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KEYS ASSOCIATED WITH THE SYMMETRIC GROUP $\mathcal{S}_4$, FRANK WORDS AND MATRIX REALIZATIONS OF PAIRS OF TABLEAUX

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ABSTRACT: A variant of the dual RSK correspondence [10, 12] gives a bijection between classes of skew-tableaux and tableau-pairs of conjugate shapes. The problem of a matrix realization, over a local principal ideal domain with prime p, of the pair $(\mathcal{T}, \mathcal{K}(\sigma))$ with $\mathcal{K}(\sigma)$ a key associated with the permutation $\sigma \in \mathcal{S}_t$, and $\mathcal{T}$ a skew-tableau with the same evaluation as $\mathcal{K}(\sigma)$, is addressed. If $\mathcal{T}$ corresponds by this variant of the dual RSK to the tableau-pair $(\mathcal{P}, \mathcal{Q})$ of conjugate shapes, there exists a matrix realization for $(\mathcal{T}, \mathcal{K}(\sigma))$, $\sigma \in \mathcal{S}_t$, only if $\mathcal{P} = \mathcal{K}(\sigma)$ [2, 4, 5, 6]. This necessary condition has also been proved to be sufficient [7], by exhibiting an explicit matrix realization, in the case the frank word $\sigma\mathcal{Q}$ is a union of row words whose lengths define the conjugate shape of $\mathcal{Q}$. Here, we extend the matrix realization given in [7] to any tableau-pair $(\mathcal{K}(\sigma), \mathcal{Q})$ of conjugate shapes, with $\sigma \in \mathcal{S}_4$. This is carried out by *stretching* the frank words with shape $(2, 1, 1, 2)$ which are not the union of one row of length four with one of length two, those associated to $431421 \equiv \mathcal{K}(\epsilon, (1, 0, 1, 0))$, with $\epsilon \in \{1423, 1432, 4123, 4132\}$, to row words of length six associated with the key 654321 in $\mathcal{S}_6$.

KEYWORDS: Biwords, frank words, keys, dual Robinson-Schensted-Knuth correspondence, matrix realization, shuffle of words.
AMS SUBJECT CLASSIFICATION (2000): Primary 05E10, 68R15, 15A33; Secondary 05A17, 15A23.

1. Introduction

A variant of the dual RSK correspondence ([10], Appendix $A.4.3$) defines a bijection between a tableau-pair $(\mathcal{P}, \mathcal{Q})$, $\mathcal{P} \in [t]^*$, $\mathcal{Q} \in [n]^*$, of conjugate shapes and a class of skew-tableaux whose word is congruent with $\mathcal{P}$, and the word $u = u_t \cdots u_1$ with u_i the column word defined by the places of the letter i in $\mathcal{T}$, is congruent with $\mathcal{Q}$. We observe that french notation is used, see Section 2. Given $\sigma \in \mathcal{S}_t$, $t \geq 1$, a key $\mathcal{K}(\sigma)$ associated with $\sigma \in \mathcal{S}_t$ [9, 16], is a tableau whose columns are the t reordered left factors of σ with multiplicity

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$(l_t,\dots,l_1)$ assigned. A tableau-pair $(\mathcal{K}(\sigma),\mathcal{Q})$, $\sigma\in\mathcal{S}_t$, of conjugate shapes is in bijective correspondence with the class of skew-tableaux whose word is in the Knuth class of $\mathcal{K}(\sigma)$ and u is the frank word in $\mathcal{Q}$ whose column lengths are the σ permutation of the column lengths of $\mathcal{Q}$ in reverse order. Moreover, the frank word u is a union of rows with lengths given by the conjugate shape of $\mathcal{Q}$ iff the word of the skew-tableau is in the shuffle of the columns of $\mathcal{K}(\sigma)$. The Knuth class of a key $\mathcal{K}(\sigma)$, $\sigma\in\mathcal{S}_t$, contains the shuffle of their columns. In general, unless the permutation $\sigma\in\mathcal{S}_t$, $t\le 3$, or the permutation word $\sigma\in\mathcal{S}_t$, $t\ge 4$, satisfies certain conditions, we do not have equality. In $\mathcal{S}_4$, equality fails only for the permutations $\epsilon\in\{1423,1432,4123,4132\}$. The words congruent to the key $\mathcal{K}(\epsilon,(l_4,l_3,l_2,l_1))$ with $l_4,l_2>0$, either are in the shuffle of the columns or in the shuffle of the columns and 431421. Their associated frank words are therefore either the union of rows or the union of rows and frank words associated with 431421. The frank words associated with 431421 are those of shape $(2,1,1,2)$ which can not be written as a union of one row of length four with one of length two. However the entries of those frank words satisfy a row condition which allows us to stretch them to a row of six columns associated with the key 654321 in $\mathcal{S}_6$.

Given a pair $(\mathcal{P},\mathcal{Q})$ of tableaux of conjugate shapes which corresponds by the variant of the dual RSK to a skew-tableau $\mathcal{T}$, we consider the problem of a matrix realization, over a local principal ideal domain with prime p, of a pair $(\mathcal{T},\mathcal{F})$ with $\mathcal{F}$ a tableau with the same evaluation as $\mathcal{T}$ (Section 4, Definition 4.1). The set of tableaux in $[t]^*$ with evaluation $(m_1,\dots,m_t)$ is a rooted tree with root the unique row tableau of evaluation $(m_1,\dots,m_t)$ and where the key of evaluation $(m_1,\dots,m_t)$ is a leaf [13]. We focus our study in the case $\mathcal{F}$ is a key associated with $\sigma\in\mathcal{S}_t$. However, in example 4.1, a matrix realization is given when $\mathcal{F}$ is the root of the tree of the tableaux of evaluation (m,n), and, in this case, $\mathcal{P}$ is running over the tableaux in this tree. It has been shown in [5] that there exists a matrix realization for $(\mathcal{T},\mathcal{K}(\sigma))$ only if $\mathcal{P}=\mathcal{K}(\sigma)$. Equivalently, only if u is the frank word in $\mathcal{Q}$ whose column lengths are in the reverse order of the permutation σ of the column lengths of $\mathcal{Q}$. This necessary condition has also been proved to be sufficient [7], by exhibiting an explicit matrix realization, in the case the frank word u in the class of $\mathcal{Q}$ is a union of row words whose lengths define the conjugate shape of $\mathcal{Q}$. By stretching the frank words of shape $(2,1,1,2)$, associated with 431421, to a row word of length six associated with the key

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654321 in $\mathcal{S}_6$, we extend the matrix realization given in [7] to $(\mathcal{T}, \mathcal{K}(\sigma))$ with $\sigma \in \mathcal{S}_4$.

The paper is divided in five sections. In section 2, definition and properties of keys and frank words are given and a variant of the dual RSK correspondence for skew-tableaux is considered [10], Appendix $A.4.3$. In section 3, we study in detail the words congruent with keys associated with $\mathcal{S}_4$ and the frank words with four columns. Special attention is addressed to the words congruent with keys $\mathcal{K}(\epsilon)$, $\epsilon \in \{1423, 1432, 4123, 4132\}$ and their associated frank words. In section 4, regarding the correspondence between tableau-pairs of conjugate shape and skew-tableaux given by a variant of the dual RSK, the concept of matrix realization, over a local principal ideal domain with prime p, of a pair $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T}$ a skew-tableau and $\mathcal{F}$ a tableau with the same evaluation as $\mathcal{T}$, is discussed. A matrix realization for the pair $(\mathcal{T}, \mathcal{K}(\sigma))$, with $\sigma \in \mathcal{S}_4$, is exhibited, reducing its construction to the case the frank word $\sigma\mathcal{Q}$ is a union of rows whose lengths define the conjugate shape of $\mathcal{Q}$. In the last section, remarks and extensions of this matrix construction, in some special cases, are made for $\sigma \in \mathcal{S}_t$, $t \geq 5$.

2. Keys, frank words and a variant of the dual RSK correspondence

2.1. Keys and frank words. Let $\mathbb{N}$ be the set of positive integers with the usual order " $\leq$ ". Given $k, t \in \mathbb{N}$, $k \leq t$, $[k, t]$ denotes the set $\{k, \ldots, t\}$ in $\mathbb{N}$. When $k = 1$, we put $[t] := [1, t]$. We denote by $[t]^*$ the free monoid in the alphabet $[t]$ and by λ the empty word.

A partition is a sequence of nonnegative integers $a = (a_1, a_2, \ldots)$, all but a finite number of which are nonzero, such that $a_1 \geq a_2 \geq \cdots$ The maximum value of i for which $a_i > 0$ is called the *length* of a. If the length of a is zero, we have the null partition $a = (0, 0, \ldots)$. If $a_i = 0$, for $i > k$, we write $a = (a_1, \ldots, a_k)$ as well. Sometimes it is convenient to use the notation $a = (a_1^{m_1}, a_2^{m_2}, ..., a_k^{m_k})$, where $a_1 > a_2 > ... > a_k$ and $a_i^{m_i}$, with $m_i \geq 0$, means that a_i appears m_i times as a part of a. Thus, every partition can be written as $a = (t^{l_t}, \ldots, 2^{l_2}, 1^{l_1})$ for some $t \geq 1$ and nonnegative integers l_i, $1 \leq i \leq t$. Given the sequence $(l_t, \ldots, l_1)$ of nonnegative integers, we associate the partition $m = (\sum_{k=1}^{t} l_k, \ldots, l_{t-1} + l_t, l_t)$, the conjugate partition of $(t^{l_t}, \ldots, 2^{l_2}, 1^{l_1})$. Let $\mathcal{S}_t$ denote the symmetric group of degree $t \geq 1$. We define the action of $\sigma \in \mathcal{S}_t$ on the partition m by putting $\sigma m := (m_1, \ldots, m_t)$, where $m_{\sigma(i)} = \sum_{k=i}^{t} l_k$, $i = 1, \ldots, t$. We have $(t^{l_t}, \ldots, 2^{l_2}, 1^{l_1}) = \sum_{i=1}^{t} (1^{m_{\sigma(i)}})$. The *reverse* sequence

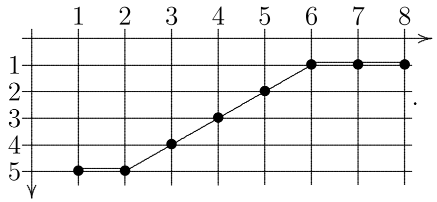
of $(m_1,\dots,m_t)$ is $(m_1,\dots,m_t)^{rev} := (rev\,\sigma)m = (m_t,\dots,m_1)$, where $rev = t\cdots 21$ denotes the reverse permutation in $\mathcal{S}_t$.

Given a word $w = x_1\cdots x_k$ over the alphabet $[t]$, we denote by $|w|_j$ the multiplicity of the letter $j\in[t]$ in w. Here k is the length of w, denoted by $|w|$. The sequence $(|w|_1,\dots,|w|_t)$ is called the *evaluation* of w, denoted by $ev(w)$. The length and evaluation of the empty word are zero. The word w, with $k\ge 1$, is said a *row* if $x_1\le\cdots\le x_k$, and a column if $x_1>\cdots>x_k$. If w is a column, in the planar representation, the letters are displayed in a column by decreasing order from top to bottom. For example, $\begin{matrix}5\\2\\1\end{matrix}$ is the planar representation of 521. Let V denote the set of all columns in $[t]^*$. Every word in $[t]^*$ has a unique factorization as a product of a minimal number of columns $w = v_1v_2\cdots v_r$, with $v_i\in V$. We call it the *column factorization* of w and denote it occasionally by $v_1\cdot v_2\cdot\ldots\cdot v_r$. The *shape* of w is the sequence $\|w\| = (|v_1|,\dots,|v_r|)$ of the lengths of the column factors v_i of w. For instance, $w = 43\cdot 32\cdot 21$ is the column factorization of w. Given the sequence of nonnegative integers $u=(u_1,\cdots,u_r)$, we define the word $uM := u_1\dots 1\cdot u_1+u_2\dots u_1+1\cdot\ldots\cdot u_r+\ldots+u_1\dots u_{r-1}+\ldots+u_1+1$ whose shape is the vector obtained by suppressing in u the null entries [16]. We identify u with the shape of uM. For instance, $(3,1,2)M = 321\cdot 4\cdot 65$.

The underlying set of a column defines a bijection $v\to\{v\}$ between the set V and the family $2^{[t]}$ of subsets of $[t]$. According to this bijection we often identify a column with its underlying set. This bijection allows to extend to V the order $\le$ on $2^{[t]}$ by letting $u\le v$ if and only if there is an injection $i:\{u\}\to\{v\}$, $x\le i(x)$. For instance, $52\le 542\le 6432$. In particular, if $\{u\}\subseteq\{v\}$ we have $u\le v$. We define another order $\triangleright$ on $2^{[t]}$, and extend it to V, putting $\{u\}\triangleright\{v\}$ if and only if there is an injection $i:\{v\}\to\{u\}$, $x\ge i(x)$ [16]. For instance, $5431\triangleright 542\triangleright 3$. A word $w = v_1\cdot v_2\cdot\ldots\cdot v_r$, $v_i\in V$, is called a *tableau* if $v_1\triangleright v_2\triangleright\cdots\triangleright v_r$. The shape of a tableau is, therefore, a partition. For instance,

$$5321\triangleright 41\triangleright 42\triangleright 4 = \begin{array}{llll}5&&&\\3&&&\\2&4&4&\\1&1&2&4\end{array}$$

Notice that $w = q_4$. Next, consider the biword $\overline{\Pi} = \overline{\Sigma} = \begin{pmatrix}11123455\\87654321\end{pmatrix}$, represented below, whose bottom row is $r_{id,8}$:



(28)

Applying algorithm 1 to $\overline{\Sigma}$, choosing the permutation $\sigma_1 = (15)\in\mathcal{S}_5$, we obtain the sequence $A_0, B_1,\dots,B_8$, where $A_0 = diag_p(\overline{a})$, with $\overline{a}$ a partition for which $\overline{\mathcal{T}} = (\overline{a},\overline{\Sigma})$ is a skew-tableau, $B_1 = P_{\sigma_1}D_{[1]}$, $B_2 = D_{[1]}$, $B_3 = T_{14}(1)T_{41}(-1)D_{[1]}$, $B_4 = T_{13}(1)T_{31}(-1)D_{[1]}$, $B_5 = T_{12}(1)T_{21}(-1)D_{[1]}$, $B_6 = T_{15}(1)T_{51}(-1)D_{[1]}$, and $B_7 = B_8 = D_{[1]}$.

By theorem 4.3, the sequence $A_0, B_1,\dots,B_8$ is a matrix realization for $(\overline{\mathcal{T}},\mathcal{K}(id,(1,0,0,\dots,0)))$. Let a be a partition such that $\mathcal{T} = (a,\Sigma)$ is a skew-tableau. The sequence $A_0', B_1',\dots,B_6'$, defined by $A_0' = diag_p(a)$, $B_1' = B_1$, $B_2' = D_{[2]}$, $B_3' = B_3$, $B_4' = T_{13}(1)T_{31}(-1)\;T_{12}(1)T_{21}(-1)\;T_{15}(1)\;T_{51}(-1)D_{[1]}$, $B_5' = D_{[1]}$ and $B_6' = D_{[2]}$, is a matrix realization for $(\mathcal{T},\mathcal{K}(\sigma,(1,0,0,0,1,0)))$.

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where $a_{i+1} < a_i$. Replacing the bottom row of Σ by $r_{id,t+2} \in \mathcal{S}_{t+2}$, we obtain the biword $\overline{\Sigma} = \begin{pmatrix} a_{t+2} & a_{t+1} & \cdots & a_{i+2} & a_{i+1} & a_i & a_{i-1} & \cdots & a_1 \\ t+2 & t+1 & \cdots & i+2 & i+1 & i & i-1 & \cdots & 1 \end{pmatrix}$, which clearly has no critical vertices.

Let $\overline{a}$ be a partition for which $\overline{\mathcal{T}} = (\overline{a}, \overline{\Sigma})$ is a skew-tableau, and let $A_0, B_1, \ldots, B_{t+2}$ be the sequence of matrices obtained applying algorithm 1 to $\overline{\Sigma}$, choosing the permutation $\sigma_1 = (1a_1)(2a_{i+1}) \in \mathcal{S}_n$ in step 1 of this algorithm. We have $A_0 = diag_p(\overline{a})$, $B_1 = P_{\sigma_1} D_{[1]}$, and $B_j = S_j D_{[1]}$, $j = 2, \ldots, t+2$. Also, $\sigma_j = (a_{j-1} a_j)\sigma_{j-1}$, for $j = 2, \ldots, t+2$. By theorem 4.3, this sequence is a matrix realization for the pair $(\overline{\mathcal{T}}, \mathcal{K}(id, (1, 0, \cdots, 0)))$.

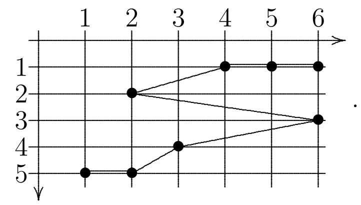
Let a be a partition such that $\mathcal{T} = (a, \Sigma)$ is a skew-tableau. Consider now the sequence

$$A'_0, B'_1, \cdots, B'_t, \tag{26}$$

where $A'_0 = diag_p(a)$, $B'_1 = P_{\sigma_1} D_{[1]}$, $B'_i = B_i$, for $i = 2, \ldots, k-1$, $B'_k = S_k D_{[2]}$, $B'_j = B_j$, for $j = k+1, \ldots, i-1$, $B'_i = S_i S_{i+1} S_{i+2} D_{[1]}$, $B'_j = B_{j+2}$, for $j = i+1, \ldots, k+l-1$, $B'_{k+l} = S_{k+l+2} D_{[2]}$, and $B'_j = B_{j+2}$, for $j = k+l+1, \ldots, t$.

We claim that the sequence $A'_0, B'_1, \cdots, B'_t$ is a matrix realization for the pair $(\mathcal{T}, \mathcal{K}(\sigma, (1, 0, \ldots, 0, 1, 0)))$. In fact, from the definition of σ_1 and since $A_0, B_1, \ldots, B_{t+2}$ is a matrix realization for $(\overline{\mathcal{T}}, \mathcal{K}(id, (1, 0, \cdots, 0)))$, we find that $A'_0 B'_1 \cdots B'_j \sim diag_p(a + \chi^{J_1} + \cdots + \chi^{J_j})$, for $j = 1, \ldots, t$. Moreover, using lemma 4.1, and following the proof of theorem 4.5, we find that (26) $\sim D_{[m_1]} \cdots D_{[m_t]}$. Therefore, (26) is a matrix realization for $[(\mathcal{P}, \mathcal{Q}); (\mathcal{K}(\sigma, (1, 0, \ldots, 0, 1, 0)), \mathcal{Y})]$. □

Example 5.1. Let $\sigma = \epsilon_1 \epsilon_2 \in S_6$, where ϵ_1 is a permutation word on $2, 6$, and ϵ_2 a permutation word on $1, 3, 4, 5$, and consider the key $\mathcal{K}(\sigma, (1, 0, 0, 0, 1, 0)) = 654321\ 62$, whose plactic class is the set $Sh(654321, 62) \cup \{q_5 = 65264321, q_4 = 65426321\}$. Let $\Pi = \Sigma = \begin{pmatrix} 11123455 \\ 65426321 \end{pmatrix}$, and $\Sigma' = \begin{pmatrix} 31\,1\,1\,4\,52\,5 \\ 66\,5\,4\,3\,22\,1 \end{pmatrix}$, represented by



. (27)

is a tableau of shape $(4, 2, 2, 1) = (4^1, 3^0, 2^2, 1^1)$. The conjugate partition $(4, 3, 1, 1)$ defines the length of the rows of the tableau.

The plactic or Knuth congruence $\equiv$ on words, over the alphabet $[t]$, [12, 13, 14], is obtained by means of Schensted's construction [19]. As usual $P(w)$ denotes the unique tableau congruent with $w \in [t]^*$ and $Q(w)$ the Q-symbol of w, the recording tableau of the row insertion of w in the Schensted's construction. The RS-correspondence $w \leftrightarrow (P(w), Q(w))$ is summarized as follows: each plactic class contains a unique tableau $\mathcal{P}$; and the elements of the plactic class of a tableau $\mathcal{P}$ are in bijection with the set of standard tableaux of the same shape as $\mathcal{P}$.

A tableau whose columns are pairwise comparable for the inclusion order is called a *key* [16]. That is, a tableau $u_r \cdot \ldots \cdot u_1$ is a key if $\{u_r\} \supseteq \cdots \supseteq \{u_2\} \supseteq \{u_1\}$. Equivalently, a key $\mathcal{K}$ is the tableau whose shape is the conjugate of its evaluation by nonincreasing order. This means that the key of evaluation $(m_1, \ldots, m_t)$ is the unique tableau such that $std(\mathcal{K})^T \equiv (ev(\mathcal{K}))M$, where "*std*" stands for standardization and "^{T}" for transposition. For instance,

$$\mathcal{K} = 5321\ 51\ 51\ 5 = \begin{matrix} 5 & & & \\ 3 & & & \\ 2 & 5 & 5 & \\ 1 & 1 & 1 & 5 \end{matrix} \tag{1}$$

is the key with evaluation $(3, 1, 1, 0, 4)$, the unique tableau with evaluation $(3, 1, 1, 0, 4)$ such that $std(\mathcal{K})^T \equiv 321 \cdot 4 \cdot 5 \cdot 9876$.

Keys are also tableaux whose columns are the left reordered factors of a permutation with multiplicity assigned. For each pair consisting of a permutation $\sigma \in \mathcal{S}_t$, written as a word $\sigma = a_1 \cdots a_t \in [t]^*$, and a sequence of nonnegative integers $(l_t, \ldots, l_1)$, Ehresmann [9] associated a key, here denoted by $\mathcal{K}(\sigma, (l_t, \ldots, l_1))$, putting

$$\mathcal{K}(\sigma, (l_t, \ldots, l_1)) := (r_{\sigma,t})^{l_t} (r_{\sigma,t-1})^{l_{t-1}} \cdots (r_{\sigma,1})^{l_1},$$

where $r_{\sigma,k}$ is the column with underlying set $\{a_1, \ldots, a_k\}$, $1 \le k \le t$. This key is the tableau with shape $(t^{l_t}, \ldots, 2^{l_2}, 1^{l_1})$ and evaluation σm. When $\sigma = id$, $\mathcal{K}(id, (l_t, \ldots, l_1))$ is said the Yamanouchi tableau of evaluation m, that is, the tableau whose shape is the conjugate of the evaluation. The congruent words are called Yamanouchi words of evaluation m.

A word $w \in [t]^*$ is said *frank* [16], [10], Appendix A.5, if its shape is a permutation of the shape of $P(w)$. The following theorem, proved by Lascoux

and Schützenberger in [16], shows that the frank words, in a plactic class, are in bijection with the set of permutations of the shape of the tableau in that class. Frank words are completely determined by the conditions imposed on their Q symbols.

Theorem 2.1. *Let $\mathcal{Q}$ be a tableau with shape m. For each permutation $\sigma \in \mathcal{S}_t$, there exists one and only one word $\sigma\mathcal{Q} \equiv \mathcal{Q}$ with shape σm. $\sigma\mathcal{Q}$ is such that the Q-symbol is $\equiv (\sigma m)M$.*

Keys and frank words are therefore related as follows.

Corollary 2.2. *The frank words J' with shape σm are those whose transposition of the Q-symbol is the standardization of the key $\mathcal{K}$ with evaluation σm, that is, $Q(J') = std(\mathcal{K})^T$. The frank words J of shape $(\sigma m)^{rev}$ are those such that $Q(J) = evac\, Q(J') = std(\mathcal{K}')^T$, with $\mathcal{K}' = evac\, \mathcal{K}$ the key of evaluation $(\sigma m)^{rev}$, where "evac" stands for the evacuation operation.*

A *skew-tableau* $\mathcal{T}$ in $[t]^*$ [15] is a tableau on the alphabet $[t] \cup \{\varnothing\}$, where the extra letter $\varnothing$ is such that $\varnothing < \varnothing < 1 < 2 < \cdots < t$. The word $w(\mathcal{T})$ of the skew-tableau $\mathcal{T}$ is the word in $[t]^*$ obtained by eliminating from $\mathcal{T}$ the extra letter $\varnothing$, and the evaluation of $\mathcal{T}$ is the evaluation of $w(\mathcal{T})$.

Let a be the partition defined by the number of letters $\varnothing$ in each column of $\mathcal{T}$. Then, if c is the shape of $\mathcal{T}$, c/a, called the *skew-shape* of $\mathcal{T}$, denotes the sequence of number of letters of $w(\mathcal{T})$ in each column of $\mathcal{T}$, from left to right. In particular, a tableau in $[t]^*$ is a skew-tableau with $a = 0$. For example, $\mathcal{T} = 43\varnothing\varnothing\varnothing\varnothing \cdot 43\varnothing\varnothing\varnothing\varnothing \cdot 421\varnothing\varnothing \cdot 1\varnothing\varnothing \cdot 4\varnothing\varnothing \cdot 21$ is a skew-tableau of skew-shape $(6,6,5,3,3,2)/(4,4,2,2,2,0) = (2,2,3,1,1,2)$, and its planar representation is

$$\begin{array}{cccccc} 4 & 4 & & & & \\ 3 & 3 & 4 & & & \\ \varnothing & \varnothing & 2 & & & \\ \varnothing & \varnothing & 1 & 1 & 4 & \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & 2 \\ \varnothing & \varnothing & \varnothing & \varnothing & \varnothing & 1 \end{array}. \tag{2}$$

A skew-tableau with word $w_1 \cdots w_n$ and skew-shape $(|w_1|, \ldots, |w_n|)$ is in the *compact form* if the inner shape $a = (\sum_{i=2}^{n} |w_i| - |v_i|, \ldots, |w_n| - |v_n|, 0)$ where v_i is the left factor of w_i of maximal length satisfying $w_{i-1} \rhd v_i$. Using

5. Final remarks

For $t > 4$, the number of words needed to add to the columns of a key in order to generate, by shuffling operations, the plactic class of that key depends, in general, on the multiplicities of their columns. For instance, in $\mathcal{S}_5$, by theorem 3.1, the Knuth class of $\mathcal{K}(15234, (1,0,0,1,0)) = 54321\,51$ is larger than $Sh(54321, 51)$, but is described, in terms of shuffling, by adding two words, 5431521 and 5415321, to the their columns. But if we consider the key $\mathcal{K}(15234, (2,0,0,1,0)) = 54321\,54321\,51$, we need to add seventeen new words to the set of columns of that key in order to describe its Knuth class in terms of shuffling. For some particular cases, however, these words have a "similar" behaviour to the word 431421, and we may use algorithm 1 to give a matrix realization for $[(\mathcal{K}, \mathcal{Q}); (\mathcal{K}, \mathcal{Y})]$, two pairs of tableaux of conjugate shapes as before such that $\mathcal{K}$ is a key. We illustrate this with a simple example.

Let $k, k+l$, $l \geq 3$, be integers in $[t]$, $t \geq 4$, and let $\sigma = k\, k+l\, \epsilon_2$ or $k+l\, k\, \epsilon_2 \in S_t$, written in the word notation, where ϵ_2 denotes a permutation word on the alphabet $[t] \setminus \{k, k+l\}$. Then, $r_{\sigma,2} = k+l\, k$. We consider the key

$$\mathcal{K}(\sigma, (1,0,\ldots,0,1,0)) = r_{\sigma,t} r_{\sigma,2}.$$

The analysis of the effect of a Knuth transformation on a shuffle of $r_{\sigma,t}$ and $r_{\sigma,2}$ shows that the plactic class of $\mathcal{K}(\sigma, (1,0,\ldots,0,1,0))$ is the set

$$Sh(r_{\sigma,t}, r_{\sigma,2}) \cup \{q_i : i = k+l-1, \ldots, k+2\},$$

where $q_i := t\, t-1 \cdots i\, k\, k+l\, i-1 \cdots 2\, 1$. For example, when $t = 4$, $k = 1$ and $k+l = 4$, we have $q_3 = 431421$, and the plactic class of $\mathcal{K}(\sigma, (1,0,1,0))$ is the set $Sh(r_{\sigma,t}, r_{\sigma,2}) \cup \{q_3\}$ (see theorem 3.2).

Theorem 5.1. *The pair $[(\mathcal{P}, \mathcal{Q}); (\mathcal{K}(\sigma, (1,0,\ldots,0,1,0)), \mathcal{Y})]$ is admissible if and only if $\mathcal{P} = \mathcal{K}(\sigma, (1,0,\ldots,0,1,0))$.*

Proof: Let $\Sigma = \begin{pmatrix} \mathcal{Q} \uparrow \\ w \end{pmatrix}$ corresponds by RSK* to $(\mathcal{P}, \mathcal{Q})$. As we have already seen, the condition is necessary [5], and is also sufficient when w is a shuffle of $r_{\sigma,t}$ and $r_{\sigma,2}$ [7]. So, assume $w = q_i$, for some $i = k+l-1, \ldots, k+2$, and

$$\Sigma = \begin{pmatrix} a_{t+2} & a_{t+1} & \cdots & a_{i+2} & a_{i+1} & a_i & a_{i-1} & \cdots & a_1 \\ t & t-1 & \cdots & i & k & k+l & i-1 & \cdots & 1 \end{pmatrix},$$

Following step 1 of algorithm 1, we may consider $\sigma_1 = (16)(23) \in \mathcal{S}_6$, since $s^0_{(6,1)} = 6, s^0_{(3,1)} = 3, s^0_{(3,6)} = 2$, and $(4,4)$ is a vertex of the shuffle component $\begin{pmatrix} 114566 \\ 654321 \end{pmatrix}$. Next, define the matrices

$$\begin{aligned}
I + S^2_{3,3} &= I + S^2_{6,6} = I, \\
I + S^3_{3,2} &= I + S(3,2,\sigma_1) = T_{23}(1), \\
I + S^3_{6,5} &= I + S(6,5,(3\,2)\sigma_1) = T_{15}(1), \\
I + S^4_{5,4} &= I + S(5,4,(6\,5)(3\,2)\sigma_1) = T_{14}(1), \\
I + S^5_{4,1} &= I + S(4,1,(5\,4)(6\,5)(3\,2)\sigma_1) = T_{16}(1), \\
I + S^6_{1,1} &= I + S^2_{2,2} = I,
\end{aligned}$$

and the permutations $\sigma_2 := \sigma_1$, $\sigma_3 := (6\,5)(3\,2)\sigma_2$, $\sigma_4 := (5\,4)\sigma_3$, $\sigma_5 := (4\,1)\sigma_4$, and $\sigma_6 := \sigma_5$. Finally, define $A_0 = diag_p(\overline{a})$, $B_1 = P_{\sigma_1} D_{[2]}$, $B_2 = D_{[2]}$, $B_3 = T_{24}(1)T_{42}(-1)T_{15}(1)\, T_{51}(-1)D_{[1]}$, $B_4 = T_{14}(1)T_{41}(-1)D_{[1]}$, $B_5 = T_{16}(1)T_{61}(-1)D_{[2]}$, and $B_6 = D_{[3]}$.

Then, by theorem 4.3, the sequence $A_0, B_1, \ldots, B_6$ is a matrix realization for the pair $(\overline{\mathcal{T}}, \mathcal{K}(\overline{\sigma}, (1,0,1,0,0,1)))$. Consider now the sequence A_0, B_1', B_2', B_3', B_4', where $A_0 = diag_p(a)$, with a a partition such that (a, Σ) is a skew tableau, $B_1' = P_\sigma D_{\{1,2,4\}}$, $B_2' = B_2$,

$$B_3' = T_{24}(1)T_{42}(-1)T_{15}(1)T_{51}(-1)T_{14}(1)T_{41}(-1)T_{16}(1)T_{61}(-1)D_{[2]},$$

and $B_4' = D_{[4]}$.

By the definition, $\sigma_1(\{1,2,4\}) = J_1$, thus $A_0 B_1' \sim diag_p(a + \chi^{J_1})$. Also, we have $A_0 B_1' B_2' \sim_L diag_p(a^0 + \chi^{J_1} + \chi^{J_2})$, since $A_0 B_1 B_2 \sim_L diag_p(a^0 + \chi^{\overline{J}_1}) P_{\sigma_2} D_{[2]} \sim_R diag_p(a^0 + \chi^{\overline{J}_1} + \chi^{\overline{J}_2})$ and $\overline{J}_2 = J_2$. In a similar way, we find that $A_0 B_1' B_2' B_3' \sim diag_p(a^0 + \chi^{J_1} + \chi^{J_2} + \chi^{J_3})$ and $A_0 B_1' B_2' B_3' B_4' \sim diag_p(a^0 + \chi^{J_1} + \chi^{J_2} + \chi^{J_3} + \chi^{J_4})$. Therefore, the sequence $A_0, B_1', B_2', B_3', B_4'$ satisfy conditions I and II of definition 4.2. It remains to show that it also satisfy condition III. So, bearing in mind lemma 4.1, we may write

$$\begin{aligned}
B_1' B_2' B_3' B_4' &\sim_L D_{\{1,2,4\}} D_{[2]} T_{42}(-1) T_{51}(-1) T_{54}(-1) T_{41}(-1) T_{61}(-1) D_{[2]} D_{[4]} \\
&\sim_R D_{\{1,2,4\}} D_{[2]} D_{[2]} D_{[4]} \sim D_{[3]} D_{[2]} D_{[2]} D_{[4]}.
\end{aligned}$$

Therefore, the sequence $A_0, B_1', B_2', B_3', B_4'$ is a matrix realization for $[(P(w), P(J_4 J_3 J_2 J_1)), (P(w), \mathcal{Y})]$.

jeu de taquin [10, 20] in consecutive columns from right to left, every skew-tableau can be put in the compact form. We identify skew-tableaux having the same compact form. The skew-tableau (2) is in the compact form.

2.2. A variant of the dual RSK-correspondence. Regarding the matrix problem addressed in the Introduction, we consider a variant of the dual Robinson-Schensted-Knuth correspondence [12], [10], Appendix A.4.3, to establish a bijection between tableau-pairs $(\mathcal{P}, \mathcal{Q})$ of conjugate shapes and skew-tableaux in the compact form.

Let $\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_1 & \cdots & u_k \\ v_1 & \cdots & v_k \end{pmatrix}$ be a biword with no repeated biletters, where $u_1, \ldots, u_k \in [n]$ and $v_1, \ldots, v_k \in [t]$. Sorting the biletters of $\begin{pmatrix} u \\ v \end{pmatrix}$ by nondecreasing rearrangement with respect to the anti-lexicographic order with priority on the first row, we get

$$\Sigma = \begin{pmatrix} 1^{f_1} & \cdots & n^{f_n} \\ w_1 & \cdots & w_n \end{pmatrix}, \tag{3}$$

where $ev(u) = (|w_1|, \ldots, |w_n|)$ and $w = w_1 \cdots w_n \in [t]^*$, $w_i \in V \cup \{\lambda\}$; and by nonincreasing rearrangement of the biletters of $\begin{pmatrix} u \\ v \end{pmatrix}$ for the lexicographic order with priority on the second row, we get

$$\Sigma' = \begin{pmatrix} J_t & \cdots & J_1 \\ t^{m_t} & \cdots & 1^{m_1} \end{pmatrix}, \tag{4}$$

where $ev(v) = (|J_1|, \ldots, |J_t|)$ and $J = J_t \cdots J_1 \in [n]^*$, $J_i \in V \cup \{\lambda\}$.

Consider the transformation $\Sigma \leftrightarrow \Sigma'$ defined by sorting the biletters of Σ in nonincreasing rearrangement with respect to the lexicographic order with priority on the second row, and by sorting the biletters of Σ' in nondecreasing rearrangement with respect to the anti-lexicographic order with priority on the first row. ¿From Greene's theorem, we have

Lemma 2.3. (*a*) *The transformation $\Sigma \leftrightarrow \Sigma'$ establishes a bijective correspondence between the k-tuples of disjoint nondecreasing subwords of $J = J_t \cdots J_1$ and those of decreasing subwords of $w = w_1 \cdots w_n$.*

(*b*) *The tableaux $P(w)$ and $P(J)$ have conjugate shapes with $(ev(w))^{rev} = (|J_t|, \ldots, |J_1|)$ and $ev(J) = (|w_1|, \ldots, |w_n|)$.*

A biword $\begin{pmatrix} u \\ v \end{pmatrix}$ without repeated biletters determines a unique pair of biwords $\Sigma = \begin{pmatrix} u\uparrow \\ w \end{pmatrix}$, $\Sigma' = \begin{pmatrix} J \\ v\downarrow \end{pmatrix}$ defined as above (we write $v\downarrow$ ($u\uparrow$) for v by nonincreasing (for u by nondecreasing) order). Two biwords are said equivalent if they consist of the same biletters. We consider the variant of the dual RSK-correspondence, here denoted by RSK*, [10], Appendix A.4.3, for an arbitrary biword without repeated billeters by

$$\begin{pmatrix} u \\ v \end{pmatrix} \stackrel{\text{RSK}^*}{\longrightarrow} (\mathcal{P}, \mathcal{Q}),$$

where $\mathcal{P} = P(w)$ and $\mathcal{Q} = P(J)$. This pair of tableaux is related as follows.

Theorem 2.4. *Let* $\Sigma = \begin{pmatrix} u\uparrow \\ w \end{pmatrix}$ *and* $\Sigma' = \begin{pmatrix} J \\ v\downarrow \end{pmatrix}$ *as before.*

(a) $P(J)$ is the unique tableau of evaluation $(|w_1|, \ldots, |w_n|)$ such that $Q(w) = std(P(J))^T$.

(b) $P(w)$ is the unique tableau of evaluation $(|J_1|, \ldots, |J_t|)$ such that $Q(J) = std(evac\, P(w))^T$, where evac stands for the evacuation operation.

The tableau-pair $(\mathcal{K}, \mathcal{Q})$ of conjugate shapes with $\mathcal{K}$ a key and the frank words with t columns are characterized as follows.

Theorem 2.5. *Let $(\mathcal{K}, \mathcal{Q})$, with $\mathcal{K} \in [t]^*$, $\mathcal{Q} \in [n]^*$, be a pair of tableaux of conjugate shapes such that $ev(\mathcal{K}) = \sigma m$ and $\sigma \in \mathcal{S}_t$. Let* $\begin{pmatrix} \mathcal{Q}\uparrow \\ w \end{pmatrix}$, $\begin{pmatrix} J \\ \mathcal{K}\downarrow \end{pmatrix}$ *correspond by RSK* to the pair $(\mathcal{K}, \mathcal{Q})$. The following statements are equivalent*

(a) $\mathcal{K}$ is the key associated with σ and $(l_t, \ldots, l_1)$.

(b) J is the frank word of shape $(\sigma m)^{rev}$ in the class of $\mathcal{Q}$.

(c) $std(\mathcal{K})^T \equiv [ev(\mathcal{K})]M \equiv Q(J') = evac\, Q(J)$, where J' is the frank word of shape σm in the class of $\mathcal{Q}$.

J is called a frank word associated with w.

The frank words J with t columns and shape $(\sigma m)^{rev}$ are those associated with some $w \equiv \mathcal{K}(\sigma, (l_t, \ldots, l_1))$ with $l_t > 0$.

We give now an interpretation of the RSK* correspondence in terms of compact skew-tableaux. Given a word $w \in [t]^*$ with evaluation $(m_1, \ldots, m_t)$, let $\mathcal{T}$ be a skew-tableau in the compact form with word w and skew-shape

for some $\nu_l \geq 0$. Notice that we may eliminate by left equivalence any upper triangular matrix $T_{ij}(p^\nu)$, $\nu > 0$, using lemma 4.1. This operation may create new elementary matrices $T_{uv}(p^{\nu'})$, but, as we have mentioned, using lemma 4.1, we may assume without loss of generality that this is not the case. Thus, we may write

$$(24) \sim_L D_{[\Gamma_1]} \prod_{l=2}^{s_1} T_{j_l i_l}(-1) D_{[m_2]} C_3 D_3 \prod_{l=1}^{s_3} T_{j_l i_l}(-1) D_{[m_3]} \prod_{l=1}^{s_4} T_{j_l i_l}(-1) D_{[\Gamma_4]}. \tag{25}$$

Use again lemma 4.1 to eliminate by left equivalence all upper triangular matrices present in C_3. New elementary matrices may be created. Among these ones, eliminate by left equivalence all those which are upper triangular. Finally, starting from right and moving to left, eliminate by right equivalence all lower triangular matrices left in the product. It is now clear that $B_1 \cdots B_4$ is equivalent to the diagonal matrix $D_{[\Gamma_1]} D_{[m_2]} D_{[m_3]} D_{[\Gamma_4]}$. Since $m_k \leq n_5 + n_4 + n_3 + n_2$, $k = 2, 3$, and $\Gamma_1 \subseteq \Gamma_4$ or $\Gamma_4 \subseteq \Gamma_1$, we find that this last diagonal matrix is equivalent to $D_{[m_1]} \cdots D_{[m_4]}$. $\square$

Attending to the necessary condition for the admissibility of a pair $[(\mathcal{P}, \mathcal{Q}); (\mathcal{K}, \mathcal{Y})]$ with $\mathcal{K}$ a key associated with a permutation $\mathcal{S}_4$ [5], and to corollary 3.2, theorem 4.3 and the main theorem above, we have the following characterization for the admissibility of two pairs of tableaux $[(\mathcal{P}, \mathcal{Q}); (\mathcal{K}, \mathcal{Y})]$ with $\mathcal{K}$ a key associated with a permutation $\mathcal{S}_4$.

Theorem 4.6. *Let $[(\mathcal{P}, \mathcal{Q}); (\mathcal{K}, \mathcal{Y})]$ be two pairs of tableaux of conjugate shapes with $ev(\mathcal{P}) = ev(\mathcal{K})$ such that $\mathcal{K}$ is a key associated with a permutation in $\mathcal{S}_4$, and $\mathcal{Y}$ a Yamanouchi tableau. Then, $[(\mathcal{P}, \mathcal{Q}); (\mathcal{K}, \mathcal{Y})]$ is admissible if and only if $\mathcal{P} = \mathcal{K}$.*

Example 4.2. Consider again the biwords Σ and Σ' (7), and recall that the bottom word of Σ is a shuffle of $Sh(\widehat{r}_5, r_{\sigma,4}, r_{\sigma,1})$, with $\sigma = 4123$. Take the biword $\Pi' = \begin{pmatrix} 114566 & 2233 & 3 \\ 431421 & 4321 & 4 \end{pmatrix}$ in examples 3.3, 3.5 and using the map ρ, consider the correspondent biword

$$\overline{\Pi'} = \begin{pmatrix} 114566 & 2233 & 3 \\ 654321 & 6521 & 6 \end{pmatrix},$$

whose graphical representation is exhibited in example 3.5. Let $\overline{a}$ be a partition such that $\overline{\mathcal{T}} = (\overline{a}, \overline{\Sigma})$ is a skew-tableau.

means that

$$\sigma_1([m_1-n_5]) = J_1 \setminus \{c^i : i \in [n_5]\},\ \sigma_2([m_2]) = J_2, \tag{21}$$

$$\sigma_3([n_5]) = \{d^i : i \in [n_5]\},\ \sigma_4([n_5]) = \{c^i : i \in [n_5]\}, \tag{22}$$

$$\sigma_5([m_3]) = J_3,\ \text{and}\ \ \sigma_6([m_4-n_5]) = J_4 \setminus \{d^i : i \in [n_5]\}. \tag{23}$$

Consider the sequence of $n \times n$ nonsingular matrices

$$A'_0, B'_1, B'_2, B'_3, B'_4,$$

defined by $A'_0 = diag_p(a)$, $B'_1 := P_{\sigma_1} D_{\Gamma_1}$, $B'_2 := B_2$, $B'_3 := S_3 S_4 S_5 D_{[m_3]}$, and $B'_4 := S_6 D_{\Gamma_4}$, where $\Gamma_1 = [m_1], \Gamma_4 = [m_4-n_5] \cup \{m_4+n_1+1, \ldots, m_4+n_1+n_5\}$ if $\sigma \in \{1423, 1432\}$, and $\Gamma_1 = [m_1-n_5] \cup \{m_1+n_1+1, \ldots, m_1+n_1+n_5\}, \Gamma_4 = [m_4]$ otherwise.

By (21) and definition of σ_1, it is clear that $A'_0 B'_1 = diag_p(a) P_{\sigma_1} D_{\Gamma_1} = diag_p(a+\chi^{J_1}) P_{\sigma_1}$, and $A'_0 B'_1 B'_2 \sim_L diag_p(a+\chi^{J_1}) P_{\sigma_2} D_{[m_2]} = diag_p(a+\chi^{J_1}+\chi^{J_2}) P_{\sigma_2}$. Next, by (22) and (23), we find that $A'_0 B'_1 B'_2 B'_3 \sim_L diag_p(a+\chi^{J_1}+\chi^{J_2}) P_{\sigma_5} D_{[m_3]} = diag_p(a+\chi^{J_1}+\chi^{J_2}+\chi^{J_3}) P_{\sigma_5}$. Finally, consider the product $A'_0 B'_1 B'_2 B'_3 B'_4 \sim diag_p(a+\chi^{J_1}+\chi^{J_2}+\chi^{J_3}) P_{\sigma_6} D_{\Gamma_4}$, and notice that for each $i = 1, \ldots, n_5$, we may write $\sigma_6 = \alpha_2^i (c^i d^i) \alpha_1^i$, with $(c^i d^i)$ the transposition of c^i with d^i, for some $\alpha_2^i, \alpha_1^i \in \mathcal{S}_n$ satisfying $\alpha_1^i(c^i) = c^i$ and $\alpha_2^i(d^i) = d^i$, since by lemma 3.7 there are no vertices of $\overline{\Pi}$ neither in row c^i to the left of $(c^i, 4)$, nor in row d^i to the right of $(d^i, 3)$. Therefore, $\sigma_6(\{m_4+n_1+1, \ldots, m_4+n_1+n_5\}) = \{d^i : i \in [n_5]\}$, and by (23), we must have

$$A'_0 B'_1 B'_2 B'_3 B'_4 \sim_L diag_p(a+\chi^{J_1}+\chi^{J_2}+\chi^{J_3}+\chi^{J_4}).$$

Thus, $A'_0, B'_1, B'_2, B'_3, B'_4$ satisfy conditions I and III of definition 4.2. It remains to show that $B'_1 \cdots B'_4$ is equivalent to the diagonal matrix $D_{[m_1]} \cdots D_{[m_4]}$. We start by using lemmas 4.1 and 4.4 to write for $k = 2, 3, 4$,

$$B'_k = \prod_{l=1}^{s_k} T_{i_l j_l}(1) C_k D_k \prod_{l=1}^{s_k} T_{j_l i_l}(-1) D_{\Gamma_k},$$

where $\Gamma_l = [m_l]$ for $l = 2, 3$, $C_k D_k$ is the identity matrix for $k = 2, 4$, and C_3 [respectively, D_3] is a product of upper [respectively, lower] elementary matrices $T_{jj'}(\tau)$, for some integers $j, j' \in [m_2]$, and $\tau = \pm 1$. Next, use again lemma 4.1 to write $B'_1 \cdots B'_4$ as

$$\begin{aligned} &\textstyle\prod_{l=1}^{s_2} T_{i_l j_l}(p^{\nu_l}) D_{[\Gamma_1]} \prod_{l=2}^{s_1} T_{j_l i_l}(-1) \prod_{l=1}^{s_2} T_{i_l j_l}(p) D_{[m_2]} C_3 D_3 \prod_{l=1}^{s_3} T_{j_l i_l}(-1) \cdot \\ &\quad \cdot \textstyle\prod_{l=1}^{s_4} T_{i_l j_l}(p) D_{[m_3]} \prod_{l=1}^{s_4} T_{j_l i_l}(-1) D_{[\Gamma_4]}, \end{aligned} \tag{24}$$

$(f_1, \ldots, f_n) = (|w_1| \ldots, |w_n|)$ such that $w = w_1 \cdots w_n, w_i \in V \cup \{\lambda\}$. Define the biword

$$\Sigma = \begin{pmatrix} \pi_1 & \cdots & \pi_k \\ x_1 & \cdots & x_k \end{pmatrix} = \begin{pmatrix} 1^{f_1} & \cdots & n^{f_n} \\ w_1 & \cdots & w_n \end{pmatrix}. \tag{5}$$

Then the top word $\pi_1 \pi_2 \cdots \pi_k = 1^{f_1} 2^{f_2} \cdots n^{f_n}$ is such that π_j is the column index, counting from left to right, of the letter x_j in $\mathcal{T}$, $1 \le j \le k$. Thus, the billeter $\begin{pmatrix} \pi_j \\ x_j \end{pmatrix}$ means that the letter x_j is placed in the column π_j of $\mathcal{T}$. For each i in $[t]$, let $J_i = y_1^i > \cdots > y_{m_i}^i \in [n]^*$ defined by the indices of the columns of the m_i letters i in $\mathcal{T}$. The columns words $J_1, \ldots, J_t$ are said the *indexing sets* of $\mathcal{T}$ and, as we have just seen, each J_i records the indices of the columns where the m_i letters i of w are placed, with respect to the planar representation of $\mathcal{T}$. Indeed we have another biword

$$\Sigma' = \begin{pmatrix} J_t & \cdots & J_2 & J_1 \\ t^{m_t} & \cdots & 2^{m_2} & 1^{m_1} \end{pmatrix}, \tag{6}$$

where $\begin{pmatrix} J_i \\ i^{m_i} \end{pmatrix}$ is the biword with bottom word i^{m_i} and top word the column $J_i = y_1^i \cdots y_{m_i}^i$.

For example, the biwords Σ and Σ' of the skew-tableau (2) are, respectively,

$$\Sigma = \begin{pmatrix} 11\ 22\ 333\ 4\ 5\ 66 \\ 43\ 43\ 421\ 1\ 4\ 21 \end{pmatrix} \quad \text{and} \quad \Sigma' = \begin{pmatrix} 5321\ 21\ 63\ 643 \\ 4444\ 33\ 22\ 111 \end{pmatrix}. \tag{7}$$

Regarding this analysis, we write often $\mathcal{T} = (a, \Sigma) = (a, \Sigma')$ or (a, Π) with Π any other equivalent biword with Σ. Certainly skew-tableaux with the same compact form are characterized by the same class of biwords. We are now in conditions to introduce another definition of skew-tableau which relates the combinatorial and matrix settings. Given $J \subseteq [t]$, we define the characteristic function of J by $(\chi^J)_i = 1$, if $i \in J$, and $(\chi^J)_i = 0$ otherwise. Given a skew-tableau $\mathcal{T} = (a, \Sigma')$, we may associate the sequence of partitions $(a^0, a^1, \ldots, a^t)$ by setting $a^0 := a$ and $a^i := a_{i-1} + \chi^{J_i}$, $i = 1, \ldots, t$. Clearly, each $a^i = (a_1^i, \ldots, a_n^i)$ is a partition and satisfy

$$a_l^i \le a_l^{i+1} \le a_l^i + 1, \tag{8}$$

for $i=0,1,\dots,t-1$, and $l=1,\dots,n$. Conversely, any sequence of partitions $(a^0,a^1,\dots,a^t)$ satisfying (8) gives rise to a skew-tableau $\mathcal{T}$ with biword defined by the sets $J_i=\{l: a_l^i=a_l^{i-1}+1\}$, $i=1,\dots,t$. For instance, the skew-tableau (2) is defined by the sequence of partitions $\mathcal{T}=(a^0,\dots,a^4)$, where $a^0=(4,4,2,2,2)$, $a^1=(4,4,3,3,2,1)$, $a^2=(4,4,4,3,2,2)$, $a^3=(5,5,4,3,2,2)$ and $a^4=(6,6,4,3,3,2)$.

¿From this and theorem 2.4 we have

Theorem 2.6. *The RSK* correspondence defined above sets up a one-to-one correspondence between pairs $(\mathcal{P},\mathcal{Q})$, with $\mathcal{P}\in[t]^*$, $\mathcal{Q}\in[n]^*$, of tableaux of conjugate shapes and*

(*a*) [10, 12] *biwords* Σ (Σ').

(*b*) [10, 12] $n\times t$ 0-1 *matrices, where the entry* (i,j) *is* 1 *iff* $\binom{i}{j}$ *is a biletter of* Σ (Σ').

(*c*) *skew-tableaux in the compact form with* n *columns and word in* $[t]^*$.

In the case of a tableau-pair $(\mathcal{P},\mathcal{Q})$ of conjugate shapes, with $\mathcal{Q}$ a Yamanouchi tableau, we have

Corollary 2.7. *Let* $(\mathcal{P},\mathcal{Q})$, *with* $\mathcal{P}\in[t]^*$, $\mathcal{Q}\in[n]^*$, *be a pair of tableaux of conjugate shapes. Let* $\begin{pmatrix}\mathcal{Q}_\downarrow\\ w\end{pmatrix}$, $\begin{pmatrix}J\\ \mathcal{P}\downarrow\end{pmatrix}$ *correspond by RSK* to the pair* $(\mathcal{P},\mathcal{Q})$. *The following statements are equivalent*

(*a*) $w=\mathcal{P}$ *is of shape* $b=ev(\mathcal{Q})$.

(*b*) $ev(\mathcal{Q})=||\mathcal{P}||=b$.

(*c*) $\mathcal{Q}$ *is the Yamanouchi tableau of evaluation* b.

(*d*) J *is a Yamanouchi word of evaluation* b.

(*e*) $J=J_t\cdots J_1$ *is such that* $J_1,\cdots,J_t$ *are the idexing sets of* $\mathcal{P}$.

In particular, if $\mathcal{P}=\mathcal{K}(\sigma,(l_t,\cdots,l_1))$, *the indexing sets of* $\mathcal{P}$ *are defined by the columns of the frank word* $[m_t]\cdot\ldots\cdot[m_1]$ *congruent with the Yamanouchi tableau of shape* m. *(If* $l_t=0$, *some* $m_i=0$.*)*

Since there is a bijection between biwords Σ and Σ' and $n\times t$ 0-1 matrix $A=(a_{yi})$, where the entry $a_{yi}=1$ exactly when the biletter $\binom{y}{i}$ occurs in the biword, we may represent them in a lattice of points of $\mathbb{N}^2$ according to the bijection $\binom{y}{i}\mapsto(y,i)\in\mathbb{N}^2$ such that $y\in J_i$, $1\le i\le t$. In drawing such a lattice of points, we shall adopt the convention, as with matrices, that the first coordinate, the row index, increases as one goes downwards, and the second coordinate, the column index, increases as one goes from left to right.

Main Theorem 4.5. *Let* $\sigma\in\mathcal{S}_4$ *and* $(l_4,\dots,l_1)$ *be a sequence of nonnegative integers. Let* $[(\mathcal{K}(\sigma,(l_4,\dots,l_1)),Q);(\mathcal{K}(\sigma,(l_4,\dots,l_1)),\mathcal{Y})]$ *be two pairs of tableaux of conjugate shapes with* $\mathcal{Y}$ *a Yamanouchi tableau. Then, the pair* $[(\mathcal{K}(\sigma,(l_4,\dots,l_1)),\mathcal{Q});(\mathcal{K}(\sigma,(l_4,\dots,l_1)),\mathcal{Y})]$ *is admissible.*

Proof: Assume that $\sigma\in\{1423,1432,4123,4132\}$ and $l_4,l_2>0$. Recall that in any other case, the plactic class of $\mathcal{K}(\sigma,(l_4,\dots,l_1))$ is shuffle of their columns and, thus, by the theorem above, the pair of tableaux $[(\mathcal{K}(\sigma,(l_t,\dots,l_1)),\mathcal{Q});(\mathcal{K}(\sigma,(l_t,\dots,l_1)),\mathcal{Y})]$ is admissible.

Let $\Sigma=\begin{pmatrix}\mathcal{Q}\uparrow\\ w\end{pmatrix}$ and $\Sigma'=\begin{pmatrix}J_4&\cdots&J_1\\ 4^{m_4}&\cdots&1^{m_1}\end{pmatrix}$ be the biwords associated with the pair $(\mathcal{K}(\sigma,(l_4,\dots,l_1)),\mathcal{Q})$, and assume without loss of generality that w is in the set $Sh(\widehat{R}_5^{n_5},R_{\sigma,4}^{n_4},\dots,R_{\sigma,1}^{n_1})$, for some $0<n_5\le\min\{l_2,l_4\}$, where $n_i=l_i$, $i=1,3$, and $n_i=l_i-n_5$, $i=2,4$. Notice that either $m_1=2n_5+n_4+n_3+n_2+n_1$ and $m_4=2n_5+n_4+n_3+n_2$, if $\sigma\in\{1423,1432\}$, or $m_4=2n_5+n_4+n_3+n_2+n_1$ and $m_1=2n_5+n_4+n_3+n_2$ otherwise. Let a be a partition such that $\mathcal{T}=(a,\Sigma)$ is a skew-tableau. Let Π (11) be a biword corresponding by RSK* to $(\mathcal{K}(\sigma,(l_4,\dots,l_1)),\mathcal{Q})$ and satisfying lemma 3.5. Consider the injection ρ and the correspondent biword $\overline{\Pi}$ (15), whose bottom word is $\mathcal{K}(\overline{\sigma},(n_5,0,n_4,n_3,n_2,n_1))$. Let $\overline{a}$ be a partition such that $\overline{\mathcal{T}}=(\overline{a},\overline{\Sigma})$ is a skew-tableau, where $\overline{\Sigma}$ and $\overline{\Sigma}'$ are the biwords equivalent to $\overline{\Pi}$.

For each $i=1,\dots,n_5$, let $\begin{pmatrix}a^ib^ic^id^ie^if^i\\ 6\,5\,4\,3\,2\,1\end{pmatrix}$ be a shuffle component of $\overline{\Pi}$. Then, if $\overline{J}_6\cdots\overline{J}_1$ is the top word of $\overline{\Sigma}'$, we have $\overline{J}_1=J_1\setminus\{c^i: i\in[n_5]\}$, $\overline{J}_2=J_2$, $\overline{J}_5=J_3$, $\overline{J_3}=\{d^i: i\in[n_5]\}$, $\overline{J}_4=\{c^i: i\in[n_5]\}$, and $\overline{J}_6=J_4\setminus\{d^i: i\in[n_5]\}$. Note also that, by proposition 3.7, there are no vertices in row c^i to the left of $(c^i,4)$, neither in row d^i, to the right of $(d^i,3)$. Then, we may apply algorithm 1 to $\overline{\Pi}$, choosing a permutation $\sigma_1\in\mathcal{S}_n$ in the conditions of step 1, and satisfying additionally

$$\sigma_1(n_1+\cdots+n_5+i)=c^i,$$

for $i=1,\dots,n_5$. Denote by $A_0,B_1,\dots,B_6$ the sequence of $n\times n$ nonsingular matrices obtained with this procedure. Then, $A_0=diag_p(\overline{a})$, $B_1=P_{\sigma_1}D_{[m_1-n_5]}$, $B_2=S_2D_{[m_2]}$, $B_k=S_kD_{[n_5]}$, for $k=3,4$, $B_5=S_5D_{[m_3]}$, and $B_6=S_6D_{[m_4-n_5]}$. By theorem 4.3, this sequence is a matrix realization for the pair $(\overline{\mathcal{T}},\mathcal{K}(\overline{\sigma},(n_5,0,n_4,n_3,n_2,n_1)))$. In particular, using lemma 4.4, this

$j=1,\ldots,q_k$, let (y_j^k,k_j) be the vertex negatively-linked to (x_j^k,k), and define inductively

$$S^{k+1}_{x_j^k y_j^k} := S(x_j^k, s^k_{(y_j^k,k_j)}, \nu^k_{j-1}\sigma_k), \text{ and } \nu_j^k := (x_j^k\ s^k_{(y_j^k,k_j)})\nu^k_{j-1}.$$

Define $\theta_{k+1} := \nu^k_{q_k}$, $\sigma_{k+1} := \theta_{k+1}\sigma_k$, and $S_{k+1} := \prod_{i=1}^{q_k}(I+S^{k+1}_{x_i^k y_i^k})(I-S^{k+1}_{x_i^k y_i^k})^T$.

Step 3. Let a be a partition of length $\leq n$ such that (a,Π) is a skew-tableau. Put $A_0 := diag_p(a)$ and $B_k := S_k D_{[m_k]}$, for $k=1,\ldots,t$, with $S_1 := P_{\sigma_1}$, and define inductively

$$A_k := A_{k-1}B_k.$$

Theorem 4.3. [7] *Let $\sigma \in \mathcal{S}_t$ and $(l_t,\ldots,l_1)$ a sequence of nonnegative integers. Let $\mathcal{T}$ be a skew-tableau with the same evaluation as $\mathcal{K}(\sigma,(l_t,\ldots,l_1))$ such that $w(\mathcal{T}) \in Sh(R^{l_t}_{\sigma,t},\ \ldots, R^{l_1}_{\sigma,1})$. Then, there is a biword Π (20) corresponding by RSK* to $\mathcal{T}$ such that the sequence A_0, $B_1,\ldots$, B_t, given by the application of algorithm 1 to Π, is a matrix realization for $(\mathcal{T},\mathcal{K}(\sigma,(l_t,\ldots,l_1)))$.*

If Π (20) has no critical vertices (see [7]), the conditions imposed on the biword Π in order the algorithm 1 to produce a matrix realization for $(\mathcal{T},\mathcal{K}(\sigma,(l_t,\ \ldots,l_1)))$, are satisfied. However, we notice that in general this is not the case, as we can see in [7], example 2.5. When the biword Π (20) has no critical vertices, the sequence of matrices produced by algorithm 1 has the following properties.

Lemma 4.4. *In the conditions of the algorithm 1, if Π (20) has no critical vertices, we have:*

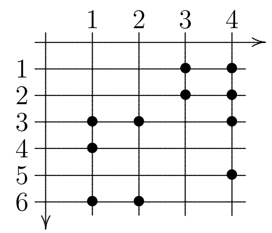
(a) $A_0B_1\cdots B_k$ is left equivalent to $diag_p(a+\chi^{J_1}+\cdots+\chi^{J_{k-1}})P_{\sigma_k}$, $k=1,\ldots,t$;

(b) for $k=1,\ldots,t-1$, there are integers $i=\sigma^{-1}(u)\in[m_k]$ and $j\notin[m_k]$ such that $I+S^{k+1}_{x_j^k y_j^k}=T_{ij}(1)$, where $(u,1)$ is the leftmost vertex of the shuffle component containing (x_j^k,k). Moreover, if $I+S^{k+1}_{x_{j'}^k y_{j'}^k}=T_{i'j'}(1)$ is another matrix, then $i\neq i'$.

Proof: See [7]. □

As we have already seen in lemma 3.5 and proposition 3.7, when σ is $1423, 1432, 4123$ or 4132, there are always biwords Π (11) and $\overline{\Pi}$ (15) without critical vertices. We are now in conditions to prove

The points (y,i), $y\in J_i$, $1\leq i\leq t$, in this lattice are said the *vertices* of the biwords Σ, Σ' or any equivalent biword. For example, the vertices of the biwords (7) are represented in the following grid:



. (9)

The word w is read, in the grid, along rows, from right to left, starting in the top downwards to the bottom one, and the word J along columns, from bottom to top, starting in the rightmost one to the left one. Compare with (2).

3. The keys associated with $\mathcal{S}_4$, their associated frank words and graphical representation

3.1. Keys associated with $\mathcal{S}_4$ and their associated frank words. The algorithm in section 4 as well as previous algorithms for matrix realizations of pairs of tableaux [2, 4, 6, 7] have been based on frank words $\sigma\mathcal{Q}$ which are the union of rows whose lengths define the conjugate shape of $\mathcal{Q}$. Frank words with two, three columns or, in general, for certain permutations defining the shape, this property holds. In the case of four columns, there are frank words of shape $(2,1,1,2)$ which can not be splitted into a union of one row of length four with one of length two. However, those frank words satisfy a row condition which enables us to *stretch* them to a row of length six. Regarding the RSK* correspondence, this phenomenon is related with words which in the shuffle of the columns of a key.

Let $w=x_1\cdots x_k\in[t]^*$ and let I be a subset of $[k]$. We denote by $w|I$ the word $x_{i_1}\cdots x_{i_l}$, if $I=\{i_1<i_2<\cdots<i_l\}$. Such a word $w|I$ is called a *subword* of w. Given q words $u_1,\ldots,u_q\in[t]^*$ of lengths $k_1,\ldots,k_q$, respectively, put $k=k_1+\cdots+k_q$ and let $[k]=\cup_{j=1}^q I_j$, where $(I_1,\ldots,I_q)$ is a q-tuple of pairwise disjoint subsets of $[k]$ with $|I_j|=k_j$, $j=1,\ldots,q$. The word $w|(I_1,\ldots,I_q)$ defined by $w|I_j=u_j$, for $j=1,\ldots,q$, [11, 18], is called a *shuffle* of $u_1,\ldots,u_q$. The words $u_1,\ldots,u_q$ are said the *shuffle components* of

$w|(I_1,\dots,I_q)$. Notice that we may have $w|(I_1,\dots,I_q)=w|(J_1,\dots,J_q)$, with $(J_1,\dots,J_q)$ another q-tuple in the conditions above.

The *shuffle* of q words $u_1,\dots,u_q$ is the set

$$Sh(u_1,\dots,u_q)=\{w|(I_1,\dots,I_q):\cup_{j=1}^{q}I_j=[k],\ |I_j|=k_j,\ w|I_j=u_j,\ j\in[q]\},$$

where $(I_1,\cdots,I_q)$ is a q-tuple of pairwise disjoint subsets of $[k]$. Given a multiset $A=\{u_1,\ \dots,\ u_q\}\subseteq[t]^*$, we put $Sh(A)=Sh(u_1,\dots,u_q)$. If C is another multiset, we put $Sh(A,C)=Sh(A\cup C)$.

Let $\sigma\in\mathcal{S}_t$, $t\ge1$, and $(l_t,\dots,l_1)$ a sequence of nonnegative integers. For $i=1,\dots,t$, let $R_{\sigma,i}^{l_i}$ be the multiset defined by l_i columns $r_{\sigma,i}$. The shuffle of the columns of $K(\sigma,(l_t,\dots,l_1))$, $Sh(R_{\sigma,t}^{l_t},\cdots,R_{\sigma,1}^{l_1})$, is a subset of its plactic class [5, 7]. To characterize the frank words associated with these words described by the shuffle operation, we introduce the notion of *union* of frank words.

Definition 3.1. Let I,J be multisets in $[k]$ with its elements by nonincreasing order, and let $x,y\in[n]^*$ be frank words with $|x|=|I|$ and $|y|=|J|$ such that the biword $\begin{pmatrix}x&y\\I&J\end{pmatrix}$ has no repeated billeters. By sorting the biletters, consider the transformation

$$\begin{pmatrix}x&y\\I&J\end{pmatrix}\leftrightarrow\Sigma'=\begin{pmatrix}J_k&\cdots&J_1\\k^{m_k}&\cdots&1^{m_1}\end{pmatrix}.$$

We say that the word $x_I\cup y_J:=J_k\cdots J_1$ is the $[I,J]$-union of x,y.

For instance, the $[321,21]$-union of the frank words 244 and 13 is $\begin{matrix}2&4&4\\&1&3\end{matrix}$.

A word congruent with $K(\sigma,(l_t,\dots,l_1))$ is in the shuffle of its columns iff any associated frank word $J_t\cdot\ldots\cdot J_1$ is the $[(r_{\sigma,t})^{l_t},\dots,(r_{\sigma,1})^{l_1}]$-union of l_j row words of length j, $1\le j\le t$, that is,

$$J_t\cdot\ldots\cdot J_1=\cup_{j=1}^{t}\cup_{i=1}^{l_j}I_{r_{\sigma,j}}^{j,i},$$

for some rows $I^{j,i}$ with length j, for $i=1,\dots,l_j$, $j=1,\dots t$ [7]. This means that, for $k=t,\dots,1$, there are sets $A_{\sigma(i)}^{k}\subseteq J_{\sigma(i)}\setminus(A_{\sigma(i)}^{k-1}\cup\cdots\cup A_{\sigma(i)}^{1}\cup A_{\sigma(i)}^{0})$, with $A_{\sigma(i)}^{0}:=\emptyset$, such that $|A_{\sigma(i)}^{k}|=l_k$, for $i=1,\dots,k$, and $A_{\sigma(i)}^{k}\le A_{\sigma(j)}^{k}$, if $\sigma(j)<\sigma(i)$ [1, 2, 3, 4, 6].

$\mathcal{K}(\sigma,(l_t,\dots,l_1)))$ in the case the word of $\mathcal{T}$ is in the shuffle of the columns of $\mathcal{K}(\sigma,(l_t,\dots,l_1))$, equivalently, the word formed by the indexing sets of $\mathcal{T}$ is a union of l_j rows of length j, $1\le j\le t$. Then we may write $\mathcal{T}=(a,\Pi)$ such that

$$\Pi=\begin{pmatrix}I^{t,l_t}&\cdots&I^{t,1}&\cdots&I^{2,l_2}&\cdots&I^{2,1}&I^{1,l_1}&\cdots&I^{1,1}\\r_{\sigma,t}&\cdots&r_{\sigma,t}&\cdots&r_{\sigma,2}&\cdots&r_{\sigma,2}&r_{\sigma,1}&\cdots&r_{\sigma,1}\end{pmatrix},\qquad(20)$$

with $I^{j,i}$ a row of length j, for $1\le i\le l_j$, $j=1,\dots t$, is in RSK* correspondence with $\mathcal{T}$.

First we need

Definition 4.3. Let $\sigma\in\mathcal{S}_n$ and consider two integers $x\ge y$ in $[n]$. We define the $n\times n$ matrix $S(x,y,\sigma)$ whose s_{ij} entry satisfy

$$s_{ij}=\begin{cases}1&\text{, if }\sigma(i)=x\text{ and }\sigma(j)=y\neq x\\0&\text{, otherwise.}\end{cases}$$

Clearly, $I+S(x,y,\sigma)$ is an elementary matrix $T_{ij}(1)$, for some integers $i,j\in[n]$. When $x=y$, $S(x,x,\sigma)$ is the null matrix.

Recall that there is a bijection between pairs of tableaux of conjugate shapes and skew-tableaux in the compact form. Our construction in the next algorithm only depends on the class of the biwords that is on the class of the skew-tableaux and not on the particular partition a defining the inner shape of our skew-tableau.

Algorithm 1. Let Π as in (20) and let $\Sigma'=\begin{pmatrix}J_t&\cdots&J_1\\t^{m_t}&\cdots&1^{m_1}\end{pmatrix}$ equivalent to Π. Our algorithm is presented as a three-step definition:

Step 1. For each $k=1,\dots,t$, let $\mathrm{X}_k\subseteq\mathbb{N}^2$ be the set of the leftmost vertices of the l_k shuffle components $\begin{pmatrix}I^{k,i}\\r_{\sigma,k}\end{pmatrix}$, $1\le i\le l_k$, and define

$$s(\mathrm{X}_k):=\{s_{(x,j)}^{0}:(x,j)\in\mathrm{X}_k\}=\{s_{L_{k+1}+1}^{0}<\cdots<s_{L_{k+1}+l_k}^{0}\}\subseteq[n].$$

Let $\sigma_1\in\mathcal{S}_n$ such that $\sigma_1(i)=s_i^0$, for $i\in[L_1]$.

Step 2. For $k=1,\dots,t-1$, let $J_k':=\{x\in J_k:(x,k)\text{ is positively-linked}\}=\{x_1^k<\cdots<x_{q_k}^k\}\subseteq J_k$ and $\nu_0^k:=id\in\mathcal{S}_n$. For each $k=1,\dots,t-1$ and

Definition 4.1. Let $(\mathcal{P},\mathcal{Q})$ and $(\mathcal{P}_B,\mathcal{Q}_B)$, $\mathcal{P},\mathcal{P}_B\in[t]^*$, $\mathcal{Q},\mathcal{Q}_B\in[n]^*$, be two pairs of tableaux of conjugate shapes such that $ev(\mathcal{P})=ev(P_B)=(m_1,\dots,m_t)$, and $\mathcal{Q}_B$ is the Yamanouchi tableau of evaluation $||\mathcal{P}_B||$. Let $\mathcal{T}=(a^0,a^1,\dots,a^t)$ be a skew-tableau which corresponds by RSK* to $(\mathcal{P},\mathcal{Q})$ and let $\mathcal{P}_B=(0,b^1,\dots,b^t)$. We say that a sequence of $n\times n$ nonsingular matrices $A_0,B_1,\dots,B_t$ is a matrix realization of $[(\mathcal{P},\mathcal{Q});(\mathcal{P}_B,\mathcal{Q}_B)]$ or $(\mathcal{T},\mathcal{P}_B)$ if:

I. For each $r\in\{1,...,t\}$, the matrix B_r has invariant partition (1^{m_r}).
II. For each $r\in\{1,...,t\}$, the matrix $B_1...B_r$ has invariant partition b^r.
III. For each $r\in\{0,1,...,t\}$, the matrix $A_r:=A_0B_1...B_r$ has invariant partition a^r.

The pair $[(\mathcal{P},\mathcal{Q});(\mathcal{P}_B,\mathcal{Q}_B)]$, or $(\mathcal{T},\mathcal{P}_B)$, is called *admissible.*

For the purpose of this paper, we shall consider only the pairs $(\mathcal{P},\mathcal{Q})$ and $(\mathcal{K},\mathcal{Y})$ of tableaux of conjugate shapes with $ev(\mathcal{P})=ev(\mathcal{K})$ such that $\mathcal{K}$ is the key of evaluation $(m_1,\dots,m_t)$ and $\mathcal{Y}$ is the Yamanouchi tableau of shape $(m_1,\dots,m_t)$ by nonincreasing order. Thus, in order to verify property II, it is sufficient to show that $B_1\cdots B_t$ has invariant partition $(1^{m_1})+\cdots+(t^{m_t})=\|\mathcal{K}\|$. Given the sequence $(m_1,\dots,m_t)$ of nonnegative integers, the tableau with this evaluation and shape $\sum_{i=1}^t(1^{m_i})$ is the key $\mathcal{K}=(0,(1^{m_1}),(1^{m_1})+(1^{m_2}),\dots,\sum_{i=1}^t(1^{m_i}))$. Therefore, definition 4.1 becomes:

Definition 4.2. Let $(\mathcal{P},\mathcal{Q})$ and $(\mathcal{K},\mathcal{Y})$, $\mathcal{K}\in[t]^*$, $\mathcal{Q}\in[n]^*$, be two pairs of tableaux of conjugate shapes such that $ev(\mathcal{P})=ev(\mathcal{K})$, $\mathcal{K}$ is the key of evaluation $(m_1,\dots,m_t)$ and $\mathcal{Y}$ the Yamanouchi tableau of shape $(m_1,\dots,m_t)$ by nonincreasing order. Let $\mathcal{T}=(a^0,a^1,\dots,a^t)$ be a skew-tableau which corresponds by RSK* to $(\mathcal{P},\mathcal{Q})$. We say that a sequence of $n\times n$ nonsingular matrices $A_0,B_1,\dots,B_t$ is a matrix realization of $[(\mathcal{P},\mathcal{Q});(\mathcal{K},\mathcal{Y})]$ or $(\mathcal{T},\mathcal{K})$ if:

I. For each $r\in\{1,...,t\}$, the matrix B_r has invariant partition (1^{m_r}).
II. The matrix $B_1...B_t$ has invariant partition $\|\mathcal{K}\|$.
III. For each $r\in\{0,1,...,t\}$, the matrix $A_r:=A_0B_1...B_r$ has invariant partition a^r.

It has been shown in [5] that $[(\mathcal{P},\mathcal{Q});(\mathcal{K},\mathcal{Y})]$ is an admissible pair only if $\mathcal{P}=\mathcal{K}$.

Let $\sigma\in\mathcal{S}_t$ and $(l_t,\dots,l_1)$ be a sequence of nonnegative integers. Let L_{t+1}:=0 and $L_k:=L_{k+1}+l_k$, $k=1,\dots,t$. The next algorithm, described in [7], gives a procedure to obtain a matrix realization for the pair $(\mathcal{T},$

In general, $Sh(R^{l_t}_{\sigma,t},\cdots,R^{l_1}_{\sigma,1})$ is a proper subset of the plactic class of the key $\mathcal{K}(\sigma,(l_t,\dots,l_1))$. By imposing conditions on $\sigma\in\mathcal{S}_t$ and on the multiplicity sequence $(l_t,\dots,l_1)$ the following result allows us to check whether the plactic class of a key either is the shuffle of its columns or not.

Theorem 3.1. [5] *Let $\sigma\in\mathcal{S}_t$ and $(l_t,\dots,l_1)$, $l_t>0$. The plactic class of $\mathcal{K}(\sigma,(l_t,\dots,l_1))$ is $Sh(R^{l_t}_{\sigma,t},\dots,R^{l_1}_{\sigma,1})$ if and only if, for $k=2,\dots,t-1$ with $l_k>0$ and $r_{\sigma,k}=a_k\cdots a_1$, the difference a_k-a_1 is at most k.*

Whenever a permutation σ satisfy the conditions of this theorem the same is true for $rev\,\sigma$. Therefore, the plactic class of a key associated with the identity or with the reverse permutation in $\mathcal{S}_t$, $t\geq1$, or with any permutation in $\mathcal{S}_t$, $t\leq3$, coincides with the shuffle of its columns and thus associated frank words are union of rows with length given by the shape of the key. In $\mathcal{S}_4$ we find the first examples where the shuffle of the columns of an associated key is not all the plactic class. For the permutations $\sigma\in\{1423,1432,4123,4132\}$ we may describe the plactic class of any associated key with the shuffle operation, by adding, in those cases where the columns of the key are not enough, the single word 431421. Denote by $\widehat{R}^{n_5}_5$ the multiset consisting of n_5 words $\widehat{r}_5:=431421\equiv\mathcal{K}(\sigma,(1,0,1,0))$.

Theorem 3.2. [5] *Let $(l_4,\dots,l_1)$ be a sequence of nonnegative integers.*

(a) Let $\sigma\in\mathcal{S}_4$. The plactic class of $\mathcal{K}(\sigma,(l_4,\dots,l_1))$ does not coincide with $Sh(R^{l_4}_{\sigma,4},\dots,R^{l_1}_{\sigma,1})$ if and only if $\sigma\in\{1423,1432,4123,4132\}$ and $l_2,l_4>0$.

(b) Let $\sigma\in\{1423,1432,4123,4132\}\subseteq\mathcal{S}_4$. The plactic class of the key $\mathcal{K}(\sigma,(l_4,\dots,l_1))$ is the union of the sets

$$Sh(\widehat{R}^{n_5}_5,R^{n_4}_{\sigma,4},\dots,R^{n_1}_{\sigma,1}),$$

where $0\leq n_5\leq\min\{l_2,l_4\}$, $n_i=l_i$, $i=1,3$, and $n_i=l_i-n_5$, $i=2,4$.

To describe the frank words with four columns, that is, the frank words associated with words congruent with keys associated with $\mathcal{S}_4$ and multiplicity $(l_4,\cdots,l_1)$, $l_4>0$, it remains to characterize the frank words associated with $\widehat{r}_5=431421\equiv\mathcal{K}(\sigma,(1,0,1,0))$. From theorem 2.5 we have

Proposition 3.3. *Let $J=dabefc$ be a word. The following conditions are equivalent*

(a) J is a frank word associated with 431421.
(b) J satisfies $a\leq b\leq c<d\leq e\leq f$.

If $w \in Sh(\widehat{R}_5^{n_5}, R_{\sigma,4}^{n_4}, \ldots, R_{\sigma,1}^{n_1})$, $n_5 > 0$, define a biword Σ as in (3), and fix a shuffle decomposition of $w = w|(X_5^{n_5}, \ldots, X_5^1, \ldots, X_1^{n_1}, \ldots, X_1^1)$, where $(X_5^{n_5}, \ldots, X_5^1, \ldots, X_1^{n_1}, \ldots, X_1^1)$ is a $(n_1 + \cdots + n_5)$-tuple of pairwise disjoint subsets of $[|\sum_{i=1}^4 in_i + 6n_5|]$, with $w|X_j^i = r_{\sigma,j}$, $i \in [n_j]$, $j \in [4]$, and $w|X_5^i = \widehat{r}_5$, $i \in [n_5]$. Denote by v the first row of Σ and let $v|X_j^i = I^{j,i}$, $i \in [n_j]$, $j \in [5]$. Then, $I^{j,i}$ must be a row with length j, for $j \in [4]$, and we must have $I^{5,i} = a^i b^i c^i d^i e^i f^i$ with $a^i \leq b^i \leq c^i < d^i \leq e^i \leq f^i$, for $i \in [n_5]$.

Therefore, Σ is a shuffle of the biwords $\begin{pmatrix} I^{j,i} \\ r_{\sigma,j} \end{pmatrix}$, $i \in [n_j]$, $j \in [4]$, and $\begin{pmatrix} I^{5,i} \\ \widehat{r}_5 \end{pmatrix}$, $i \in [n_5]$, and we may consider the equivalent biword

$$\Pi := \left(\begin{array}{ccc|ccccccc} I^{5,n_5} & \cdots & I^{5,1} & I^{4,n_4} & \cdots & I^{4,1} & \cdots & I^{1,n_1} & \cdots & I^{1,1} \\ \widehat{r}_5 & \cdots & \widehat{r}_5 & r_{\sigma,4} & \cdots & r_{\sigma,4} & \cdots & r_{\sigma,1} & \cdots & r_{\sigma,1} \end{array}\right), \qquad (10)$$

with bottom row the word $(\widehat{r}_5)^{n_5}(r_{\sigma,4})^{n_4} \cdots (r_{\sigma,1})^{n_1}$. In general, there is more than one biword Π associated with Σ, each corresponding to a different shuffle decomposition of w in $Sh(\widehat{R}_5^{n_5}, \ldots, R_{\sigma,1}^{n_1})$. On the other hand, sorting the biletters of Π by nondecreasing rearrangement for the antilexicographic order with priority of the first row, we get the biword Σ. Notice that this amounts to shuffle appropriately the biwords $\begin{pmatrix} I^{j,i} \\ r_{\sigma,j} \end{pmatrix}$, $1 \leq j \leq 4$, and $\begin{pmatrix} I^{5,i} \\ \widehat{r}_5 \end{pmatrix}$ of Π. Sorting the biletters of Π by nonincreasing rearrangement for the lexicographic order with priority of the second row we get Σ'. This rearrangement transforms the biword $\begin{pmatrix} I^{5,i} \\ \widehat{r}_5 \end{pmatrix}$ into $\begin{pmatrix} \widehat{J}^i \\ \widehat{r} \end{pmatrix} := \begin{pmatrix} d^i & a^i & b^i & e^i & f^i & c^i \\ 4 & 4 & 3 & 2 & 1 & 1 \end{pmatrix}$. Thus, Σ' is also obtained from Π by transforming each factor $\begin{pmatrix} I^{5,i} \\ r_{\sigma,5} \end{pmatrix}$ into $\begin{pmatrix} \widehat{J}^i \\ \widehat{r} \end{pmatrix}$, and then shuffling appropriately the biwords $\begin{pmatrix} I^{j,i} \\ r_{\sigma,j} \end{pmatrix}$, $1 \leq j \leq 4$, and $\begin{pmatrix} \widehat{J}^i \\ \widehat{r} \end{pmatrix}$:

$$\Pi \leftrightarrow \left(\begin{array}{ccc|ccccccc} \widehat{J}^{n_5} & \cdots & \widehat{J}^1 & I^{4,n_4} & \cdots & I^{4,1} & \cdots & I^{1,n_1} & \cdots & I^{1,1} \\ \widehat{r} & \cdots & \widehat{r} & r_{\sigma,4} & \cdots & r_{\sigma,4} & \cdots & r_{\sigma,1} & \cdots & r_{\sigma,1} \end{array}\right) \leftrightarrow \Sigma'.$$

Proposition 3.4. *Let $\sigma \in \mathcal{S}_t$, $(l_4, \ldots, l_1)$, $l_4 > 0$ a sequence of nonnegative integers, and $m = (\sum_{i=1}^4 l_i, \ldots, l_4 + l_3, l_4)$. Then*

observe in the next example, where $\mathcal{P}_B$ is the root of the tree, the unique row tableau of evaluation $(m_1, \ldots, m_t)$.

Example 4.1. (a) Let $B_1 = P_{(24)}P_{(35)}D_{\{1,2,3\}}, B_2 = D_{\{4,5\}}$. The sequence $B_1 B_2 \sim_L D_{\{1,2,3\}}D_{\{4,5\}}$ gives rise to the biword $\Sigma'_B = \begin{pmatrix} 54 & 321 \\ 22 & 111 \end{pmatrix}$, and hence to the pair $(\mathcal{P}_B, \mathcal{Q}_B)$, where $\mathcal{P}_B = 11122$. On the other hand, the sequence $\Delta_a B_1 B_2 \sim_R \Delta_a D_{\{1,4,5\}}D_{\{2,3\}}$ leads to the skew-tableau $\left(a, \Sigma' = \begin{pmatrix} 32 & 541 \\ 22 & 111 \end{pmatrix}\right)$, and corresponds to the tableau-pair $(\mathcal{P}, \mathcal{Q})$, where $\mathcal{P} = 21 \cdot 21 \cdot 1$.

(b) Let $B_1 = P_{(34)}D_{\{1,2,3\}}, B_2 = D_{\{4,5\}}$. The sequence $B_1 B_2 \sim_L D_{\{1,2,3\}}D_{\{4,5\}}$ gives rise to the biword $\Sigma'_B = \begin{pmatrix} 54 & 321 \\ 22 & 111 \end{pmatrix}$, and to the pair $(\mathcal{P}_B, \mathcal{Q}_B)$, where $P_B = 11122$. On the other hand, the sequence $\Delta_a B_1 B_2 \sim_R \Delta_a D_{\{1,2,4\}}D_{\{3,5\}}$ leads to the skew-tableau $\left(a, \Sigma' = \begin{pmatrix} 53 & 421 \\ 22 & 111 \end{pmatrix}\right)$ and corresponds to the tableau-pair $(\mathcal{P}, \mathcal{Q})$, with $\mathcal{P} = 21 \cdot 1 \cdot 1 \cdot 2$.

(c) Let $B_1 = D_{\{1,2,3\}}, B_2 = D_{\{4,5\}}$. The sequence $B_1 B_2 = D_{\{1,2,3\}}D_{\{4,5\}}$ gives rise to the biword $\Sigma'_B = \begin{pmatrix} 54 & 321 \\ 22 & 111 \end{pmatrix}$, and to the pair $(\mathcal{P}_B, \mathcal{Q}_B)$, where $\mathcal{P}_B = 11122$. But the sequence $\Delta_a B_1 B_2 = \Delta_a D_{\{1,2,3\}}D_{\{4,5\}}$ gives the skew-tableau $\left(a, \Sigma' = \begin{pmatrix} 54 & 321 \\ 22 & 111 \end{pmatrix}\right)$ and corresponds to the tableau-pair $(\mathcal{P}, \mathcal{Q})$, with $\mathcal{P} = 11122 = \mathcal{P}_B$.

Therefore the sequences $\Delta_a, UD_{\{1,2,3\}}, D_{\{4,5\}}$, with U running over $\mathcal{U}_5$, give rise to skew tableaux with words congruent with $\mathcal{P}$ running over the set $\{11122; 21 \cdot 1 \cdot 1 \cdot 2; 21 \cdot 21 \cdot 1\}$ the set of all tableaux of evaluation $(3, 2)$. This example can be easily generalized to all tableaux of evaluation (m, n), with $m \geq n$. The sequences $\Delta_a, UD_{\{1,\ldots,m\}}, D_{\{m+1,\ldots,m+n\}}$, with U running over $\mathcal{U}_{m+n}$, give rise to words congruent with $\mathcal{P}$ running over the tableaux of evaluation (m, n).

Regarding the RSK* correspondence between skew-tableaux and pairs of tableaux of conjugate shapes, following [2, 4, 6], we introduce the definition of a matrix realization of a pair of tableaux $(\mathcal{T}, \mathcal{F})$ with $\mathcal{T}$ a skew-tableau and $\mathcal{F}$ a tableau with the same evaluation as $\mathcal{T}$.

contradicting condition (iii). Hence there is not a matrix $U \in \mathcal{U}_2$ satisfying conditions $(i), (ii)$ and (iii).

The sequence (17) gives rise to the biwords $\Sigma'_B = \begin{pmatrix} F_t & \cdots & F_1 \\ t^{m_t} & \cdots & 1^{m_1} \end{pmatrix}$, $\Sigma_B = \begin{pmatrix} 1^{b_1} & \cdots & n^{b_n} \\ w_1 & \cdots & w_n \end{pmatrix}$, with $w_1 \cdot \ldots \cdot w_n$ a tableau of shape $b^t = (b_1, \ldots, b_n)$ and indexing sets $F_1, \ldots, F_t$, and corresponds by RSK* to the tableau-pair $(\mathcal{P}_B, \mathcal{Q}_B)$ of conjugate shapes, where $\mathcal{P}_B = w_1 \cdot \ldots \cdot w_n \in [t]^*$ and $\mathcal{Q}_B = P(F_t \ldots F_1) \in [n]^*$ the Yamanouchi tableau of evaluation $||\mathcal{P}_B||$.

Let a be a partition, and consider now the sequence $\Delta_a, B_1, ..., B_t$. Using (17), (18), we may assume without loss of generality, that the sequence $\Delta_a, B_1, ..., B_t$ has the form

$$\Delta_a, V_1 D_{F_1}, V_2 D_{F_2}, ..., V_t D_{F_t}. \tag{19}$$

Using again theorem 4.2, there exist matrices $U'_1, \ldots, U'_t \in \mathcal{U}_n$ and sets $J_1, \ldots, J_t \subseteq [n]$, with $|J_i| = |F_i| = m_i$, $1 \le i \le t$, such that

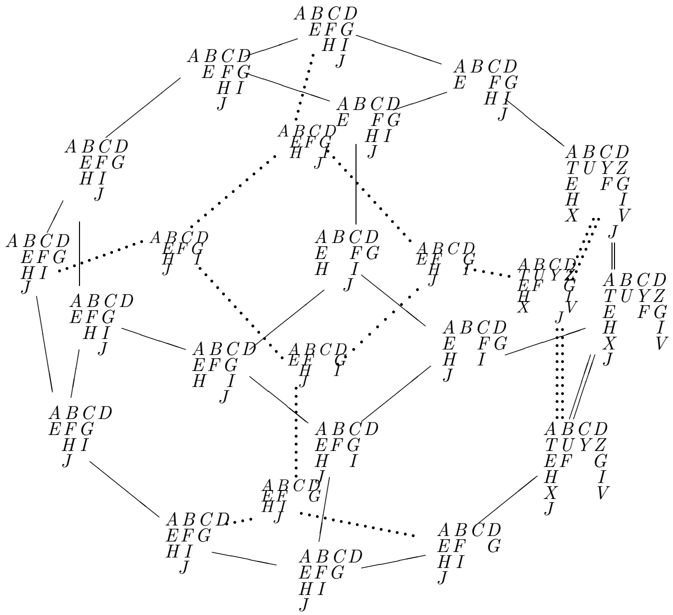
$$\begin{aligned} \Delta_a B_1 ... B_k &\sim_R \Delta_a V_1 D_{F_1} V_2 D_{F_2} ... V_k D_{F_k} \\ &\sim_R \Delta_a U'_1 D_{J_1} U'_2 D_{J_2} \cdots U'_k D_{J_k} \\ &\sim_L diag_p(a + \chi^{J_1} + \chi^{J_2} + \cdots + \chi^{J_k}), \end{aligned}$$

with $a + \chi^{J_1} + \chi^{J_2} + \cdots + \chi^{J_k}$ a partition, for $k = 1, \ldots, t$. The sequence (19) gives rise to the skew-tableau $\mathcal{T} := \left(a, \Sigma' = \begin{pmatrix} J_t & \cdots & J_1 \\ t^{m_t} & \cdots & 1^{m_1} \end{pmatrix}\right)$ and corresponds by RSK* to the tableau-pair $(\mathcal{P}, \mathcal{Q})$ of conjugate shapes, where $\mathcal{Q} = P(J_t \cdots J_1)$ and $\mathcal{P}$ is the tableau congruent with the word of $\mathcal{T}$, and simultaneously to the pair $(\mathcal{P}_B, \mathcal{Q}_B)$ of tableaux of conjugate shapes, where $\mathcal{P}$ and $\mathcal{P}_B$ have the same evaluation $(m_1, \ldots, m_t)$ and $\mathcal{Q}_B$ is the Yamanouchi tableau of evaluation $||\mathcal{P}_B||$.

We have proved in [5] that when (17) has the form $UD_{[m_1]} \cdots D_{[m_t]}$, then $\mathcal{P} = \mathcal{P}_B$ is the key of evaluation $(m_1, \cdots, m_t)$, and $\mathcal{Q}_B$ is the Yamanouchi tableau of shape $(m_1, \ldots, m_t)$ by nonincreasing order. Thus the sequence $\Delta_a UD_{[m_1]} \cdots D_{[m_t]}$ gives rise to the pairs $(\mathcal{P}, \mathcal{Q})$ and $(\mathcal{P}, \mathcal{Q}_B)$ of tableaux of conjugate shapes such that $\mathcal{P}$ is the key of evaluation $(m_1, \cdots, m_t)$ and $\mathcal{Q}_B$ the Yamanouchi tableau of shape $(m_1, \ldots, m_t)$ by nonincreasing order. The key of evaluation $(m_1, \ldots, m_t)$ is a leaf of the rooted tree of all tableaux of evaluation $(m_1, \ldots, m_t)$. In general, we do not have $\mathcal{P} = \mathcal{P}_B$, as we may

(a) *If* $\sigma \in \{1423, 1432, 4123, 4132\}$, $J_4 \cdot J_3 \cdot J_2 \cdot J_1$ *with shape* $(\sigma m)^{rev}$ *is frank iff there exist rows* $I^{j,i}$ *of length* j , $j \in [4]$, $i \in [n_j]$, *and words* $\widehat{J}^i = d^i a^i b^i e^i f^i c^i$, *with* $a^i \le b^i \le c^i < d^i \le e^i \le f^i$, $i \in [n_5]$, *such that* $J_4 \cdots J_1 = (\cup_{i=1}^{n_5} \widehat{J}^i_{\widehat{r}}) \cup (\cup_{j=1}^{4} \cup_{i=1}^{n_j} I^{j,i}_{r_{\sigma,j}})$, *where* $\widehat{r} = 443211$, *and* $0 \le n_5 \le \min\{l_2, l_4\}$, $n_i = l_i$, $i = 1, 3$, *and* $n_i = l_i - n_5$, $i = 2, 4$.

(c) $J_4 \cdot J_3 \cdot J_2 \cdot J_1$ *with shape* σm *is frank iff there is a set decomposition of the columns according to the following diagram,*



where $A \le B \le C \le D$, $E \le F \le G$, $H \le I$, *and* $T \le U \le V < X \le Y \le Z$, *with* $|A| = |B| = |C| = |D|$, $|E| = |F| = |G|$, $|H| = |I|$, $|T| = |U| = |V| = |X| = |Y| = |Z|$, *and in each column the sets are pairwise disjoints.*

We have outlined the frank words with shape σm, for each permutation $\sigma \in \{4123, 4132, 1423, 1432\}$, by linking them with double lines. When $l_2, l_4 > 0$, some of them are not only union of rows of length j, $1 \le j \le 4$, but also

of frank words of shape $(2,1,1,2)$, which can not be splitted into a union of rows of length four and two, giving rise to the component $\begin{array}{cccc} X & & & \\ T & \cdot U & \cdot Y & \cdot Z \\ & & & V \end{array}$.

3.2. Graphical representation of words in $Sh(\widehat{R}_5^{n_5},\cdots,R_{\sigma,1}^{n_1})$ and their associated frank words. Let $w\in Sh(\widehat{R}_5^{n_5},\cdots,R_{\sigma,1}^{n_1})$, with σ a permutation in $\{1423,1432,4123,4132\}$, and consider a biword Σ, as in (3). Fix a shuffle decomposition for w and consider the correspondent biword Π equivalent to Σ

$$\Pi=\left(\begin{array}{ccc|ccccccc} I^{5,n_5} & \cdots & I^{5,1} & I^{4,n_4} & \cdots & I^{4,1} & \cdots & I^{1,n_1} & \cdots & I^{1,1} \\ \widehat{r}_5 & \cdots & \widehat{r}_5 & r_{\sigma,4} & \cdots & r_{\sigma,4} & \cdots & r_{\sigma,1} & \cdots & r_{\sigma,1} \end{array}\right), \tag{11}$$

where each $I^{j,i}$ is a row with $|I^{j,i}|=j$, $i\in[n_j]$, $j\in[4]$, and $I^{5,i}=a^ib^ic^id^ie^if^i$ with $a^i\le b^i\le c^i<d^i\le e^i\le f^i$, $i\in[n_5]$.

Consider the graphical representation of the vertices of Π. Linking, by a straight line, the vertices of consecutive biletters of each factor $\begin{pmatrix} I^{j,i} \\ r_{\sigma,j}\end{pmatrix}$ and $\begin{pmatrix} I^{5,i} \\ \widehat{r}_5\end{pmatrix}$, we get a graphical representation of each shuffle component of w. Therefore (11) is graphically represented by $n_5+\cdots+n_1$ polygonal lines, n_j polygonal lines of nonnegative slope corresponding to the shuffle component $\begin{pmatrix} I^{j,i} \\ r_{\sigma,j}\end{pmatrix}$, $i\in[n_j]$, $j\in[4]$, and n_5 to $\begin{pmatrix} I^{5,i} \\ \widehat{r}_5\end{pmatrix}$, $i\in[n_5]$, where the line linking the vertices $\begin{pmatrix} c^i \\ 1\end{pmatrix}$ and $\begin{pmatrix} d^i \\ 4\end{pmatrix}$ have negative slope.

Example 3.1. Consider the biword Σ (7), whose bottom word is a shuffle of $431421, 4321$ and 4. We may sort the billeters of Σ in several ways, in order to obtain a biword Π. Take, for instance, the biword

$$\Pi=\begin{pmatrix} 113566 & 2234 & 3 \\ 431421 & 4321 & 4\end{pmatrix} \tag{12}$$

Theorem 4.2. *Let $U\in\mathcal{U}_n$, and $1\le m\le n$. Given a partition a of length $\le n$, there exist $J\subseteq[n]$, with $|J|=m$, and $V\in\mathcal{U}_n$ such that $\Delta_aUD_{[m]}\sim_R \Delta_aVD_J\sim_L\Delta_aD_J=diag_p(a+\chi^J)$, with $a+\chi^J$ a partition.*

Proof: See the proof of theorem 3.7 in [6]. □

We recall now the discussion in section 2.2 and put it into our matrix framework. Given a sequence of $n\times n$ nonsingular matrices $B_1,...,B_t$, where B_r has elementary invariant partition (1^{m_k}), for $k=1,...,t$, there exist $U_1,\dots,U_t\in\mathcal{U}_n$ such that $B_1...B_k\sim_R U_1D_{[m_1]}U_2D_{[m_2]}...U_kD_{[m_k]}$, for $k=1,\dots,t$. Using previous theorem, we also find matrices $V_2,\dots,V_t\in\mathcal{U}_n$, and sets $F_2,\dots,F_t\subseteq[n]$, with $|F_i|=m_i$, $2\le i\le t$, such that, for $k=1,\dots,t$,

$$\begin{aligned} B_1B_2\cdots B_k &\sim_R U_1D_{[m_1]}U_2D_{[m_2]}\cdots U_kD_{[m_k]} \\ &\sim_R U_1D_{[m_1]}V_2D_{F_2}\cdots V_kD_{F_k} \\ &\sim_L D_{[m_1]}D_{F_2}\cdots D_{F_k} \\ &= diag_p(\chi^{[m_1]}+\chi^{F_2}+\cdots+\chi^{F_k}), \end{aligned} \tag{17}$$

with $b^k:=\chi^{[m_1]}+\chi^{F_2}+\cdots+\chi^{F_k}$ a partition. Therefore, we may assume without loss of generality, that our sequence $B_1,...,B_t$ has the form

$$V_1D_{F_1},\dots,V_tD_{F_t}, \tag{18}$$

where $V_1,\dots,V_t\in\mathcal{U}_n$ and $F_1:=[m_1],F_2,\dots,F_t\subseteq[n]$, with $|F_i|=m_i$, $i\le i\le t$, such that $diag_p(\chi^{[m_1]}+\chi^{F_2}+\cdots+\chi^{F_k})$ is a partition, $k=1,\dots,t$. The sets $F_1,\dots,F_t$ are uniquely determined by the sequence $B_1,B_2,\dots,B_t$. In particular, when $F_i=[m_i]$, $1\le i\le t$, it has been proved [2, 4, 6] that we may consider in (18), $V_2=\cdots=V_t=I_n$. In general, however, this is not the case. For instance, consider the sequence B_1,B_2,B_3, where $B_1:=D_{\{1\}}$, $B_i:=D_{\{2\}}$, $i=2,3$, and note that $B_1B_2B_3\sim D_{\{1\}}D_{\{2\}}D_{\{1\}}$. Assume the existence of an unimodular matrix $U\in\mathcal{U}_2$ such that $(i)\,UD_{\{1\}}\sim_R D_{\{1\}}$, $(ii)\,UD_{\{1\}}D_{\{2\}}\sim_R D_{\{1\}}D_{\{2\}}$ and $(iii)\,UD_{\{1\}}D_{\{2\}}D_{\{2\}}\sim_R D_{\{1\}}D_{\{2\}}D_{\{1\}}$. By theorem 3.5 in [6], we may write $U=T_nP_\sigma QL$, for some $\sigma\in\mathcal{S}_2$, where L is lower triangular with units along the main diagonal, Q is upper triangular with 1's along the main diagonal and multiples of p above it, and $T_n=\begin{bmatrix}1 & xp^n\\ 0 & 1\end{bmatrix}$, for some $n\ge 0$. Since $UD_{\{1\}}\sim_R D_{\{1\}}$, we must have $\sigma=id$ and $n>0$. But then,

$$UD_{\{1\}}D_{\{2\}}D_{\{2\}}\sim_R T_nD_{\{1\}}D_{\{2\}}D_{\{2\}}\sim_R D_{\{1\}}D_{\{2\}}D_{\{2\}}\not\sim_R D_{\{1\}}D_{\{2\}}D_{\{1\}},$$

equivalent to A (written $B \sim_R A$) if $B = AV$ for some unimodular matrix V; and B is *equivalent* to A (written $B \sim A$) if $B = UAV$ for some unimodular matrices U, V. The relations $\sim_L$, $\sim_R$ and $\sim$ are equivalence relations in the set of all $n \times n$ matrices over $\mathcal{R}_p$.

Let A be an $n \times n$ nonsingular matrix. By the Smith normal form theorem [8, 17], there exist nonnegative integers $a_1, ..., a_n$ with $a_1 \geq ... \geq a_n$ such that A is equivalent to

$$diag(p^{a_1}, ..., p^{a_n}).$$

The sequence $a = (a_1, ..., a_n)$, of the exponents of the p-powers in the Smith normal form of A, is a partition of length $\leq n$, uniquely determined by the matrix A. We call a the *invariant partition* of A. More generally, if we are given a sequence of nonnegative integers $e_1, ..., e_n$, the following notation for p-powered diagonal matrices will be used:

$$diag_p(e_1, ..., e_n) := diag(p^{e_1}, ..., p^{e_n}).$$

Given a subset $J \subseteq [n]$, we put $D_J := diag_p(\chi^J)$.

We denote by E_{ij} the $n \times n$ matrix having 1 in position (i, j) and $0's$ elsewhere, and define the elementary unimodular matrices $T_{ij}(x)$ as follows:

$$\begin{aligned} T_{ij}(x) &= I + xE_{ij}, \quad \text{where } i \neq j \text{ and } x \in \mathcal{R}_p; \\ T_{ii}(v) &= I + (v-1)E_{ii}, \quad \text{where } v \text{ is a unit of } \mathcal{R}_p. \end{aligned}$$

It is obvious, that $E_{ij}E_{rs} = \delta_{jr}E_{is}$, where δ_{jr} denotes the Kronecker symbol, that is, $\delta_{jr} = 1$ if $j = r$, and equals 0 otherwise. In the lemma below we state some basic properties of the these elementary matrices $T_{ij}(x)$, which will be used later.

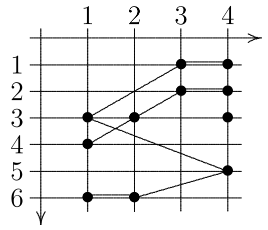
Lemma 4.1. *Let $i, j, r, s, m \in [n]$, and $x, y, v \in \mathcal{R}_p$, such that v is a unit. Then,*

(i) $T_{ij}(x)T_{rs}(y) = T_{rs}(y)T_{ij}(x)$, whenever $i \neq s$ and $j \neq r$.
(ii) $T_{ij}(x)T_{js}(y) = T_{js}(y)T_{ij}(x)T_{is}(xy)$, if $i \neq s$.
(iii) $T_{ii}(v)T_{rs}(x) = T_{rs}(ux)T_{ii}(v)$, for some unit u.
(iv) $T_{ij}(x)D_{[m]} = D_{[m]}T_{ij}(x)$, if $i, j \notin [m]$.
(v) $T_{ij}(x)D_{[m]} = D_{[m]}T_{ij}(xp)$, if $i \notin [m]$ and $j \in [m]$.

Proof: Straightforward. □

Given a partition a, let $\Delta_a := diag_p(a)$.

Linking, respectively, the vertices of the consecutive billeters of $\begin{pmatrix} 113566 \\ 431421 \end{pmatrix}$ and $\begin{pmatrix} 2234 \\ 4321 \end{pmatrix}$ by a polygonal line, we get the following graphical representation of the bottom word $43\underline{4}\,\underline{3}\,\overline{4}\,\underline{2}1\underline{1}421$ of Σ where the underlined letters indicate the shuffle component 4321, and the upperlined letter indicates the shuffle component 4:



. (13)

Since the shuffle components $r_{\sigma,j}$, $j = 1, \ldots, 4$, are columns, the rightmost letter of $r_{\sigma,j}$ corresponds, in the graphical representation, to the leftmost vertex of its polygonal line. By the leftmost vertex of a polygonal line of $\widehat{r}_5 = 431421$, we mean the vertex in column 1, in that polygonal line, having the biggest row value, which corresponds to the rightmost letter of $\widehat{r}_5$. For instance, in (13), the leftmost vertex $(6, 1)$ of the polygonal line of 431421 corresponds to the rightmost letter 1 of this shuffle component.

Two vertices $(a, b), (x, y)$ of Π are *linked* if they are consecutive vertices of a polygonal line. In this case, if $b < y$ and $a \geq x$, (a, b) is said *positively-linked* to (x, y), and (x, y) is said *negatively-linked* to (a, b). For instance, in the graphical representation (13), the vertex $(5, 4)$ is negatively-linked to $(6, 2)$, but is not positively-linked to any vertex.

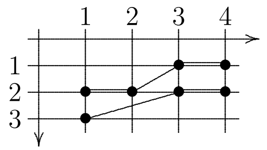
Definition 3.2. Let u, v be two shuffle components of w. A vertex $(b, y) \in u$ is said a critical vertex of $\{u, v\}$ if one of the following conditions holds:

(*i*) (b, y) is negatively-linked to a vertex (b', y') with $y - y' > 1$, or it represents the right most letter of u, and there is a pair of linked vertices $(b, x), (a, y)$ in v, with $y > x \geq y'$.

(*ii*) (b, y) is positively-linked to (a, x) and there is a vertex (a, y) in v.

Notice that if $(b, y) \in u$ is a critical vertex satisfying condition (i) above, then we must have $b = 4$ or $b = 3$, and u is either $r_{\sigma,1} = 4$, or $r_{\sigma,2} = 41$, or $r_{\sigma,3} = 421$, or $r_{\sigma,3} = 431$, or $\widehat{r}_5 = 431421$.

Example 3.2. In (13), the vertex $(4,1) \in 4321$ is a critical vertex of $\{431421, 4321\}$, since it is positively-linked to $(3,2)$ and there is a vertex $(3,1) \in 431421$. Another example is given by the biword $\begin{pmatrix} 1122 & 223 \\ 4321 & 431 \end{pmatrix}$, where $(2,3)$ is a critical vertex, since it is negatively-linked to $(3,1)$, with $3-1>1$, and the vertices $(2,2)$ and $(1,3)$ are linked:



.

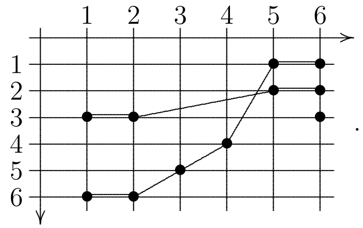
A word w in $Sh(\widehat{R}_5^{n_5}, \ldots, R_{\sigma,1}^{n_1})$, has, in general, several shuffle decompositions. Given a shuffle decomposition of w, it is a simple task to adjust, if necessary, the links between the vertices of the biword Π, and, therefore, the shuffle decomposition itself, in order to form a new biword $\widehat{\Pi}$ satisfying the conditions of the following lemma.

Lemma 3.5. *There is a shuffle decomposition of w such that the corresponding biword Π (11) satisfies the following conditions:*
(a) Any two shuffle components have no critical vertices.
(b) If $\begin{pmatrix} I^{5,i} \\ \widehat{r}_5 \end{pmatrix} = \begin{pmatrix} abcdef \\ 431421 \end{pmatrix}$ is a shuffle component of Π, then:
(i) in row c there is, at most, one more vertex, placed in column 4, which must be negatively-linked to a vertex in column 3;
(ii) if $d \neq e$, in row d there is, at most, one more vertex, placed in column 1, which must be positively-linked;
(iii) if $d \neq e$, in row e, to the right of $(e,2)$, there are no vertices.

Proof: We prove only condition $(b)(iii)$. All other conditions are proven in a similar way. Recall that we must have $a \leq b \leq c < d \leq e \leq f$. If $(e,3)$ is also a vertex of Π, then it must belong to one of the words $431421, 4321$ or 431. In these cases, Π must have a sub-biword either of the form

$$\alpha = \begin{pmatrix} abcdef & gehjkl \\ 431421 & 431421 \end{pmatrix}, \text{ or } \beta = \begin{pmatrix} abcdef & gehi \\ 431421 & 4321 \end{pmatrix} \text{ or } \gamma = \begin{pmatrix} abcdef & geh \\ 431421 & 431 \end{pmatrix},$$

$\overline{\overline{\Pi}} = \begin{pmatrix} 114566 & 2233 & 3 \\ 654321 & 6521 & 6 \end{pmatrix}$, whose graphical representation is given below



. (16)

The bottom word of the biword $\overline{\overline{\Sigma}}$ associated with $\overline{\overline{\Pi}}$ is $\rho(w) = 65656214321$, a shuffle of the columns 654321, 6521 and 6, precisely the words read along the polygonal lines, now all having nonnegative slope. The frank word $\begin{array}{cccccc} 3 & & & & & \\ 2 & 2 & & & & \\ 1 & 1 & 4 & 5 & 6 & 6 \\ & & & & 3 & 3 \end{array}$ associated with $\rho(w)$ is now a $[654321, 6521, 6]$-union of row frank words 114566, 2233 and 3.

In the next proposition, we state some properties of the biword $\overline{\Pi}$.

Proposition 3.7. *Let $\overline{\Pi}$ be the biword defined above, and $\begin{pmatrix} abcdef \\ 654321 \end{pmatrix}$ one of its shuffle components. Then,*
(a) $\overline{\Pi}$ has no critical vertices.
(b) There are no vertices neither in row c to the left of $(c,4)$, nor in row d to the right of $(d,3)$, nor in row e to the right of $(e,2)$.

Proof: It is a consequence of the definition of $\overline{\Pi}$ and lemma 3.5. □

We are now ready to get into the matrix framework.

4. Matrix realizations of pairs of tableaux

4.1. An algorithm and statement of results. Let $\mathcal{R}_p$ be a local principal ideal domain with maximal ideal (p), and let $\mathcal{U}_n$ be the group of $n \times n$ unimodular matrices over $\mathcal{R}_p$. All the matrices in this paper are $n \times n$ nonsingular matrices with entries over $\mathcal{R}_p$. Given a matrix A, A^T will denote the transpose of A. Given matrices A and B, we say that B is *left equivalent* to A (written $B \sim_L A$) if $B = UA$ for some unimodular matrix U; B is *right*

the correspondent biword Π (10). Let

$$\overline{\Pi}=\begin{pmatrix} I^{5,n_5} & \cdots & I^{5,1} & I^{4,n_4} & \cdots & I^{4,1} & \cdots & I^{1,n_1} & \cdots & I^{1,1}\\ r_{\overline{\sigma},6} & \cdots & r_{\overline{\sigma},6} & r_{\overline{\sigma},4} & \cdots & r_{\overline{\sigma},4} & \cdots & r_{\overline{\sigma},1} & \cdots & r_{\overline{\sigma},1}\end{pmatrix} \qquad (15)$$

be the biword obtained by transforming each shuffle component $\begin{pmatrix} I^{j,i}\\ r_{\sigma,j}\end{pmatrix}$ and $\begin{pmatrix} a^ib^ic^id^ie^if^i\\ 4\,3\,1\,4\,2\,1\end{pmatrix}$ of Π, into $\begin{pmatrix} I^{j,i}\\ r_{\overline{\sigma},j}\end{pmatrix}$ and $\begin{pmatrix} a^ib^ic^id^ie^if^i\\ 6\,5\,4\,3\,2\,1\end{pmatrix}$ respectively, where $a^i\leq b^i\leq c^i<d^i\leq e^i\leq f^i$. Let $\overline{\Sigma}$ be the biword obtained by sorting the biletters of $\overline{\Pi}$ by nondecreasing rearrangement for the antilexicographic order with priority on the first row. The second row of $\overline{\Sigma}$, denoted $\rho(w)$, is a shuffle of columns $r_{\overline{\sigma},j}$, $j=6,4,3,2,1$. After this injection the frank word $d^ia^i\cdot b^i\cdot e^i\cdot f^ic^i$, satisfying $a^i\leq b^i\leq c^i<d^i\leq e^i\leq f^i$, associated with 431421, is stretched to a row word $a^i\,b^i\,c^i\,d^i\,e^i\,f^i$ associated with $\rho(431421)=654321$. Thus the frank word $J_4\cdot J_3\cdot J_2\cdot J_1$ with four columns having $d^ia^i\cdot b^i\cdot e^i\cdot f^ic^i$ as a component is transformed into one of six columns $\overline{J}_6\cdot\overline{J}_5\cdots\overline{J}_1$, now a *union* of rows having the row $a^ib^ic^id^ie^if^i$ as a component, such that $\overline{J}_1=J_1\setminus\{c^i: i\in[n_5]\}$, $\overline{J}_2=J_2$, $\overline{J}_5=J_3$, $\overline{J_3}=\{d^i: i\in[n_5]\}$, $\overline{J}_4=\{c^i: i\in[n_5]\}$, and $\overline{J}_6=J_4\setminus\{d^i: i\in[n_5]\}$.

Example 3.5. Let $\sigma=4123$, consider the biword Σ (7), whose bottom word is a shuffle in $Sh(\widehat{r}_5,r_{\sigma,4},r_{\sigma,1})$, and the biword $\widehat{\Pi}=\begin{pmatrix}114566 & 2233 & 3\\ 431421 & 4321 & 4\end{pmatrix}$ graphically represented in example 3.3. The frank word $\begin{array}{cccc} 5 & & & \\ 3 & & & \\ 2 & 2 & 6 & 6\\ 1 & 1 & 3 & 4\\ & & & 3\end{array}$ associated with w is a $[44211,4321,4]$-union of the frank words $51\cdot 1\cdot 6\cdot 64$, 2233 and 3. We have $\overline{\sigma}=612534$. Applying the map ρ, we obtain the biword

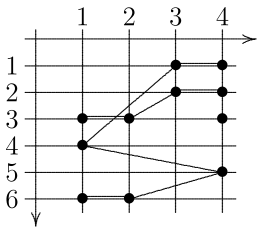
with $g\leq e\leq h\leq i<j\leq k\leq l$. We may re-link the vertices of these shuffle components by replacing in Π the biword α,β or γ with the biword

$$\begin{pmatrix} abcjkl & geef & dh\\ 431421 & 4321 & 41\end{pmatrix},\begin{pmatrix} abcdhi & geef\\ 431421 & 4321\end{pmatrix}\ \text{or}\ \begin{pmatrix} geef & abc & dh\\ 4321 & 431 & 41\end{pmatrix},$$

respectively. In any case, the vertex $(e,2)$ belongs now to the new shuffle component 4321, and is linked to $(e,3)$. Notice also that we have only changed the links between the two shuffle components. Therefore, we may assume, without loss of generality, that if $(e,2)$ is a vertex of a shuffle component $\widehat{r}_5$, there are no vertex in position $(e,3)$.

Assume now that $(e,4)$ is a vertex, but $(e,3)$ is not. An analysis similar to the one done above shows that if $(e,4)$ represents one of the letters 4 of 431421, or belongs to the columns 4321, 431, 421, 41 or 4, then we may re-link the vertices of these shuffle components in such a way that the vertex $(e,2)$ is linked to $(e,4)$. Therefore, we may assume that $(e,4)$ is not a vertex of Π. $\square$

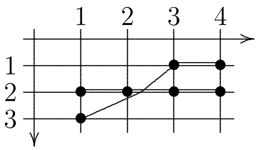
Example 3.3. The biword Π (12), graphically represented in (13), fails to satisfy condition $(b)(i)$ of the lemma above, since the middle letter 1 of $43\overline{1}421$ is represented by the vertex $(3,1)$, and there is a vertex in row 3, column 2. Rearranging the links between the vertices, and therefore the shuffle decomposition itself, we obtain the biword $\widehat{\Pi}=\begin{pmatrix}114566 & 2233 & 3\\ 431421 & 4321 & 4\end{pmatrix}$, which satisfy all the required conditions of lemma 3.5.



. (14)

Example 3.4. In example 3.2, the biword $\Pi=\begin{pmatrix}1122 & 223\\ 4321 & 431\end{pmatrix}$ does not satisfy condition (i), since $(2,3)$ is a critical vertex. Rearranging the links between the vertices, we obtain the biword $\widehat{\Pi}=\begin{pmatrix}2222 & 113\\ 4321 & 431\end{pmatrix}$, represented

below, which satisfy all the required conditions of lemma 3.5:



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In what follows, we assume that our shuffle decomposition of w satisfies the conditions of lemma 3.5.

Definition 3.3. [7] Consider the biword Π (11). For each vertex (a,b) of Π, define the map

$$\begin{array}{rcl}[0,b] & \longrightarrow & [1,a]\\ n & \longmapsto & s^n_{(a,b)}\end{array}$$

defined as follows:
(I) If there is no vertex of Π in row a to the left of column b, then let $s^n_{(a,b)} := a$.
(II) Otherwise, let (a,b'), $b'<b$, be the rightmost vertex of Π, in row a, to the left of (a,b).
 (a) If $b'\le n$, put $s^n_{(a,b)} := a$.
 (b) If $b'>n$ and (a,b') is positively-linked to a vertex (x,y) of Π, with $x<a$, put $s^n_{(a,b)} := s^n_{(x,y)}$.
 (c) Else, put $s^n_{(a,b)} := s^n_{(a,b')}$.

The number $s^n_{(a,b)}$, with $n<b$, indicates, according to a certain path, a row $x\le a$ with a vertex (x,y), $n\le y\le b$, such that there are no vertices in the interval $]n,y[$. Since by lemma 3.5 (a), there are no critical vertices in our fixed shuffle decomposition, we find that $s^n_{(a,b)}\neq a$ only if $b=4$ and $(a,4)$ is either negatively-linked to a vertex in column 1 or it corresponds to the shuffle component $r_{\sigma,1}=4$.

For instance, consider the biword $\widehat{\Pi}$ displayed in example 3.3. Since the vertex $(6,1)$ is placed in column 1, by rule (I) we have $s^0_{(6,1)}=6$. By the same reason, $s^0_{(3,1)}=3$. To compute $s^0_{(3,4)}$, note that $(3,2)$ is the vertex closest to $(3,4)$, and it is positively-linked to $(2,3)$. Thus, by rule $(II)(b)$, $s^0_{(3,4)}=s^0_{(2,3)}=2$, since there are no vertices to the left of this last vertex. We have $s^3_{(1,4)}=1$, by rule $(II)(a)$, and $s^0_{(1,4)}=s^0_{(1,3)}=1$, by $(II)(c)$ and (I).

Corollary 3.6. (a) *If (a,i) is the leftmost vertex of a shuffle component $r_{\sigma,k}$, then $s^0_{(a,i)}\neq a$ only if $\sigma\in\{4123,4132\}$, $i=4$, $k=1$, and the nearest vertex in row a is in column 1, representing the underlined letter 1 of $43142\underline{1}$, $432\underline{1}$, $43\underline{1}$ or $42\underline{1}$, or in column 2, representing the underlined letter 2 of $43\underline{2}1$.*
(b) *If $(b,j),(a,i)$ are linked vertices of a shuffle component $r_{\sigma,k}$, then $s^j_{(a,i)}\neq a$ only if $j=1$, $i=4$, and the nearest vertex in row a is in column 2, representing the underlined letter 2 of $43\underline{2}1$.*

Proof: Follows from lemma 3.5 (a). □

3.3. An injection of the plactic class of a key associated with $\mathcal{S}_4$ into $\mathcal{S}_6$.

For each $\sigma=\sigma(1)\sigma(2)\sigma(3)\sigma(4)\in\mathcal{S}_4$, define the permutation $\overline{\sigma}(1)\overline{\sigma}(2)\overline{\sigma}(3)\overline{\sigma}(4)34\in\mathcal{S}_6$, where

$$\overline{\sigma}(k)=\begin{cases}\sigma(k), & \sigma(k)=1,2\\ \sigma(k)+2, & \sigma(k)=3,4.\end{cases}$$

This correspondence is a bijection between $\mathcal{S}_4$ and the set

$$\{\overline{\sigma}=\alpha 34\in\mathcal{S}_6 : \alpha \text{ a permutation on } \{1256\}\}\subseteq\mathcal{S}_6.$$

In particular, the set $S:=\{1423,1432,4123,4132\}$ is transformed into $\{162534, 165234, 612534, 615234\}$. Let $\rho:\{\widehat{r}_5, r_{\sigma,j} : j\in[4]\}\to\{r_{\overline{\sigma},j} : j\in[6]\}$, such that $\rho(\widehat{r}_5)=r_{\overline{\sigma},6}$ and $\rho(r_{\sigma,j})=r_{\overline{\sigma},j}$, $j=1,2,3,4$.

Given $\sigma\in\mathcal{S}_4$ and a sequence $(l_4,\dots,l_1)$ of nonnegative integers with $l_4>0$,

$$\begin{aligned}&[\mathcal{K}(\sigma,(l_4,\dots,l_1))] := \{w\in[4]^* : w\equiv\mathcal{K}(\sigma,(l_4,\dots,l_1))\} =\\ &=\begin{cases} Sh(R^{n_4}_{\sigma,4},\dots,R^{n_1}_{\sigma,1}), & \text{if } \sigma\in\mathcal{S}_4\setminus S, \text{ or } \sigma\in S \text{ and } l_2=0;\\ \bigcup_{n_5=0}^{\min\{l_2,l_4\}} Sh(\widehat{R}^{n_5}_5, R^{n_4}_{\sigma,4},\dots,R^{n_1}_{\sigma,1}), & \text{if } \sigma\in S \text{ and } l_2\neq 0,\end{cases}\end{aligned}$$

where $n_i=l_i$, $i=1,3$, and $n_i=l_i-n_5$, $i=2,4$.

For each $n_5=0,\dots,\min\{l_2,l_4\}$, the map ρ can be extended, by shuffling, to a bijection between $Sh(\widehat{R}^{n_5}_5, R^{n_4}_{\sigma,4},\dots,R^{n_1}_{\sigma,1})$ and $Sh(\widehat{R}^{n_5}_{\overline{\sigma},6}, R^{n_4}_{\overline{\sigma},4},\dots,R^{n_1}_{\overline{\sigma},1})\subsetneq[\mathcal{K}(\overline{\sigma},(n_5,0,n_4,n_3,n_2,n_1))]$. Thus every word in the plactic class of the key $\mathcal{K}(\sigma,(l_4,\dots,l_1))$ has a copy in the shuffle of the columns of some key, associated with $\mathcal{S}_6$.

Let $w\in Sh(\widehat{R}^{n_5}_5, R^{n_4}_{\sigma,4},\dots,R^{n_1}_{\sigma,1})$, and Σ a biword with bottom row w. Fix a shuffle decomposition satisfying the conditions of lemma 3.5 and consider