi_\alpha)$.

Proposition 6.5. *We have the following relation $\pounds_V E_L = \pounds_{V'} E'_l$.*

Proof. From the relation $\overline{\Pi}^* E'_l = E_L$, we have

$$\pounds_V E_L = d_{\mathcal{T}A} E_L(V) = d_{\mathcal{T}A}(\overline{\Pi}^* E'_l)(V).$$

The map $\overline{\Pi}$ is a homomorphism of Lie algebroids, then

$$\pounds_V E_L = \overline{\Pi}^*(d_{\mathcal{T}\widehat{A}}E'_l)(V) = d_{\mathcal{T}\widehat{A}}E'_l(V'),$$

that is, $\pounds_V E_L = \pounds_{V'} E'_l$. □

As an immediate consequence we have that:

Corollary 6.6. *The energy of the constrained system in A is conserved iff the energy of the reduced system in $\widehat{A}$ is conserved.*

6.3. Example: *Non-Abelian Čaplygin systems*. A non-Abelian *Čaplygin system* is a constrained system whose configuration space is a principal fibre bundle $\pi : P \to M = P/G$ endowed with a connection given by the constraint distribution H such that $TP = H \oplus V$, where V is the vertical bundle; therefore, the constraints ϕ_α are linear in the velocities and the energy of the system is conserved (see [3] and references therein). The Lagrangian $L \in C^\infty(TP)$ of the system is supposed to be regular and invariant for the lifted action of the Lie group G on P, i.e. $L = l \circ \Pi$ where $\Pi : TP \to TP/G$ is the canonical projection defined by the lift action. The constrained system in TP can be formulated as follows:

$$\begin{cases} i(V)\omega_L - d_{T(TP)}E_L \in S^*((\mathcal{T}H)^0) \\ V|_H \in \mathcal{T}H \end{cases} . \tag{6.7}$$

The solution of the system is of the form $V = V_L + \lambda^a Z_a$ where V_L is the solution of the system without constraints and Z_a is a vertical section of $T(TP)$ such that $i(\lambda^a Z_a)\omega_L = \lambda^a S^*(d_{T(TP)}\phi_a)$ and $\pounds_V \phi_a = 0$.

The canonical projection $\Pi : TP \to TP/G$ maps the subbundle H of $A = TP$ onto the subbundle $H' = H/G \cong TM$ of $\widehat{A} = TP/G$, and it is a homomorphism of Lie algebroids, [4]. Thus, the reduced constrained system on TP/G has a solution that satisfies

$$\begin{cases} i(V')\omega'_l - d_{\mathcal{T}(TP/G)}E'_l \in S'^*([T(TM)]^0) \\ V'|_{TM} \in T(TM) \,. \end{cases} \tag{6.8}$$

The horizontal lift of a vector field in M into a section of TP/G, $\iota' : TM \to TP/G$, is a Lie algebroid homomorphism because TM is an integrable distribution of TP/G, then TM is a Lie subalgebroid of TP/G. So we can also formulate the system (6.8) as

$$i(V'|_{TM})\delta\widetilde{\theta}'_l = -\delta\widetilde{E}'_l + \overline{\iota'}^*(\Phi'),$$

where $\Phi' = i(\Pi(\lambda^a Z_a))\omega'_l \in S'^*([\mathcal{T}H']^0) \subset \mathcal{T}^*(TP/G)$ is a semibasic 1-form, $\overline{\iota'} = I^{TP/G} \circ I' : \mathcal{T}H' = T(TM) \to \mathcal{T}(TP/G)$ with $I' : T(TM) \to \mathcal{T}^{TP/G}(TM)$ defined by $I'(a,b,v) = (a, \iota'(b), v)$ and $I^{TP/G} : \mathcal{T}^{TP/G}(TM) \to \mathcal{T}(TP/G)$ defined by $I^{TP/G}(a,b,v) = (\iota'(a), b, T\iota'(v))$, and $\delta = I'^* \circ d_{\mathcal{T}^{TP/G}(TM)} = d_{T(TM)} \circ I'^*$. So we have,

$$i(V'|_{TM})d_{TTM}(\overline{\iota'}^*\theta'_l) = -d_{T(TM)}(\overline{\iota'}^* E'_l) + \overline{\iota'}^*(\Phi').$$

Note that, $\overline{\iota'}^*\theta'_l = \theta_{\overline{l}}$ and $\overline{\iota'}^* E'_l = E_{\overline{l}}$ with $\overline{l} : TM \to M$ defined by $\overline{l}(Y_{\overline{q}}) = L(Y_q^H)$ for all $\overline{q} = \pi(q)$, $Y_{\overline{q}} \in T_{\overline{q}}M$, where Y^H denotes the horizontal lift to P of a vector field Y on M. Therefore,

$$i(V'|_{TM})\omega_{\overline{l}} = d_{T(TM)}(\overline{\iota'}^* E'_l) - \overline{\iota'}^*(\Phi'),$$

with $\omega_{\overline{l}} = -d_{TTM}(\overline{\iota'}^*\theta'_l)$. As in [3] we have $i(V'|_{TM})\overline{\iota'}^*(\Phi') = 0$, because

$$\begin{aligned} i(V'|_{TM})\overline{\iota'}^*(\Phi') &= \Phi'(\iota' \circ V'|_{TM}) = \Phi'(V') = \omega'_l(\overline{\Pi}(\lambda^a Z_a), \overline{\Pi}(V')) \\ &= \omega_L(\lambda^a Z_a, V') = 0. \end{aligned}$$

Once this work has been finished we have realized that some similar results had been announced in [6], and are given in [7] and [19]. We thank the authors J. Cortés *et.al* for sending us the preprint [7] with their results.

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