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SEMI-STABLE AND EXTREMAL SOLUTIONS OF REACTION EQUATIONS INVOLVING THE p-LAPLACIAN

XAVIER CABRÉ AND MANEL SANCHÓN

ABSTRACT: We consider nonnegative solutions of $-\Delta_p u = f(x,u)$, where $p > 1$ and Δ_p is the p-Laplace operator, in a smooth bounded domain of $\mathbb{R}^N$ with zero Dirichlet boundary conditions. We introduce the notion of semi-stability for a solution (perhaps unbounded). We prove that certain minimizers, or one-sided minimizers, of the energy are semi-stable, and study the properties of this class of solutions.

Under some assumptions on f that make its growth comparable to u^m, we prove that every semi-stable solution is bounded if $m < m_{cs}$. Here, $m_{cs} = m_{cs}(N,p)$ is an explicit exponent which is optimal for the boundedness of semi-stable solutions. In particular, it is bigger than the critical Sobolev exponent $p^* - 1$.

We also study a type of semi-stable solutions called extremal solutions, for which we establish optimal L^∞ estimates. Moreover, we characterize singular extremal solutions by their semi-stability property when the domain is a ball and $1 < p < 2$.

1. Introduction

Let Ω be a smooth bounded domain of $\mathbb{R}^N$ and $p > 1$. We consider the nonlinear elliptic problem

$$\begin{cases} -\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(x,u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1_p}$$

where Δ_p is the p-Laplace operator, $f(x,t)$ is nonnegative, measurable in $x \in \Omega$, and C^1 in $t \in [0,+\infty)$ for a.e. $x \in \Omega$. In most of our results we will assume that there exist positive constants m and c such that

$$0 \leq f(x,t) \leq c(1+t)^m \quad \text{and} \quad 0 \leq f_t(x,t) \tag{1.2}$$

for all $t \geq 0$ and a.e. $x \in \Omega$. Here f_t denotes the partial derivative of f with respect to its second variable. In some results we will make further growth assumptions on f. They will be always satisfied by our model nonlinearity $f(u) = \lambda(1+u)^m$, where λ and m are positive constants (with, in some results, $m > p-1$).

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Throughout the paper, we say that u is a *solution* of (1.1_p) if $u \in W_0^{1,p}(\Omega)$, $u \geq 0$ a.e., $f(x,u) \in L^1(\Omega)$, and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi = \int_\Omega f(x,u)\varphi \quad \text{for all } \varphi \in C_c^\infty(\Omega), \tag{1.3}$$

that is, for all C^∞ functions φ with compact support in Ω. These solutions, which may be unbounded, are usually called weak energy solutions. We will refer to them simply as solutions, for short. Note that for a solution u, (1.3) holds for every $\varphi \in W_0^{1,p}(\Omega)$, by a standard density argument. In addition, since u is p-superharmonic we have that if $u \not\equiv 0$ then $u > 0$ a.e. in Ω, by the strong maximum principle (see [Mo99, Tr67, Va84]).

On the other hand, we say that $u \in W_0^{1,p}(\Omega)$ is a *regular solution* of (1.1_p) if u is a solution and $f(x,u) \in L^\infty(\Omega)$. By well known regularity results for degenerate elliptic equations, one has that every regular solution belongs to $C^{1,\beta}(\overline{\Omega})$ for some positive $\beta \in (0,1]$ (see for instance [Lie88]).

Consider the critical exponent

$$m_c(p) := \begin{cases} +\infty & \text{if } N \leq p, \\ p^* - 1 & \text{if } N > p, \end{cases} \tag{1.4}$$

where $p^* := Np/(N-p)$ corresponds to the critical Sobolev embedding. Recalling hypothesis (1.2) on the nonlinearity f, if $m \leq m_c(p)$ then every solution u of (1.1_p) belongs to $L^\infty(\Omega)$, and therefore $u \in C^{1,\beta}(\overline{\Omega})$. In the subcritical case ($m < m_c$), this is a consequence of the results in [Se64, DiB83, To84, Lie88]. The critical case ($m = m_c$) is more delicate and a proof can be found in [Pe97]. Moreover, it is also known that in the supercritical case ($m > m_c$), u is not necessarily bounded (see Proposition 1.3 below for an example).

In this article we are concerned with a certain type of solutions: those which are semi-stable. Formally, a solution u is said to be semi-stable if the second variation of energy at u (defined below) is nonnegative. In this paper we find another critical exponent $m_{cs} = m_{cs}(p)$ for which every semi-stable solution u of (1.1_p) is bounded if $m < m_{cs}$, while there exist singular semi-stable solutions for every $m \geq m_{cs}$. Of course, the exponent m_{cs} will be greater than m_c — whenever m_c is finite. Our result, which requires a further growth assumption on f besides (1.2), extends work for $p = 2$ by Crandall and Rabinowitz [CR75] and by Mignot and Puel [MP80] concerning certain solutions called extremal solutions. For general $p > 1$ optimal bounds for the extremal solution have been obtained when $f(u) = \lambda e^u$ by García-Azorero, Peral, and Puel [GP92, GPP94]. All these results will be explained in more detail below.

Other of our results are inspired by the methods developed by Brezis and Vázquez in [BV97] to study extremal solutions for the Laplace operator. We extend to the case $p \neq 2$ some of their results on regularity and characterization of such solutions, as well as a result on nonexistence of singular solutions from [BCMR96].

An important aspect of our work relies on giving an appropriate general definition of semi-stability of a solution, specially when $1 < p < 2$. To our knowledge, this task is undertaken here for the first time when $p \neq 2$. Our definition of semi-stability allows the solution to be unbounded, and this is important for some applications. For instance, we establish that the class of semi-stable solutions includes certain minimizers (possibly unbounded) of the energy, as well as minimal and extremal solutions (these are solutions of problem (1.1_p) when f is replaced by λf and f satisfies certain assumptions described below). Several of the ideas used here already appear in [GP92, GPP94], which treated the case $f(u) = \lambda e^u$ and $p > 1$.

Formally, the semi-stability of a solution u means the nonnegativeness of the second variation of the energy functional J associated to (1.1_p), defined by

$$J(u) := \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega F(x,u), \tag{1.5}$$

where

$$F(x,t) = \int_0^t f(x,s)ds. \tag{1.6}$$

But a precise definition of this notion is needed since, in general, the energy functional is not twice differentiable (or not even well defined) in all of $W_0^{1,p}(\Omega)$. The reason for this is that in (1.2) we allow supercritical growth for the reaction term f.

Definition 1.1. Assume that $0 \leq f(x,t)$ is nondecreasing and C^1 in t for a.e. $x \in \Omega$. Let $u \in W_0^{1,p}(\Omega)$ be a solution of (1.1_p). Define

$$\mathcal{A}_u := W_0^{1,p}(\Omega) \qquad \text{if } p \geq 2,$$

and

$$\begin{aligned} \mathcal{A}_u := \ & \{\psi \in W_0^{1,p}(\Omega) : |\psi| \leq Cu \text{ and } |\nabla\psi| \leq C|\nabla u| \\ & \text{in } \Omega, \text{ for some constant } C\} \qquad \text{if } 1 < p < 2. \end{aligned} \tag{1.7}$$

We say that u is *semi-stable* if

$$\int_{\{\nabla u\neq 0\}}|\nabla u|^{p-2}\left\{(p-2)(\frac{\nabla u}{|\nabla u|}\cdot\nabla\psi)^2+|\nabla\psi|^2\right\}-\int_\Omega f_u(x,u)\psi^2\geq 0 \quad (1.8)$$

for all $\psi\in\mathcal{A}_u$.

Note that the left hand side of (1.8) is formally the second variation of J at u. The first integral in (1.8) is well defined and finite since its integrand belongs to L^1. This follows from Hölder inequality when $p\geq 2$ (and in this case the integral can be computed in all of Ω instead of $\{\nabla u\neq 0\}$), and from the pointwise bound for $|\nabla\psi|$ in (1.7) when $1<p<2$. On the other hand, the second integral in (1.8) is well defined in $[0,+\infty]$ since $f_u\geq 0$ by hypothesis. Therefore, the left hand side of (1.8) is a well defined quantity in $[-\infty,+\infty)$. In particular, if u satisfies inequality (1.8) then the second integral will be finite.

For $1<p<2$ we have introduced a class $\mathcal{A}_u$ of admissible functions in order that the second variation of energy is well defined. We have found that the class given by (1.7) is appropriate in all of our arguments, but there could be other good classes. Since the set of test functions $\mathcal{A}_u$ is smaller than usual, the class of semi-stable solutions could seem to be too large. However, by using adequate test functions in $\mathcal{A}_u$, we will prove existence and uniqueness results for semi-stable solutions (Theorem 1.4 and Theorem 1.5), and also obtain sharp regularity results for these solutions (Theorem 1.2).

The first and second variation of energy is analyzed in detail in section 2. We will see that in the presence of sub and supersolutions (perhaps unbounded), J is well defined in a closed convex set M, even for general reaction terms f (not necessarily with power growth). Moreover, we will prove that the infimum of J in M is achieved at some $u\in M$. This minimizer u will be a solution of (1.1_p) and, in addition, it will be semi-stable in the sense of Definition 1.1.

Our first result establishes an $L^\infty(\Omega)$ bound for every semi-stable solution of (1.1_p), with $p>1$ arbitrary, whenever the growth exponent m for the reaction term is smaller than a certain exponent $m_{cs}(p)$ defined below. The estimate requires an additional power growth assumption on f related to the exponent m. It extends a result (that we describe below) obtained for $p=2$ by Crandall and Rabinowitz [CR75] and by Mignot and Puel [MP80].

Theorem 1.2. *For* $p > 1$ *define*

$$m_{cs}(p) := \begin{cases} +\infty & \textit{if } N \le p + \dfrac{4p}{p-1}, \\ \dfrac{(p-1)N - 2\sqrt{(p-1)(N-1)} + 2 - p}{N - (p+2) - 2\sqrt{\frac{N-1}{p-1}}} & \textit{if } N > p + \dfrac{4p}{p-1}. \end{cases} \tag{1.9}$$

Let $u \in W_0^{1,p}(\Omega)$ *be a semi-stable solution of* (1.1_p)*. Assume that* f *satisfies* (1.2) *and*

$$\liminf_{t\to+\infty} \frac{f_t(x,t)t}{f(x,t)} \ge m \tag{1.10}$$

uniformly in a.e. $x \in \Omega$*, for some* $m > p-1$.

Then,

$$\|u\|_{W_0^{1,p}} \le C \tag{1.11}$$

for some constant C *depending only on* N*,* m*,* p*,* $|\Omega|$*, and* f.

If in addition $m < m_{cs}(p)$*, then* $u \in L^\infty(\Omega)$ *and*

$$\|u\|_\infty \le C, \tag{1.12}$$

where C *is a constant depending only on* N*,* m*,* p*,* $|\Omega|$*, and* f.

The way in which the constants C in (1.12) and (1.11) depend on the nonlinearity f is explained in detail in Remark 3.4 and will be important for other results and proofs in the article.

Theorem 1.2 applies to the nonlinearity $f = a(x)(1+u)^m$, and also $f = a(x)u^m$, for every positive and bounded function a.

It can be easily checked that $N > p + 4p/(p-1)$ is necessary and sufficient for the denominator in the expression (1.9) to be positive and define a finite exponent $m_{cs}(p)$. It is also easy to verify that, whenever the Sobolev critical exponent $m_c(p)$ defined in (1.4) is finite, we then have $m_c(p) < m_{cs}(p)$. One can verify also that, if $m > p-1$, then $m < m_{cs}(p)$ is equivalent to

$$N < G(p,m) := \frac{p}{p-1}\left(1 + \frac{pm}{m-(p-1)} + 2\sqrt{\frac{m}{m-(p-1)}}\right), \tag{1.13}$$

an inequality that we will use in some proofs.

Our next result establishes the optimality of the exponent $m_{cs}(p)$ in Theorem 1.2 for the boundedness of semi-stable solutions.

Proposition 1.3. *Let* $\Omega = B_1$*. Assume* $N > p$ *and* $m > (p-1)N/(N-p)$*. Let*

$$U^{\#}(x) = |x|^{\frac{-p}{m-(p-1)}} - 1, \tag{1.14}$$

$$\lambda^{\#} = \left(\frac{p}{m-(p-1)}\right)^{p-1} \left[N - \frac{mp}{m-(p-1)}\right], \tag{1.15}$$

and

$$f(u) = \lambda^{\#}(1+u)^m.$$

We then have:

(i) $U^{\#} \in W_0^{1,p}(\Omega)$ *if and only if* $m > m_c(p)$. *In such case,* $U^{\#}$ *is a solution of* (1.1$_p$).

(ii) *Assume* $m > m_c(p)$. *Then,* $U^{\#}$ *is a semi-stable solution of* (1.1$_p$) *if and only if* $m \geq m_{cs}(p)$.

Throughout the paper we consider solutions in $W_0^{1,p}(\Omega)$. In Theorem 1.2 this assumption is necessary. Indeed, for a certain range of exponents m with $(p-1)N/(N-p) < m \leq m_c(p) \leq m_{cs}(p)$, the function $U^{\#}$ of Proposition 1.3 is an entropy solution (but not in $W_0^{1,p}(\Omega)$), it satisfies the semi-stability condition (1.8), and however it is unbounded. See Remark 5.2 for more details and Theorem 6.2 in [BV97] for the case $p = 2$.

Theorem 1.2 will be proved in two steps. Following a method of [CR75] for $p = 2$, we first obtain an *a priori* $L^q(\Omega)$ estimate for semi-stable solutions of (1.1$_p$) using hypothesis (1.10) on f and the semi-stability condition (1.8). We then improve this regularity using assumption (1.2) on f and a bootstrap argument. On the other hand, the proof of Proposition 1.3 is simple and relies on a Hardy type inequality.

The two previous results establish that semi-stable solutions of (1.1$_p$) enjoy better regularity properties than general solutions. This fact has been already studied in relation with the so called extremal solutions —a class of solutions which turn out to be semi-stable in most cases, for instance when f is convex or when $p \geq 2$. To introduce the concept of extremal solution, consider the problem

$$\begin{cases} -\Delta_p u = \lambda f(u) & \text{in } \Omega, \\ u \geq 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.16$_{\lambda,p}$}$$

where $\lambda > 0$ and f is an increasing C^1 function with $f(0) > 0$ and

$$\lim_{t\to+\infty} \frac{f(t)}{t^{p-1}} = +\infty. \tag{1.17}$$

For $p = 2$ it is well known the existence of an extremal parameter $\lambda^* \in (0, +\infty)$ such that: if $\lambda \in (0, \lambda^*)$ then problem (1.16$_{\lambda,2}$) admits a regular solution u_λ which is minimal among all other solutions, while if $\lambda > \lambda^*$ then problem $(1.16_{\lambda,2})$ admits no regular solution. It is known that the minimal solutions u_λ are semi-stable. Their increasing limit $u^* := \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a weak solution of $(1.16_{\lambda^*,2})$; u^* is called the extremal solution. Brezis and Vázquez [BV97] proved, under an additional hypothesis on f, that u^* belongs to $W_0^{1,2}(\Omega)$ and that it is a semi-stable solution.

Define

$$m_{cs}(2) := \begin{cases} +\infty & \text{if } N \le 10, \\ \dfrac{N - 2\sqrt{N-1}}{N - 4 - 2\sqrt{N-1}} & \text{if } N > 10, \end{cases}$$

as in (1.9). For $p = 2$, Crandall and Rabinowitz [CR75] and Mignot and Puel [MP80] studied the case $f(u) = (1 + u)^m$ and proved that u^* is bounded if $m < m_{cs}(2)$. Joseph and Lundgren [JL73] used phase plane techniques to make a detailed analysis of all solutions when the domain Ω is a ball. In particular, they showed that if $m \ge m_{cs}(2)$ then u^* is unbounded. More recently, Brezis and Vázquez [BV97] have introduced a simpler approach to this question based on PDE techniques (and not in phase plane analysis). They characterized singular extremal solutions by their semi-stability property. In this paper we extend the PDE techniques of [BV97] to certain cases where $p \ne 2$.

First we state our result on existence and properties of minimal and extremal solutions for every $p > 1$. Point (i) of the following theorem uses ideas on existence of solutions from [GP92, GPP94]. Part of point (ii) extends a $W_0^{1,2}$ regularity result of [BV97]. Point (iii) on nonexistence of energy (perhaps singular) solutions extends a result for $p = 2$ from [BCMR96].

Theorem 1.4. *Let $p > 1$ and assume that $f = f(u)$ is an increasing C^1 function satisfying $f(0) > 0$ and (1.17). Then, there exists $\lambda^* \in (0, \infty)$ such that:*

(i) *If $\lambda \in (0, \lambda^*)$, then problem $(1.16_{\lambda,p})$ admits a minimal regular solution u_λ. Minimal means that it is smaller than any other supersolution of the problem. In particular, the family $\{u_\lambda\}$ is increasing in λ. Moreover, every u_λ is semi-stable.*

If $\lambda > \lambda^*$, then problem $(1.16_{\lambda,p})$ admits no regular solution.

(ii) *Assume that in addition f satisfies (1.2) and (1.10) for some $m > p - 1$. Then:*

(ii1) *$u^* := \lim_{\lambda\uparrow\lambda^*} u_\lambda$ belongs to $W_0^{1,p}(\Omega)$ and it is a solution of $(1.16_{\lambda^*,p})$.*

(ii2) *If either $p \ge 2$, or $1 < p < 2$ and f is convex, then u^* is semi-stable.*

(ii3) *If $m < m_{cs}(p)$, then $u^* \in L^\infty(\Omega)$.*

(iii) *If in addition* $f(t)^{\frac{1}{p-1}}$ *is a convex function satisfying*

$$\int_0^\infty \frac{dt}{f(t)^{\frac{1}{p-1}}} < +\infty, \tag{1.18}$$

then $(1.16_{\lambda,p})$ *admits no solution for* $\lambda > \lambda^*$.

The family of minimal solutions is a continuous branch when $p = 2$ and f is, in addition, convex. In the generality of the previous theorem, the family may be discontinuous and have jumps at some parameters λ (see [CC05] for an example when $p = 2$ and f is not convex).

Minimal and extremal solutions of $(1.16_{\lambda,p})$ for $p > 1$ have been studied, when $f(u) = e^u$, by García-Azorero, Peral, and Puel [GP92, GPP94]. They established the boundedness of the extremal solution when $N < p + 4p/(p-1)$, and showed that this condition is optimal. Recently we have learned about the work of Ferrero [Fe04], carried out independently of ours, where problem $(1.16_{\lambda,p})$ is studied for the model case $f(u) = (1+u)^m$. [Fe04] establishes the sufficiency of condition $m < m_{cs}(p)$ for the extremal solution of $(1.16_{\lambda,p})$ to be bounded. In Remark 1.7 we describe further regularity results on semi-stable and extremal solutions.

While the nonexistence of *regular* solutions for $\lambda > \lambda^*$ is an immediate fact, part (iii) of Theorem 1.4 establishes the nonexistence of $W_0^{1,p}$ solutions (possibly unbounded). It uses the ideas of Brezis et al. [BCMR96] for the Laplacian case.

Our following result extends Theorem 3.1 of [BV97] (that dealt with $p = 2$, convex nonlinearities f, and smooth bounded domains Ω) to the case $1 < p < 2$ and $\Omega = B_1$. It is a characterization of singular extremal solutions of $(1.16_{\lambda,p})$ by their semi-stability property.

Theorem 1.5. *Assume that* $\Omega = B_1 \subset \mathbb{R}^N$, $1 < p < 2$, *and that* f *is a* C^1, *increasing, and convex function satisfying* $f(0) > 0$. *Then:*

(i) *For* $\lambda < \lambda^*$, *the minimal solution* u_λ *of* $(1.16_{\lambda,p})$ *is the unique radially nonincreasing and semi-stable solution of* $(1.16_{\lambda,p})$.

(ii) *Assume that* f *satisfies in addition* (1.2), (1.10), *and* (1.18), *for some* $m > p-1$. *Assume that* $v \in W_0^{1,p}(\Omega)$ *is an unbounded, radially nonincreasing, and semi-stable solution of* $(1.16_{\lambda,p})$ *for some* $\lambda > 0$. *Then,* $\lambda = \lambda^*$ *and* $v = u^*$.

The nonlinearity $f(u) = (1+u)^m$, with $m \geq 1$, satisfies all the assumptions in parts (i) and (ii) of Theorem 1.5.

For $\Omega = B_1$, Damascelli and Sciunzi [DS04] recently used the moving planes method to show that every regular solution of $(1.16_{\lambda,p})$ is radially decreasing if f is nonnegative, continuous in $[0,\infty)$, and locally Lipschitz in $(0,\infty)$. As a consequence, the minimal solution u_λ of $(1.16_{\lambda,p})$ is radially decreasing. Letting $\lambda \uparrow \lambda^*$, it follows that the extremal solution is radially nonincreasing.

As a consequence of Theorem 1.5 and Proposition 1.3, we can identify the extremal solution and parameter for the pure power nonlinearity when $1 < p < 2$ and $m \geq 1$ through a pure PDE argument. In the general case $p > 1$, the same result has been proved, independently of ours, in [Fe04] by using phase plane techniques.

Corollary 1.6. *Assume that* $\Omega = B_1 \subset \mathbb{R}^N$*,* $1 < p < 2$*,* $f(u) = (1+u)^m$*, and* $m \geq \max\{1, m_{cs}(p)\}$*. Then, the extremal solution and parameter of* $(1.16_{\lambda,p})$ *are* $u^* = U^\#$ *and* $\lambda^* = \lambda^\#$*, where* $U^\#$ *and* $\lambda^\#$ *are given by* (1.14) *and* (1.15).

Remark 1.7. In [San05] the second author studies the regularity of the extremal solution to problem $(1.16_{\lambda,p})$ in smooth bounded domains when $p \geq 2$ and $(f(u) - f(0))^{1/(p-1)}$ is a positive, increasing, and convex function satisfying (1.17). [San05] establishes the boundedness of the extremal solution whenever $N < p + p/(p-1)$, extending an important work of Nedev [Ne00] for $p = 2$. Note that these two works make no additional growth assumption on f besides the convexity hypothesis above.

The first author and Capella [CC05] prove optimal results for the regularity of semi-stable solutions of (1.1_2) when $\Omega = B_1$ and $f = f(u)$ is a general locally Lipschitz function. For instance, [CC05] establishes that every radial semi-stable solution is bounded if $N \leq 9$. In general bounded domains of $\mathbb{R}^N$ it is still an open problem to prove (or disprove) the boundedness of every semi-stable solution when $4 \leq N \leq 9$ and $p = 2$. On the other hand, the authors and Capella [CCS05] extend the radial results of [CC05] for $p = 2$ to the case $p > 1$ obtaining, for instance, the boundedness of every radial semi-stable solution when $N < p + 4p/(p-1)$.

The paper is organized as follows. In section 2 we study the first and second variation of energy in appropriate closed convex sets of $W_0^{1,p}(\Omega)$. Section 3 is concerned with some regularity results for the p-Laplacian and with the proof of the L^∞ estimate of Theorem 1.2. In section 4 we establish Theorem 1.4 on minimal and extremal solutions. Finally, in section 5 we prove Proposition 1.3, Theorem 1.5, and Corollary 1.6.

2. First and second variation of energy

We consider

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $f(x,t)$ is a nonnegative and nondecreasing function of t for a.e. $x \in \Omega$.

We say that $u \in W_0^{1,p}(\Omega)$ is a supersolution of (2.1) if $f(x,u) \in L^1(\Omega)$ and $-\Delta_p u \geq f(x,u)$ in the weak sense. Reversing the inequality one defines the notion of subsolution.

Assume that there exist two $W_0^{1,p}(\Omega)$ functions $\underline{u}$ and $\overline{u}$ (perhaps unbounded) which are a sub and a supersolution of (2.1), respectively, such that $\underline{u} \leq \overline{u}$, and consider the closed convex set

$$M_{\underline{u},\overline{u}} := \{v \in W_0^{1,p}(\Omega) : \underline{u} \leq v \leq \overline{u} \text{ a.e.}\}. \tag{2.2}$$

We consider the energy functional J associated to (2.1)

$$J(v) := \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega \underline{F}(x,v), \tag{2.3}$$

where

$$\underline{F}(x,t) := \int_{\underline{u}(x)}^t f(x,s)ds. \tag{2.4}$$

We note that the functional J defined here may be different from the one defined in (1.5) and (1.6). We use the same notation for both functionals since there is not risk of confusion. We also note that both functionals coincide when $\underline{u} \equiv 0$.

In the following result, we show that J is well defined and bounded from below in $M_{\underline{u},\overline{u}}$ and that it attains its infimum at some $u \in M_{\underline{u},\overline{u}}$, which is a solution of (2.1). This result is standard when $\underline{u}$ and $\overline{u}$ are bounded. Instead, here we allow these functions to be unbounded. The case $p = 2$ has been studied in [CM96], a paper that, in addition, introduces a truncated energy functional which satisfies the Palais-Smale condition in $W_0^{1,2}(\Omega)$.

Proposition 2.1. *Assume that $f(x,t)$ is nonnegative and nondecreasing in t for a.e. $x \in \Omega$. Let $\underline{u}$ and $\overline{u}$ be a subsolution and a supersolution of* (2.1)*, respectively, such that $\underline{u} \leq \overline{u}$. Let $M_{\underline{u},\overline{u}}$ be defined by* (2.2) *and consider $J : M_{\underline{u},\overline{u}} \to \mathbb{R}$ defined by* (2.3) *and* (2.4)*. Then, the following assertions hold:*

(i) *J is well defined in $M_{\underline{u},\overline{u}}$, bounded from below, coercive, and weakly lower semicontinuous in $M_{\underline{u},\overline{u}}$. Moreover, J attains its infimum at some $u \in M_{\underline{u},\overline{u}}$, which is a solution of* (2.1).

(ii) *There exists a solution* u_m *of* (2.1)*, with* $\underline{u} \le u_m \le \overline{u}$*, which is minimal among all possible supersolutions* v *satisfying* $\underline{u} \le v$*.*

Proof: (i) First, we claim that $v \longmapsto f(x,v)$ defines a (uniformly) bounded map from $M_{\underline{u},\overline{u}}$ to $W^{-1,p'}(\Omega)$. Indeed, let $v \in M_{\underline{u},\overline{u}}$ and $\varphi \in C_c^\infty(\Omega)$. Using that $f(x,t)$ is nonnegative and nondecreasing in t for a.e. $x \in \Omega$, that $\overline{u}$ is a supersolution of (2.1), and Hölder inequality, we get

$$\begin{aligned} \left| \int_\Omega f(x,v)\varphi \right| &\le \int_\Omega f(x,\overline{u})|\varphi| \le \int_\Omega |\nabla \overline{u}|^{p-2} \nabla \overline{u} \cdot \nabla |\varphi| \\ &\le \|\nabla \overline{u}\|_p^{p-1} \|\nabla \varphi\|_p. \end{aligned} \tag{2.5}$$

Hence, the claim and

$$\|f(x,v)\|_{W^{-1,p'}(\Omega)} \le \|\nabla \overline{u}\|_p^{p-1}$$

follow by a standard density argument.

Using the definition of $\underline{F}$ and (2.5) with $0 \le \varphi = v - \underline{u}$, we get $\underline{F}(x,v) \ge 0$ and

$$\int_\Omega \underline{F}(x,v) \le \int_\Omega f(x,v)(v - \underline{u}) \le \|\nabla \overline{u}\|_p^{p-1} \|\nabla (v - \underline{u})\|_p. \tag{2.6}$$

From (2.6) it is easy to show that J is well defined in $M_{\underline{u},\overline{u}}$, bounded from below, and coercive. To prove that J is weakly lower semicontinuous, it suffices to show that

$$\int_\Omega \underline{F}(x,v_m) \to \int_\Omega \underline{F}(x,v)$$

if $v_m \in M_{\underline{u},\overline{u}}$, $v_m \rightharpoonup v$ weakly in $M_{\underline{u},\overline{u}}$. Noting that

$$|\underline{F}(x,v_m)| = \int_{\underline{u}}^{v_m} f(x,s)ds \le f(x,\overline{u})(\overline{u} - \underline{u}),$$

and that the right hand side of the last inequality belongs to $L^1(\Omega)$ by (2.6) applied with $v = \overline{u}$, we may appeal to the dominated convergence theorem to conclude. As a consequence, we obtain that J attains its infimum at some $u \in M_{\underline{u},\overline{u}}$.

Finally, we prove that every minimizer u in $M_{\underline{u},\overline{u}}$ is a solution of (2.1) following a method used for $p = 2$ in [Stru90]. For this, we need the following preliminary observation.

Let $v \in M_{\underline{u},\overline{u}}$, $\psi \in W_0^{1,p}(\Omega)$ with $\psi \not\equiv 0$, and assume that $v + t\psi \in M_{\underline{u},\overline{u}}$ for all $t \in [0, t_0]$, with $t_0 > 0$. In such case, it is easy to prove that $J(v + t\psi)$ is differentiable with respect to t for $t \in [0, t_0]$, and its derivative is given by

$$\frac{d}{dt}J(v + t\psi) = \int_\Omega |\nabla(v + t\psi)|^{p-2}\nabla(v + t\psi) \cdot \nabla\psi - \int_\Omega f(x, v + t\psi)\psi \quad (2.7)$$

for all $t \in [0, t_0]$. That is, J is differentiable at v in the direction $t\psi$, $t \geq 0$, and its directional derivative is given by

$$J'(v)\psi := \int_\Omega |\nabla v|^{p-2}\nabla v \cdot \nabla\psi - \int_\Omega f(x, v)\psi. \quad (2.8)$$

The notation $J'(v)$ that we have introduced is not meant to be understood as that the functional J is differentiable (note that J is not even defined in an open set of $W_0^{1,p}(\Omega)$). What we mean is first that the directional derivative exists and that it is given by the right hand side of (2.8). Second, that the right hand side of (2.8) defines a continuous linear form, which we denote by $J'(v)$, on $W_0^{1,p}(\Omega)$. Note also that $J'(v) = 0$ (in the sense of forms) means exactly that v is a solution of (2.1).

To show that every minimizer u in $M_{\underline{u},\overline{u}}$ is a solution, let $\varphi \in C_c^\infty(\Omega)$ and $\varepsilon > 0$. Consider

$$v_\varepsilon = \min\{\overline{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in M_{\underline{u},\overline{u}}$$

with $\varphi^\varepsilon = (u+\varepsilon\varphi-\overline{u})^+$ and $\varphi_\varepsilon = (u+\varepsilon\varphi-\underline{u})^-$. Noting that $u+t(v_\varepsilon-u) \in M_{\underline{u},\overline{u}}$ for all $t \in [0, 1]$ we obtain that J is differentiable at u in the direction $v_\varepsilon - u$. Since u minimizes J in $M_{\underline{u},\overline{u}}$, we have

$$0 \leq J'(u)(v_\varepsilon - u),$$

and hence, since $J'(u)$ is a linear form on $W_0^{1,p}(\Omega)$,

$$0 \leq \varepsilon J'(u)\varphi - J'(u)\varphi^\varepsilon + J'(u)\varphi_\varepsilon.$$

As a consequence, we obtain

$$J'(u)\varphi \geq \frac{1}{\varepsilon}[J'(u)\varphi^\varepsilon - J'(u)\varphi_\varepsilon]. \quad (2.9)$$

Next we show that $J'(u)\varphi^\varepsilon \geq \mathrm{o}(\varepsilon)$, meaning that $\liminf_{\varepsilon\to 0} \varepsilon^{-1}J'(u)\varphi^\varepsilon \geq 0$. Indeed, since $\overline{u}$ is a supersolution of (2.1) and $\varphi^\varepsilon \geq 0$, we have

$$J'(\overline{u})\varphi^\varepsilon \geq 0.$$

Therefore

$$
\begin{aligned}
J'(u)\varphi^\varepsilon &\geq (J'(u) - J'(\overline{u}))\varphi^\varepsilon \\
&= \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}) \cdot \nabla\varphi^\varepsilon - \int_\Omega (f(x,u) - f(x,\overline{u}))\varphi^\varepsilon \\
&\geq \int_\Omega (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}) \cdot \nabla\varphi^\varepsilon, \qquad (2.10)
\end{aligned}
$$

where in the last inequality we have used that f is nondecreasing. Using $\varphi^\varepsilon = (u + \varepsilon\varphi - \overline{u})^+$ in (2.10), and also

$$
(|x|^{p-2}x - |y|^{p-2}y) \cdot (x - y) \geq 0 \quad \text{for all } x, y \in \mathbb{R}^N
$$

(see for instance Appendix A in [Pe97] for a proof of this inequality) with $x = \nabla u$ and $y = \nabla \overline{u}$, we obtain

$$
J'(u)\varphi^\varepsilon \geq \varepsilon \int_{\Omega_\varepsilon} (|\nabla u|^{p-2}\nabla u - |\nabla \overline{u}|^{p-2}\nabla \overline{u}) \cdot \nabla\varphi,
$$

where $\Omega_\varepsilon = \{x \in \Omega : u(x) < \overline{u}(x) \leq u(x) + \varepsilon\varphi(x)\}$.

Noting that the measure of Ω_ε tends to zero as ε goes to zero, we obtain that $J'(u)\varphi^\varepsilon \geq \mathrm{o}(\varepsilon)$. Proceeding in an analogous way, we also obtain that $J'(u)\varphi_\varepsilon \leq \mathrm{o}(\varepsilon)$. Therefore, from (2.9) we obtain $J'(u)\varphi \geq 0$ for all $\varphi \in C_c^\infty(\Omega)$. The previous inequality applied to $-\varphi$ instead of φ gives $J'(u)\varphi = 0$ for all $\varphi \in C_c^\infty(\Omega)$. That is, u is a solution of (2.1).

(ii) Let $u^0 := \underline{u}$ and remember that $f(x, \underline{u}) \in W^{-1,p'}(\Omega)$. Let u^1 be the solution of

$$
\begin{cases} -\Delta_p u^1 = f(x, u^0) & \text{in } \Omega, \\ u^1 = 0 & \text{on } \partial\Omega. \end{cases}
$$

Since $-\Delta_p u^0 \leq f(x, u^0) \leq f(x, \overline{u}) \leq -\Delta_p \overline{u}$, we have that $\underline{u} = u^0 \leq u^1 \leq \overline{u}$ by the weak comparison principle (see for instance Appendix A in [Pe97] for a proof). Moreover, since $f(x,t)$ is nondecreasing in t, one has that u^1 is a subsolution of (2.1). In addition, we know that $f(x, u^1) \in W^{-1,p'}(\Omega)$.

Now, given u^{n-1}, we take the solution u^n of

$$
\begin{cases} -\Delta_p u^n = f(x, u^{n-1}) & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega. \end{cases} \qquad (2.11)
$$

We obtain a nondecreasing sequence $\{u^n\}$ such that $\underline{u} \le u^n \le \overline{u}$ and $f(x,u^n)$ belongs to $W^{-1,p'}(\Omega)$ for all $n \ge 0$. Moreover, since $\{u^n\}$ is nondecreasing and

$$\int_\Omega |\nabla u^n|^p = \int_\Omega f(x,u^{n-1})u^n \le \int_\Omega f(x,\overline{u})\overline{u} \le \|f(x,\overline{u})\|_{W^{-1,p'}(\Omega)} \|\overline{u}\|_{W_0^{1,p}(\Omega)}$$

for all n, we have that $u^n \rightharpoonup u_m$ weakly in $W_0^{1,p}(\Omega)$, for some function $u_m \in M_{\underline{u},\overline{u}}$. In addition, we have monotone convergence a.e. of u^n towards u_m, and hence also monotone convergence of $f(x,u^n)$ to $f(x,u_m)$. This last convergence is also in the $L^1(\Omega)$ sense, since $0 \le f(x,u^n) \le f(x,\overline{u}) \in L^1(\Omega)$ for all n.

In this situation, the results of Boccardo and Murat [BM92] (Theorem 2.1 and Remark 2.1 in [BM92]) establish that ∇u^n converges strongly in L^q for all $q < p$. This allows to pass to the limit in the left hand side (2.11) and deduce that that u_m is a solution of (2.1). Note that since the p-Laplacian is a nonlinear operator, the weak convergence of the gradients is not enough to pass to the limit.

It is also clear that u_m is minimal, since every supersolution of (2.1) could be taken as $\overline{u}$ in the iterative scheme above (note that the u_m constructed does not depend on the choice of $\overline{u}$). ∎

Our following result concerns the second variation of J. Under some conditions on f, it establishes that every absolute minimizer of J in $M_{0,\overline{u}}$ is semi-stable in the sense of Definition 1.1.

Proposition 2.2. *Assume that $f(x,t)$ is nonnegative, nondecreasing and C^1 in t for a.e. $x \in \Omega$, and that $f(\cdot,0)$ is not identically zero. Let $\overline{u}$ be a supersolution of* (1.1_p)*, and assume that there exists $h \in L^1(\Omega)$ such that*

$$f_u(x,w)\overline{u}^2 \le h \quad \text{in } \Omega \text{ for all } w \in M_{0,\overline{u}}, \tag{2.12}$$

where $M_{0,\overline{u}}$ is defined by (2.2).

We then have that every absolute minimizer u of J in $M_{0,\overline{u}}$ is semi-stable. In addition, the minimal solution of (1.1_p) *in $M_{0,\overline{u}}$ is semi-stable.*

Proof: By Proposition 2.1 we know that every absolute minimizer u of J in $M_{0,\overline{u}}$ is a solution of (1.1_p) (and also that at least one minimizer always exists). In addition, $u \not\equiv 0$ since $f(\cdot,0) \not\equiv 0$ by hypothesis. We consider two cases:

Case 1. Assume $p \ge 2$ and let $\psi \in C_c^\infty(\Omega)$ be nonnegative. Since the support of ψ is a compact subset of Ω, we have that $\overline{u} \ge u \ge c$ a.e. in supp ψ for some positive constant c. This follows from the weak Harnack inequality of Trudinger (see [Tr67], or [MZ97] when $p \le N$; note that if $p > N$ then u is continuous and positive, thus the statement is trivial). It also follows from a quantitative version of the strong maximum principle in [Mo99].

Therefore there exists $t_0 > 0$ such that $u + t(-\psi) \in M_{0,u} \subset M_{0,\overline{u}}$ for all $t \in [0, t_0]$. As a consequence, we have (2.7), *i.e.*,

$$\frac{d}{dt}J(u-t\psi) = -\int_\Omega |\nabla(u-t\psi)|^{p-2}\nabla(u-t\psi)\cdot\nabla\psi + \int_\Omega f(x,u-t\psi)\psi, \quad (2.13)$$

for all $t \in [0, t_0]$. It is easy to show, using Hölder inequality, that the first integral in (2.13) is a continuously differentiable function of t in $[0, t_0]$. Moreover, since $\psi \in C_c^\infty(\Omega)$ and

$$0 \le f_u(x, u - t\psi)\psi^2 \le \left(\frac{\psi}{c}\right)^2 f_u(x, u - t\psi)\overline{u}^2 \le \left(\frac{\psi}{c}\right)^2 h(x) \in L^1(\Omega)$$

by hypothesis (2.12), we have that the second integral in (2.13) is also continuously differentiable. Hence, $J(u-t\psi)$ is twice continuously differentiable respect to t for all $t \in [0, t_0]$ and

$$\begin{aligned} &\frac{d^2}{dt^2}J(u - t\psi) \\ &\quad = \int_\Omega |\nabla(u - t\psi)|^{p-2}\left\{(p-2)\Big(\frac{\nabla(u - t\psi)}{|\nabla(u - t\psi)|}\cdot\nabla\psi\Big)^2 + |\nabla\psi|^2\right\} \\ &\qquad - \int_\Omega f_u(x, u - t\psi)\psi^2 \quad \text{for all } t \in [0, t_0]. \end{aligned}$$

Since u is an absolute minimizer of J in $M_{0,\overline{u}}$ and $J'(u) = 0$, we obtain (1.8) (that is, $J''(u)(\psi, \psi) \ge 0$, where $J''(u)$ is the quadratic form on $W_0^{1,p}(\Omega)$ given by the left hand side of (1.8)) for all nonnegative $\psi \in C_c^\infty(\Omega)$. By density, $J''(u)(\psi, \psi) \ge 0$ also holds for all nonnegative $\psi \in W_0^{1,p}(\Omega)$. Now, writing any $\psi \in W_0^{1,p}(\Omega)$ as its positive part minus its negative part and using that $J''(u)$ is a quadratic form, we conclude that (1.8) also holds for all $\psi \in W_0^{1,p}(\Omega) = \mathcal{A}_u$.

Case 2. Assume $1 < p < 2$ and let $\psi \in \mathcal{A}_u$ be nonnegative. By definition of $\mathcal{A}_u$ there exists a positive constant C such that $0 \le \psi \le Cu$ and $|\nabla\psi| \le C|\nabla u|$. In particular,

$$(1 - Ct)u \le u - t\psi \le u \quad \text{and} \quad (1 - Ct)|\nabla u| \le |\nabla(u - t\psi)| \le (1 + Ct)|\nabla u|$$

for all $t \ge 0$. Therefore, for all $t \in [0, 1/C)$, $u - t\psi \in M_{0,u} \subset M_{0,\overline{u}}$, and $\nabla(u - t\psi) = 0$ if and only if $\nabla u = 0$. Thus, from (2.7) with $t_0 = 1/C$, we obtain

$$\frac{d}{dt}J(u - t\psi) = -\int_{\{\nabla u\ne 0\}} |\nabla(u - t\psi)|^{p-2}\nabla(u - t\psi)\cdot\nabla\psi + \int_\Omega f(x, u - t\psi)\psi,$$

for all $t \in [0, 1/C)$. Now, using that

$$|\nabla(u - t\psi)|^{p-2}|\nabla\psi|^2 \leq \frac{C^2}{(1 - Ct)^{2-p}}|\nabla u|^p$$

and

$$f_u(x, u - t\psi)\psi^2 \leq C^2 f_u(x, u - t\psi)\overline{u}^2 \leq C^2 h(x) \in L^1(\Omega)$$

by hypothesis, it is easy to show, using the dominated convergence theorem, that $J(u - t\psi)$ is twice continuously differentiable in $t \in [0, 1/C)$ and

$$\begin{aligned} &\frac{d^2}{dt^2}J(u - t\psi) \\ &= \int_{\{\nabla u \neq 0\}} |\nabla(u - t\psi)|^{p-2}\left\{(p-2)\left(\frac{\nabla(u - t\psi)}{|\nabla(u - t\psi)|} \cdot \nabla\psi\right)^2 + |\nabla\psi|^2\right\} \\ &\quad - \int_\Omega f_u(x, u - t\psi)\psi^2 \quad \text{for all } t \in [0, 1/C). \end{aligned} \tag{2.14}$$

Since u is an absolute minimizer of J in $M_{0,\overline{u}}$ and $J'(u) = 0$, we obtain that (2.14) is nonnegative at $t = 0$ for all $0 \leq \psi \in \mathcal{A}_u$. That is, $J''(u)(\psi, \psi) \geq 0$ for all nonnegative $\psi \in \mathcal{A}_u$, where $J''(u)$ is the quadratic form defined by the left hand side of (1.8). Noting, as in case 1, that $J''(u)$ is a quadratic form, and that the positive and negative parts of a function $\psi \in \mathcal{A}_u$ also belong to $\mathcal{A}_u$, we conclude that (1.8) holds for all $\psi \in \mathcal{A}_u$. Hence, u is a semi-stable solution of (2.1).

Finally, since the minimal solution u_m of (1.1_p) (obtained in Proposition 2.1(ii)) is the unique solution of this problem in M_{0,u_m}, and therefore the absolute minimizer of J in M_{0,u_m}, we conclude from the previous result (applied with $\overline{u} = u_m$) that u_m is semi-stable. ∎

3. L^∞ estimate: proof of Theorem 1.2

To prove our regularity result we will use the following lemma from [Gre02] and [ABFOT03].

Lemma 3.1. *Assume that $g \in L^q(\Omega)$ for some $q \geq 1$ and that u is a solution of*

$$\begin{cases} -\Delta_p u = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The following assertions hold:

(i) *If $q > N/p$ then $u \in L^\infty(\Omega)$. Moreover,*

$$\|u\|_\infty \leq C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, q, and $|\Omega|$.

(ii) *If $q = N/p$ then $u \in L^r(\Omega)$ for all $1 \le r < +\infty$. Moreover,*

$$\|u\|_r \le C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, r, and $|\Omega|$.

(iii) *If $1 \le q < N/p$ then $|u|^r \in L^1(\Omega)$ for all $0 < r < r_1$, where $r_1 := (p-1)Nq/(N-qp)$. Moreover,*

$$\||u|^r\|_1^{1/r} \le C\|g\|_q^{\frac{1}{p-1}},$$

where C is a constant depending only on N, p, q, r, and $|\Omega|$.

Note that in part (iii), r may be less than 1. This case, $0 < r < 1$, is not considered in Corollary 1 of [Gre02], but it follows easily from Theorem 1 of [Gre02]. This case is however considered in [ABFOT03]. Here, for the sake of completeness, we include the proof of Lemma 3.1. We have slightly modified the proofs in [Gre02] and [ABFOT03], in the spirit of Talenti [Ta79], using Jensen inequality instead of Hölder inequality.

Proof of Lemma 3.1: Let $u \in W_0^{1,p}(\Omega)$ be a solution of (3.1). A consequence of the Fleming-Rishel formula [FR60] and the isoperimetric inequality for functions of bounded variation (and hence for functions in $W_0^{1,p}(\Omega)$) is the following inequality:

$$CV(t)^{(N-1)/N} \le P(t) = \frac{d}{dt}\int_{\{|u|\le t\}} |\nabla u| dx \qquad \text{for a.e. } t > 0, \tag{3.2}$$

where $C = N|B_1|^{1/N}$, $V(t) = |\{x \in \Omega : |u| > t\}|$, and $P(t)$ stands for the perimeter in the sense of De Giorgi, *i.e.*, $P(t)$ is the total variation of the characteristic function of $\{|u| > t\}$. A proof of this inequality can be found in [Ta79], page 172. We also note that $V(t)$ is differentiable almost everywhere since it is a nonincreasing function.

Let

$$\theta_{h,t}(s) := \begin{cases} 0 & \text{if } 0 \le s \le t, \\ (s-t)/h & \text{if } t < s < t+h, \\ 1 & \text{if } s \ge t+h, \end{cases}$$

and $\theta_{h,t}(-s) := -\theta_{h,t}(s)$ for $s \geq 0$. Multiplying (3.1) by $\theta_{h,t}(u)$ and using Jensen inequality we obtain

$$\left(\frac{h}{|E_{h,t}|}\right)^{p-1}\left(\frac{1}{h}\int_{E_{h,t}}|\nabla u|dx\right)^{p} \leq \frac{1}{h}\int_{E_{h,t}}|\nabla u|^{p}\,dx = \int_{\{|u|>t\}} g(x)\theta_{h,t}(u)\,dx,$$

where $E_{h,t} := \{t < u \leq t+h\}$. Letting $h \downarrow 0$ and using Hölder inequality, we have

$$\frac{1}{(-V'(t))^{p-1}}\left(\frac{d}{dt}\int_{\{|u|\leq t\}}|\nabla u|dx\right)^{p} \leq \int_{\{|u|>t\}}|g(x)|\,dx \leq \|g\|_{q}V(t)^{1/q'}. \quad (3.3)$$

Therefore, from (3.2) and (3.3), we obtain

$$\frac{C^{p}V(t)^{p(N-1)/N}}{(-V'(t))^{p-1}} \leq \frac{P(t)^{p}}{(-V'(t))^{p-1}} \leq \|g\|_{q}V(t)^{1/q'},$$

or equivalently,

$$1 \leq \left(\frac{\|g\|_{q}}{C^{p}}\right)^{\frac{1}{p-1}} V(t)^{-1+\frac{1}{p-1}(\frac{p}{N}-\frac{1}{q})}(-V'(t)) \quad (3.4)$$

for a.e. $t > 0$.

Case 1. If $q > N/p$, then (3.4) leads to $V(t) = 0$ for all

$$t \geq t_{0} = -r_{1}\left(\frac{\|g\|_{q}}{C^{p}}\right)^{\frac{1}{p-1}}|\Omega|^{-\frac{1}{r_{1}}},$$

where $r_1 < 0$ is defined in statement (iii) of Lemma 3.1. This yields assertion (i).

Case 2. If $q = N/p$, then (3.4) leads to

$$V(t) \leq |\Omega| \exp\left(-\left(\frac{C^{p}}{\|g\|_{q}}\right)^{\frac{1}{p-1}} t\right).$$

Case 3. If $1 \leq q < N/p$, then (3.4) gives that

$$V(t) \leq (C_{1}t + C_{2})^{-r_{1}},$$

where

$$C_{1} = \frac{1}{r_{1}}\left(\frac{C^{p}}{\|g\|_{q}}\right)^{\frac{1}{p-1}} \quad \text{and} \quad C_{2} = |\Omega|^{-\frac{1}{r_{1}}}.$$

We conclude the proof in cases 2 and 3 noting that

$$\int_{\Omega}|u|^{r}dx = r\int_{0}^{\infty} t^{r-1}V(t)dt,$$

and using the estimates obtained for $V(t)$. ■

To establish Theorem 1.2 we will first prove an extension of Lemma 1.17 in [CR75] to the case $p > 1$.

Lemma 3.2. *Assume that f satisfies* (1.2). *Let u be a solution of* (1.1_p). *If $f(x, u) \in L^{q_0}(\Omega)$ for some $q_0 \geq 1$ satisfying*

$$\left(1 - \frac{p-1}{m}\right) N < q_0 p, \tag{3.5}$$

then

$$\|u\|_\infty \leq C, \tag{3.6}$$

where C is a constant depending only on N, m, p, q_0, $|\Omega|$, c, and $\|f(x, u)\|_{q_0}$. Here c is the constant in (1.2).

Proof: If $N < q_0 p$ then Lemma 3.1(i) leads automatically to (3.6). If $N \geq q_0 p$ then Lemma 3.1(ii)-(iii) gives that

$$|u|^r \in L^1(\Omega) \quad \text{for all } 0 < r < r_1 = (p-1)\frac{Nq_0}{N - q_0 p}.$$

From (1.2) it follows that

$$f(x, u) \in L^q(\Omega) \quad \text{for all } 1 \leq q < q_1 := \frac{p-1}{m}\frac{Nq_0}{N - q_0 p}.$$

Note that $q_1 > q_0 \geq 1$ thanks to (3.5). If $N = q_0 p$ then we have that $f(x, u) \in L^q(\Omega)$ for all $1 \leq q < +\infty$. By Lemma 3.1(i) we obtain (3.6).

Assume $N > q_0 p$. By (3.5) we have that $q_1 > q_0$, and then the previous argument may be repeated successively to obtain an increasing sequence

$$q_{k+1} := \frac{p-1}{m}\frac{Nq_k}{N - q_k p}, \quad 0 \leq k \leq k_0, \tag{3.7}$$

for some $k_0 \leq +\infty$, such that $f(x, u) \in L^q$ if $q < q_k$ for some $k \leq k_0$. Here $k_0 + 1$ denotes the number of times that we can apply this algorithm. Note that we can construct q_{k+1} whenever $N > q_k p$.

We claim that $k_0 < +\infty$. More precisely, there exists $k_0 = k_0(N, m, p, q_0)$ such that $N \leq q_{k_0} p$. Indeed, otherwise $\{q_k\}_{k \in \mathbb{N}}$ is an increasing sequence with limit

$$q_\infty := \lim_{k \to +\infty} q_k = \left(1 - \frac{p-1}{m}\right)\frac{N}{p}$$

by (3.7). Since $\{q_k\}$ is increasing we have that $q_0 \leq q_\infty$, a contradiction with (3.5). Therefore we may assume the existence of $k_0 < +\infty$ such that $N \leq q_{k_0}p$ and $f(x,u) \in L^q(\Omega)$ for all $1 \leq q < q_{k_0}$.

If $N = q_{k_0}p$ then we have that $f(x,u) \in L^q(\Omega)$ for all $1 \leq q < +\infty$. By Lemma 3.1(i) we obtain (3.6). If $N < q_{k_0}p$ one can choose $1 \leq q < q_{k_0}$ such that $N < qp$ and apply again Lemma 3.1(i) to conclude the proof. ■

Remark 3.3. If u is a solution of (1.1_p) then, by definition, $f(x,u) \in L^1(\Omega)$. As a consequence, if $m \leq p-1$ then condition (3.5) holds with $q_0 = 1$, and hence, in this case, every solution of (1.1_p) is bounded.

Using the semi-stability condition (1.8) and Lemma 3.2 we now prove Theorem 1.2. For future results and proofs in the article, it is important to state how the constant C in (1.12) and (1.11) depends on the nonlinearity f.

Remark 3.4. In (1.12) and (1.11), C depends on f only through the exponent m in (1.2), the constant c in (1.2), and the constant L defined as follows. L is the smallest constant such that, for a.e. $x \in \Omega$, we have

$$\frac{f_t(x,t)t}{f(x,t)} \geq \overline{m} \qquad \text{for all } t \geq L, \tag{3.8}$$

where $\overline{m} \in (p-1, m)$ is a constant depending only on N, m, and p, which will be obtained in the proof of Theorem 1.2. Of course, the existence of L is guaranteed by hypothesis (1.10), since $\overline{m} < m$.

Proof of Theorem 1.2: We assume that $m > p-1$ and that u is a semi-stable solution of (1.1_p). For a given $k > 0$ we define the truncation function

$$T_k(s) := \begin{cases} s & \text{if } |s| \leq k, \\ k\ \text{sign}(s) & \text{if } |s| > k. \end{cases}$$

For $\alpha > 1$ (which will be chosen later depending only on N, m, and p), let $\varphi = uT_k(u)^{2\alpha-2}/(2\alpha-1)$ and $\psi = uT_k(u)^{\alpha-1}$, $\alpha > 1$. We note that $\varphi, \psi \in W_0^{1,p}(\Omega)$,

$$0 \leq \psi \leq k^{\alpha-1}u \quad \text{and} \quad |\nabla\psi| \leq \alpha k^{\alpha-1}|\nabla u|.$$

In particular $\psi \in \mathcal{A}_u$.

Multiplying (1.1_p) by φ and integrating, we obtain

$$\int_{\{u\leq k\}} |\nabla u|^p u^{2\alpha-2} + \frac{k^{2\alpha-2}}{2\alpha-1}\int_{\{u>k\}} |\nabla u|^p = \frac{1}{2\alpha-1}\int_\Omega f(x,u)uT_k(u)^{2\alpha-2}.$$

Using this equality and the semi-stability condition (1.8) applied with the previous choice of $\psi \in \mathcal{A}_u$, we obtain

$$\begin{aligned}
&(p-1)\frac{\alpha^2}{2\alpha-1}\int_\Omega f(x,u)uT_k(u)^{2\alpha-2} \\
&\quad = (p-1)\alpha^2\left(\int_{\{u\le k\}}|\nabla u|^p u^{2\alpha-2} + \frac{k^{2\alpha-2}}{2\alpha-1}\int_{\{u>k\}}|\nabla u|^p\right) \\
&\quad \ge (p-1)\left(\alpha^2\int_{\{u\le k\}}|\nabla u|^p u^{2\alpha-2} + k^{2\alpha-2}\int_{\{u>k\}}|\nabla u|^p\right) \\
&\quad \ge \int_\Omega f_u(x,u)u^2T_k(u)^{2\alpha-2}.
\end{aligned} \tag{3.9}$$

For $\overline{m} \in (p-1,m)$ (that we will choose later depending only on N, m, and p) let L be the smallest constant satisfying (3.8) for a.e. $x \in \Omega$. By assumption (1.2) and the definition of L we have

$$f_t(x,t)t \ge \overline{m}f(x,t) - \overline{m}c(1+L)^m\chi_{\{t\le L\}} \quad \text{for all } t \ge 0 \text{ and a.e. } x \in \Omega.$$

Using this inequality in (3.9) we obtain

$$(\overline{m} - (p-1)\frac{\alpha^2}{2\alpha-1})\int_\Omega f(x,u)uT_k(u)^{2\alpha-2} \le \overline{m}c(1+L)^m L^{2\alpha-1}|\Omega|,$$

for every $k > 0$. Now, we note that $(p-1)\alpha^2/(2\alpha-1) < \overline{m}$ for every $\alpha \in (\overline{m}/(p-1), \alpha(\overline{m}))$, where

$$\alpha(\overline{m}) := \frac{\overline{m} + \sqrt{\overline{m}(\overline{m}-(p-1))}}{p-1}. \tag{3.10}$$

Therefore

$$\int_\Omega f(x,u)u^{2\alpha-1} \le C \quad \text{for all } \alpha \in (\overline{m}/(p-1), \alpha(\overline{m})), \tag{3.11}$$

where C, here and in the rest of the proof, is a constant depending only on N, m, p, $|\Omega|$, L, and c (remember that α and $\overline{m}$ will be chosen later depending only on N, m, and p).

By hypothesis (1.2), (3.11) leads to

$$\int_\Omega |f(x,u)|^q \le C \quad \text{for all } 1 \le q < \frac{2\alpha(\overline{m}) + \overline{m} - 1}{\overline{m}}. \tag{3.12}$$

Choose first $\overline{m}$ to be any number in $(p-1,m)$, and choose α to be any number in $(\overline{m}/(p-1), \alpha(\overline{m}))$. Take any $q > 1$ satisfying (3.12). Multiplying (1.1_p) by u, and using (3.11) and (3.12), we have

$$\begin{aligned}\int_\Omega |\nabla u|^p &= \int_\Omega f(x,u)u \le \int_{\{u\le 1\}} f(x,u) + \int_{\{u\ge 1\}} f(x,u)u^{2\alpha-1} \\ &\le \|f(x,u)\|_q |\Omega|^{1/q'} + \int_\Omega f(x,u)u^{2\alpha-1} \le C,\end{aligned}$$

that establishes (1.11).

To prove (1.12), assume in addition that $m < m_{cs}(p)$. As we said in the introduction this condition is equivalent to (1.13), and a simple computation shows that it is also equivalent to

$$\left(1 - \frac{p-1}{m}\right) N < p\frac{2\alpha(m)+m-1}{m}, \tag{3.13}$$

where $\alpha(m)$ is defined by expression (3.10). Choose $\overline{m} = \overline{m}(N,m,p) \in (p-1, m)$ sufficiently close to m such that (3.13) holds when replacing m by $\overline{m}$ in its right hand side. Using (3.12), we can choose $q_0 = q_0(N,m,p)$ such that

$$1 \le q_0 < \frac{2\alpha(\overline{m}) + \overline{m} - 1}{\overline{m}},$$

$f(x,u) \in L^{q_0}(\Omega)$, and

$$\left(1 - \frac{p-1}{m}\right) N < pq_0.$$

Using Lemma 3.2 and (3.12) we obtain that $\|u\|_\infty \le C$. ■

The following remark will be useful in future sections.

Remark 3.5. Using (3.9) and (3.11) (recall that they hold for every $\overline{m} \in (p-1, m)$), we have that

$$\int_\Omega f_u(x,u)u^{2\alpha} \le (p-1)\frac{\alpha^2}{2\alpha-1}\int_\Omega f(x,u)u^{2\alpha-1} \le C$$

for all $\alpha \in (m/(p-1), \alpha(m))$. Hence, choosing any $\alpha \in (m/(p-1), \alpha(m))$ and noting that $m > p-1$, we obtain

$$\int_\Omega f_u(x,u)u^2 \le C.$$

Moreover, as a consequence of (3.10) and (3.12) we obtain that

$$\int_\Omega |f(x,u)|^{(p^*)'} \le C, \tag{3.14}$$

since $(p^*)' < (2\alpha(\overline{m}) + \overline{m} - 1)/\overline{m}$ for every $\overline{m} > p - 1$.

In the last two inequalities C is a constant depending only on N, m, p, $|\Omega|$, L, and c.

4.Minimal and extremal solutions: proof of Theorem 1.4

For $p = 2$ and f convex, the existence of the family of minimal solutions can be obtained using the Implicit Function Theorem. Due to the degeneracy of the p-Laplacian, it is not clear that this method can be used for $p \neq 2$. Instead, we use a monotone iteration argument following the ideas of [GP92, GPP94], which study ($1.16_{\lambda,p}$) with $f(u) = e^u$. To prove Theorem 1.4(i) we will also use the results from section 2 on the first and second variation, as well as the fact that the first eigenvalue of the p-Laplacian is isolated.

To establish Theorem 1.4(ii) we first prove that $u^* = \lim_{\lambda\uparrow\lambda^*} u_\lambda$ is a solution of the extremal problem. This will be consequence of the $W^{1,p}$ estimates of Theorem 1.2 for u_λ, which will turn out to be independent of λ. Then we will simply apply Theorem 1.2 and Proposition 2.2 to u^*.

Proof of Theorem 1.4: Assume that $f = f(u)$ is an increasing C^1 function satisfying $f(0) > 0$ and (1.17). We will prove the result on several steps.

Step 1. Since $f(0) > 0$, we have that 0 is a subsolution of ($1.16_{\lambda,p}$) and it is not a solution. We consider the problem

$$\begin{cases} -\Delta_p u^0 = f(0) \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Since $f(0) \in L^\infty(\Omega)$ this problem has a unique positive regular solution $u^0 \in C^{1,\beta}(\overline{\Omega})$. Let $M = \max_{\overline{\Omega}} u^0$ and take $\lambda < f(0)/f(M)$. Then

$$\begin{cases} -\Delta_p u^0 = f(0) > \lambda f(M) \geq \lambda f(u^0) \quad \text{in } \Omega, \\ u^0 = 0 \quad \text{on } \partial\Omega, \end{cases}$$

i.e., u^0 is a supersolution of ($1.16_{\lambda,p}$) if λ is small enough. We use Propositions 2.1 and 2.2 with $\underline{u} = 0$ and $\overline{u} = u^0 \in L^\infty(\Omega)$ to obtain the existence of the minimal solution $u_\lambda \in W_0^{1,p}(\Omega)$ and its semi-stability. Since $0 \leq u_\lambda \leq u^0 \in L^\infty(\Omega)$, we have that u_λ is a regular solution of ($1.16_{\lambda,p}$).

Moreover, we note that each regular solution of ($1.16_{\lambda_0,p}$) is a supersolution to problem ($1.16_{\lambda,p}$) for $\lambda \in (0, \lambda_0)$. Hence, by the previous argument the set of $\lambda \in (0, \infty)$ such that problem ($1.16_{\lambda,p}$) has a regular solution is an interval. In addition u_λ is increasing in λ, by minimality.

Step 2. Now we will prove that

$$\lambda^* := \sup\{\lambda \ : \ (1.16_{\lambda,p}) \text{ admits a regular solution}\} < +\infty. \tag{4.1}$$

For this, we prove that problem $(1.16_{\lambda,p})$ has no regular solution if $\lambda > \tilde{\lambda} := \max\{\lambda_1, \lambda_1/\alpha\}$, where λ_1 is the first eigenvalue of the p-Laplacian and

$$\alpha := \inf_{t \geq 0} \frac{f(t)}{t^{p-1}} > 0.$$

We argue by contradiction, that is, we assume that $(1.16_{\lambda,p})$ admits a regular solution u for $\lambda > \tilde{\lambda}$. Let $v_1 \in C^{1,\beta}(\overline{\Omega})$ be a positive eigenfunction associated with the first eigenvalue λ_1 of the p-Laplacian, *i.e.*,

$$\begin{cases} -\Delta_p v_1 = \lambda_1 |v_1|^{p-2} v_1 & \text{in } \Omega, \\ v_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

such that $\|v_1\|_\infty \leq f(0)^{1/(p-1)}$. Note that

$$-\Delta_p v_1 = \lambda_1 v_1^{p-1} \leq \lambda_1 f(0) < \lambda f(0) \leq \lambda f(u) = -\Delta_p u.$$

By the weak comparison principle for the p-Laplacian (see for instance Appendix A of [Pe97] for a proof) we have that $v_1 \leq u$. Let v_2 be the solution to problem

$$\begin{cases} -\Delta_p v_2 = (\lambda_1 + \varepsilon) v_1^{p-1} & \text{in } \Omega, \\ v_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

For ε small enough we obtain

$$-\Delta_p v_2 = (\lambda_1 + \varepsilon) v_1^{p-1} \leq (\lambda_1 + \varepsilon) u^{p-1} \leq \lambda f(u) = -\Delta_p u$$

and

$$-\Delta_p v_1 \leq (\lambda_1 + \varepsilon) v_1^{p-1} = -\Delta_p v_2.$$

Using the weak comparison principle again we obtain $v_1 \leq v_2 \leq u$. Now, let us consider the solutions of

$$\begin{cases} -\Delta_p v_n = (\lambda_1 + \varepsilon) v_{n-1}^{p-1} & \text{in } \Omega, \\ v_n = 0 & \text{on } \partial\Omega, \end{cases}$$

obtaining an increasing sequence $\{v_n\}$ such that $v_1 \leq v_{n-1} \leq v_n \leq u \in C^{1,\beta}(\overline{\Omega})$. The increasing limit $w \in W_0^{1,p}(\Omega)$ of the sequence $\{v_n\}$ is also the limit in the L^q sense for all $q < +\infty$. As a consequence (see part (ii) below for a more general argument), we deduce that w solves the problem

$$\begin{cases} -\Delta_p w = (\lambda_1 + \varepsilon) w^{p-1} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

This is impossible if ε is small enough since the first eigenvalue for the p-Laplacian is isolated (see [A87] or [Ba88]). Therefore $\lambda^* \leq \tilde{\lambda} < +\infty$.

(ii) Assume that f satisfies (1.2) and (1.10) for some $m > p-1$. We will prove that $u^* := \lim_{\lambda \uparrow \lambda^*} u_\lambda$ is a solution of $(1.16_{\lambda^*,p})$. Applying Remark 3.5 to u_λ for $\lambda \in (0, \lambda^*)$ (and, of course, with f replaced by λf) we obtain that $f(u_\lambda)$ converges to $f(u^*)$ in $L^{(p^*)'}(\Omega)$, since $\|f(u_\lambda)\|_{(p^*)'} \leq C$ for some constant C independent of λ. Noting that $L^{(p^*)'}(\Omega) \subset W^{-1,p'}(\Omega)$ and that $(-\Delta_p)^{-1}$ is a continuous operator from $W^{-1,p'}(\Omega)$ onto $W_0^{1,p}(\Omega)$ (see for instance [Pe97]), we obtain that u_λ converges strongly to u^* in $W_0^{1,p}(\Omega)$. Therefore we obtain that for each $\varphi \in C_c^\infty(\Omega)$,

$$\begin{aligned}\int_\Omega |\nabla u^*|^{p-2} \nabla u^* \cdot \nabla \varphi &= \lim_{\lambda \uparrow \lambda^*} \int_\Omega |\nabla u_\lambda|^{p-2} \nabla u_\lambda \cdot \nabla \varphi \\ &= \lim_{\lambda \uparrow \lambda^*} \lambda \int_\Omega f(u_\lambda)\varphi = \lambda^* \int_\Omega f(u^*)\varphi.\end{aligned}$$

That is, u^* is a solution of $(1.16_{\lambda^*,p})$.

Finally, let u_{λ^*} be the minimal solution of $(1.16_{\lambda^*,p})$. Noting that

$$u_\lambda \leq u_{\lambda^*} \leq u^* = \lim_{\lambda \uparrow \lambda^*} u_\lambda \quad \text{for all } \lambda < \lambda^*,$$

we obtain that $u^* = u_{\lambda^*}$. This proves part (ii1).

To establish (ii2) note that minimal solutions u_λ are semi-stable for every $\lambda \in (0, \lambda^*)$ by Proposition 2.2, that is,

$$\int_{\{\nabla u_\lambda \neq 0\}} |\nabla u_\lambda|^{p-2} \left\{ (p-2)\left(\frac{\nabla u_\lambda}{|\nabla u_\lambda|} \cdot \nabla \psi\right)^2 + |\nabla \psi|^2 \right\} - \lambda \int_\Omega f'(u_\lambda)\psi^2 \geq 0$$

for every $\psi \in \mathcal{A}_{u_\lambda}$. If $p \geq 2$ then $\mathcal{A}_{u_\lambda} = W_0^{1,p}(\Omega)$. Noting that $f' \geq 0$, using Fatou's lemma and the convergence in $W_0^{1,p}$ proved above, and taking the limit as $\lambda \to \lambda^*$, we obtain that u^* is a semi-stable solution of the extremal problem.

Assume $1 < p < 2$ and that f is convex. Note that

$$\lambda^* f'(w)(u^*)^2 \leq \lambda^* f'(u^*)(u^*)^2 =: h \quad \text{for all } w \in M_{0,u^*},$$

where $M_{0,u^*} = \{v \in W_0^{1,p}(\Omega) : 0 \leq v \leq u^*\}$. Since $h = \lambda^* f'(u^*)(u^*)^2$ belongs to $L^1(\Omega)$ by monotone convergence and Remark 3.5, Proposition 2.2 gives that u^* is semi-stable.

To show (ii3), we simply apply Theorem 1.2. Since $m < m_{cs}(p)$, we have $u^* \in L^\infty(\Omega)$.

(iii) Assume that $f(t)^{\frac{1}{p-1}}$ is a convex function satisfying (1.18). By Proposition 4.1 (given below), if v is a solution of $(1.16_{\lambda,p})$ then there exists a regular solution of $(1.16_{(1-\varepsilon)\lambda,p})$ for each $\varepsilon \in (0,1)$. By the definition (4.1) of λ^* we deduce statement (iii). ■

In order to prove Theorem 1.4(iii) we have used the following result. Its proof follows the ideas of Theorem 3 in [BCMR96].

Proposition 4.1. *Under the assumptions of Theorem* 1.4(iii)*, if there exists a solution U of* $(1.16_{\lambda,p})$ *then, for every $\varepsilon \in (0,1)$, problem* $(1.16_{(1-\varepsilon)\lambda,p})$ *admits a regular solution.*

Proof: Let us define $g(u) := f(u)^{\frac{1}{p-1}}$,

$$h(u) := \int_0^u \frac{ds}{g(s)}, \quad \tilde{h}(u) := \frac{h(u)}{(1-\varepsilon)^{\frac{1}{p-1}}}, \quad \text{and} \quad \Phi(u) := \tilde{h}^{-1}(h(u)).$$

We note that $\Phi(0) = 0$, $0 \le \Phi(u) \le u$, $\Phi(+\infty) < +\infty$,

$$\Phi'(u) = (1-\varepsilon)^{\frac{1}{p-1}} \frac{g(\Phi(u))}{g(u)} \le 1,$$

and

$$\begin{aligned} \Phi''(u) &= (1-\varepsilon)^{\frac{1}{p-1}} \frac{g'(\Phi(u))\Phi'(u)g(u) - g(\Phi(u))g'(u)}{g(u)^2} \\ &= (1-\varepsilon)^{\frac{1}{p-1}} g(\Phi(u)) \frac{(1-\varepsilon)^{\frac{1}{p-1}} g'(\Phi(u)) - g'(u)}{g(u)^2}. \end{aligned}$$

Using the convexity of g and $0 \le \Phi(u) \le u$, we obtain that $\Phi'' \le 0$, and therefore Φ is a concave bounded function. Let $V := \Phi(U)$. By Lemma 3.2 in [AP03] we have

$$-\Delta_p V = -\Delta_p \Phi(U) \ge \Phi'(U)^{p-1}(-\Delta_p U) = (1-\varepsilon)\lambda f(V)$$

in the weak sense. Then V is a bounded supersolution of $(1.16_{(1-\varepsilon)\lambda,p})$. It follows from a monotone iteration argument (see the proof of Proposition 2.1(ii)) that there exists a regular solution u of $(1.16_{(1-\varepsilon)\lambda,p})$ satisfying $0 \le u \le V = \Phi(U)$. ■

5. Characterization of singular extremal solutions

The rest of the paper is devoted to prove Proposition 1.3, Theorem 1.5, and Corollary 1.6. In order to prove Proposition 1.3 we will use a Hardy type inequality which is an immediate consequence of the Caffarelli-Kohn-Nirenberg inequalities (see for instance [ACP04]).

Proposition 5.1. *Let B_1 be the unit ball of $\mathbb{R}^N$ and let $\mathcal{D}^1_{0,\alpha}(B_1)$ be the completion of $C^\infty_c(B_1)$ with respect to the norm*

$$\|\phi\|_\alpha := \left(\int_{B_1} |x|^{-2\alpha}(|\phi|^2 + |\nabla\phi|^2)dx\right)^{1/2}.$$

If $\alpha \in (-\infty, (N-2)/2)$, then

$$\left(\frac{N-2(\alpha+1)}{2}\right)^2 \int_{B_1} |x|^{-2(\alpha+1)}\varphi^2 dx \le \int_{B_1} |x|^{-2\alpha}\varphi_r^2 dx, \qquad (5.1)$$

for all $\varphi \in \mathcal{D}^1_{0,\alpha}(B_1)$, where φ_r denotes the radial derivative, and the constant appearing in (5.1) *is optimal* (*even among radial functions*) *and it is not achieved.*

Proof: Even that (5.1) is standard and well known, we give the idea of the proof. Let $r = |x|$ and $x = r\sigma$. Integrating by parts, using $\alpha < (N-2)/2$, and the Cauchy-Schwarz inequality, we have

$$\int_0^1 r^{-2(\alpha+1)}\varphi(r\sigma)^2 r^{N-1}dr = -\frac{2}{N-2(\alpha+1)}\int_0^1 r^{-2(\alpha+1)+N}\varphi(r\sigma)\varphi_r(r\sigma)dr$$

$$\le \frac{2}{N-2(\alpha+1)}\left(\int_0^1 r^{-2(\alpha+1)}\varphi(r\sigma)^2 r^{N-1}dr\right)^{\frac12}\left(\int_0^1 r^{-2\alpha}\varphi_r(r\sigma)^2 r^{N-1}dr\right)^{\frac12}.$$

Hence,

$$\left(\frac{N-2(\alpha+1)}{2}\right)^2\int_0^1 r^{-2(\alpha+1)}\varphi(r\sigma)^2 r^{N-1}dr \le \int_0^1 r^{-2\alpha}\varphi_r(r\sigma)^2 r^{N-1}dr.$$

Finally integrate with respect to σ to obtain (5.1). The optimality of the constant appearing in the last inequality can be found in [ACP04]. ■

Proof of Proposition 1.3: Assume $N > p$ and $m > (p-1)N/(N-p)$. Let $U = U^\#$ and $\lambda = \lambda^\#$ be given by (1.14) and (1.15), respectively, and let $f(u) = \lambda(1+u)^m$.

(i) An easy computation shows that $U \in W^{1,p}_0(B_1)$ if and only if $m > m_c = p^* - 1$.

Assume $m > m_c$ and note that $U \in C^2(B_1 \setminus \{0\})$ satisfies (in the classical sense) (1.1$_p$) in $B_1 \setminus \{0\}$. Take $\xi \in C^\infty(\mathbb{R}^N)$ such that $\xi \equiv 0$ in B_1, $0 \le \xi \le 1$ in $B_2 \setminus B_1$, and $\xi \equiv 1$ in $\mathbb{R}^N \setminus B_2$. Let $\xi_\delta(\cdot) = \xi(\cdot/\delta)$ for every $\delta > 0$ and let

$\varphi \in C_c^\infty(B_1)$. Multiplying (1.1_p) in $B_1 \setminus \{0\}$ by $\xi_\delta \varphi$ and integrating by parts, we have

$$\int_{B_1} \xi_\delta |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi + \int_{B_{2\delta}\setminus B_\delta} \varphi |\nabla U|^{p-2} \nabla U \cdot \nabla \xi_\delta = \lambda \int_{B_1} f(U) \xi_\delta \varphi. \quad (5.2)$$

Since $U \in W_0^{1,p}(B_1)$, $0 \le \xi_\delta \le 1$ tends to 1 a.e. in B_1 as δ goes to zero, and $f(U) \in L^1(B_1)$, we obtain that the first and third integrals clearly converge, as δ goes to zero, to

$$\int_{B_1} |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi \quad \text{and} \quad \int_{B_1} f(U)\varphi,$$

respectively. Since $|\varphi \nabla \xi_\delta| \le C/\delta$ and $N > p$, the second integral in (5.2) converges to zero as $\delta \to 0$. Therefore, U is a solution of (1.1_p).

(ii) Assume $m > m_c(p)$, or equivalently,

$$N > N_c := \frac{p(m+1)}{m-(p-1)}. \quad (5.3)$$

By Theorem 1.2 we have that every semi-stable solution of (1.1_p) is bounded if $m < m_{cs}(p)$. Hence, if U is semi-stable then $m \ge m_{cs}(p)$.

Assume $m \ge m_{cs}(p)$ and note that

$$|\nabla U|^{p-2} = \left(\frac{p}{m-(p-1)}\right)^{p-2} |x|^{-2\alpha} \quad \text{and} \quad (1+U)^{m-1} = |x|^{-2(\alpha+1)},$$

where

$$\alpha := \frac{(m+1)(p-2)}{2(m-(p-1))}. \quad (5.4)$$

We will prove that U is semi-stable, that is,

$$\int_{B_1} |x|^{-2\alpha} [(p-2)(\frac{x}{|x|} \cdot \nabla \psi)^2 + |\nabla \psi|^2] \ge C(N,m,p) \int_{B_1} |x|^{-2\alpha-2} \psi^2 \quad (5.5)$$

for all $\psi \in \mathcal{A}_U$, where $\mathcal{A}_U$ is defined in Definition 1.1 and

$$C(N,m,p) := \frac{mp}{m-(p-1)} \left(N - \frac{mp}{m-(p-1)} \right). \quad (5.6)$$

First we note that $\mathcal{A}_U \subset \mathcal{D}^1_{0,\alpha}(B_1)$. Indeed, for $p = 2$ this is obvious, for $1 < p < 2$ one can use the definition of $\mathcal{A}_U$, and for $p > 2$ follows from Hölder inequality. Moreover, by (5.3) one obtains that $\alpha < (N-2)/2$, and therefore applying Proposition 5.1 we have

$$\begin{aligned}\int_{B_1} &|x|^{-2\alpha}[(p-2)(\frac{x}{|x|}\cdot\nabla\psi)^2+|\nabla\psi|^2] \\ &\geq (p-1)\int_{B_1}|x|^{-2\alpha}\psi_r^2 \\ &\geq (p-1)\left(\frac{N-2(\alpha+1)}{2}\right)^2\int_{B_1}|x|^{-2(\alpha+1)}\psi^2\end{aligned} \tag{5.7}$$

for all $\psi \in \mathcal{D}^1_{0,\alpha}(B_1)$, and hence for all $\psi \in \mathcal{A}_U$.

Finally, we note that

$$(p-1)\left(\frac{N-2(\alpha+1)}{2}\right)^2 \geq C(N,m,p) \tag{5.8}$$

since $m \geq m_{cs}(p)$ (or equivalently (1.13) with reverse inequality). Hence (5.5) follows immediately from (5.7) and (5.8). ■

Remark 5.2. Assume $(p-1)N/(N-p) < m \leq m_c(p)$. Let $f(t)=(1+t)^m$, and $\lambda^\#$ be defined in (1.15). In this case, the explicit function $U^\#$ defined in (1.14) is not in $W_0^{1,p}$. It is easy to check that $f(U^\#) \in L^1(B_1)$ since $(p-1)N/(N-p) < m$. Hence $U^\#$ is an entropy solution of $(1.16_{\lambda^\#,p})$ (see [ABFOT03] for the definition of entropy solution). However, for $p>1$ small enough $|\nabla U^\#| \notin L^1(B_1)$ and therefore it is not a solution in the weak sense.

Let α and $C(N,m,p)$ be defined in (5.4) and (5.6). Since

$$m > (p-1)\frac{N}{N-p} > p-1,$$

we have that $\alpha < (N-2)/2$, and therefore (5.7) holds for all $\psi \in \mathcal{D}^1_{0,\alpha}(B_1)$. We also note that (5.8) (in this case) is equivalent to

$$m \leq \tilde{m}(p) := \frac{(p-1)N+2\sqrt{(p-1)(N-1)}+2-p}{N-(p+2)+2\sqrt{(N-1)/(p-1)}}.$$

In particular, if $m \leq \tilde{m}(p)$ then (5.5) holds for all $\psi \in \mathcal{D}^1_{0,\alpha}(B_1)$. On the other hand, if $\tilde{m}(p) < m \leq m_c(p)$ then (5.5) does not hold for some $\psi \in \mathcal{D}^1_{0,\alpha}(B_1)$ by the optimality of the constant appearing on Hardy inequality (5.1).

In order to prove Theorem 1.5 we will use the following result.

Lemma 5.3. *Assume that $\Omega = B_1$, $p > 1$, and that f is increasing. Let u and U be two radial nonincreasing solutions of* $(1.16_{\lambda,p})$ *such that $u \le U$. Then $U - u$ is radially nonincreasing. Therefore, $|\nabla u| = -u' \le -U' = |\nabla U|$ in $B_1 \setminus \{0\}$.*

Proof: Let $\varepsilon > 0$. We note that $u, U \in L^\infty(\overline{B}_1 \setminus B_\varepsilon)$, since both are radially nonincreasing solutions of $(1.16_{\lambda,p})$, and satisfy

$$\begin{cases} -\Delta_p v = \lambda f(v) & \text{in } B_1 \setminus B_\varepsilon, \\ v = 0 & \text{on } \partial B_1, \\ v = v(\varepsilon) & \text{on } \partial B_\varepsilon. \end{cases}$$

In particular, $u, U \in C^1(\overline{B}_1 \setminus \{0\})$. Moreover, by hypothesis,

$$\int_{B_1} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi = \lambda \int_{B_1} f(u) \varphi \tag{5.9}$$

and

$$\int_{B_1} |\nabla U|^{p-2} \nabla U \cdot \nabla \varphi = \lambda \int_{B_1} f(U) \varphi \tag{5.10}$$

for all $\varphi \in C_c^\infty(B_1)$.

We argue by contradiction. Assume that there exist $r_0, r_1 \in (0,1)$ such that $U'(r) - u'(r) > 0$ for all $r \in (r_0, r_1)$. Let $\varphi \in C_0^\infty(B_1)$ be a radially nonincreasing and nonnegative function such that $\varphi \equiv c$ in $[0, r_0]$ (for a positive constant c) and $\varphi \equiv 0$ in $[r_1, 1]$.

Subtracting (5.10) from (5.9), and using that $u \le U$ and $\nabla \psi \cdot \nabla \varphi = |\nabla \psi||\nabla \varphi|$ for $\psi = U$ and $\psi = u$, we obtain

$$\begin{aligned} 0 &\le \lambda \int_{B_1} (f(U) - f(u)) \varphi \\ &= \int_{B_1} (|\nabla U|^{p-2} \nabla U - |\nabla u|^{p-2} \nabla u) \cdot \nabla \varphi \\ &= \int_{B_{r_1} \setminus B_{r_0}} (|\nabla U|^{p-1} - |\nabla u|^{p-1}) |\nabla \varphi| < 0, \end{aligned}$$

a contradiction. ∎

Using Lemma 5.3 we can now prove Theorem 1.5.

Proof of Theorem 1.5: Assume $1 < p < 2$ and that f is a C^1, increasing, and convex function satisfying $f(0) > 0$. From the convexity assumption and $1 < p < 2$ we obtain that (1.17) holds.

(i) Let $\lambda \in (0, \lambda^*)$ and let u_λ be the minimal solution of $(1.16_{\lambda,p})$ given by Theorem 1.4(i). We note that u_λ is a radially decreasing function (see [DS04]).

Let U be any radially nonincreasing semi-stable solution of $(1.16_{\lambda,p})$. We need to prove that $u_\lambda = U$. Indeed, the following proof also holds when $\lambda = \lambda^*$, and u^* and U are solutions of the extremal problem $(1.16_{\lambda^*,p})$, establishing in this case $u^* = U$.

Since u_λ is the minimal solution of $(1.16_{\lambda,p})$ we have $\eta := U - u_\lambda \geq 0$. Let $M_{0,U}$ be defined by (2.2). We note that $U - t\eta \in M_{0,U}$ for all $t \in [0,1]$. By Lemma 5.3 we have

$$|\eta| = U - u_\lambda \leq U \quad \text{and} \quad |\nabla\eta| = |\nabla U| - |\nabla u_\lambda| \leq |\nabla U|,$$

and therefore $\eta \in \mathcal{A}_U$.

Moreover, using the convexity of f and the semi-stability condition (1.8) for $U \in W_0^{1,p}(\Omega)$ with $\psi = U \in \mathcal{A}_U$, we obtain $\lambda f'(w)U^2 \leq \lambda f'(U)U^2$ for all $w \in M_{0,U}$ and

$$\int_\Omega \lambda f'(U)U^2 \leq (p-1)\int_\Omega |\nabla U|^p < +\infty.$$

Therefore, we are under the assumptions of Proposition 2.2 (taking $\overline{u} = U$ and $h = \lambda f'(U)U^2$). Hence, if $g(t) := J(U - t\eta)$ then g is twice continuously differentiable in $[0,1]$ (see the proof of Proposition 2.2 and note that the constant C appearing in (2.14) is equal to 1). Moreover, $g'(0) = g'(1) = 0$ since both u_λ and U are solutions of $(1.16_{\lambda,p})$. By (2.14) and Lemma 5.3, we get

$$g''(t) = (p-1)\int_{B_1} |\nabla(U - t\eta)|^{p-2}|\nabla\eta|^2 - \lambda\int_{B_1} f'(U - t\eta)\eta^2, \tag{5.11}$$

for all $t \in [0,1]$.

If $U \not\equiv u_\lambda$ then $|\nabla U| > |\nabla u_\lambda|$ in a set of positive measure. We know that $|\nabla U| \geq |\nabla u_\lambda|$ everywhere. Note that $f'(U - t\eta)$ is nonincreasing in t a.e., and that the first integral in (5.11) is an increasing function, since

$$|\nabla(U - t\eta)| = |\nabla U| - t(|\nabla U| - |\nabla u_\lambda|)$$

is nonincreasing everywhere (and decreasing in a set of positive measure) and $1 < p < 2$. Therefore $g''(t)$ is an increasing function. It follows that

$$\begin{aligned} 0 &= g'(1) - g'(0) = \int_0^1 g''(s)ds \\ &> g''(0) = (p-1)\int_{B_1} |\nabla U|^{p-2}|\nabla\eta|^2 - \lambda\int_{B_1} f'(U)\eta^2, \end{aligned}$$

obtaining a contradiction, since the last expression is nonnegative (remember that U is radially nonincreasing and semi-stable). Therefore $\eta = U - u_\lambda \equiv 0$, proving (i).

(ii) Assume that f satisfies in addition (1.2), (1.10), and (1.18). By Theorem 1.4(ii) we have that $u^* \in W_0^{1,p}(\Omega)$ is a semi-stable solution of $(1.16_{\lambda^*,p})$. In part (i) we have established that u^* is indeed the unique radially nonincreasing and semi-stable solution of $(1.16_{\lambda^*,p})$.

Let $v \in W_0^{1,p}(\Omega)$ be an unbounded radially nonincreasing and semi-stable solution of $(1.16_{\lambda,p})$ for some $\lambda > 0$. First, we note that $\lambda \leq \lambda^*$ by Theorem 1.4(iii). Second, by part (i) we obtain that $\lambda = \lambda^*$ since minimal solutions are bounded for $\lambda < \lambda^*$. Finally, since u^* is the unique radially nonincreasing and semi-stable solution of $(1.16_{\lambda^*,p})$ we obtain that $v = u^*$. ■

Finally, we prove Corollary 1.6 as an immediate consequence of Proposition 1.3 and Theorem 1.5(ii).

Proof of Corollary 1.6: Let $U^\#$ and $\lambda^\#$ be given by (1.14) and (1.15). Let $f(u) = (1+u)^m$ with $m \geq \max\{1, m_{cs}(p)\}$. We note that f is convex and satisfies (1.2), (1.10), and (1.18). By Proposition 1.3 we have that $U^\# \in W_0^{1,p}(\Omega)$ is an unbounded semi-stable solution of $(1.16_{\lambda^\#,p})$. Using Theorem 1.5(ii) we obtain $\lambda^* = \lambda^\#$ and $u^* = U^\#$. ■

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XAVIER CABRÉ
ICREA AND UNIVERSITAT POLITÈCNICA DE CATALUNYA, DEPARTAMENT DE MATEMÀTICA APLICADA I, DIAGONAL 647, 08028 BARCELONA, SPAIN

E-mail address: xavier.cabre@upc.edu

MANEL SANCHÓN
CENTRO DE MATEMÁTICA, UNIVERSIDADE DE COIMBRA, 3001-454 COIMBRA,PORTUGAL

E-mail address: msanchon@mat.uc.pt