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ON THE ALGEBRAIC REPRESENTATION OF SEMICONTINUITY

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ABSTRACT: The concepts of upper and lower semicontinuity in pointfree topology were introduced and first studied by Li and Wang in 1997. However Li and Wang's treatment does not faithfully reflect the original classical notion. In this note, we present algebraic descriptions of upper and lower semicontinuous real functions, in terms of frame homomorphisms, that suggest the right alternative to the definitions of Li and Wang. This fixes the discrepancy between the classical and the pointfree notions and turns out to be the appropriate notion that makes the Katětov-Tong Theorem provable in the pointfree context without any restrictions.

KEYWORDS: frame, locale, congruence lattice, biframe, congruence biframe, upper and lower semicontinuous real functions, frame of reals, upper and lower frames of reals.
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1. Introduction

Since the usual space $\mathbb{R}$ of real numbers is sober, continuous real functions $X \to \mathbb{R}$ on a space X are completely described by frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to \mathcal{O}X$, defined on the frame $\mathfrak{L}(\mathbb{R})$ of reals. Upper semicontinuous real functions (that is, continuous maps $X \to \mathbb{R}_l$, where $\mathbb{R}_l$ denotes the space $(\mathbb{R}, \mathcal{T}_l)$ of real numbers with the lower topology) and lower semicontinuous real functions (that is, continuous maps $X \to \mathbb{R}_u$, where $\mathbb{R}_u$ denotes the space $(\mathbb{R}, \mathcal{T}_u)$ of real numbers with the upper topology) are also important classes of continuous maps.

In the category of locales, the concepts of upper and lower semicontinuous real functions were introduced and first studied by Li and Wang [8]. However, Li and Wang's treatment does not faithfully reflect the original classical notion: an upper (resp. lower) semicontinuous real function on the frame $\mathcal{O}X$ of open sets of a space X does not necessarily describe an upper (resp. lower) semicontinuous real function on X. This explains the need to

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insert some assumption in the statements of the pointfree generalizations of some classical results dealing with semicontinuous real functions (cf. [10] and [3]).

Indeed, since the spaces $\mathbb{R}_l$ and $\mathbb{R}_u$ are not sober, upper and lower semicontinuous real functions on X are not represented by, respectively, frame homomorphisms $\mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$ and $\mathfrak{L}_u(\mathbb{R}) \to \mathcal{O}X$, defined on the *lower frame* $\mathfrak{L}_l(\mathbb{R})$ and the *upper frame* $\mathfrak{L}_u(\mathbb{R})$ of reals.

This raises the question whether there are nice algebraic descriptions of upper and lower semicontinuity, in terms of frame homomorphisms. In these notes we show that such descriptions do indeed exist. These descriptions lead us to the appropriate pointfree notions of semicontinuous real functions.

The crucial idea behind our approach is to take the bitopological point of view: spaces $\mathbb{R}_l$ and $\mathbb{R}_u$ are not sober but the bispace $(\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u)$ of reals is sober.

2. Background

Pointfree topology regards the points of a space as subsidiary to its open sets and deals with "lattices of open sets" abstractly defined as follows:

A *frame* (also *locale*) is a complete lattice L satisfying the infinite distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for every $x \in L$ and every $S \subseteq L$, and a *frame homomorphism* is a map $h : L \to M$ between frames which preserves the respective operations $\wedge$ (including the top element 1) and $\bigvee$ (including the bottom element 0). Frm is then the corresponding category of frames and their homomorphisms. For general information on frames and locales we refer to [5] and [9].

By the algebraic nature of frames, there is the notion of a *congruence* on a frame L, as an equivalence relation θ on L which is a subframe of $L \times L$ in the obvious sense, and the corresponding quotient frame L/θ is then defined just as quotients are always defined for algebraic systems, making the map $L \to L/\theta$ taking each $x \in L$ to its θ-block a frame homomorphism. The lattice of frame congruences on L under set inclusion is a frame, denoted by $\mathfrak{C}L$. This is the analogue, in the pointfree context, of the Skula modification of a topological space. A good presentation of the congruence frame is given by Frith [4]. Here, we shall need the following properties:

(1) For any $x \in L$, ∇_x and Δ_x are, respectively, the congruences defined by $\{(a,b) \in L \times L \mid a \vee x = b \vee x\}$ and $\{(a,b) \in L \times L \mid a \wedge x = b \wedge x\}$. The ∇_x are called *closed* and the Δ_x *open*. Each ∇_x is complemented in $\mathfrak{C}L$ with complement Δ_x. We use the symbol $\neg$ to denote complementation in $\mathfrak{C}L$.
(2) $\nabla L := \{\nabla_x \mid x \in L\}$ is a subframe of $\mathfrak{C}L$. Let ΔL denote the subframe of $\mathfrak{C}L$ generated by $\{\Delta_x \mid x \in L\}$. The triple $(\mathfrak{C}L, \nabla L, \Delta L)$ is a biframe. This is the analogue, for frames, of the *Salbany bitopological space* $(X, \mathcal{O}X, \mathcal{C}X)$, defined for every topological space $(X, \mathcal{O}X)$ (where $\mathcal{C}X$ denotes the topology on X generated by the closed sets of $(X, \mathcal{O}X)$).
(3) The correspondence $x \mapsto \nabla_x$ defines an isomorphism $L \to \nabla L$, whereas the map $x \mapsto \Delta_x$ is a dual poset embedding $L \to \Delta L$ taking finitary meets to finitary joins and arbitrary joins to arbitrary meets.

The fact that Frm is an algebraic category (in particular, one has free frames and quotient frames) also permits a procedure familiar from traditional algebra, namely, the definition of a frame by *generators and relations*: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u,v) for the given relations $u = v$. So, in the context of pointfree topology the reals may be introduced independent of any notion of real number, by defining the following suitable frame [6] (cf. [1]):

The *frame of reals* is the frame $\mathfrak{L}(\mathbb{R})$ generated by all ordered pairs (α, β) where $\alpha, \beta \in \mathbb{Q}$, subject to the relations

(R_1) $(\alpha,\beta) \wedge (\gamma,\delta) = (\alpha \vee \gamma, \beta \wedge \delta)$,
(R_2) $(\alpha,\beta) \vee (\gamma,\delta) = (\alpha,\delta)$ whenever $\alpha \leq \gamma < \beta \leq \delta$,
(R_3) $(\alpha,\beta) = \bigvee\{(\gamma,\delta) \mid \alpha < \gamma < \delta < \beta\}$,
(R_4) $1 = \bigvee\{(\alpha,\beta) \mid \alpha,\beta \in \mathbb{Q}\}$.

By the familiar adjoint situation between frames and topological spaces

$$\mathsf{Top} \underset{\Sigma}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{Frm} \tag{2.1}$$

we have a natural isomorphism

$$\mathsf{Frm}(L, \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \Sigma L). \tag{2.2}$$

For $L = \mathfrak{L}(\mathbb{R})$, since the spectrum $\Sigma\mathfrak{L}(\mathbb{R})$ is homeomorphic to the usual space $\mathbb{R}$ of reals ([1], Proposition 1), one obtains

$$\mathsf{Frm}(\mathfrak{L}(\mathbb{R}), \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \mathbb{R}).$$

This shows that continuous real functions on a space X may be represented as frame homomorphisms $h : \mathfrak{L}(\mathbb{R}) \to \mathcal{O}X$, and hence regarding the frame homomorphisms $\mathfrak{L}(\mathbb{R}) \to L$, for a general frame L, as the *continuous real functions* on L provides a natural extension of the classical notion (see [1] for a detailed account).

3. Semicontinuous real functions

Let $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ denote the subframes of $\mathfrak{L}(\mathbb{R})$ generated by, respectively, elements $(-,\alpha) := \bigvee_{\beta\in\mathbb{Q}}(\beta,\alpha)$ and $(\alpha,-) := \bigvee_{\beta\in\mathbb{Q}}(\alpha,\beta)$ $(\alpha \in \mathbb{Q})$. Note that $\mathfrak{L}_l(\mathbb{R}) \cong \mathcal{T}_l$ and $\mathfrak{L}_u(\mathbb{R}) \cong \mathcal{T}_u$. It should be also pointed out that, in [8], $\mathfrak{L}_l(\mathbb{R})$ and $\mathfrak{L}_u(\mathbb{R})$ are denoted by $\overline{\mathfrak{L}}_u(\mathbb{R})$ and $\overline{\mathfrak{L}}_l(\mathbb{R})$, respectively. Here we interchange, with respect to the notation used by Li and Wang (and also by the second author in [10]) the upper frame and the lower frame of reals, in order to be in accordance with the usual terminology for spaces. Li and Wang [8] defined upper (resp. lower) semicontinuous real functions on a frame L as frame homomorphisms $h : \mathfrak{L}_l(\mathbb{R}) \to L$ (resp. $h : \mathfrak{L}_u(\mathbb{R}) \to L$).

Since $\mathbb{R}_l$ is not sober, $\Sigma\mathfrak{L}_l(\mathbb{R}) \not\cong \mathbb{R}_l$. Indeed, besides the points $\xi_x : \mathfrak{L}_l(\mathbb{R}) \to 2$ $(x \in \mathbb{R})$, there is the point $\xi_{-\infty} : \mathfrak{L}_l(\mathbb{R}) \to 2$ given by $\xi_{-\infty}(-,\alpha) = 1$ for every $\alpha \in \mathbb{Q}$. (Recall that a point of a frame L is a frame homomorphism $\xi : L_0 \to 2$ where 2 denotes the two-element frame $\{0 < 1\}$). So, the spectrum $\Sigma\mathfrak{L}_l(\mathbb{R})$ of $\mathfrak{L}_l(\mathbb{R})$ is homeomorphic to the space $\mathbb{R}_{-\infty} = \mathbb{R} \cup \{-\infty\}$ with opens $[-\infty,\alpha)$ $(\alpha \in \mathbb{R})$. Of course, $\mathbb{R}_l$ being not sober, there is no frame L such that $\Sigma L \cong \mathbb{R}_l$; the frame $\mathfrak{L}_l(\mathbb{R})$, defined by Li and Wang, is the frame whose spectrum best approximates the space $\mathbb{R}_l$.

The following examples show that semicontinuous real functions on spatial frames do not necessarily represent semicontinuous real functions on the corresponding space:

Examples 3.1. (1) For any space X, the upper semicontinuous real function $h : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$, defined by $h(-,\alpha) = X$ for every $\alpha \in \mathbb{Q}$, corresponds to the continuous mapping $f : X \to \mathbb{R}_{-\infty}$ given by $f(x) = -\infty$ $(x \in X)$.
(2) For X being the usual space of reals $\mathbb{R}$, the upper semicontinuous real function $h : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$, defined by $h(-,\alpha) = (-e^{\alpha}, e^{\alpha})$ for every $\alpha \in \mathbb{Q}$,

corresponds to the continuous mapping $f : X \to \mathbb{R}_{-\infty}$ given by $f(x) = \log |x|$ if $x \neq 0$, $f(0) = -\infty$.

Recall that a space is *sober* if for each meet-irreducible $U \subseteq X$ there is exactly one $x \in X$ such that $U = X \setminus \overline{\{x\}}$. This condition is a conjunction of two requirements, namely the T_0 condition and the *weak soberness* condition [11] "for each meet-irreducible $U \subseteq X$ there exists an $x \in X$ such that $U = X \setminus \overline{\{x\}}$".

It is well known that sober spaces Y are characterized by the fact that continuous mappings $X \to Y$ are in a natural bijection with the frame homomorphisms $\mathcal{O}Y \to \mathcal{O}X$. More precisely [11]:

A space Y is T_0 if and only if for each frame homomorphism $h : \mathcal{O}Y \to \mathcal{O}X$ there is at **most** *one continuous map $f : X \to Y$ such that $h = \mathcal{O}(f)$.*

A space Y is weakly sober if and only if for each frame homomorphism $h : \mathcal{O}Y \to \mathcal{O}X$ there is at **least** *one continuous map $f : X \to Y$ such that $h = \mathcal{O}(f)$.*

Therefore, since $\mathbb{R}_u$ and $\mathbb{R}_l$ are both T_0 but not sober, the correspondences

$$X \xrightarrow{f} \mathbb{R}_u \in \mathsf{Top} \quad \overset{\mathcal{O}}{\rightsquigarrow} \quad \mathcal{O}\mathbb{R}_u \xrightarrow{\mathcal{O}(f)} \mathcal{O}X \in \mathsf{Frm} \tag{3.1}$$

and

$$X \xrightarrow{g} \mathbb{R}_l \in \mathsf{Top} \quad \overset{\mathcal{O}}{\rightsquigarrow} \quad \mathcal{O}\mathbb{R}_l \xrightarrow{\mathcal{O}(g)} \mathcal{O}X \in \mathsf{Frm} \tag{3.2}$$

are one-to-one but not onto.

Remark 3.2. For each $x \in X$, let $p_x : \mathcal{O}X \to 2$ be given by $p_x(U) = 1$ if and only if $x \in U$. It is obvious that, for each upper semicontinuous $f : X \to \mathbb{R}$ and for each $x \in X$, the set $\{\alpha \in \mathbb{Q} \mid p_x(\mathcal{O}(f)(-, \alpha)) = 1\}$ is bounded below.

Conversely, for each map $h : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$ let $\tilde{h} : X \to \mathbb{R}_{-\infty}$ be defined by

$$\tilde{h}(x) = \inf\{\alpha \in \mathbb{Q} \mid p_x(h(-, \alpha)) = 1\}, \tag{3.3}$$

where the inf means the infimum in $\mathbb{R} \cup \{-\infty\}$.

Proposition 3.3. *If h preserves arbitrary joins then:*

(1) $\tilde{h}^{-1}([-\infty, \alpha)) = h(-, \alpha)$.

(2) *If $\{\alpha \in \mathbb{Q} \mid p_x(h(-, \alpha)) = 1\}$ is bounded below for every $x \in X$, then $\tilde{h} : X \to \mathbb{R}$ and it is upper semicontinuous.*

Proof.

(1) The inclusion $\tilde{h}^{-1}([-\infty,\alpha)) \subseteq h(-,\alpha)$ is obvious. The reverse inclusion is also obvious since, by hypothesis, $h(-,\alpha) = h(\bigvee_{\beta<\alpha}(-,\beta)) = \bigcup_{\beta<\alpha} h(-,\beta)$.

(2) By (1), it remains to show that $\tilde{h}$ is bounded. Let $x \in X$. Of course $\tilde{h}(x) > -\infty$, because $\{\alpha \in \mathbb{Q} \mid p_x(h(-,\alpha)) = 1\}$ is bounded below. On the other hand, since $X = h(1) = h(\bigvee_{\alpha\in\mathbb{Q}}(-,\alpha)) = \bigcup_{\alpha\in\mathbb{Q}} h(-,\alpha)$, the set $\{\alpha \in \mathbb{Q} \mid x \in h(-,\alpha)\}$ is non-empty, that is, $\tilde{h}(x) < +\infty$, for every $x \in X$. $\square$

As an immediate consequence of Proposition 3.3 and Remark 3.2 we then have the following:

Corollary 3.4. *Upper semicontinuous mappings $f : X \to \mathbb{R}$ are in a bijective correspondence (via $\mathcal{O}$) with the frame homomorphisms $h : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$ such that $\{\alpha \in \mathbb{Q} \mid p_x(h(-,\alpha)) = 1\}$ is bounded below for every $x \in X$.* $\square$

Remark 3.5. If X is sober then the points of $\mathcal{O}X$ are precisely $\{p_x \mid x \in X\}$. In this case, the condition, in Corollary 3.4, that $\{\alpha \in \mathbb{Q} \mid p_x(h(-,\alpha)) = 1\}$ is bounded below for every $x \in X$ means that $\{\alpha \in \mathbb{Q} \mid p(h(-,\alpha)) = 1\}$ is bounded below for every point p of $\mathcal{O}X$.

Putting $L = \mathfrak{L}_l(\mathbb{R})$ in (2.2), we get $\mathsf{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X) \xrightarrow{\sim} \mathsf{Top}(X, \mathbb{R}_{-\infty})$. Since any upper semicontinuous mapping $f : X \to \mathbb{R}$ may be seen as a continuous map $f : X \to \mathbb{R}_{-\infty}$, we may embed $\mathsf{Top}(X, \mathbb{R}_l)$ in $\mathsf{Top}(X, \mathbb{R}_{-\infty})$.

In conclusion, we have

$$\begin{tikzcd}
\mathsf{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X) \arrow[rr, leftrightarrow, "\simeq"] & & \mathsf{Top}(X, \mathbb{R}_{-\infty}) \\
\mathsf{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X)_{\mathsf{b}} \arrow[u, "\subset"] \arrow[rr, leftrightarrow, "\simeq"] & & \mathsf{Top}(X, \mathbb{R}_l) \arrow[u, "\subset"']
\end{tikzcd}$$

where $\mathsf{Frm}(\mathfrak{L}_l(\mathbb{R}), \mathcal{O}X)_{\mathsf{b}}$ denotes the family of all frame homomorphisms $h : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X$ for which $\{\alpha \in \mathbb{Q} \mid p_x(h(-,\alpha)) = 1\}$ is bounded below for every $x \in X$. This shows why the definitions introduced by Li and Wang are more general than the classical ones.

4. The bitopological approach

Recall that a *bitopological space* [7] (briefly, *bispace*) is a triple $(X, \mathcal{T}_1, \mathcal{T}_2)$ in which X is a set and the $\mathcal{T}_i$ are topologies on X. A *bicontinuous map* $f : (X, \mathcal{T}_1, \mathcal{T}_2) \to (Y, \mathcal{U}_1, \mathcal{U}_2)$ is a map $f : X \to Y$ such that $f : (X, \mathcal{T}_i) \to (Y, \mathcal{U}_i)$ is continuous for $i = 1, 2$. The bispaces with these maps form the category BiTop.

Recall also that a *biframe* [2] is a triple (L_0, L_1, L_2) where L_1 and L_2 are subframes of the frame L_0, which together generate L_0. A *biframe homomorphism,* $f : (L_0, L_1, L_2) \longrightarrow (M_0, M_1, M_2)$, is a frame homomorphism $f : L_0 \longrightarrow M_0$ which maps L_i into $M_i\,(i = 1, 2)$ and BiFrm denotes the resulting category.

There is a contravariant functor $\mathcal{O} : \mathsf{BiTop} \to \mathsf{BiFrm}$ given as follows: for a bispace $(X, \mathcal{T}_1, \mathcal{T}_2)$, $\mathcal{O}((X, \mathcal{T}_1, \mathcal{T}_2)) = (\mathcal{T}_1 \vee \mathcal{T}_2, \mathcal{T}_1, \mathcal{T}_2)$, where $\mathcal{T}_1 \vee \mathcal{T}_2$ is the coarsest topology on X finer than $\mathcal{T}_1$ and $\mathcal{T}_2$, and $\mathcal{O}$ acts on a map f by taking f-preimages of open sets.

There is also the contravariant spectrum functor $\Sigma : \mathsf{BiFrm} \to \mathsf{BiTop}$ given as follows: for a biframe $L = (L_0, L_1, L_2)$, $\Sigma(L) = (\Sigma L_0, \{\Sigma_a : a \in L_1\}, \{\Sigma_b : b \in L_2\})$, where ΣL_0 is the set of all points of the frame L_0, and $\Sigma_x = \{\xi \in \Sigma L_0 \mid \xi(x) = 1\}$; for each biframe map $h : L \to M$, the bicontinuous map $\Sigma(h) : \Sigma(M) \to \Sigma(L)$ is defined by $\Sigma(h)(\xi) = \xi \circ h$. The functor Σ is a right adjoint to $\mathcal{O}$ [2]. The fixed objects in this dual adjunction are the sober bispaces and the spatial biframes, respectively.

Additional information concerning bispaces and biframes may be found in [2].

The following basic result suggests to look at semicontinuity from a bitopological point of view.

Proposition 4.1. *For each topological space $(X, \mathcal{O}X)$ and each $f : X \to \mathbb{R}$, the following are equivalent:*

(i) *f is upper semicontinuous.*
(ii) *The map $f : (X, \mathcal{O}X, \mathcal{C}X) \to (\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u)$ is bicontinuous.*

Proof. Let $f : X \to \mathbb{R}$ be upper semicontinuous. Then, of course, $f : (X, \mathcal{O}X) \to \mathbb{R}_l$ is continuous. On the other hand, $f^{-1}((\alpha, +\infty))$ is equal to

$$f^{-1}(\bigcup_{\beta>\alpha}[\beta, +\infty)) = \bigcup_{\beta>\alpha} f^{-1}([\beta, +\infty)) = \bigcup_{\beta>\alpha}(X \setminus f^{-1}((-\infty, \beta))) \in \mathcal{C}X,$$

thus $f : (X, \mathcal{C}X) \to \mathbb{R}_u$ is also continuous.

The converse is obvious. $\square$

Let us denote the bitopological spaces $(X, \mathcal{O}X, \mathcal{C}X)$ and $(\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u)$ briefly by, respectively, $S(X)$ and $\mathbb{R}$. The proposition above asserts that

$$\mathsf{Top}(X, \mathbb{R}_l) \simeq \mathsf{BiTop}(S(X), \mathbb{R}). \tag{4.1}$$

But, by the adjoint situation between biframes and bitopological spaces

$$\mathsf{BiTop} \underset{\Sigma}{\overset{\mathcal{O}}{\rightleftarrows}} \mathsf{BiFrm} \tag{4.2}$$

we have a natural isomorphism

$$\mathsf{BiTop}((X, \mathcal{T}_1, \mathcal{T}_2), \Sigma(L, L_1, L_2)) \xrightarrow{\sim} \mathsf{BiFrm}((L, L_1, L_2), \mathcal{O}(X, \mathcal{T}_1, \mathcal{T}_2)). \tag{4.3}$$

Combining this, for

$$(L, L_1, L_2) = (\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R}))$$

and

$$(X, \mathcal{T}_1, \mathcal{T}_2) = (X, \mathcal{O}X, \mathcal{C}X),$$

with the isomorphism $\Sigma(\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})) \simeq (\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u)$ (now the bispace $(\mathbb{R}, \mathcal{T}_l, \mathcal{T}_u)$ is sober), we obtain

$$\mathsf{BiTop}(S(X), \mathbb{R}) \xrightarrow{\sim} \mathsf{BiFrm}((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), \mathcal{O}S(X)). \tag{4.4}$$

On the other hand, $\mathcal{O}S(X) = (\mathcal{O}X \vee \mathcal{C}X, \mathcal{O}X, \mathcal{C}X)$ is isomorphic to the congruence biframe $(\mathfrak{C}(\mathcal{O}X), \nabla(\mathcal{O}X), \Delta(\mathcal{O}X))$ of the frame $\mathcal{O}X$ [4]. Hence

$$\mathsf{Top}(X, \mathbb{R}_l) \simeq \mathsf{BiFrm}((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), (\mathfrak{C}(\mathcal{O}X), \nabla(\mathcal{O}X), \Delta(\mathcal{O}X))). \tag{4.5}$$

But, for a general frame L, we have:

Proposition 4.2. *For any frame L, there is a bijection from*

$$\mathcal{A} := \mathsf{BiFrm}\Big((\mathfrak{L}(\mathbb{R}), \mathfrak{L}_l(\mathbb{R}), \mathfrak{L}_u(\mathbb{R})), (\mathfrak{C}L, \nabla L, \Delta L)\Big)$$

into

$$\mathcal{B} := \Big\{ f : \mathfrak{L}_l(\mathbb{R}) \to \nabla L \in \mathsf{Frm} \mid \bigvee_{\alpha \in \mathbb{Q}} \neg f(-, \alpha) = 1 \Big\},$$

given by $\Phi : h \longmapsto h_{|\mathfrak{L}_l(\mathbb{R})}$.

Proof. Let $h \in \mathcal{A}$. Then $\Phi(h) = h_{|\mathfrak{L}_l(\mathbb{R})} \in \mathcal{B}$. Indeed,

$$1 = h(\bigvee_{\alpha\in\mathbb{Q}}(\alpha,-)) = \bigvee_{\alpha\in\mathbb{Q}} h(\alpha,-) \leq \bigvee_{\alpha\in\mathbb{Q}} \neg\Phi(h)(-,\alpha),$$

since

$$h(\alpha,-)\wedge\Phi(h)(-,\alpha) = h(\alpha,-)\wedge h(-,\alpha) = h(0) = 0.$$

Conversely, consider $f \in \mathcal{B}$ and let $\Psi(f) : \mathfrak{L}(\mathbb{R}) \to \mathfrak{C}L$ be defined by

$$\Psi(f)(\alpha,\beta) = f(-,\beta)\wedge\bigvee_{\gamma>\alpha}\neg f(-,\gamma).$$

This is a frame homomorphism from $\mathfrak{L}(\mathbb{R})$ into $\mathfrak{C}L$, since it transforms the relations (R_1)-(R_4) into identities in $\mathfrak{C}L$:

(R_1)

$$\begin{aligned}
\Psi(f)(\alpha,\beta)\wedge\Psi(f)(\gamma,\delta) &= f(-,\beta)\wedge\Big(\bigvee_{\alpha'>\alpha}\neg f(-,\alpha')\Big)\wedge f(-,\delta)\wedge\Big(\bigvee_{\gamma'>\gamma}\neg f(-,\gamma')\Big)\\
&= f(-,\beta\wedge\delta)\wedge\bigvee_{\alpha'>\alpha,\gamma'>\gamma}(\neg f(-,\alpha')\wedge\neg f(-,\gamma'))\\
&= f(-,\beta\wedge\delta)\wedge\bigvee_{\alpha'>\alpha,\gamma'>\gamma}\neg f(-,\alpha'\vee\gamma')\\
&= f(-,\beta\wedge\delta)\wedge\bigvee_{\alpha'>\alpha\vee\gamma}\neg f(-,\alpha') = \Psi(f)(\alpha\vee\gamma,\beta\wedge\delta).
\end{aligned}$$

(R_2) Let $\alpha\leq\gamma<\beta\leq\delta$. Then

$$\begin{aligned}
\Psi(f)(\alpha,\beta)\vee\Psi(f)(\gamma,\delta) &= \Big(f(-,\beta)\wedge\bigvee_{\alpha'>\alpha}\neg f(-,\alpha')\Big)\vee\Big(f(-,\delta)\wedge\bigvee_{\gamma'>\gamma}\neg f(-,\gamma')\Big)\\
&= f(-,\delta)\wedge\Big(f(-,\delta)\vee\bigvee_{\alpha'>\alpha}\neg f(-,\alpha')\Big)\wedge\\
&\quad\wedge\Big(f(-,\beta)\vee\bigvee_{\gamma'>\gamma}\neg f(-,\gamma')\Big)\wedge\Big(\bigvee_{\alpha'>\alpha}\neg f(-,\alpha')\Big)\\
&= f(-,\delta)\wedge(\bigvee_{\alpha'>\alpha}\neg f(-,\alpha')) = \Psi(f)(\alpha,\delta)
\end{aligned}$$

since $f(-,\beta)\vee\bigvee_{\gamma'>\gamma}\neg f(-,\gamma')\geq f(-,\beta)\vee\neg f(-,\beta) = 1$.

$$(\mathrm{R}_3)\quad \bigvee_{\alpha<\gamma<\delta<\beta} \Psi(f)(\gamma,\delta) = \bigvee_{\alpha<\gamma<\delta<\beta} \Big(f(-,\delta) \wedge \bigvee_{\gamma'>\gamma} \neg f(-,\gamma')\Big)$$
$$= \bigvee_{\alpha<\gamma'<\delta<\beta} (f(-,\delta) \wedge \neg f(-,\gamma'))$$
$$= \bigvee_{\alpha<\gamma'<\beta} \bigvee_{\gamma'<\delta<\beta} (f(-,\delta) \wedge \neg f(-,\gamma'))$$
$$= \Big(\bigvee_{\alpha<\gamma'<\beta} \neg f(-,\gamma')\Big) \wedge \Big(\bigvee_{\delta<\beta} (f(-,\delta)\Big)$$
$$= \Big(\bigvee_{\alpha<\gamma'<\beta} \neg f(-,\gamma')\Big) \wedge f(-,\beta) = \Psi(f)(\alpha,\beta).$$

$$(\mathrm{R}_4)\quad \bigvee_{\alpha,\beta} \Psi(f)(\alpha,\beta) = \bigvee_{\alpha,\beta}\Big(f(-,\beta) \wedge \bigvee_{\gamma>\alpha} \neg f(-,\gamma)\Big)$$
$$= \bigvee_{\beta}\Big(f(-,\beta) \wedge \bigvee_{\alpha} \bigvee_{\gamma>\alpha} \neg f(-,\gamma)\Big).$$

But $\bigvee_\alpha \bigvee_{\gamma>\alpha} \neg f(-,\gamma) = \bigvee_\gamma \neg f(-,\gamma) = 1$, hence

$$\bigvee_{\alpha,\beta} \Psi(f)(\alpha,\beta) = \bigvee_{\beta} f(-,\beta) = f(\bigvee_{\beta}(-,\beta)) = f(1) = 1.$$

Further, $\Phi\Psi(f) = f$. Finally, $\Psi\Phi(h) = h$. In fact, for any $\alpha \in \mathbb{Q}$, $\Psi\Phi(h)(-,\alpha)$ is clearly equal to $h(-,\alpha)$ and $\Psi\Phi(h)(\alpha,-) = \bigvee_{\beta>\alpha} \neg h(-,\beta)$ is equal to $h(\alpha,-)$:

- $h(-,\beta)$ is complemented in $\mathfrak{C}L$ and $h(-,\beta) \vee h(\alpha,-) = 1$, thus $h(\alpha,-) \geq \neg h(-,\beta)$;
- $h(\alpha,-) = h(\bigvee_{\beta>\alpha}(\beta,-)) = \bigvee_{\beta>a} h(\beta,-)$; since $h(-,\beta) \wedge h(\beta,-) = 0$, then $h(\beta,-) \leq \neg h(-,\beta)$ and, consequently,
$$\bigvee_{\beta>a} h(\beta,-) \leq \bigvee_{\beta>\alpha} \neg h(-,\beta).$$
□

Since the correspondence $\nabla_a \mapsto a$ gives an isomorphism $\nabla L \cong L$, we may rewrite $\mathcal{B}$ as

$$\Big\{f : \mathfrak{L}_l(\mathbb{R}) \to L \in \mathsf{Frm} \mid \bigvee_{\alpha\in\mathbb{Q}} \Delta_{f(-,\alpha)} = 1\Big\}. \tag{4.6}$$

From (4.1), (4.4), (4.6) and Proposition 4.2, it follows immediately that:

Corollary 4.3. $\mathsf{Top}(X, \mathbb{R}_l) \simeq \{f : \mathfrak{L}_l(\mathbb{R}) \to \mathcal{O}X \in \mathsf{Frm} \mid \bigvee_{\alpha\in\mathbb{Q}} \Delta_{f(-,\alpha)} = 1\}$. □

Similarly,

$$\mathsf{Top}(X, \mathbb{R}_u) \simeq \{g : \mathfrak{L}_u(\mathbb{R}) \to \mathcal{O}X \in \mathsf{Frm} \mid \bigvee_{\alpha\in\mathbb{Q}} \Delta_{g(\alpha,-)} = 1\}.$$

Hence, the following are the right generalizations of the classical semicontinuous real functions, making them the natural substitute for the latter in the context of pointfree topology:

Definition 4.4.

(1) An *upper semicontinuous real function* on a frame L is a frame homomorphism $f : \mathfrak{L}_l(\mathbb{R}) \to L$ satisfying $\bigvee_{\alpha\in\mathbb{Q}} \Delta_{f(-,\alpha)} = 1$.

(2) A *lower semicontinuous real function* on a frame L is a frame homomorphism $g : \mathfrak{L}_u(\mathbb{R}) \to L$ satisfying $\bigvee_{\alpha\in\mathbb{Q}} \Delta_{g(\alpha,-)} = 1$.

Remarks 4.5. (1) In particular, $f : \mathfrak{L}_l(\mathbb{R}) \to 2$ is upper (resp. lower) semicontinuous if and only if f is a point of $\mathfrak{L}_l(\mathbb{R})$ (resp. $\mathfrak{L}_u(\mathbb{R})$) different from $\xi_{-\infty}$.

(2) For any upper semicontinuous f and any lower semicontinuous g, let $f \leq g$ if $f(-,\alpha) \vee g(\beta,-) = 1$ whenever $\beta < \alpha$, and let $g \leq f$ if $f(-,\alpha) \wedge g(\alpha,-) = 0$ for every $\alpha \in \mathbb{Q}$. By Proposition 2.2 of [10], any continuous real function $h : \mathfrak{L}(\mathbb{R}) \to L$ gives rise to an upper semicontinuous real function $f := h_{|\mathfrak{L}_l(\mathbb{R})}$ and a lower semicontinuous real function $g := h_{|\mathfrak{L}_u(\mathbb{R})}$, satisfying $f \leq g$ and $g \leq f$. It is easy to see that continuous real functions on L are completely represented by these pairs (f, g), with f upper semicontinuous and g lower semicontinuous, such that $f \leq g$ and $g \leq f$.

Under these definitions, the localic Katětov-Tong Theorem (Theorem 4.6 of [10]), as well as the results concerning the semicontinuous quasi-uniformity of a frame (cf. [3]), has now precisely the same formulation as in the classical context (for this recall that a *normal* frame is one in which $x \vee y = 1$ implies the existence of $a, b \in L$ such that $x \vee a = 1 = y \vee b$ and $a \wedge b = 0$):

> *A frame L is normal if and only if for every upper semicontinuous real function $f : \mathfrak{L}_l(\mathbb{R}) \to L$ and every lower semicontinuous real function $g : \mathfrak{L}_u(\mathbb{R}) \to L$ with $f \leq g$, there exists a continuous real function $h : \mathfrak{L}(\mathbb{R}) \to L$ such that $f \leq h \leq g$.*

This shows that the classical Katětov-Tong Theorem for normal spaces, which is known to be the most important result concerning semicontinuous real functions, is ultimately a result about normal frames, from which the classical version readily follows.

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