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ON SOME CHARACTERIZATIONS OF BESOV SPACES OF GENERALIZED SMOOTHNESS

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Abstract: We present characterizations of the Besov spaces of generalized smoothness $B^{\sigma,N}_{p,q}(\mathbb{R}^n)$ via approximation and by means of differences.

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1. Introduction

Function spaces of generalized smoothness have been introduced and considered, in particular since the middle of the seventies of the last century, by several authors from different points of view and different degrees of generality. We refer, for instance, to Goldman [9, 10], Kalyabin [13, 14], Merucci [16], Cobos and Fernandez [4], the survey by Kalyabin and Lizorkin [12], Leopold [15], Edmunds and Triebel [6, 7] and Moura [17, 18].

From the standpoint of a Fourier analytic characterization, the recent work by Farkas and Leopold [8] represents a unified and general approach on this topic. Following their approach, we see that, under certain conditions, the Besov spaces of generalized smoothness can be analogously characterized via approximation and by means of differences.

2. Preliminaries

2.1. General notation. In this paper we shall adopt the following general notation: $\mathbb{N}$ denotes the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\mathbb{R}^n$, $n \in \mathbb{N}$, denotes the n-dimensional real Euclidean space and $\mathbb{R} = \mathbb{R}^1$. We use the equivalence "$\sim$" in

$$a_k \sim b_k \qquad \text{or} \qquad \phi(r) \sim \psi(r)$$

always to mean that there are two positive numbers c_1 and c_2 such that

$$c_1 a_k \leq b_k \leq c_2 a_k \qquad \text{or} \qquad c_1\phi(x) \leq \psi(x) \leq c_2\phi(x)$$

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for all admitted values of the discrete variable k or the continuous variable x, where $(a_k)_k$, $(b_k)_k$ are non-negative sequences and ϕ, ψ are non-negative functions. If $a \in \mathbb{R}$ then $a_+ := \max(a, 0)$. We consider here only function spaces defined on $\mathbb{R}^n$; so for convenience we shall usually omit the "$\mathbb{R}^n$" from their notation.

2.2. Sequences. In this subsection we explain the class of sequences we shall be interested in and some related basic results.

A sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$ of positive real numbers is said to be *admissible* if there exist two positive constants d_0 and d_1 such that

$$d_0\,\gamma_j \le \gamma_{j+1} \le d_1\,\gamma_j, \qquad j \in \mathbb{N}_0. \tag{1}$$

For an admissible sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$, let

$$\underline{\gamma}_j := \inf_{k\ge 0} \frac{\gamma_{j+k}}{\gamma_k} \qquad \text{and} \qquad \overline{\gamma}_j := \sup_{k\ge 0} \frac{\gamma_{j+k}}{\gamma_k}, \qquad j \in \mathbb{N}_0. \tag{2}$$

Then clearly $\underline{\gamma}_j\gamma_k \le \gamma_{j+k} \le \gamma_k\overline{\gamma}_j$ for any $j, k \in \mathbb{N}_0$. In particular, $\underline{\gamma}_1$ and $\overline{\gamma}_1$ are the best possible constants d_0 and d_1 in (1), respectively. Then, the lower and upper Boyd indices of the sequence γ are defined, respectively, by

$$\underline{s}(\gamma) := \lim_{j\to\infty} \frac{\log \underline{\gamma}_j}{j} \qquad \text{and} \qquad \overline{s}(\gamma) := \lim_{j\to\infty} \frac{\log \overline{\gamma}_j}{j}. \tag{3}$$

The above definition is well posed: the sequence $(\log \overline{\gamma}_j)_{j\in\mathbb{N}}$ is sub-additive and hence the right-hand side limit in (3) exists, it is finite (since γ is an admissible sequence) and it coincides with $\inf_{j>0} \log \overline{\gamma}_j/j$. The corresponding assertions for the lower counterpart $\underline{s}(\gamma)$ can be read off observing that $\log \underline{\gamma}_j = -\log(\overline{\gamma^{-1}})_j$.
The Boyd index $\overline{s}(\gamma)$ of an admissible sequence γ describes the asymptotic behaviour of the $\overline{\gamma}_j$'s and provides more information than simply $\overline{\gamma}_1$ and, what is more, is stable under the equivalence of sequences: if $\gamma \sim \tau$, then $\overline{s}(\gamma) = \overline{s}(\tau)$ as one readily verifies. The same applies to the lower counterpart.

Observe also that for each $\varepsilon > 0$ there are two positive constants $c_1 = c_1(\varepsilon)$ and $c_2 = c_2(\varepsilon)$ such that

$$c_1\;2^{(\underline{s}(\gamma)-\varepsilon)j} \le \underline{\gamma}_j \le \overline{\gamma}_j \le c_2\;2^{(\overline{s}(\gamma)+\varepsilon)j}, \qquad j \in \mathbb{N}_0. \tag{4}$$

Examples 2.1. We consider some examples of admissible sequences.
(i) The sequence $\gamma = (\gamma_j)_{j\in\mathbb{N}_0}$,

$$\gamma_j = 2^{sj}\,(1+j)^b\,(1+\log(1+j))^c$$

with arbitrary fixed real numbers s, b and c is a standard example of an admissible sequence with $\underline{s}(\gamma) = \overline{s}(\gamma) = s$.
(ii) Let $\Phi : (0, 1] \to \mathbb{R}$ be a slowly varying function (or equivalent to a slowly varying one) in the sense of [1]. Then, for each $s \in \mathbb{R}$ the sequence $\gamma = (2^{sj}\Phi(2^{-j}))_{j\in\mathbb{N}_0}$ is an admissible sequence. Also here we have $\underline{s}(\gamma) = \overline{s}(\gamma) = s$.
(iii) In view of Proposition 1.9.7 of [2], the case $\gamma = (2^{sj}\Psi(2^{-j}))_{j\in\mathbb{N}_0}$, where now Ψ is an admissible function in the sense of [6] (i.e. a positive monotone function defined on $(0, 1]$ such that $\Psi(2^{-2j}) \sim \Psi(2^{-j})$, $j \in \mathbb{N}_0$), can be regarded as a special case of (ii).

Remark 2.2. The examples above have in common the fact that their upper and lower Boyd indices coincide. However, this is not in general the case. Examples 2.2.6 and 2.2.7 of [8], due to Kalyabin, show that an admissible sequence has not necessarily a fixed main order.

2.3. Function spaces of generalized smoothness. Let $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\underline{N}_1 > 1$ (recall (2)). In particular N is a so-called strongly increasing sequence (cf. [8, Def. 2.2.1]), which guarantees the existence of a number $k_0 \in \mathbb{N}_0$ such that

$$N_k \geq 2N_j \quad \text{for any} \quad k,\, j \quad \text{such that} \quad k \geq j + k_0. \tag{5}$$

It should be noted that the sequence $N = (N_j)_{j\in\mathbb{N}_0}$ plays the same role as the sequence $(2^j)_{j\in\mathbb{N}_0}$ in the classical construction of the spaces B^s_{pq} and F^s_{pq}. This will be clear from the following considerations.

For a fixed sequence $N = (N_j)_{j\in\mathbb{N}_0}$, as above, we define the associated covering $\Omega^N = (\Omega^N_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$ by

$$\Omega^N_j = \{\xi \in \mathbb{R}^n : |\xi| \leq N_{j+k_0}\}, \quad j = 0, \cdots, k_0 - 1,$$

and

$$\Omega^N_j = \{\xi \in \mathbb{R}^n : N_{j-k_0} \leq |\xi| \leq N_{j+k_0}\}, \quad j \geq k_0,$$

with k_0 according to (5).

Definition 2.3. *For a fixed admissible sequence $N = (N_j)_{j\in\mathbb{N}_0}$ with $\underline{N}_1 > 1$, and for the associated covering $\Omega^N = (\Omega^N_j)_{j\in\mathbb{N}_0}$ of $\mathbb{R}^n$, a system $\varphi^N = (\varphi^N_j)_{j\in\mathbb{N}_0}$ will be called a (generalized) partition of unity subordinated to Ω^N if:*
(i)

$$\varphi^N_j \in C^\infty_0 \quad \textit{and} \quad \varphi^N_j(\xi) \geq 0 \quad \textit{if} \quad \xi \in \mathbb{R}^n \quad \textit{for any} \quad j \in \mathbb{N}_0;$$

(ii)

$$\text{supp } \varphi_j^N \subset \Omega_j^N \quad \textit{for any} \quad j \in \mathbb{N}_0;$$

(iii) *for any* $\alpha \in \mathbb{N}_0^n$ *there exists a constant* $c_\alpha > 0$ *such that for any* $j \in \mathbb{N}_0$

$$|D^\alpha \varphi_j^N(\xi)| \le c_\alpha (1 + |\xi|^2)^{-|\alpha|/2} \quad \textit{for any} \quad \xi \in \mathbb{R}^n;$$

(iv) *there exists a constant* $c_\varphi > 0$ *such that*

$$0 < \sum_{j=0}^{\infty} \varphi_j^N(\xi) = c_\varphi < \infty \quad \textit{for any} \quad \xi \in \mathbb{R}^n.$$

Before turning to the definition of the Besov spaces of generalized smoothness let us recall that $\mathcal{S}$ denotes the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ equipped with the usual topology and by $\mathcal{S}'$ we denote its topological dual, the space of all tempered distributions on $\mathbb{R}^n$. If $f \in \mathcal{S}'$, then $\mathcal{F}f$ and $\mathcal{F}^{-1}f$ denote the Fourier and the inverse Fourier transform of f, respectively. For $\varphi \in \mathcal{S}$ and $f \in \mathcal{S}'$ we will use the notation $\varphi(D)f = [\mathcal{F}^{-1}(\varphi \mathcal{F} f)]$. Furthermore, if $0 < p \le \infty$ and $0 < q \le \infty$ then L_p and ℓ_q have the standard meaning and, if $(f_j)_{j \in \mathbb{N}_0}$ is a sequence of complex-valued Lebesgue measurable functions on $\mathbb{R}^n$, then

$$\|(f_j)_{j \in \mathbb{N}_0} | \ell_q(L_p)\| = \Big(\sum_{j=0}^{\infty} \|f_j \mid L_p\|^q \Big)^{1/q}$$

with the appropriate modification if $q = \infty$.

Definition 2.4. *Let* $N = (N_j)_{j \in \mathbb{N}_0}$ *be an admissible sequence with* $\underline{N}_1 > 1$ *and* φ^N *a system of functions as in Definition 2.3. Let* $0 < p, q \le \infty$ *and* $\sigma = (\sigma_j)_{j \in \mathbb{N}_0}$ *be an admissible sequence. Then the Besov space of generalized smoothness* $B^{\sigma,N}_{p,q}$ *is the set of all tempered distributions* f *such that the quasi-norm*

$$\|f | B^{\sigma,N}_{p,q}\| := \|(\sigma_j \, \varphi_j^N(D) f)_{j \in \mathbb{N}_0} | \ell_q(L_p)\|$$

is finite.

Remark 2.5. Note that if $0 < p < \infty$ then the Triebel-Lizorkin space of generalized smoothness $F^{\sigma,N}_{p,q}$ is defined in an analogous way, by interchanging the roles of the L_p and ℓ_q quasi-norms.

If $\sigma = (2^{sj})_{j \in \mathbb{N}_0}$, with s a real number, and $N = (2^j)_{j \in \mathbb{N}_0}$ then the spaces $B^{\sigma,N}_{p,q}$ coincide with the usual Besov spaces $B^s_{p,q}$ and $F^{\sigma,N}_{p,q}$ coincide with the usual Triebel-Lizorkin spaces $F^s_{p,q}$. If we let $\sigma = (2^{sj}\Psi(2^{-j}))_{j \in \mathbb{N}_0}$, where

Ψ is an admissible function in the sense of [6, 7](see Example 2.1–(iii)), the corresponding Besov space coincide with the space $B^{(s,\Psi)}_{p,q}$ introduced by Edmunds and Triebel in [6, 7] and also considered by Moura in [17, 18]. A similarly observation holds for the F-counterpart.

3. Characterization by approximation

Let $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\underline{N}_1 > 1$ (recall (2)) and $k_0 \in \mathbb{N}_0$ as in (5). Let $0 < p \le \infty$ and

$$\mathcal{U}^N_p := \Big\{ a = (a_j)_{j\in\mathbb{N}_0} : a_j \in \mathcal{S}' \cap L_p,\ \operatorname{supp} \mathcal{F}a_j \subset \{y : |y| \le N_{j+k_0}\},\ j \in \mathbb{N}_0 \Big\}.$$

Theorem 3.1. *Let $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ be an admissible sequence, $0 < p, q \le \infty$, and $N = (N_j)_{j\in\mathbb{N}_0}$ be an admissible sequence with $\underline{N}_1 > 1$, as above.*

Suppose that $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > n(1/p - 1)_+$. Then

$$B^{\sigma,N}_{p,q} = \Big\{ f \in \mathcal{S}' : \exists\, a = (a_j)_{j\in\mathbb{N}_0} \in \mathcal{U}^N_p \ \text{ such that } f = \lim_{k\to\infty} a_k \ \text{ in } \mathcal{S}' \text{ and}$$
$$\|f|B^{\sigma,N}_{p,q}\|^a := \|a_0|L_p\| + \|(\sigma_k(f - a_k))_{k\in\mathbb{N}}|\ell_q(L_p)\| < \infty \Big\}.$$

Furthermore,

$$\|f|B^{\sigma,N}_{p,q}\|^X := \inf \|f|B^{\sigma,N}_{p,q}\|^a$$

where the infimum is taken over all admissible systems $a \in \mathcal{U}^N_p$, is an equivalent quasi-norm in $B^{\sigma,N}_{p,q}$.

Proof: First of all let us remark that, taking into account the inequalities in (4), if $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > \delta \ge 0$ then

$$\big(\underline{\sigma}_j^{-1}\overline{N}_j^{\delta}\big)_{j\in\mathbb{N}} \in \ell_u \quad \text{for any} \quad u \in (0, \infty].$$

Step 1. In this step we prove that $\|\cdot|B^{\sigma,N}_{p,q}\|^X$ can be estimated from above by $\|\cdot|B^{\sigma,N}_{p,q}\|$. Let $f \in B^{\sigma,N}_{p,q}$ and let $(\varphi^N_j)_{j=0}^{\infty}$ be a generalized partition of unity, according to Definition 2.3. Then

$$a_k := \sum_{j=0}^{k} \varphi^N_j(D) f \longrightarrow f \quad \text{in} \quad \mathcal{S}' \quad \text{as} \quad k \to \infty.$$

Moreover, $a_k \in \mathcal{S}' \cap L_p$ and $\operatorname{supp} \mathcal{F} a_k \subset \{y : |y| \le N_{k+k_0}\}$, $k \in \mathbb{N}_0$. Hence, $a = (a_k)_{k\in\mathbb{N}_0} \in \mathcal{U}_p^N$. Let $\bar{p} := \min(1,p)$ and put $\varphi_l^N \equiv 0$ if $-l \in \mathbb{N}$. Then

$$\begin{aligned} \sigma_k \, \|f - a_k|L_p\| &\le \sigma_k \Big(\sum_{j=k+1}^{\infty} \|\varphi_j^N(D) f|L_p\|^{\bar{p}} \Big)^{1/\bar{p}} \\ &\le \Big(\sum_{m=1}^{\infty} \underline{\sigma}_m^{-\bar{p}} \, \sigma_{k+m}^{\bar{p}} \, \|\varphi_{k+m}^N(D) f|L_p\|^{\bar{p}} \Big)^{1/\bar{p}}. \end{aligned} \tag{6}$$

If $0 < q \le \bar{p}$, from (6) and using the fact that $(\underline{\sigma}_j^{-1})_{j\in\mathbb{N}_0} \in \ell_q$, we get

$$\begin{aligned} \Big(\sum_{k=1}^{\infty} \sigma_k^q \, \|f - a_k|L_p\|^q \Big)^{1/q} &\le \Big(\sum_{k=1}^{\infty} \sum_{m=1}^{\infty} \underline{\sigma}_m^{-q} \, \sigma_{k+m}^q \, \|\varphi_{k+m}^N(D) f|L_p\|^q \Big)^{1/q} \\ &\le \|(\underline{\sigma}_j^{-1})_{j\in\mathbb{N}_0}|\ell_q\| \; \|f|B_{p,q}^{\sigma,N}\| \le c \, \|f|B_{p,q}^{\sigma,N}\|. \end{aligned}$$

If $q > \bar{p}$, from (6) by using the triangle inequality with respect to $\ell_{q/\bar{p}}$ and $(\underline{\sigma}_j^{-1})_{j\in\mathbb{N}_0} \in \ell_{\bar{p}}$, we obtain

$$\begin{aligned} \Big(\sum_{k=1}^{\infty} \sigma_k^q \, \|f - a_k|L_p\|^q \Big)^{1/q} &\le \Big\{ \sum_{m=1}^{\infty} \underline{\sigma}_m^{-\bar{p}} \, \Big(\sum_{k=1}^{\infty} \sigma_{k+m}^q \, \|\varphi_{k+m}^N(D) f|L_p\|^q \Big)^{\bar{p}/q} \Big\}^{1/\bar{p}} \\ &\le \|(\underline{\sigma}_j^{-1})_{j\in\mathbb{N}_0}|\ell_{\bar{p}}\| \; \|f|B_{p,q}^{\sigma,N}\| \le c \, \|f|B_{p,q}^{\sigma,N}\| \end{aligned}$$

(with the usual modification if $q = \infty$). Since we also have

$$\|a_0|L_p\| = \|\varphi_0^N(D) f|L_p\| \le c \, \|f|B_{p,q}^{\sigma,N}\|,$$

then

$$\|f|B_{p,q}^{\sigma,N}\|^a \le c \, \|f|B_{p,q}^{\sigma,N}\|, \quad f \in B_{p,q}^{\sigma,N}.$$

Step 2. Let $f \in \mathcal{S}'$ and $a = (a_k)_{k\in\mathbb{N}_0} \in \mathcal{U}_p^N$ be such that

$$f = \lim_{k\to\infty} a_k \quad \text{in} \quad \mathcal{S}' \qquad \text{and} \qquad \|f|B_{p,q}^{\sigma,N}\|^a < \infty.$$

Let $l_0 = l_0(N) \in \mathbb{N}$ be such that $\underline{N}_1^{l_0} > \overline{N}_1$ and let $\mu_0, \mu \in \mathcal{S}$ be positive functions such that

$$\mu_0(\xi) = 1 \quad \text{if} \quad |\xi| \le N_1, \quad \operatorname{supp} \mu_0 \subset \{\xi \in \mathbb{R}^n : |\xi| \le \underline{N}_1 N_1\}$$

and

$$\mu(\xi) = 1 \quad \text{if} \quad \overline{N}_1^{-1} \le |\xi| \le \overline{N}_1, \quad \operatorname{supp} \mu \subset \{\xi \in \mathbb{R}^n : \underline{N}_1^{-l_0} \le |\xi| \le \underline{N}_1^{l_0}\}.$$

For each $j \in \mathbb{N}$ define $\mu_j(\xi) := \mu(N_j^{-1}\xi)$, $\xi \in \mathbb{R}^n$. Putting $a_l \equiv 0$ if $-l \in \mathbb{N}$, we have

$$\mu_k(D)f = \sum_{j=1-l_0-k_0}^{\infty} \mathcal{F}^{-1}[\mu_k \mathcal{F}(a_{k+j} - a_{k+j-1})], \quad k \in \mathbb{N}_0,$$

and hence, with $\bar{p}$ as in step 1, we get

$$\|\mu_k(D)f|L_p\| \le \Big(\sum_{j=-1}^{\infty} \|\mathcal{F}^{-1}[\mu_k \mathcal{F}(a_{k+j} - a_{k+j-1})]|L_p\|^{\bar{p}}\Big)^{1/\bar{p}}, \quad k \in \mathbb{N}_0. \tag{7}$$

Recall that $\operatorname{supp} \mathcal{F}(a_{k+j} - a_{k+j-1}) \subset \{\xi : |\xi| \le N_{k+j+k_0}\}$. By hypothesis $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > n(1/p-1)_+$, so that we can choose a real number λ satisfying $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > \lambda - n/2 > n(1/p-1)_+$. Then the inequality (13) of [19, 1.5.2, p. 28] yields

$$\begin{aligned}\|\mathcal{F}^{-1}[\,\mu_k \mathcal{F}(a_{k+j} - a_{k+j-1})]|L_p\| \\ \le c\,\|\mu_k(N_{k+j+k_0}\cdot)|H_2^\lambda\|\;\|a_{k+j} - a_{k+j-1}|L_p\|, \quad k \in \mathbb{N}_0,\end{aligned} \tag{8}$$

where c is a positive constant independent of k and j. Using Proposition 1 of [5, p. 33] we obtain, for $k \in \mathbb{N}$,

$$\begin{aligned}\|\mu_k(N_{k+j+k_0}\cdot)|H_2^\lambda\| &= \|\mu(N_{k+j+k_0}N_k^{-1}\cdot)|H_2^\lambda\| \\ &\le\; c_1\left(N_{k+j+k_0}N_k^{-1}\right)^{\lambda-\frac{n}{2}}\;\|\mu(a\cdot)|H_2^\lambda\| \le c_2\,\overline{N}_j^{\lambda-\frac{n}{2}}\end{aligned} \tag{9}$$

where $a = a(N)$ and c_1, c_2 are positive constants independent of j and k. In an analogous way, for $k = 0$ we also have

$$\|\mu_0(N_{j+k_0}\cdot)|H_2^\lambda\| \le c\,\overline{N}_j^{\lambda-\frac{n}{2}}. \tag{10}$$

Inserting (8), (9) and (10) in (7), leads us to

$$\|\mu_k(D)f|L_p\| \le c\,\Big(\sum_{j=1-l_0-k_0}^{\infty} \overline{N}_j^{(\lambda-\frac{n}{2})\bar{p}}\;\|a_{k+j} - a_{k+j-1}|L_p\|^{\bar{p}}\Big)^{1/\bar{p}}$$

(with the understanding that $\overline{N}_j \equiv 1$ if $-j \in \mathbb{N}$). If $0 < q \leq \bar{p}$, it follows

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sigma_k^q \, \|\mu_k(D) f | L_p\|^q \\
&\leq \; c_1 \sum_{k=0}^{\infty} \sigma_k^q \sum_{j=1-l_0-k_0}^{\infty} \overline{N}_j^{(\lambda-\frac{n}{2})q} \, \|a_{k+j} - a_{k+j-1} | L_p\|^q \\
&\leq \; c_2 \, \|(\overline{N}_j^{(\lambda-\frac{n}{2})} \underline{\sigma}_j^{-1})_{j\in\mathbb{N}_0} | \ell_q\|^q \; \sum_{m=0}^{\infty} \sigma_m^q \, \|a_m - a_{m-1} | L_p\|^q \\
&\leq \; c_3 \, \|a_0 | L_p\| + c_4 \sum_{m=0}^{\infty} \sigma_m^q \, \|f - a_m | L_p\|^q, \qquad (11)
\end{aligned}$$

where we have used the fact that $(\overline{N}_j^{(\lambda-\frac{n}{2})} \underline{\sigma_j}^{-1})_{j\in\mathbb{N}_0} \in \ell_q$, which is a consequence of the choice of λ. On the other hand, if $q > \bar{p}$, reasoning as in step 1, we obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sigma_k^q \, \|\mu_k(D) f | L_p\|^q \\
&\leq \; c_1 \, \|(\overline{N}_j^{(\lambda-\frac{n}{2})} \underline{\sigma}_j^{-1})_{j\in\mathbb{N}_0} | \ell_{\bar{p}}\|^q \; \sum_{m=0}^{\infty} \sigma_m^q \, \|a_m - a_{m-1} | L_p\|^q \\
&\leq \; c_2 \, \|a_0 | L_p\| + c_3 \sum_{m=0}^{\infty} \sigma_m^q \, \|f - a_m | L_p\|^q \qquad (12)
\end{aligned}$$

(with the usual modification if $q = \infty$). Using now the Theorem 4.2.2 of [8], by (11) and (12) we get, for any $q \in (0, \infty]$,

$$\|f | B_{p,q}^{\sigma,N}\| \leq c \, \|f | B_{p,q}^{\sigma,N}\|^a, \quad f \in B_{p,q}^{\sigma,N},$$

where c is a positive constant independent of the chosen system $a = (a_k)_{k\in\mathbb{N}_0}$. Therefore,

$$\|f | B_{p,q}^{\sigma,N}\| \leq c \, \|f | B_{p,q}^{\sigma,N}\|^X, \quad f \in B_{p,q}^{\sigma,N}.$$

■

Remark 3.2. For $N = (2^j)_{j\in\mathbb{N}_0}$ and $\sigma = (2^{sj})_{j\in\mathbb{N}_0}$ or $\sigma = (2^{sj}\Phi(2^{-j}))_{j\in\mathbb{N}_0}$ (cf. (i) and (ii) of Examples 2.1), the above theorem coincide with Theorem 2.5.3(i) of [19] and Theorem 2.4 of [11], respectively.

A characterization as above is valid for the space $F^{\sigma,N}_{p,q}$, replacing the condition $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > n(1/p-1)_+$ by $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > n(1/\min(p,q)-1)_+$ and interchanging the roles of the L_p and ℓ_q quasi-norms.

4. Characterization by differences

Next we recall the definition of differences of functions. If f is an arbitrary function on $\mathbb{R}^n$, $h \in \mathbb{R}^n$ and $k \in \mathbb{N}$, then

$$(\Delta^k_h f)(x) := \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} f(x+jh), \quad x \in \mathbb{R}^n.$$

Note that Δ^k_h can also be defined iteratively via

$$(\Delta^1_h f)(x) = f(x+h) - f(x) \quad \text{and} \quad (\Delta^{k+1}_h f)(x) = \Delta^1_h(\Delta^k_h f)(x), \quad k \in \mathbb{N}.$$

Furthermore, the *k-th modulus of smoothness* of a function $f \in L_p$, $1 \le p \le \infty$, $k \in \mathbb{N}$, is defined by

$$\omega_k(f,t)_p = \sup_{|h|\le t} \|\Delta^k_h f|L_p\|, \quad t > 0.$$

Theorem 4.1. *Let $0 < p,q \le \infty$, $\sigma = (\sigma_j)_{j\in\mathbb{N}_0}$ and $N = (N_j)_{j\in\mathbb{N}_0}$ be admissible sequences, the latter satisfying $\underline{N}_1 > 1$, such that $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > n(1/p-1)_+$. Let $M \in \mathbb{N}$ with $M > \overline{s}(\sigma)\,\underline{s}(N)^{-1}$. Then*

$$B^{\sigma,N}_{p,q} = \Big\{ f \in L_{\max(1,p)} : \\ \|f|B^{\sigma,N}_{p,q}\|^{(M)} := \|f|L_p\| + \Big(\sum_{j=0}^{\infty} \sigma_j^q\, \omega_M(f,N_j^{-1})_p^q\Big)^{1/q} < \infty \Big\}.$$

(with the usual modification if $q=\infty$) in the sense of equivalent quasi-norms.

Remark 4.2. It can be easily verified that, for any $a \in (0,+\infty)$,

$$\|f|B^{\sigma,N}_{p,q}\|^{(M)} \sim \|f|L_p\| + \bigg(\int_0^a \Big(\Lambda(t^{-1})\,\omega_M(f,t)_p\Big)^q \frac{\mathrm{d}t}{t}\bigg)^{1/q}$$

where $\Lambda : (0,\infty) \to (0,\infty)$ is an admissible function – according to [3, Def. 2.2.1] – such that $\Lambda(z) \sim \Lambda(N_j) = \sigma_j,\ z \in [N_j, N_{j+1}],\ j \in \mathbb{N}_0$.

Proof: *Step 1.* In this step we shall prove that there exists a positive constant c such that

$$\|f \mid B^{\sigma,N}_{p,q}\|^{(M)} \le c\,\|f \mid B^{\sigma,N}_{p,q}\| \tag{13}$$

holds for all $f \in B^{\sigma,N}_{p,q}$. If $f \in B^{\sigma,N}_{p,q}$ with $\underline{s}(\sigma)\,\overline{s}(N)^{-1} > n(1/p-1)_+$, then $f \in L_{\max(1,p)}$ and

$$\|f \mid L_p\| \le c\,\|f \mid B^{\sigma,N}_{pq}\|, \tag{14}$$

for some positive constant c which does not depend on f. This is a consequence of Corollary 3.18 of [3], as $(\sigma_j^{-1} N_j^{n(\frac{1}{p}-1)_+}) \in \ell_{q'}$ in this case.
With $(\varphi_j^N)_{j\in\mathbb{N}_o}$ being a generalized partition of unity according to Definition 2.3 we have, with convergence in L_p,

$$f = \sum_{j=0}^{\infty} \varphi_j(D) f. \tag{15}$$

Let $a > n/p$. If $k \in \mathbb{N}_0$ then for each $\varrho \in \mathbb{R}^n$ with $|\varrho| \le N_k^{-1}$ and $j \in \{0,\cdots,k\}$ we have

$$\begin{aligned}
|\Delta^M_\varrho\big(\varphi_j^N(D)f\big)(x)| &= \Big|\sum_{\ell=0}^{M}\binom{M}{\ell}(-1)^{M-\ell}\big(\varphi_j^N(D)f\big)(x+\ell\varrho)\Big| \\
&\le \Big|\sum_{\ell=0}^{M}\binom{M}{\ell}(-1)^{M-\ell}\sum_{|\beta|\le M-1}\frac{D^\beta\big(\varphi_j^N(D)f\big)(x)}{\beta!}\,(\ell\varrho)^\beta\Big| \\
&\quad +\Big|\sum_{\ell=0}^{M}\binom{M}{\ell}(-1)^{M-\ell}\sum_{|\alpha|=M}\frac{D^\alpha\big(\varphi_j^N(D)f\big)(x_{j,\varrho})}{\alpha!}\,(\ell\varrho)^\alpha\Big| \\
&\le c_1\,N_k^{-M}\sup_{|y-x|\le N_k^{-1}}\sum_{|\alpha|=M}\big|D^\alpha\big(\varphi_j^N(D)f\big)(y)\big| \\
&\le c_2\,N_j^M\,N_k^{-M}\sup_{|y-x|\le N_k^{-1}}\sup_{z\in\mathbb{R}^n}\frac{\big|\big(\varphi_j^N(D)f\big)(y-z)\big|}{1+|N_{j+1}z|^a} \\
&\le c_3\,\underline{N}_{k-j}^{-M}\,(\varphi_j^{N*}f)_a(x),
\end{aligned} \tag{16}$$

where $(\varphi_j^{N*}f)_a$ denotes the Peetre's maximal functions. Besides Taylor's expansion we have used the fact that $\sum_{\ell=0}^{M}\binom{M}{\ell}(-1)^{M-\ell}\ell^m = 0$ for any $m \in \mathbb{N}_0$ with $m < M$, and Theorem 1.3.1 of [19, p. 16] for the third inequality. Hence, for $j \in \mathbb{N}_0$,

$$\sup_{|\varrho| \le N_k^{-1}} \|\Delta_\varrho^M \big(\varphi_j^N(D) f\big) \mid L_p\| \le c \min(1, \underline{N}_{k-j}^{-M}) \, \|(\varphi_j^{N*} f)_a \mid L_p\| \tag{17}$$

where $c > 0$ is a constant independent of f.

Suppose that $0 < p \le 1$ (otherwise one has to modify the calculations below in an obvious way – similarly to the proof of the previous theorem), then, by (15) and (17), we get

$$\begin{aligned} \sup_{|\varrho| \le N_k^{-1}} \|\Delta_\varrho^M f \mid L_p\|^p &\le \sum_{j=0}^{\infty} \sup_{|\varrho| \le N_k^{-1}} \|\Delta_\varrho^M (\varphi_j^M(D) f) \mid L_p\|^p \\ &\le c \sum_{j=0}^{k} \underline{N}_{k-j}^{-Mp} \, \|(\varphi_j^{N*} f)_a \mid L_p\|^p + c \sum_{j=k+1}^{\infty} \|(\varphi_j^{N*} f)_a \mid L_p\|^p. \end{aligned} \tag{18}$$

With Λ as in Remark 4.2, it follows

$$\begin{aligned} \|f \mid B_{p,q}^{\sigma,N}\|^{(M)q} &\le c_1 \|f \mid L_p\|^q + c_1 \int_0^{N_0^{-1}} \Big(\Lambda(t^{-1}) \, \omega_M(f,t)_p\Big)^q \, \frac{\mathrm{d}t}{t} \\ &\le c_1 \|f \mid L_p\|^q + c_1 \sum_{k=0}^{\infty} \int_{N_{k+1}^{-1}}^{N_k^{-1}} \Big(\Lambda(t^{-1}) \, \omega_M(f,t)_p\Big)^q \, \frac{\mathrm{d}t}{t} \\ &\le c_1 \|f \mid L_p\|^q + c_2 \sum_{k=0}^{\infty} \sigma_k^q \, \omega_M(f, N_k^{-1})_p^q \\ &\le c_1 \|f \mid L_p\|^q + c_3 \sum_{k=0}^{\infty} \sigma_k^q \Big(\sum_{j=0}^{k} \underline{N}_{k-j}^{-Mp} \, \|(\varphi_j^{N*} f)_a \mid L_p\|^p\Big)^{q/p} \\ &\quad + c_3 \sum_{k=0}^{\infty} \sigma_k^q \Big(\sum_{j=k+1}^{\infty} \|(\varphi_j^{N*} f)_a \mid L_p\|^p\Big)^{q/p} \end{aligned} \tag{19}$$

(with the usual modification if $q=\infty$). By hypothesis $M > \overline{s}(\sigma)\,\underline{s}(N)^{-1}$, so that (4) entails $(\overline{\sigma}_j \underline{N}_j^{-M})_{j\in\mathbb{N}_0} \in \ell_u$, for any $u \in (0,\infty]$. Then we obtain

$$\sum_{k=0}^{\infty} \sigma_k^q \Big(\sum_{j=0}^{k} \underline{N}_{k-j}^{-Mp} \, \|(\varphi_j^{N*} f)_a \mid L_p\|^p \Big)^{q/p}$$
$$\leq c \, \|(\overline{\sigma}_j \underline{N}_j^{-M})_{j\in\mathbb{N}_0} \mid \ell_{\min(1,p,q)}\|^q \; \sum_{j=0}^{\infty} \sigma_j^q \, \|(\varphi_j^{N*} f)_a \mid L_p\|^q \tag{20}$$

and also

$$\sum_{k=0}^{\infty} \sigma_k^q \Big(\sum_{j=k+1}^{\infty} \|(\varphi_j^{N*} f)_a \mid L_p\|^p \Big)^{q/p}$$
$$\leq c \, \|(\underline{\sigma}_j^{-1})_{j\in\mathbb{N}_0} \mid \ell_{\min(1,p,q)}\|^q \; \sum_{j=0}^{\infty} \sigma_j^q \, \|(\varphi_j^{N*} f)_a \mid L_p\|^q. \tag{21}$$

We insert (20) and (21) in (19). Then (13) is a consequence of Theorem 4.3.4 in [3] and (14).
Step 2. Let $f \in B_{p,q}^{\sigma,N}$. By Theorem 3.1,

$$\|f \mid B_{p,q}^{\sigma,N}\| \leq c \, \|f \mid L_p\| + c \Big(\sum_{j=1}^{\infty} \sigma_j^q \; E_p(N_{j+k_0}, f)^q \Big)^{1/q} \tag{22}$$

for some positive constant c and with

$$E_p(b,f) := \inf \|f - g \mid L_p\|$$

where the infimum is taken over all $g \in \mathcal{S}' \cap L_p$ such that $\operatorname{supp} \widehat{g} \subset \{y : |y| \leq b\}$. As there is a positive constant c (which depends on p and k) such that

$$E_p(b,f) \leq c \sup_{|h| \leq b^{-1}} \|\Delta_h^M f \mid L_p\|$$

for all $b \geq 1$ and $f \in L_p$ – cf. Prop. 2.5.12 of [19, p. 109] –, from (22) we get

$$\|f \mid B_{p,q}^{\sigma,N}\| \leq c_1 \, \|f \mid L_p\| + c_2 \Big(\sum_{j=1}^{\infty} \sigma_j^q \; \omega^M(f, N_{j+k_0}^{-1})_p^q \Big)^{1/q} \leq c_3 \, \|f \mid B_{p,q}^{\sigma,N}\|^{(M)},$$

which completes the proof. ∎

Remark 4.3. For $N = (2^j)_{j\in\mathbb{N}_0}$ and $\sigma = (2^{sj})_{j\in\mathbb{N}_0}$ or $\sigma = (2^{sj}\Phi(2^{-j}))_{j\in\mathbb{N}_0}$ (cf. Examples 2.1(i),(ii)), the above theorem is covered by Theorem 2.5.12 of [19] and Theorem 2.5 of [11], respectively. For $1 < p < \infty$, $1 < q \leq \infty$ this type of characterization has been obtained by Kalyabin in [14].

References

[1] N. H. Bingham, C. M. Goldie, and J. L. Teugels, *Regular variation*, Cambridge Univ. Press, 1987.
[2] M. Bricchi, *Tailored function spaces and h-sets*, Ph.D. thesis, Univ. Jena, 2002.
[3] A. M. Caetano and W. Farkas, *Local growth envelopes of Besov spaces of generalized smoothness*, Z. Anal. Anwendungen, to appear.
[4] F. Cobos and D. Fernandez, *Hardy-Sobolev spaces and Besov spaces with a function parameter*, Lecture Notes in Math., vol. 1302, Springer, 1988, pp. 158–170.
[5] D. E. Edmunds and H. Triebel, *Function Spaces, Entropy Numbers and Differential Operators*, Cambridge Univ. Press, 1996.
[6] D. E. Edmunds and H. Triebel, *Spectral theory for isotropic fractal drums*, C. R. Acad. Sci. Paris **326(I)** (1998), 1269–1274.
[7] ———, *Eigenfrequencies of isotropic fractal drums*, Operator Theory: Advances and Applications **110** (1999), 81–102.
[8] W. Farkas and H.-G. Leopold, *Characterisations of function spaces of generalised smoothness*, Annali di Mat. Pura Appl. , to appear.
[9] M. L. Goldman, *A description of the traces of some function spaces*, Trudy Mat. Inst. Steklov **150** (1979), 99–127, (Russian) English transl.: Proc. Steklov Inst. Math. 1981, no. 4 (150).
[10] M. L. Gold'man, *A method of coverings for describing general spaces of Besov type*, Trudy Mat. Inst. Steklov **156** (1980), 47–81, (Russian) English transl.: Proc. Steklov Inst. Math. 1983, no. 2 (156).
[11] D. D. Haroske and S. D. Moura, *Continuity envelopes of spaces of generalised smoothness, entropy and approximation numbers*, J. Approx. Theory **128** (2004), no. 2, 151–174.
[12] G. A. Kalyabin and P. I. Lizorkin, *Spaces of generalized smoothness*, Math. Nachr. **133** (1987), 7–32.
[13] G. A. Kalyabin, *Characterization of spaces of generalized liouville differentiation.*, Mat. Sb. Nov. Ser. **104** (1977), 42–48, (Russian).
[14] ———, *Description of functions from classes of Besov-Lizorkin-Triebel type*, Tr. Mat. Inst. Steklova **156** (1980), 82–109, (Russian) English translation: Proc. Steklov Inst. Math. 1983, no. 2 (156).
[15] H.-G. Leopold, *Embeddings and entropy numbers in Besov spaces of generalized smoothness*, "Function Spaces: The fifth conference", Lecture notes in pure and applied math. **213**, (ed. H. Hudzig and L. Skrzypczak) Marcel Dekker, 2000, pp. 323–336.
[16] C. Merucci, *Applications of interpolation with a function parameter to Lorentz, Sobolev and Besov spaces*, Lecture Notes in Math., vol. 1070, Springer, 1984, pp. 183–201.
[17] S. Moura, *Function spaces of generalised smoothness*, Diss. Math. **398** (2001), 88 pp.
[18] ———, *Function spaces of generalised smoothness, entropy numbers, applications*, Ph.D. thesis, University of Coimbra, 2001.
[19] H. Triebel, *Theory of function spaces*, Birkhäuser, Basel, 1983.

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