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INTERNAL DEFORMATION OF LIE ALGEBROIDS AND SYMPLECTIC REALIZATIONS

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ABSTRACT: Given a Lie algebroid and a bundle over its base which is endowed with a localizable Poisson structure and a flat connection, we construct an extended bundle whose dual is endowed with an almost-Poisson structure that is a quadratic Poisson structure when a certain compatibility property is satisfied. This new formalism on Lie algebroids describes systems with internal degrees of freedom.

KEYWORDS: Lie algebroids, internal dynamics, Poisson structures.
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1. Introduction

In the last twenty years the Lie algebroids were shown to be an important instrument in the geometrical formulation of many problems in mathematics, mechanics and theoretical physics. Roughly speaking, a Lie algebroid is a generalization of both a Lie algebra and a tangente bundle, these being the simplest examples of Lie algebroids. For a detailed study of this subject we refer the books of Cannas *et. al.* [1] and of Mackenzie [18].

Since Pradines [22], who introduced the Lie algebroids as infinitesimal objects corresponding to Lie grupoids, several authors have studied the theory of Lie algebroids giving important contributions for the knowledge of their properties and applications. Among others, Higgins *et.al.* [10] introduced the notion of prolongation of a Lie algebroid over a map; Weinstein [23] was the first to study Lagrangian mechanics on Lie algebroids (see also [17]) and obtained the Euler-Lagrange equations using the natural linear Poisson structure existing on the dual of a Lie algebroid and the Legendre transformation of a regular Lagrangian; Martínez [21] answered a question posed by Weinstein of whether it was possible to develop a Lagrangian mechanics on Lie algebroids similar to Klein's formalism for classical Lagrangian mechanics, and developed the formalism for the Lagrangian mechanics on Lie

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algebroids, generalizing the fundamental geometrical elements of Lagrangian mechanics (see also [15] and references therein).

In this paper we generalize the concept of Lie algebroid in such a way that its dual vector bundle is endowed with a non-linear Poisson structure, such that the Poisson bracket of basic functions is non-zero and moreover the dual of the anchor map of this generalized Lie algebroid is a symplectic realization of its dual bundle. In classical mechanics these non-linear Poisson structures are usually related with dynamical systems that have additional degrees of freedom (associated to *spin* and *isospin*) to which we call internal degrees of freedom (see [4]). We will study in a future work dynamical systems on Lie algebroids with internal degrees of freedom, as well as we shall solve the generalized Feynman problem on Lie algebroids, i.e. the problem of given a second order differential equation dynamical system, finding all Poisson tensors on a Lie algebroid such that it is a Hamiltonian vector field.

This article is organized in the following way. In Section 2 we describe the motivation for introducing generalized Lie algebroids and recall the definition of symplectic realization. In section 3 the definition and some properties of Lie algebroids are reviewed. In section 4, we "deform" the linear Poisson structure on a Lie co-algebroid A^* by a Poisson structure on a fibre bundle $\mathcal{F}$, using a flat connection on the bundle. This process is called an internal deformation. We will show that when the connection satisfies a certain compatibility condition the deformed structure is a quadratic Poisson structure on the extended vector bundle $\mathcal{F} \bowtie A^*$. The Poisson manifold $\mathcal{F} \bowtie A^*$ is called a quadratic co-algebroid. Then, given a flat connection on the fibre bundle $\mathcal{F}$, we define on the extended vector bundle $\mathcal{F} \bowtie A$ a structure to which we call generalized Lie algebroid and, by imposing the compatibility condition, we obtain a quadratic algebroid. In a parallel way with the case of Lie algebroids, we study the properties of these generalized Lie algebroids. We prove that the dual bundle of a generalized Lie algebroid is endowed with a linear Poisson structure and that the dual of a quadratic algebroid is a quadratic co-algebroid. In section 5, we show that the anchor of a quadratic algebroid is a Poisson morphism between quadratic co-algebroids. Finally, in the last section some examples of internal deformation of Lie algebroids are given. The paper closes with an Appendix where we review the concept of a connection of a surjective submersion.

2. Lie algebroids

We recall that a Lie algebroid over a differentiable manifold M is a vector bundle $p : A \to M$ whose linear space of sections is endowed with a real Lie algebra structure $[\cdot,\cdot]_A$ and a vector bundle morphism $\rho : A \to TM$ over the identity map on M, called the anchor, that induces a map between the space of sections, represented by the same symbol, and that satisfy the following Leibniz-like compatibility condition:

$$[v_1, f v_2]_A = f[v_1, v_2]_A + (\rho(v_1)f)\, v_2, \tag{1}$$

for all pairs (v_1, v_2) of sections of the bundle A and for all differentiable functions f on the base M. The Lie algebroid is denoted by $(A, \rho, [\cdot,\cdot]_A)$ or simply by A whenever is clear to which Lie algebroid we refer to.

The space $\Gamma(A)$ of sections of the bundle $p : A \to M$ is a $C^\infty(M)$-module and the anchor map ρ is a $C^\infty(M)$-linear map from the space $\Gamma(A)$ into the space $\mathfrak{X}(M)$ of vector fields on M. From the condition (1) and the Jacobi identity of the Lie bracket $[\cdot,\cdot]_A$, one can easily show that the anchor is a Lie algebra homomorphism.

Given a Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ over a manifold M, let $\{x^i \mid i : 1, ..., m\}$ be a local coordinate set on the open set $\mathcal{U}$ of M and $\{e_\alpha \mid \alpha : 1, ..., r\}$ a basis of local sections on $\mathcal{U}$ of the vector bundle A. In such local coordinates, the anchor map and the Lie bracket on $\Gamma(A)$ are given by*:

$$[e_\alpha, e_\beta]_A = c_{\alpha\beta}{}^{\gamma}\, e_\gamma, \qquad \rho(e_\alpha) = \rho^i{}_\alpha\, \partial_{x^i},$$

where $c_{\alpha\beta}{}^{\gamma}$ and $\rho^i{}_\alpha$ are differentiable functions on an open set of M, that we call *structure functions of the Lie algebroid.* Since ρ is a homomorphism of Lie algebras, these functions satisfy:

$$\rho^j{}_\alpha \frac{\partial \rho^i{}_\beta}{\partial q^j} - \rho^j{}_\beta \frac{\partial \rho^i{}_\alpha}{\partial q^j} = \rho^i{}_\gamma\, c_{\alpha\beta}{}^{\gamma}\,, \qquad i = 1, \ldots, m\,. \tag{2}$$

Moreover, the Leibniz condition and the Jacobi identity imply:

$$\sum_{\mathrm{cycl}(\alpha,\beta,\gamma)} \left[\rho^i{}_\gamma \frac{\partial c_{\beta\alpha}{}^{\mu}}{\partial q^i} + c_{\alpha\beta}{}^{\nu}\, c_{\nu\gamma}{}^{\mu}\right] = 0\ , \qquad \mu = 1, \ldots, r\,. \tag{3}$$

The equations (2) and (3) are known as the *compatibility equations of the structure functions* or *structure equations* of the Lie algebroid.

*Hereafter summation in repeated indices will be understood.

The simplest (non-trivial) examples of Lie algebroids are the tangent bundle $TM \to M$, with $\rho = \mathrm{id}_{TM}$ and the Lie bracket on $\Gamma(TM)$ being the usual bracket of vector fields on M, and the finite dimensional Lie algebra $\mathfrak{g}$ considered as a vector bundle over a single point $\mathfrak{g} \to \{\cdot\}$. In this case, the anchor is the zero map and the Lie bracket on sections coincides with the bracket on the Lie algebra. Now we will recall other examples of Lie algebroids that we will use ahead in the last section.

Example 2.1 (Lie algebra bundle)**.** *Let (A, p, M) be a vector bundle and $[\cdot,\cdot]$ a bilinear and skew-symmetric bracket, defined on the fibreed product $A \times_M A$ with values in A, such that each fibre of A endowed with the bracket is a Lie algebra. In these conditions, we define a Lie bracket $[\cdot,\cdot]_A$ on the sections of A by setting*

$$[v, w]_A(x) = [v(x), w(x)],$$

for all $x \in M$ and $v, w \in \Gamma(A)$. The bundle (A, p, M) endowed with the bracket $[\cdot,\cdot]_A$ and the zero map between A and the tangent bundle TM is a Lie algebroid, which is called the Lie algebra bundle over the manifold M.

Example 2.2 (The action groupoid)**.** *Let $\phi : M \times G \to M$ be a right action of a Lie group G on the manifold M. For each X in the Lie algebra $\mathfrak{g}$ of G, there exists a vector field on M that we call the fundamental vector associated with X, defined by $X_M(m) = d/dt[\phi(m, \exp(tX))]|_{t=0} = \phi_{m*}(X)$, for all $m \in M$. Moreover, it is well known that $[X_M, Y_M] = [X, Y]_M$. We may identify the linear space $\Gamma(M \times \mathfrak{g})$ of sections of the bundle $M \times \mathfrak{g} \to M$ with the space $C^\infty(M, \mathfrak{g})$ of differentiable functions on M with values in $\mathfrak{g}$. Note that each section $V \in \Gamma(M \times \mathfrak{g})$ is of the form $V(x) = (x, \underline{V}(x))$, for all $x \in M$, where $\underline{V} \in C^\infty(M, \mathfrak{g})$, and this is a one-to-one correspondence which allows us to identify V with $\underline{V}$.*

The trivial vector bundle $M \times \mathfrak{g} \to M$ can be endowed with a Lie algebroid structure whose anchor map ρ is given by the infinitesimal action of $\mathfrak{g}$ on M. This action is considered as a bundle map from $M \times \mathfrak{g} \to M$ into $TM \to M$; $\rho(x, X) = X_M(x)$, for all $x \in M$ and $X \in \mathfrak{g}$. The Lie bracket on the space $\Gamma(M \times \mathfrak{g}) \cong C^\infty(M, \mathfrak{g})$ is given by

$$[V, W]_{M\times\mathfrak{g}}(x) = (x, [\underline{V}(x), \underline{W}(x)]_{\mathfrak{g}} + (\underline{V}(x))_M \underline{W}(x) - (\underline{W}(x))_M \underline{V}(x)) \,, \quad (4)$$

for all $V, W \in \Gamma(M \times \mathfrak{g})$. This Lie algebroid structure can also be characterized as the unique Lie algebroid structure such that $[X, Y]_{M\times\mathfrak{g}} = [X, Y]_{\mathfrak{g}}$ for any pair of constant sections and $\rho(x, X) = X_M(x)$ [3].

Example 2.3 (Poisson manifold)**.** *Let (M, Λ) be a Poisson manifold. The bivector Λ defines a Poisson structure in M by $\{f, g\} := \Lambda(df, dg)$ and induces a vector bundle morphism $\Lambda^\sharp : T^*M \to TM$ by means of $\left\langle \beta_x, \Lambda^\sharp_x(\alpha_x) \right\rangle = \Lambda_x(\alpha_x, \beta_x)$, for all $\alpha_x, \beta_x \in T^*_xM$ and $x \in M$. This vector bundle map induces a liner map between the linear spaces of their sections, to be represented by the same symbol, $\Lambda^\sharp : \Omega^1(M) \longrightarrow \mathfrak{X}(M)$. The cotangent bundle T^*M can be endowed with a Lie algebroid structure whose anchor map is $\Lambda^\sharp$ and the Lie bracket of sections is defined by (see* [9, 19]*)*

$$[\alpha, \beta]_{T^*M} \;=\; \pounds_{\Lambda^\sharp(\alpha)}\beta - \pounds_{\Lambda^\sharp(\beta)}\alpha - d(\Lambda(\alpha, \beta)). \tag{5}$$

For exact forms, this bracket reduces to $[df, dg]_{T^*M} = d\{f, g\}$, for all $f, g \in C^\infty(M)$. The triple $(T^*M, \Lambda^\sharp, [\cdot, \cdot]_{T^*M})$ is called the Lie algebroid of the Poisson manifold (M, Λ).

Another very remarkable property is that the dual bundle $\tau : A^* \to M$ of a given Lie algebroid $(A, \rho, [\cdot, \cdot]_A)$ over a m-dimensional manifold M is endowed with a natural linear Poisson structure $\{\cdot, \cdot\}_{A^*}$, i.e. the Poisson bracket of linear functions on A^* is still a linear function on A^*.

In fact, in order to describe the Poisson structure of A^* it suffices to give the Poisson brackets of a class of functions such that their differentials span the cotangent space at each point of A^*. Such a class of functions is given by functions which are affine in the fibres. Those functions which are constant on the fibres, basic functions, correspond to pull-back of functions on M. On the other hand, each section v of A can be identified as a function on A^* that is linear in the fibres, i.e. $\chi : \Gamma(A) \to L(A^*)$ maps each section v of A into the linear function $\chi(v)$ on A^* given by $\chi(v)(\alpha) = \langle \alpha, v \rangle$, for all $\alpha \in \Gamma(A^*)$, with $\langle \cdot, \cdot \rangle$ the duality bracket between $\Gamma(A^*)$ and $\Gamma(A)$. The bracket $\{\cdot, \cdot\}_{A^*}$ is given by (see e.g. [5]):

$$\{f \circ \tau, g \circ \tau\}_{A^*} = 0,\; \{\chi(v), f \circ \tau\}_{A^*} = \rho(v) f \circ \tau,\; \{\chi(v), \chi(w)\}_{A^*} = \chi([v, w]_A),$$

for all $f, g \in C^\infty(M)$ and $v, w \in \Gamma(A)$.

The linear Poisson bracket $\{\cdot, \cdot\}_{A^*}$ defines a Poisson tensor that we represent by Λ_{A^*}. Let $\left\{x^i \mid i = 1, ..., m\right\}$ be a local coordinate set on an open set $\mathcal{U}$ of the manifold M and $\{e_\alpha \mid \alpha = 1, ..., r\}$ a basis of local sections of A. In local coordinates, the Poisson tensor Λ_{A^*} is given by

$$\Lambda_{A^*} = \rho^i{}_\alpha \partial_{\mu_\alpha} \wedge \partial_{x^i} + c_{\alpha\beta}{}^\gamma \mu_\gamma\, \partial_{\mu_\alpha} \wedge \partial_{\mu_\beta},$$

where $\mu_\alpha = \chi(e_\alpha)$ is the linear function on A^* associated with the section e_α of the basis of sections of A, and $\rho^i{}_\alpha$ and $c_{\alpha\beta}{}^\gamma$ are the structure functions of the Lie algebroid A.

We shall call *Lie co-algebroid* to the pair formed by the dual bundle A^* and the linear Poisson structure.

Let $(A, \rho, [\cdot,\cdot]_A)$ be a Lie algebroid over M and let us consider the cotangent bundle T^*M endowed with its canonical symplectic structure. Then we have the following result:

Proposition 2.4 ([20]). *The dual map of the anchor of a Lie algebroid* $(A, \rho, [\cdot,\cdot]_A)$ *over* M*,* $(\rho^*, \mathrm{id}_M) : (T^*M, \pi, M) \to (A^*, \tau, M)$*, is a symplectic realization of the Lie co-algebroid* $(A^*, \{\cdot,\cdot\}_{A^*})$*.*

3. Internal deformation of Lie algebroids

In this section, we consider a Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ over a manifold M and a fibre bundle $\pi : \mathcal{F} \to M$, where $\mathcal{F}$ is endowed with both a flat connection and a Poisson structure Λ. The Poisson structure Λ on $\mathcal{F}$ will satisfy the property of *localizability* on the bundle, that is, Λ is π-projectable onto 0: if x^i represents the local coordinates on the base manifold M and (x^i, ξ^a) represent local coordinates on the fibre bundle $\mathcal{F}$, then the Poisson tensor Λ is given in local coordinates by

$$\Lambda(x, \xi) = C_{ab}(x, \xi)\, \partial_{\xi^a} \wedge \partial_{\xi^b}.$$

Note that the localizability property of the Poisson tensor Λ on $\mathcal{F}$ just means that $\Lambda(dx^i, dx^j) = \Lambda(dx^i, d\xi^a) = 0$. This property corresponds in the quantum case to the fact that the position variables x^i must be compatible ("simultaneously measurable") among themselves, and also with the variables ξ^a.

Definition 3.1 ([4]). *A localizable Poisson tensor on a fibre bundle* $\pi : \mathcal{F} \to M$ *is a bundle Poisson tensor, i.e. a smooth assignment to each fibre of a Poisson tensor* Λ*.*

When the matrix $C = (C_{ab})$ is regular, we will say that the Poisson structure Λ on $\mathcal{F}$ is *regular along the fibres.* The bundle $\mathcal{F}$ is called the *internal bundle* and the local coordinates ξ^a are called *internal variables* [4].

The main goal of this section is to "deform" the linear Poisson structure on the Lie co-algebroid A^* by the Poisson structure on $\mathcal{F}$, in such a way that we obtain a non-linear Poisson structure on the extended bundle $\mathcal{F} \bowtie A^* \to \mathcal{F}$. We call this process an *internal deformation* of the Lie co-algebroid.

3.1. Quadratic Poisson structure on $\mathcal{F} \bowtie A^*$. Let us consider the pull-back of the Lie co-algebroid $\tau_0 : A^* \to M$ by the bundle map $\pi : \mathcal{F} \to M$, $\pi^! A^* = \{(q, \alpha) \in \mathcal{F} \times A^* \mid \pi(q) = \tau_0(\alpha)\}$:

$$\begin{tikzcd} \pi^! A^* \arrow[r, "\tau_2"] \arrow[d, "\tau_1"'] & A^* \arrow[d, "\tau_0"] \\ \mathcal{F} \arrow[r, "\pi"] & M \end{tikzcd}$$

We will represent the pull-back bundle $\pi^! A^*$ by $\mathcal{F} \bowtie A^*$. This space is a vector bundle over $\mathcal{F}$ whose fibres are isomorphic to the fibres of $\tau_0 : A^* \to M$; $\tau_1^{-1}(q) \simeq A^*_x$ with $x = \pi(q)$.

The vector bundle $\tau_1 : \mathcal{F} \bowtie A^* \to \mathcal{F}$ can be endowed with a bivector $\Lambda_{\mathcal{F}\bowtie A^*}$ which is τ_1-projectable onto Λ and τ_2-projectable onto Λ_{A^*}, defined by:

$$\begin{aligned} \{f \circ \tau_1, g \circ \tau_1\}_{\mathcal{F}\bowtie A^*} &= \{f, g\} \circ \tau_1, \\ \{\chi(v) \circ \tau_2, g \circ \tau_1\}_{\mathcal{F}\bowtie A^*} &= \widetilde{\rho(v)} g \circ \tau_1, \\ \{\chi(v) \circ \tau_2, \chi(w) \circ \tau_2\}_{\mathcal{F}\bowtie A^*} &= \chi([v, w]_A) \circ \tau_2, \end{aligned} \tag{6}$$

for all $f, g \in C^\infty(\mathcal{F})$ and $v, w \in \Gamma(A)$, where $\{\cdot, \cdot\}$ is the Poisson bracket defined by the Poisson tensor Λ on $\mathcal{F}$, and

$$\widetilde{\rho(v)}(q) := (T\pi|_{H_q})^{-1}(\, \rho(v)(\pi(q))\,),$$

is the horizontal lift of the vector field $\rho(v)$ on M to a vector field on $\mathcal{F}$ defined by the *connection* of $\pi : \mathcal{F} \to M$ (see appendix). I.e., it is defined by a distribution H of $\mathcal{F}$ which is complementary to the vector bundle $T^\pi \mathcal{F}$ of π-vertical vector fields on $\mathcal{F}$, such that $T_q\pi : H_q \to T_{\pi(q)}M$ is an isomorphism for all $q \in \mathcal{F}$; therefore, $T\mathcal{F} = H \oplus T^\pi \mathcal{F}$.

We observe that the bracket $\{\cdot, \cdot\}_{\mathcal{F}\bowtie A^*}$ is completely defined by equations (6), because the cotangente space to $\mathcal{F} \bowtie A^*$ is generated by the differentials of the linear functions of the form $\chi(v) \circ \tau_2$ and the differentials of basic functions $f \circ \tau_1$ on $\mathcal{F} \bowtie A^*$. Moreover, $\{\cdot, \cdot\}_{\mathcal{F}\bowtie A^*}$ is an almost-Poisson bracket, i.e. all the properties of a Poisson bracket are satisfied except possibly the Jacobi identity.

Let now $\{(x^i, \mu_\alpha) \mid i = 1, ..., m, \alpha = 1, ..., r\}$ be a set of local coordinates on the vector bundle A^*, where the x^i's represent a set of local coordinates on the manifold M and the $\mu_\alpha = \chi(e_\alpha)$'s are the linear functions associated with the elements e_α's of a basis of local sections of A; we call this local coordinates *external variables*. In local coordinates, the bivector $\Lambda_{\mathcal{F}\bowtie A^*}$ defined by (6) is given by

$$\Lambda_{\mathcal{F}\bowtie A^*} = \Gamma^b_j\, \rho^j{}_\alpha \partial_{\mu_\alpha} \wedge \partial_{\xi^b} + \rho^i{}_\alpha \partial_{\mu_\alpha} \wedge \partial_{x^i} + c_{\alpha\beta}{}^\gamma \mu_\gamma\, \partial_{\mu_\alpha} \wedge \partial_{\mu_\beta} + C_{ab}\, \partial_{\xi^a} \wedge \partial_{\xi^b}\,,$$

where $\widetilde{\rho(e_\alpha)}(x,\xi) = \rho^i{}_\alpha(x)\partial_{x^i} + \Gamma^b_j(x,\xi)\rho^j{}_\alpha(x)\partial_{\xi^b}$, $\rho^i{}_\alpha$ and $c_{\alpha\beta}{}^\gamma$ are the structure functions of A and the symbols Γ^b_j represent the connection "coefficients".

Proposition 3.2. *The bivector $\Lambda_{\mathcal{F}\bowtie A^*}$ is a Poisson bivector, i.e. the bracket defined on $\mathcal{F} \bowtie A^*$ verifies the Jacobi identity, if and only if $\pounds_{\widetilde{\rho(v)}}\Lambda = 0$, for all $v \in \Gamma(A)$, which is equivalent to*

$$\widetilde{\rho(v)}\,\{f,g\} = \left\{\widetilde{\rho(v)}f, g\right\} + \left\{f, \widetilde{\rho(v)}g\right\}, \tag{7}$$

for all $f, g \in C^\infty(\mathcal{F})$ and $v \in \Gamma(A)$.

Proof: In order to prove the statement, we have to consider the Jacobi identity for any triple of differentiable functions on $\mathcal{F}$ or, equivalently, for: three basic functions, three linear functions, one basic function and two linear functions and finally one linear function and two basic functions. In the first case, the Jacobi identity holds because $\{\cdot,\cdot\}$ is a Poisson bracket on $\mathcal{F}$. In the second case, it also holds because $[\cdot,\cdot]_A$ is a Lie bracket. When we have two linear functions and one basic function, the Jacobi identity is satisfied because the connection is flat. In the last case, we have that

$$\begin{aligned}\{\chi(v)\circ\tau_2, \{f\circ\tau_1, g\circ\tau_1\}_{\mathcal{F}\bowtie A^*}\}_{\mathcal{F}\bowtie A^*} &= \{\{\chi(v)\circ\tau_2, f\circ\tau_1\}_{\mathcal{F}\bowtie A^*}, g\circ\tau_1\}_{\mathcal{F}\bowtie A^*} \\ &\quad + \{f\circ\tau_1, \{\chi(v)\circ\tau_2, g\circ\tau_1\}_{\mathcal{F}\bowtie A^*}\}_{\mathcal{F}\bowtie A^*}\end{aligned}$$

is equivalent, from the definition of $\{\cdot,\cdot\}_{\mathcal{F}\bowtie A^*}$, to

$$\widetilde{\rho(v)}\,\{f,g\}\circ\tau_1 = \left\{\widetilde{\rho(v)}f\circ\tau_1, g\circ\tau_1\right\}_{\mathcal{F}\bowtie A^*} + \left\{f\circ\tau_1, \widetilde{\rho(v)}g\circ\tau_1\right\}_{\mathcal{F}\bowtie A^*},$$

that is,

$$\widetilde{\rho(v)}\,\{f,g\}\circ\tau_1 = \left\{\widetilde{\rho(v)}f, g\right\}\circ\tau_1 + \left\{f, \widetilde{\rho(v)}g\right\}\circ\tau_1$$

which is condition (7). ■

In local coordinates, the compatibility condition (7) is expressed in the following way:

$$\rho^i{}_\alpha \left(\frac{\partial C_{ab}}{\partial x^i} + \Gamma_i^d \frac{\partial C_{ab}}{\partial \xi^d} - C_{db} \frac{\partial \Gamma_i^a}{\partial \xi^d} - C_{ad} \frac{\partial \Gamma_i^b}{\partial \xi^d} \right) \;=\; 0,$$

which allows one to conclude that it is satisfied when the anchor is zero.

If $\Lambda_{\mathcal{F} \bowtie A^*}$ is a Poisson tensor, then the Poisson bracket is quadratic $\{\cdot,\cdot\}_{\mathcal{F} \bowtie A^*}$, i.e. the Poisson bracket of linear functions is quadratic on linear functions:

$$\begin{aligned} &\{f(\chi(v) \circ \tau_2), g(\chi(w) \circ \tau_2)\}_{\mathcal{F} \bowtie A^*} \\ &= \{f,g\}\,(\chi(v) \circ \tau_2)(\chi(w) \circ \tau_2) - ([g\,\widetilde{\rho(w)} f] \circ \tau_1)(\chi(v) \circ \tau_2) \\ &\quad +([f\,\widetilde{\rho(v)} g] \circ \tau_1)(\chi(w) \circ \tau_2) + fg(\chi([v,w]_A) \circ \tau_2)\,, \end{aligned}$$

for all $f, g \in C^\infty(\mathcal{F})$ and $v, w \in \Gamma(A)$. Therefore, the Poisson structure $\Lambda_{\mathcal{F} \bowtie A^*}$ on $\mathcal{F} \bowtie A^*$ is non-linear.

Definition 3.3. *Given a Lie co-algebroid $\tau_0 : A^* \to M$ and a fibre bundle $\pi : \mathcal{F} \to M$ equipped with a flat connection and a localizable Poisson structure Λ that satisfies the compatibility condition (7), then the vector bundle $\mathcal{F} \bowtie A^* \to \mathcal{F}$ endowed with the non-linear Poisson structure $\Lambda_{\mathcal{F} \bowtie A^*}$ given above is called a quadratic co-algebroid.*

3.2. Generalized Lie algebroid structure on $\mathcal{F} \bowtie A$. In this section, we will relate the quadratic co-algebroid structure on $\mathcal{F} \bowtie A^*$ with a Lie algebroid structure on its dual bundle. Note that, the dual $(\mathcal{F} \bowtie A^*)^*$ of the extended bundle $\mathcal{F} \bowtie A^*$ coincides with $p_1 : \mathcal{F} \bowtie A \to \mathcal{F}$ the pull-back of the vector bundle $p_0 : A \to M$ by the projection $\pi : \mathcal{F} \to M$, because, $(\mathcal{F} \bowtie A^*)^*_q \cong (A^*)^*_{\pi(q)} = (A^*_{\pi(q)})^* = A_{\pi(q)} \cong (\mathcal{F} \bowtie A)_q$, for all $q \in \mathcal{F}$. The pull-back of the vector bundle $p_0 : A \to M$ by the projection $\pi : \mathcal{F} \to M$, $\mathcal{F} \bowtie A = \pi^! A = \{(q,v) \in \mathcal{F} \times A \mid \pi(q) = p_0(v)\}$, is a vector bundle over $\mathcal{F}$ whose fibres are isomorphic to the fibres of A, and we have the following commutative diagram:

$$\begin{tikzcd} \mathcal{F} \bowtie A \arrow[r, "p_2"] \arrow[d, "p_1"'] & A \arrow[d, "p_0"] \\ \mathcal{F} \arrow[r, "\pi"] & M \end{tikzcd}$$

where $p_1(q,v) = q$ and $p_2(q,v) = v$. Note that the space of sections of $\mathcal{F} \bowtie A$ is a $C^\infty(\mathcal{F})$-modulo generated by sections of the form $\overline{v}(q) = (q, v(\pi(q)))$

with $v \in \Gamma(A)$, and this allows us to conclude that the space $\Gamma(\mathcal{F} \bowtie A)$ is isomorphic to the space of sections of A along the map π, $\Gamma_\pi(A) = \{Z : \mathcal{F} \to A \mid p_0 \circ Z = \pi\}$ (see [10]):

$$\overline{v} \in \Gamma(\mathcal{F} \bowtie A) \mapsto v \circ \pi \in \Gamma_\pi(A).$$

Let us consider a Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ over a manifold M and a fibre bundle $\pi : \mathcal{F} \to M$ endowed with a localizable Poisson structure Λ of the form $\Lambda = C_{ab}\xi^a \wedge \xi^b$. Given a flat connection in the bundle $\mathcal{F}$, the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ is given by a vector field $\widetilde{X} \in \mathfrak{X}(\mathcal{F})$ defined by

$$\widetilde{X}(q) = (T\pi|_{H_q})^{-1}(X(\pi(q)))\,,$$

for all $q \in \mathcal{F}$. Since the connection is flat, these vector fields satisfy:

$$\widetilde{[X, Y]} = [\widetilde{X}, \widetilde{Y}]\,, \tag{8}$$

for all $X, Y \in \mathfrak{X}(M)$, where the bracket on the left hand side is the bracket of vector fields on M and the bracket on the right one is the bracket of vector fields on $\mathcal{F}$.

We are going to endow the vector bundle $\mathcal{F} \bowtie A$ over $\mathcal{F}$ with a structure that we call a generalized Lie algebroid structure. The *generalized anchor* $\varrho : \mathcal{F} \bowtie A \to \mathcal{F} \bowtie TM$ is defined by

$$\varrho(q, v(\pi(q))) = (q, \rho(v)(\pi(q))),$$

for all $q \in \mathcal{F}$ and $v \in \Gamma(A)$. This generalized anchor induces a map between the spaces of sections, that we represent by the same symbol,

$$\varrho(f\overline{v}) = f\,\overline{\rho(v)},$$

whose action on differentiable functions on $\mathcal{F}$ is defined by

$$\varrho(f\overline{v})g := f\widetilde{\rho(v)}g,$$

for all $f, g \in C^\infty(\mathcal{F})$ and $v \in \Gamma(A)$. In the space of sections of $\mathcal{F} \bowtie A$ we introduce the following bilinear and skew-symmetric bracket

$$[\overline{v}, g\overline{w}]_{\mathcal{F}\bowtie A} := g\overline{[v, w]_A} + (\varrho(\overline{v})g)\overline{w}\,, \tag{9}$$

for all $v, w \in \Gamma(A)$ and $g \in C^\infty(\mathcal{F})$.

Remark 3.4. *From the definition of the generalized anchor, we conclude that it is $C^\infty(\mathcal{F})$-linear because $\varrho(f\overline{v}) = f\overline{\rho(v)} = f\varrho(\overline{v})$, for all $f \in C^\infty(\mathcal{F})$ and $v \in \Gamma(A)$.*

Proposition 3.5. *The bracket defined by (9) on the space of sections of $\mathcal{F} \bowtie A$ is a Lie bracket.*

Proof: The bracket (9) is bilinear and skew-symmetric. To verify the Jacobi identity, we have to show that

$$\sum_{cycl}[[f\overline{v}, g\overline{w}]_{\mathcal{F}\bowtie A}, h\overline{z}]_{\mathcal{F}\bowtie A} = 0\,,$$

for all $f, g, h \in C^\infty(\mathcal{F})$ and $v, w, z \in \Gamma(A)$. From the definition of $[\cdot,\cdot]_{\mathcal{F}\bowtie A}$, we have

$$\begin{aligned}
&\sum_{cycl}[[f\overline{v}, g\overline{w}]_{\mathcal{F}\bowtie A}, h\overline{z}]_{\mathcal{F}\bowtie A} \\
&= \sum_{cycl}\Big(fgh\overline{[[v,w]_A, z]_A} + \{fg\varrho(\overline{[v,w]_A})h\}\overline{z} - h\varrho(\overline{z})(fg)\overline{[v,w]_A}\Big) \\
&+ \sum_{cycl}\Big((fh\varrho(\overline{v})g)\overline{[w,z]_A} + (f\varrho(\overline{v})g)(\varrho(\overline{w})h)\overline{z} - h\varrho(\overline{z})(f\varrho(\overline{v})g)\overline{w}\Big) \\
&- \sum_{cycl}\Big(gh(\varrho(\overline{w})f)\overline{[v,z]_A} + (g\varrho(\overline{w})f)(\varrho(\overline{v})h)\overline{z} - h\varrho(\overline{z})(g\varrho(\overline{w})f)\overline{v}\Big)\,. \qquad (10)
\end{aligned}$$

Since the bracket on the sections of A satisfies the Jacobi identity and ρ is a homomorphism of Lie algebras, we have

$$\sum_{cycl}\Big(fgh\overline{[[v,w]_A, z]_A} + \{fg\varrho(\overline{[v,w]_A})h\}\overline{z}\Big) \;=\; \sum_{cycl}\Big(fg\{[\rho(v),\rho(w)]h\}\overline{z}\Big)\,.$$

If we reorganize the terms of (10) we obtain

$$
\begin{aligned}
&\sum_{cycl}[[f\overline{v}, g\overline{w}]_{\mathcal{F}\bowtie A}, h\overline{z}]_{\mathcal{F}\bowtie A}\\
&\overset{(8)}{=}\sum_{cycl}\Big(fg\{[\widetilde{\rho(v)}, \widetilde{\rho(w)}]h\}\overline{z} - fh(\varrho(\overline{z})\varrho(\overline{v})g)\,\overline{w} + gh(\varrho(\overline{z})\varrho(\overline{w})f)\overline{v}\Big)\\
&+\sum_{cycl}\Big(fh(\varrho(\overline{v})g)\overline{[w,z]_A} - h\varrho(\overline{z})(fg)\overline{[v,w]_A} - gh(\varrho(\overline{w})f)\overline{[v,z]_A}\Big)\\
&+\sum_{cycl}(\, f(\varrho(\overline{v})g)(\varrho(\overline{w})h)\overline{z} - h(\varrho(\overline{z})f)(\varrho(\overline{v})g)\overline{w}\,)\\
&+\sum_{cycl}(\, h(\varrho(\overline{z})g)(\varrho(\overline{w})f)\overline{v} - g(\varrho(\overline{w})f)(\varrho(\overline{v})h)\overline{z}\,) = 0,
\end{aligned}
$$

because each cyclic sum is zero. Therefore, the bracket $[\cdot,\cdot]_{\mathcal{F}\bowtie A}$ satisfies the Jacobi identity. ■

Definition 3.6. *The vector bundle $p_1 : \mathcal{F}\bowtie A \to A$ endowed with the generalized anchor $\varrho : \mathcal{F}\bowtie A \to \mathcal{F}\bowtie TM$ and the Lie bracket on the sections of $\mathcal{F}\bowtie A$ defined by (9) is called a generalized Lie algebroid.*

Let $\{(x^i,\xi^a) \mid i = 1,...,m,\ a = 1,...,n\}$ be a set of local coordinates on the bundle $\pi : \mathcal{F} \to M$ and $\{e_\alpha \mid \alpha = 1,...,r\}$ a basis of local sections of the vector bundle $p_0 : A \to M$. In local coordinates, the generalized Lie algebroid structure on $\mathcal{F}\bowtie A$ is given by:

$$[\overline{e_\alpha}, \overline{e_\beta}]_{\mathcal{F}\bowtie A} = c_{\alpha\beta}{}^{\gamma}\,\overline{e_\gamma}, \quad \varrho(\overline{e_\alpha})x^i = \rho^i_\alpha, \quad \varrho(\overline{e_\alpha})\xi^a = \Gamma^a_j\rho^j_\alpha. \tag{11}$$

The local functions $c_{\alpha\beta}{}^{\gamma}$, ρ^i_α and $\Gamma^a_j\rho^j_\alpha$ are called the structure functions of the generalized Lie algebroid $\mathcal{F}\bowtie A$.

Example 3.7. *Let us consider the tangent bundle $\tau_M : TM \to M$ endowed with its usual Lie algebroid structure. The fibre bundle $\mathcal{F}\bowtie TM \to \mathcal{F}$ is endowed with a generalized Lie algebroid structure. The generalized anchor is the identity map on $\mathcal{F}\bowtie TM$,*

$$\varrho(q, X(\pi(q))) = (q, X(\pi(q))),$$

and the Lie bracket on the sections of $\mathcal{F} \bowtie TM$ *is given, for all* $X, Y \in \mathfrak{X}(M)$ *and* $f \in C^\infty(\mathcal{F})$, *by*

$$[\overline{X}, f\overline{Y}]_{\mathcal{F}\bowtie TM} = f\overline{[X,Y]} + (\widetilde{X}f)\overline{Y},$$

where $[\cdot,\cdot]$ *is the usual bracket of vector fields on* M. *In local coordinates* (x^i, ξ^a) *on the bundle* $\pi : \mathcal{F} \to M$ *and in the basis of sections* $\{\partial_{x^i} \mid i = 1, ..., n\}$ *of the tangent bundle* TM, *the generalized anchor and bracket are respectively given by:*

$$[\overline{\partial_{x^i}}, \overline{\partial_{x^j}}] = \overline{0}, \quad \varrho(\overline{\partial_{x^i}})x^j = \delta^i_j, \quad \varrho(\overline{\partial_{x^i}})\xi^a = \Gamma^a_i.$$

Since the anchor ρ of the Lie algebroid A is a homomorphism of Lie algebras, from the definition of the generalized anchor ϱ and the bracket $[\cdot,\cdot]_{\mathcal{F}\bowtie A}$, we can easily prove that the generalized anchor is a homomorphism of the Lie algebra $(\Gamma(\mathcal{F} \bowtie A), [\cdot,\cdot]_{\mathcal{F}\bowtie A})$ into the Lie algebra $(\Gamma(\mathcal{F} \bowtie TM), [\cdot,\cdot]_{\mathcal{F}\bowtie TM})$:

$$\begin{aligned}
\varrho([f\overline{v}, g\overline{w}]_{\mathcal{F}\bowtie A}) &= \varrho\Big(fg\overline{[v,w]_A} + (\varrho(f\overline{v})g)\overline{w} - (\varrho(g\overline{w})f)\overline{v} \Big) \\
&= fg\overline{\rho([v,w]_A)} + (\varrho(f\overline{v})g)\overline{\rho(w)} - (\varrho(g\overline{w})f)\overline{\rho(v)} \\
&= fg\overline{[\rho(v),\rho(w)]} + (\varrho(f\overline{v})g)\overline{\rho(w)} - (\varrho(g\overline{w})f)\overline{\rho(v)} \\
&= [\varrho(f\overline{v}), \varrho(g\overline{w})]_{\mathcal{F}\bowtie TM},
\end{aligned}$$

for all $f, g \in C^\infty(\mathcal{F})$ and $v, w \in \Gamma(A)$.

Let us recall the following result:

Proposition 3.8 ([5])**.** *The vector bundle* $p : A \to M$ *is a Lie algebroid over* M *if and only if the dual bundle admits a Poisson structure whose linear functions form a Lie subalgebra.*

As in the case of Lie algebroids, the dual vector bundle $\mathcal{F} \bowtie A^*$ of the generalized Lie algebroid can be endowed with a linear Poisson structure given by:

$$\begin{aligned}
\{f \circ \tau_1, g \circ \tau_1\}^1_{\mathcal{F}\bowtie A^*} &= 0 \\
\{\chi(v) \circ \tau_2, g \circ \tau_1\}^1_{\mathcal{F}\bowtie A^*} &= \widetilde{\rho(v)}g \circ \tau_1 \\
\{\chi(v) \circ \tau_2, \chi(w) \circ \tau_2\}^1_{\mathcal{F}\bowtie A^*} &= \chi([v,w]_A) \circ \tau_2.
\end{aligned} \tag{12}$$

This linear Poisson bracket is obtained from (6) in the case where the Poisson structure on $\mathcal{F}$ is the trivial one. Note that, in this particular case, the compatibility condition 7 is trivially satisfied.

Since $\mathcal{F} \bowtie A^*$ is equipped with a linear Poisson structure, the vector bundle $\mathcal{F} \bowtie A$ is endowed with a Lie algebroid structure over $\mathcal{F}$. So, there exists an anchor map $\widetilde{\rho} : \mathcal{F} \bowtie A \to T\mathcal{F}$ and a Lie bracket $[\cdot,\cdot]_{\mathcal{F}\bowtie A}$ that satisfy the Leibniz rule:

$$[\overline{v}, f\overline{w}]_{\mathcal{F}\bowtie A} = f[\overline{v}, \overline{w}]_{\mathcal{F}\bowtie A} + \widetilde{\rho}(\overline{v})f\,\overline{w}\,,$$

for all $\overline{v}, \overline{w} \in \Gamma(\mathcal{F} \bowtie A)$ and $f \in C^\infty(\mathcal{F})$. In fact, the anchor is just the map $\widetilde{\rho}(\overline{v}) := \widetilde{\rho(v)}$, for all $\overline{v} \in \Gamma(\mathcal{F} \bowtie A)$, and the Lie bracket $[\cdot,\cdot]_{\mathcal{F}\bowtie A}$ on sections of $\mathcal{F} \bowtie A$ is the one defined by (9). This Lie algebroid $(\mathcal{F} \bowtie A, \widetilde{\rho}, [\cdot,\cdot]_{\mathcal{F}\bowtie A})$ over $\mathcal{F}$ is an action Lie algebroid $A \ltimes \pi$, defined by an action $\Phi : \Gamma(A) \to \mathfrak{X}(\mathcal{F})$ of the Lie algebroid A over π, where Φ is the $\mathbb{R}$-linear map defined by $\Phi(v) = \widetilde{\rho(v)}$, for all $v \in \Gamma(A)$, which satisfy the following conditions:

$$\Phi(fv) = (f \circ \pi)\widetilde{\rho(v)}, \quad \Phi(v)(f \circ \pi) = \rho(v)f \circ \pi, \quad \Phi([v,w]_A) = [\Phi(v), \Phi(w)],$$

for all $v, w \in \Gamma(A)$ and $f \in C^\infty(M)$. From what we have seen so far, the only difference between the Lie algebroid $(\mathcal{F} \bowtie A, \widetilde{\rho}, [\cdot,\cdot]_{\mathcal{F}\bowtie A})$ and the generalized Lie algebroid $(\mathcal{F} \bowtie A, \varrho, [\cdot,\cdot]_{\mathcal{F}\bowtie A})$ is on the anchor map. But this is not so important, because the action on functions is the same and, therefore, the structure functions of the generalized Lie algebroid, given by (11), coincide with the structure functions of the Lie algebroid $(\mathcal{F} \bowtie A, \widetilde{\rho}, [\cdot,\cdot]_{\mathcal{F}\bowtie A})$. Furthermore, the structure equations of the Lie algebroid $\mathcal{F} \bowtie A$ coincide with equations (2) and (3). We notice that $\varrho = \widetilde{\pi} \circ \widetilde{\rho}$, where $\widetilde{\pi} : T\mathcal{F} \to \mathcal{F} \bowtie TM$ represents the projection of the tangent bundle $T\mathcal{F}$ onto the bundle $\mathcal{F} \bowtie TM$ and $\widetilde{\rho} = h \circ \varrho$, where $h : \mathcal{F} \bowtie TM \to T\mathcal{F}$ is a section of the map $\widetilde{\pi}$ associated with the flat connection of $\pi : \mathcal{F} \to M$. Therefore, given a Lie algebroid A and an internal bundle $\mathcal{F}$, a generalized Lie algebroid structure on the extended bundle $\mathcal{F} \bowtie A$ corresponds to a Lie algebroid structure on $\mathcal{F} \bowtie A$.

Since $\mathcal{F} \bowtie A$ is a Lie algebroid, we can consider its exterior derivative $d_{\widetilde{\rho}}$, which is a derivation of degree one of the exterior algebra $\Omega^\bullet(\mathcal{F} \bowtie A)$ and nilpotent of order two, $d_{\widetilde{\rho}}^2 = 0$. On $\mathcal{F} \bowtie A$-k-forms, this operator is defined in the usual way by:

$$\begin{aligned} d_{\widetilde{\rho}}\overline{\alpha}(\overline{v}_1, ..., \overline{v}_k, \overline{v}_{k+1}) \;=\; & \sum_{i=1}^{k+1}(-1)^{i+1}\widetilde{\rho}(\overline{v}_i)\overline{\alpha}(\overline{v}_1, ..., \widehat{\overline{v}_i}, ..., \overline{v}_{k+1}) \\ & + \sum_{1\le i<j\le k+1}(-1)^{i+j}\overline{\alpha}([\overline{v}_i, \overline{v}_j]_{\mathcal{F}\bowtie A}, ..., \widehat{\overline{v}_i}, ..., \widehat{\overline{v}_j}, ..., \overline{v}_{k+1}), \end{aligned} \tag{13}$$

where $\overline{\alpha} \in \Omega^k(\mathcal{F} \bowtie A)$ and $\overline{v}_1, ..., \overline{v}_k, \overline{v}_{k+1} \in \Gamma(\mathcal{F} \bowtie A)$; the symbol $\widehat{\cdot}$ means omission of such element. Let d_ρ be the exterior derivative of the Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ and d the de Rham operator on the manifold M. Then:

Proposition 3.9. *The exterior derivative $d_{\widetilde{\rho}}$ of the Lie algebroid $\mathcal{F} \bowtie A$ satisfies the following properties:*

(i) $d_{\widetilde{\rho}} \circ p_2^* = p_2^* \circ d_\rho$;
(ii) $d_{\widetilde{\rho}} \circ (\rho \circ p_2)^* = (\rho \circ p_2)^* \circ d$,

where $p_2 : \mathcal{F} \bowtie A \to A$.

Proof: (i) Since $d_{\widetilde{\rho}}$ and d_ρ are derivations of degree one on the exterior algebras $\Omega^\bullet(\mathcal{F} \bowtie A)$ and $\Omega^\bullet(A)$, respectively, we just need to prove the equality $d_{\widetilde{\rho}} \circ p_2^* = p_2^* \circ d_\rho$ for forms of degrees zero and one. With forms of degree zero we obtain

$$T\pi \circ \widetilde{\rho} = \rho \circ p_2, \tag{14}$$

and in forms of degree one we have

$$p_2 \circ [\overline{v}, \overline{w}]_{\mathcal{F} \bowtie A} = [v, w]_A \circ \pi\,, \tag{15}$$

for all $v, w \in \Gamma(A)$. From the definitions of the anchor map $\widetilde{\rho}$ and of the Lie bracket $[\cdot,\cdot]_{\mathcal{F} \bowtie A}$, we can conclude that (14) and (15) hold.

(ii) From condition (i) we have $d_{\widetilde{\rho}} \circ p_2^* \circ \rho^* = p_2^* \circ d_\rho \circ \rho^*$. Since ρ is a homomorphism of Lie algebroids then $d_\rho \circ \rho^* = \rho^* \circ d$. Therefore, $d_{\widetilde{\rho}} \circ p_2^* \circ \rho^* = p_2^* \circ \rho^* \circ d$. ■

Let $\{\overline{e_\alpha} \mid \alpha = 1, ..., r\}$ be a basis of local sections of $\mathcal{F} \bowtie A$ associated to the local basis of sections $\{e_\alpha \mid \alpha = 1, ..., r\}$ of A and let $\{\overline{e^\alpha} \mid \alpha = 1, ..., r\}$ be the corresponding dual basis of sections of $\mathcal{F} \bowtie A^*$. The exterior derivative is characterized by its value on 0-forms and 1-forms, so in local coordinates:

$$d_{\widetilde{\rho}} = \Gamma^b_i\, \rho^i_\alpha\, \partial_{\xi^b} \otimes \overline{e^\alpha} + \rho^i_\alpha\, \partial_{x^i} \otimes \overline{e^\alpha} - \frac{1}{2}\, c_{\alpha\beta}{}^\gamma \overline{e_\gamma} \otimes \overline{e^\alpha} \wedge \overline{e^\beta}.$$

Note that the structure equations (2) and (3) of the Lie algebroid $\mathcal{F} \bowtie A$ are equivalent to $d^2_{\widetilde{\rho}} x^i = d^2_{\widetilde{\rho}} \xi^a = 0$ and $d^2_{\widetilde{\rho}} \overline{e^\nu} = 0$, respectively.

One can define on the dual bundle $\mathcal{F} \bowtie A^*$ another non-linear Poisson structure by setting:

$$\begin{aligned} \{f\circ\tau_1, g\circ\tau_1\}^0_{\mathcal{F}\bowtie A^*} &= \{f,g\}\circ\tau_1 \\ \{\chi(v)\circ\tau_2, g\circ\tau_1\}^0_{\mathcal{F}\bowtie A^*} &= 0 \\ \{\chi(v)\circ\tau_2, \chi(w)\circ\tau_2\}^0_{\mathcal{F}\bowtie A^*} &= 0. \end{aligned} \tag{16}$$

This non-linear Poisson bracket is obtained from (6) in the case where the Lie algebroid structure of A is the trivial one. Note that, in this case, the compatibility condition (7) is trivially satisfied. Therefore, the almost-Poisson structure on $\mathcal{F} \bowtie A^*$ given by (6) is the sum of the linear Poisson structure (12) with the Poisson structure (16),

$$\{\cdot,\cdot\}_{\mathcal{F}\bowtie A^*} = \{\cdot,\cdot\}^0_{\mathcal{F}\bowtie A^*} + \{\cdot,\cdot\}^1_{\mathcal{F}\bowtie A^*}.$$

So, when the Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ and the internal bundle $(\mathcal{F}, \{\cdot,\cdot\})$ satisfy the compatibility condition (7), i.e. the bracket $\{\cdot,\cdot\}_{\mathcal{F}\bowtie A^*}$ is Poisson, we have that the two Poisson structures (12) and (16) are compatible.

Definition 3.10. *Given a Lie algebroid $p_0 : A \to M$ and a fibre bundle $\pi : \mathcal{F} \to M$ equipped with a flat connection and a localizable Poisson structure that satisfy the compatibility condition (7), then the vector bundle $\mathcal{F} \bowtie A \to \mathcal{F}$ endowed with the generalized Lie algebroid structure $(\varrho, [\cdot,\cdot]_{\mathcal{F}\bowtie A})$ is called a quadratic algebroid.*

From the above definition, we can conclude that the internal deformation of a Lie algebroid A by a fibre bundle $\mathcal{F}$, endows the extended bundle $\mathcal{F} \bowtie A$ with a structure of quadratic algebroid. Moreover, we conclude that the dual bundle of a quadratic algebroid is a quadratic co-algebroid. Note that a Lie algebroid is a particular case of a generalized Lie algebroid. In fact, a Lie algebroid is a generalized Lie algebroid deformed by the bundle $\pi = \mathrm{id}_M : M \to M$, where $M \bowtie A \equiv A$ and $M \bowtie TM \equiv TM$.

Example 3.11 (Tangent bundle)**.** *Let us consider the tangent bundle TM endowed with its natural Lie algebroid structure and a differential manifold N. Let be $\mathcal{F} = M \times N$ and $\pi : \mathcal{F} = M \times N \to M$ the product bundle endowed with the natural connection and a localizable Poisson structure of the form $\Lambda = C_{ab}(\xi)\,\partial_{\xi^a} \wedge \partial_{\xi^b}$. The connection is given by the horizontal distribution of $\mathcal{F}$, $H_q = T_{\pi(q)}M$, for all $q \in \mathcal{F}$ and the horizontal lift of a vector field $X \in \mathfrak{X}(M)$ to $\mathcal{F}$ is given by $\widetilde{X} = (X, 0) \equiv X$ and therefore the connection is* *flat. The vector bundle* $\kappa_1 : \mathcal{F} \bowtie TM \to \mathcal{F}$

$$\begin{tikzcd} F \bowtie TM \arrow[r, "\kappa_2"] \arrow[d, "\kappa_1"'] & TM \arrow[d, "\kappa_0"] \\ \mathcal{F} \arrow[r, "\pi"] & M \end{tikzcd}$$

is endowed with a generalized Lie algebroid structure over $\mathcal{F}$*. The bundle* $\mathcal{F} \bowtie TM$ *is a quadratic algebroid because the compatibility condition (7) is satisfied*

$$\widetilde{X}\{h_1, h_2\} = \left\{\widetilde{X}h_1, h_2\right\} + \left\{h_1, \widetilde{X}h_2\right\},$$

for all $h_1, h_2 \in C^\infty(\mathcal{F})$ *and* $X \in \mathfrak{X}(M)$*. This condition holds because the function* $C_{ab}(\xi) = \left\{\xi^a, \xi^b\right\}$ *does not depend on the local coordinates* x^i*; note that* $\widetilde{X}\left\{\xi^a, \xi^b\right\} = 0$*, for all* $X \in \mathfrak{X}(M)$*. Since* $\mathcal{F} \bowtie TM$ *is a quadratic algebroid, we may define on the dual bundle* $\mathcal{F} \bowtie T^*M$

$$\begin{tikzcd} F \bowtie T^*M \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & T^*M \arrow[d, "\pi_0"] \\ \mathcal{F} \arrow[r, "\pi"] & M \end{tikzcd}$$

a quadratic co-algebroid structure, whose non-linear Poisson structure is given in matricial form (using Darboux coordinates for the symplectic manifold T^*M*) by:*

$$\Lambda_{\mathcal{F}\bowtie T^*M} = \begin{pmatrix} 0 & 0 & -I \\ 0 & C & 0 \\ I & 0 & 0 \end{pmatrix}. \tag{17}$$

When the matrix C *is invertible the Poisson structure* $\Lambda_{\mathcal{F}\bowtie T^*M}$ *is regular, i.e. the Poisson structure is non-degenerated. In this case,* $\mathcal{F} \bowtie T^*M$ *is a symplectic manifold.*

Given a Lie algebroid $(A, \rho, [\cdot,\cdot])$ over a manifold M and an internal bundle $\pi : \mathcal{F} \to M$, it is important to observe that when the extended bundle $\mathcal{F} \bowtie TM$ is a quadratic algebroid, then the extended bundle $\mathcal{F} \bowtie A$ is also a quadratic algebroid, however the converse is false.

In the conditions of the above example, let us suppose that the Poisson structure $\Lambda = C_{ab}(\xi)\,\partial_{\xi^a} \wedge \partial_{\xi^b}$ on $\mathcal{F}$ is such that $C_{ab}(0) = 0$. Let $(\Phi, \phi) : (T^*M, \pi_0, M) \to (\mathcal{F} \bowtie T^*M, \pi_1, \mathcal{F})$ be the dual morphism of

$\kappa_2 : \mathcal{F} \bowtie TM \to TM$ where ϕ is such that $\pi \circ \phi = \mathrm{id}_M$ and satisfies $\phi^* x^i = x^i$ and $\phi^* \xi_a = 0$. Then, we can establish the following:

Proposition 3.12. *The morphism (Φ, ϕ) is a Poisson morphism, i.e.*

$$\{F \circ \Phi, G \circ \Phi\}_{T^*M} = \{F, G\}_{\mathcal{F} \bowtie T^*M} \circ \Phi\,, \tag{18}$$

for all $F, G \in C^\infty(\mathcal{F} \bowtie T^*M)$, where T^*M is endowed with its canonical symplectic structure and $\mathcal{F} \bowtie T^*M$ is equipped with the Poisson structure (17).

Proof: We just have to prove the above condition for either two basic functions, $f \circ \pi_1$, or two linear functions $\chi_0(v) \circ \pi_2$ and for one basic and a linear one. Let us first consider two basic functions on $\mathcal{F} \bowtie T^*M$, $F = f \circ \pi_1$ and $G = g \circ \pi_1$. Then, from the above definition of the bracket $\{\cdot, \cdot\}_{\mathcal{F} \bowtie T^*M}$, we have

$$\{F, G\}_{\mathcal{F} \bowtie T^*M} \circ \Phi \;=\; \{f \circ \pi_1, g \circ \pi_1\}_{\mathcal{F} \bowtie T^*M} \circ \Phi = \{f, g\} \circ \pi_1 \circ \Phi$$

and because $\pi_1 \circ \Phi = \phi \circ \pi_0$, we still have

$$\{F, G\}_{\mathcal{F} \bowtie T^*M} \circ \Phi \;=\; \{f, g\} \circ \phi \circ \pi_0 = C_{ab}(\xi) \frac{\partial f}{\partial \xi_a} \frac{\partial g}{\partial \xi_b} \circ \phi \circ \pi_0.$$

Since $C_{ab}(\xi \circ \phi) = C_{ab}(0) = 0$, we can conclude that

$$\{F, G\}_{\mathcal{F} \bowtie T^*M} \circ \Phi = 0.$$

On the other hand, from the definition of the Poisson bracket on T^*M, we have

$$\{F \circ \Phi, G \circ \Phi\}_{T^*M} \;=\; \{f \circ \phi \circ \pi_0, g \circ \phi \circ \pi_0\}_{T^*M} = 0$$

and the condition (18) holds. Let us now consider two linear functions $F = \chi_0(v) \circ \pi_2$ and $G = \chi_0(w) \circ \pi_2$, with $v, w \in \Gamma(A)$. Since $\pi_2 \circ \Phi = \mathrm{id}_{T^*M}$, we have

$$\{F \circ \Phi, G \circ \Phi\}_{T^*M} \;=\; \{\chi_0(v), \chi_0(w)\}_{T^*M} := \chi_0([v, w]),$$

that is,

$$\{F \circ \Phi, G \circ \Phi\}_{T^*M} \;=\; \chi_0([v, w]) \circ \pi_2 \circ \Phi.$$

Once again, from the definition of the bracket $\{\cdot, \cdot\}_{\mathcal{F} \bowtie T^*M}$, we have

$$\{F \circ \Phi, G \circ \Phi\}_{T^*M} \;=\; \{\chi_0(v) \circ \pi_2, \chi_0(w) \circ \pi_2\}_{\mathcal{F} \bowtie T^*M} \circ \Phi = \{F, G\}_{\mathcal{F} \bowtie T^*M} \circ \Phi.$$

Finally, we have to prove (18) in the case where $F = \chi_0(v)\circ\pi_2$ and $G = g\circ\pi_1$. Then,

$$\begin{aligned}
\{F\circ\Phi, G\circ\Phi\}_{T^*M} &= \{\chi_0(v)\circ\pi_2\circ\Phi, g\circ\pi_1\circ\Phi\}_{T^*M}\\
&= \{\chi_0(v), g\circ\phi\circ\pi_0\}_{T^*M}\\
&:= v(g\circ\phi)\circ\pi_0.
\end{aligned}$$

Since $T\phi\circ v = \widetilde{v}\circ\phi$, we obtain

$$\{F\circ\Phi, G\circ\Phi\}_{T^*M} = \widetilde{v}g\circ\phi\circ\pi_0 = \widetilde{v}g\circ\pi_1\circ\Phi\,,$$

and from the definition of $\{\cdot,\cdot\}_{\mathcal{F}\bowtie T^*M}$, we conclude that

$$\{F\circ\Phi, G\circ\Phi\}_{T^*M} = \{\chi_0(v)\circ\pi_2, g\circ\pi_1\}_{\mathcal{F}\bowtie T^*M}\circ\Phi = \{F,G\}_{\mathcal{F}\bowtie T^*M}\circ\Phi. \blacksquare$$

4. Symplectic realizations of $\mathcal{F}\bowtie A^*$

Let us consider a quadratic algebroid $(\mathcal{F}\bowtie A, \varrho, [\cdot,\cdot]_{\mathcal{F}\bowtie A})$ obtained by internal deformation of the Lie algebroid $(A,\rho,[\cdot,\cdot]_A)$ over M by the internal bundle $\mathcal{F}\to M$ endowed with a localizable Poisson tensor and a flat connection. We are going to prove that the dual map of the generalized anchor, $\varrho^*:\mathcal{F}\bowtie T^*M\to\mathcal{F}\bowtie A^*$, of the quadratic co-algebroid $\mathcal{F}\bowtie T^*M$ is a Poisson morphism.

$$\begin{tikzcd}
T^*M \arrow[d, "\pi_0"'] & \mathcal{F}\bowtie T^*M \arrow[l, "\pi_2"'] \arrow[r, "\varrho^*"] \arrow[d, "\pi_1"'] & \mathcal{F}\bowtie A^* \arrow[r, "\tau_2"] \arrow[d, "\tau_1"'] & A^* \arrow[d, "\tau_0"] \\
M & \mathcal{F} \arrow[l, "\pi"] \arrow[r, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F} \arrow[r, "\pi"'] & M
\end{tikzcd}$$

When the Poisson structure on $\mathcal{F}$ is regular along the fibres, i.e. when the matrix of entries $C_{ab} = \left\{\xi^a,\xi^b\right\}$ is invertible, the map ϱ^* provides a symplectic realization of the quadratic co-algebroid $\mathcal{F}\bowtie A^*$.

Let $H = \chi(v)$ be a linear function on A^* associated to $v\in\Gamma(A)$. Then:

$$\begin{aligned}
&H\circ\tau_2 = \chi'(\overline{v}),\\
&H\circ\tau_2\circ\varrho^* = \chi_0'(\varrho(\overline{v})) = \chi_0'(\,\overline{\rho(v)}\,) = \chi_0(\rho(v))\circ\pi_2,
\end{aligned} \tag{19}$$

where $\chi':\Gamma(\mathcal{F}\bowtie A)\to L(\mathcal{F}\bowtie A^*)$ and $\chi_0':\Gamma(\mathcal{F}\bowtie TM)\to L(\mathcal{F}\bowtie T^*M)$ are isomorphisms, and $L(\mathcal{F}\bowtie A^*)$, $L(\mathcal{F}\bowtie T^*M)$ denote the spaces of linear functions on $\mathcal{F}\bowtie A^*$ and $\mathcal{F}\bowtie T^*M$, respectively.

Proposition 4.1. *Let* $(\mathcal{F} \bowtie TM, \varrho_0, [\cdot,\cdot]_{\mathcal{F}\bowtie TM})$ *and* $(\mathcal{F} \bowtie A, \varrho, [\cdot,\cdot]_{\mathcal{F}\bowtie A})$ *be quadratic algebroids obtained by internal deformation of the Lie algebroids* TM *and* A*, respectively, by the bundle* $\mathcal{F} \to M$ *endowed with a localizable Poisson structure and a flat connection. Then,* $\varrho^* : \mathcal{F} \bowtie T^*M \to \mathcal{F} \bowtie A^*$ *is a Poisson morphism.*

Proof: We have to show that ϱ^* is a Poisson morphism, that is,

$$\{F, G\}_{\mathcal{F}\bowtie A^*} \circ \varrho^* \;=\; \{F \circ \varrho^*, G \circ \varrho^*\}_{\mathcal{F}\bowtie T^*M}, \tag{20}$$

for all affine functions $F, G \in C^\infty(\mathcal{F} \bowtie A^*)$. With $f, g \in C^\infty(\mathcal{F})$, we have

$$\{f \circ \tau_1 \circ \varrho^*, g \circ \tau_1 \circ \varrho^*\}_{\mathcal{F}\bowtie T^*M} = \{f \circ \mathrm{id}_{\mathcal{F}} \circ \pi_1, g \circ \mathrm{id}_{\mathcal{F}} \circ \pi_1\}_{\mathcal{F}\bowtie T^*M} = \{f, g\} \circ \pi_1$$

and therefore,

$$\{f \circ \tau_1 \circ \varrho^*, g \circ \tau_1 \circ \varrho^*\}_{\mathcal{F}\bowtie T^*M} = \{f, g\} \circ \tau_1 \circ \varrho^* = \{f \circ \tau_1, g \circ \tau_1\}_{\mathcal{F}\bowtie A^*} \circ \varrho^*.$$

Now, let us suppose that $F = \chi(v) \circ \tau_2$ is a linear function on $\mathcal{F} \bowtie A^*$, associated to the section v of A, and $G = g \circ \tau_1$ is a basic function on $\mathcal{F} \bowtie A^*$. Then, from (19),

$$\{F \circ \varrho^*, G \circ \varrho^*\}_{\mathcal{F}\bowtie T^*M} = \{\chi_0(\rho(v)) \circ \pi_2, g \circ \pi_1\}_{\mathcal{F}\bowtie T^*M} = \widetilde{\rho(v)} g \circ \tau_1 \circ \varrho^*.$$

On the other hand,

$$\{F, G\}_{\mathcal{F}\bowtie A^*} \circ \varrho^* = \{\chi(v) \circ \tau_2, g \circ \tau_1\}_{\mathcal{F}\bowtie A^*} \circ \varrho^* = \widetilde{\rho(v)} g \circ \tau_1 \circ \varrho^*$$

and condition (20) holds. Finally, let now $F \circ \tau_2 = \chi(v) \circ \tau_2$ and $G \circ \tau_2 = \chi(w) \circ \tau_2$ be two linear functions on $\mathcal{F} \bowtie A^*$, where v and w are sections of A. Then,

$$\begin{aligned} \{F \circ \varrho^*, G \circ \varrho^*\}_{\mathcal{F}\bowtie T^*M} &= \{\chi_0(\rho(v)) \circ \pi_2, \chi_0(\rho(w)) \circ \pi_2\}_{\mathcal{F}\bowtie T^*M} \\ &= \chi_0([\rho(v), \rho(w)]) \circ \pi_2. \end{aligned}$$

Since ρ is a Lie algebra homomorphism, we also have

$$\begin{aligned} \{F \circ \varrho^*, G \circ \varrho^*\}_{\mathcal{F}\bowtie T^*M} &= \chi_0(\rho([v, w]_A)) \circ \pi_2 \\ &= \chi([v, w]_A) \circ \tau_2 \circ \varrho^* \\ &= \{\chi(v) \circ \tau_2, \chi(w) \circ \tau_2\}_{\mathcal{F}\bowtie A^*} \circ \varrho^* \end{aligned}$$

and therefore (20) holds. ∎

As we said before, when the Poisson structure on $\mathcal{F}$ is regular along the fibres, the non-linear Poisson structure on $\mathcal{F} \bowtie T^*M$ is regular. Then $\mathcal{F} \bowtie$

T^*M is a symplectic manifold, and therefore the dual map of the generalized anchor is a symplectic realization of the quadratic co-algebroid $\mathcal{F} \bowtie A^*$.

We notice that, in general, the generalized Lie algebroid $\mathcal{F} \bowtie TM$ is not a quadratic algebroid because it does not satisfy the compatibility condition (7). Nevertheless, the vector bundle $\mathcal{F} \bowtie T^*M$ is endowed with an almost-Poisson structure given by (6), when $A = TM$. In this case, we say that the dual map of the anchor is an *almost-Poisson morphism*.

Now, let us suppose that the fibre bundle $\mathcal{F}$ satisfies the conditions of Proposition 4.1 and that the Poisson tensor on $\mathcal{F}$ is of the form $\Lambda = C_{ab}(\xi)\, \partial_{\xi^a} \wedge \partial_{\xi^b}$, regular along the fibres, with $C_{ab}(0) = 0$. Then, the morphism $(\Phi, \phi) : (T^*M, \pi_0, M) \to (\mathcal{F} \bowtie T^*M, \pi_1, \mathcal{F})$, introduced in Proposition 3.12, is a symplectic realization of the quadratic co-algebroid $\mathcal{F} \bowtie T^*M$. Moreover, we conclude that the composition $\varrho^* \circ \Phi : T^*M \to \mathcal{F} \bowtie A^*$ is a Poisson morphism over the map $\phi : M \to \mathcal{F}$ and, therefore, it is a symplectic realization of the quadratic co-algebroid $\mathcal{F} \bowtie A^*$. The dual map of $\varrho^* \circ \Phi$ is the vector bundle morphism $\sigma = \kappa_2 \circ \varrho : \mathcal{F} \bowtie A \to TM$ over the map $\pi : \mathcal{F} \to M$,

$$\begin{tikzcd}
\mathcal{F} \bowtie A \arrow[r, "\varrho"] \arrow[d, "p_1"'] & \mathcal{F} \bowtie TM \arrow[r, "\kappa_2"] \arrow[d, "\kappa_1"'] & TM \arrow[d, "\kappa_0"] \\
\mathcal{F} \arrow[r, "\mathrm{id}_{\mathcal{F}}"'] & \mathcal{F} \arrow[r, "\pi"'] & M
\end{tikzcd}$$

given by

$$\sigma(q, v(\pi(q))) = \rho(v)(\pi(q)),$$

for all $q \in \mathcal{F}$ and $v \in \Gamma(A)$. The following conditions hold

(i) $\sigma = T\pi \circ \widetilde{\rho}$,

(ii) $\sigma \circ [\overline{v}, \overline{w}]_{\mathcal{F}\bowtie A} = [\rho(v), \rho(w)] \circ \pi$,

for all $v, w \in \Gamma(A)$. Thus, σ is a Lie algebroid homomorphism.

5. Some examples of internal deformation

In this section, we will give some examples of internal deformations of a Lie algebroid $(A, \rho, [\cdot,\cdot]_A)$ over a manifold M by internal bundles $\mathcal{F} \to M$ endowed with a localizable Poisson structure Λ and a flat connection.

5.1. Lie algebra bundle. Let $\pi : \mathcal{F} \to \{\cdot\}$ be a fibre bundle with a unique fibre $\mathcal{F}$ endowed with a localizable Poisson structure of the form $\Lambda = C_{ab}\, \partial_{\xi^a} \wedge \partial_{\xi^b}$, where C_{ab} are constants. The natural connection of this bundle is defined by the distribution $H_q = \{0\}$, for all $q \in \mathcal{F}$, and the horizontal lift to $\mathcal{F}$ of the vector field $X = 0$ is $\widetilde{X} = 0$. Therefore, the connection is obviously flat. Given a finite dimensional Lie algebra $\mathfrak{g}$, we can consider $\mathfrak{g}$ as a Lie algebroid over the set $\{\cdot\}$– the anchor ρ is zero and the bracket on the sections is given by the Lie bracket of the Lie algebra. The internal deformation of $\mathfrak{g}$ by $\mathcal{F}$ is represented by the following diagram:

$$\begin{tikzcd}
\mathcal{F} \bowtie \mathfrak{g} \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathfrak{g} \arrow[d, "p_0"] \\
\mathcal{F} \arrow[r, "\pi"] & \{\cdot\}
\end{tikzcd}$$

where $p_1(q, X) = q$ and $p_2(q, X) = X$, for all $q \in \mathcal{F}$ and $X \in \mathfrak{g}$. The generalized Lie algebroid structure on $\mathcal{F} \bowtie \mathfrak{g}$ is given by:

$$\begin{aligned}
[\overline{e_\alpha}, \overline{e_\beta}]_{\mathcal{F}\bowtie\mathfrak{g}} &= c_{\alpha\beta}{}^{\gamma}\, \overline{e_\gamma}\,, \\
\varrho(\overline{e_\alpha})\xi^a &= 0\,,
\end{aligned}$$

where $c_{\alpha\beta}{}^{\gamma}$ are the structure constants of the Lie algebra $\mathfrak{g}$ and $\overline{e_\alpha}$ are the elements of a basis of sections in the space $\Gamma(\mathcal{F} \bowtie \mathfrak{g}) \simeq C^\infty(\mathcal{F}; \mathfrak{g})$. Note that the space $C^\infty(\mathcal{F}; \mathfrak{g})$ is generated by a set of constant functions $s_\alpha(q) = e_\alpha$, where the elements e_α define a basis of $\mathfrak{g}$. Since the trivial bundle $\mathcal{F} \times \mathfrak{g} \to \mathcal{F}$ coincides with the vector bundle $\mathcal{F} \bowtie \mathfrak{g} \to \mathcal{F}$, one can show that the structures of Lie algebra bundle and Lie algebroid on $\mathcal{F} \bowtie \mathfrak{g}$ are the same.

In these conditions, we easily prove the compatibility condition (7):

$$\varrho(\overline{X})\,\{f, g\} \;=\; \left\{\varrho(\overline{X})f, g\right\} + \left\{f, \varrho(\overline{X})g\right\},$$

for all $X \in \Gamma(\mathfrak{g})$ and $f, g \in C^\infty(\mathcal{F})$; note that $\varrho(\overline{X})f = \widetilde{\rho(X)}f = 0$. Therefore, the vector bundle $p_1 : \mathcal{F} \bowtie \mathfrak{g} \to \mathcal{F}$ is endowed with a quadratic algebroid structure and its dual $\tau_1 : \mathcal{F} \bowtie \mathfrak{g}^* \to \mathcal{F}$ is endowed with a quadratic co-algebroid structure, whose non-linear Poisson bracket is given by:

$$\begin{aligned}
\{\chi(\overline{e_\alpha}), \chi(\overline{e_\beta})\}_{\mathcal{F}\bowtie\mathfrak{g}^*} &= c_{\alpha\beta}{}^{\gamma}\chi(\overline{e_\gamma})\,, \\
\{\chi(\overline{e_\alpha}), \xi^a\}_{\mathcal{F}\bowtie\mathfrak{g}^*} &= 0\,, \\
\{\xi^a, \xi^b\}_{\mathcal{F}\bowtie\mathfrak{g}^*} &= C_{ab}\,.
\end{aligned}$$

In the matricial form, the Poisson structure on $\mathcal{F} \bowtie \mathfrak{g}^*$ is given by

$$\Lambda_{\mathcal{F}\bowtie\mathfrak{g}^*} = \begin{pmatrix} C & 0 \\ 0 & c \end{pmatrix},$$

where $C = (C_{ab})$ and $c = (c_{\alpha\beta})$ with $c_{\alpha\beta} = c_{\alpha\beta}{}^{\gamma}\chi(\overline{e_\gamma})$.

5.2. Poisson manifold. Let (M, Π) be a Poisson manifold and N a differentiable manifold. We will deform the Lie algebroid structure on T^*M, defined by the Poisson tensor Π, by the internal bundle $\pi : \mathcal{F} = M \times N \to M$ endowed with a localizable Poisson structure of the form $\Lambda(x, \xi) = C_{ab}(\xi)\, \partial_{\xi^a} \wedge \partial_{\xi^b}$. The natural connection of the bundle $\mathcal{F}$ is defined by the distribution $H_q = T_{\pi(q)}M$, for all $q \in \mathcal{F}$. In this case, the horizontal lift of $X \in \mathfrak{X}(M)$ to $\mathcal{F}$ is a vector field on $\mathcal{F}$ given by $\widetilde{X}(q) = (X(\pi(q)), 0) \equiv X(\pi(q))$, for all $q \in \mathcal{F}$.

Let us consider a generalized Lie algebroid structure on the vector bundle $\pi_1 : \mathcal{F} \bowtie T^*M \to \mathcal{F}$. Since the compatibility condition (7) holds, $\mathcal{F} \bowtie T^*M$ is a quadratic algebroid. The dual vector bundle $\kappa_1 : \mathcal{F} \bowtie TM \to \mathcal{F}$ is endowed with a non-linear (quadratic) Poisson structure, given in the matricial form by:

$$\Lambda_{\mathcal{F}\bowtie TM} = \begin{pmatrix} 0 & 0 & \Pi \\ 0 & C & 0 \\ \Pi & 0 & \Upsilon \end{pmatrix},$$

where $C = (C_{ab})$, $\Pi = (\Pi_{ij})$ and $\Upsilon = (\Upsilon_{ij})$ with

$$\Upsilon_{ij} = \left\{\chi(dx^i) \circ \kappa_2, \chi(dx^j) \circ \kappa_2\right\}_{\mathcal{F}\bowtie TM}.$$

If Π is a non-degenerated tensor and C is invertible, then the non-linear Poisson tensor $\Lambda_{\mathcal{F}\bowtie TM}$ is regular.

5.3. Group action. Let G be a Lie group with a Lie algebra $\mathfrak{g}$ that acts on the Poisson manifold $(M, \{\cdot,\cdot\}_M)$ and let $\pi : \mathcal{F} = M \times N \to M$ be the product bundle, where N is a differentiable manifold, endowed with a localizable Poisson structure $\{\cdot,\cdot\}$. As we have already remarked, the natural connection of the bundle $\mathcal{F}$ is defined by the distribution $H_q = T_{\pi(q)}M$, for all $q \in \mathcal{F}$. In this case, the horizontal lift of $X \in \mathfrak{X}(M)$ to $\mathcal{F}$ is a vector field on $\mathcal{F}$ given by $\widetilde{X}(q) = (X(\pi(q)), 0) \equiv X(\pi(q))$, for all $q \in \mathcal{F}$.

Let us suppose that the group action is Hamiltonian, i.e. for each $X \in \mathfrak{g}$ there is a function $H_X \in C^\infty(M)$ such that the fundamental vector field X_M on M is Hamiltonian with Hamiltonian function H_X. Such correspondence is linear. Suppose also that the Poisson bracket on M defines a Poisson bracket

$\{\cdot,\cdot\}_{\mathcal{F}}$ on $\mathcal{F}$ which is compatible with the bracket $\{\cdot,\cdot\}$:

$$\begin{aligned}\{f\circ\pi, g\circ\pi\}_{\mathcal{F}} &= \{f,g\}_M\circ\pi\,,\\ \{f\circ\pi, G\}_{\mathcal{F}} &= 0\,,\\ \{F,G\}_{\mathcal{F}} &= 0\,,\end{aligned}$$

for all $f,g\in C^\infty(M)$ and where F and G are functions on $\mathcal{F}$ that only depend on N.

The internal deformation of the Lie algebroid $M\times\mathfrak{g}\to M$ by the bundle $\mathcal{F}$ is a quadratic algebroid. Indeed,

$$\widetilde{\rho(s_X)} = \{H_X\circ\pi,\cdot\}_{\mathcal{F}}$$

because $\rho(s_X) = X_M = \{H_X,\cdot\}_M$ for all constant section $s_X(x) = (x,X)$ of the trivial bundle $M\times\mathfrak{g}\to M$. Since the Poisson brackets $\{\cdot,\cdot\}_{\mathcal{F}}$ and $\{\cdot,\cdot\}$ are compatible, we have

$$\widetilde{\rho(s_X)}\,\{F,G\} = \{H_X\circ\pi,\{F,G\}\}_{\mathcal{F}} = \{\{H_X\circ\pi,F\}_{\mathcal{F}},G\}+\{F,\{H_X\circ\pi,G\}_{\mathcal{F}}\}\,,$$

that is,

$$\widetilde{\rho(s_X)}\,\{F,G\} = \left\{\widetilde{\rho(s_X)}F,G\right\}+\left\{F,\widetilde{\rho(s_X)}G\right\},$$

for all $F,G\in C^\infty(\mathcal{F})$. The quadratic algebroid structure of $\mathcal{F}\bowtie(M\times\mathfrak{g})$ is given by:

$$\begin{aligned}[\overline{v},\overline{w}]_{\mathcal{F}\bowtie(M\times\mathfrak{g})} &= \overline{[v,w]_{M\times\mathfrak{g}}}\,,\\ \varrho(s_X)x^i &= \rho(s_X)x^i = X_Mx^i\,,\\ \varrho(s_X)\xi^a &= 0\,,\end{aligned}$$

for all $X\in\mathfrak{g}$ and $v,w\in\Gamma(M\times\mathfrak{g})$. The dual vector bundle $\mathcal{F}\bowtie(M\times\mathfrak{g}^*)\to\mathcal{F}$ is a quadratic co-algebroid, and so it is endowed with a quadratic Poisson structure.

5.4. Free motion. Let $\mathcal{F}$ be the trivial vector bundle $Q\times V\to Q$ where Q is a m-dimensional manifold with local coordinates q^i and V is a n-dimensional real linear space with coordinates I_a. The free motion on V is characterized by

$$\dot{I}_a = 0.$$

Let η be a Riemannian metric on Q. Free motion on Q would be described by the geodesics of such a metric. Consider now, the following singular Lagrangian associated to the free system

$$L = T + \widehat{c},$$

where T is the kinetic energy of the system defined by the metric η, and $\widehat{c}$ is the linear function on TV defined by a 1-form c on V, by setting $\widehat{c}(w) = \langle c_I, w\rangle$ for all $w \in T_I V \equiv V$, such that, the 2-form dc is symplectic on V. The 2-form dc defines a regular Poisson structure on V,

$$\Lambda = C_{ab}(I)\, \partial_{I_a} \wedge \partial_{I_b},$$

this Poisson structure can be consider as a localizable Poisson structure on the bundle $\mathcal{F} = Q \times V \to Q$. In local coordinates, the Lagrangian is written in the following way

$$L(q, v, I, \dot{I}) = \frac{1}{2}\eta_{ij}(q) v^i v^j + c^a(I)\dot{I}_a$$

and the Cartan forms of order one and two are given by:

$$\begin{aligned} \theta_L &= \eta_{ij} v^i dq^j + c^a(I) dI_a\,, \\ \omega_L &= v^i \left(\frac{\partial \eta_{ij}}{\partial q^k} - \frac{\partial \eta_{ik}}{\partial q^j}\right) dq^j \wedge dq^k + \eta_{kj} dq^j \wedge dv^k + C^{ab} dI_a \wedge dI_b\,, \end{aligned}$$

where C^{ab} are the entries of the inverse matrix of (C_{ab}). Let $\Pi : T\mathcal{F} \to \mathcal{F} \bowtie TQ$ be the projection of $T\mathcal{F}$ onto $\mathcal{F} \bowtie TQ$. The pre-symplectic form ω_L in $T\mathcal{F}$ is Π-projectable: there exists a symplectic form ω on $\mathcal{F} \bowtie TQ$ such that $\Pi^*\omega = \omega_L$. Thus, the fundamental Poisson brackets characterizing such symplectic structure on $\mathcal{F} \bowtie TQ$ are:

$$\left\{v^i, q^j\right\}_{F\bowtie TQ} = -\eta^{ij}, \quad \left\{v^i, v^j\right\}_{F\bowtie TQ} = \eta^{ir} a_{rl} \eta^{lj}, \quad \{I_a, I_b\}_{F\bowtie TQ} = C_{ab},$$

where $v^i = \chi'(\overline{dq^i})$ is the linear function on $\mathcal{F} \bowtie TQ$ associated to the local section dq^i of T^*Q, η^{jk} represents the entries of the inverse matrix of $\eta = (\eta_{kj})$ and $a_{rl} = v^k \left(\partial\eta_{kl}/\partial q^r - \partial\eta_{kr}/\partial q^l\right)$. Note that $\left\{q^i, q^j\right\}_{F\bowtie TQ} = 0 = \left\{q^i, I_a\right\}_{F\bowtie TQ}$.

The vector bundle $\mathcal{F} \bowtie TQ$ endowed with the above Poisson structure is a quadratic co-algebroid. Therefore, the dual bundle $\mathcal{F} \bowtie T^*Q$ is endowed with the following generalized Lie algebroid structure:

$$\begin{aligned} \left[\overline{dq^i}, \overline{dq^j}\right]_{\mathcal{F}\bowtie A} &= \eta^{ir}\left(\frac{\partial\eta_{kl}}{\partial q^r} - \frac{\partial\eta_{kr}}{\partial q^l}\right)\eta^{lj}\,\overline{dq^k}\,, \\ \varrho(\,\overline{dq^i}\,)q^j &= -\eta^{ij}\,, \\ \varrho(\,\overline{dq^i}\,)I_a &= 0. \end{aligned}$$

6. Appendix: Connection of a surjective submersion

Let us consider a fibre bundle $\pi : \mathcal{F} \to M$. We denote by $T^\pi\mathcal{F}$ the *vertical distribution* of vector fields on $\mathcal{F}$ with zero projection by the differential of π. A *connection of* π *(or Ehresmann connection)* is given by a differential distribution H of $\mathcal{F}$, that we call *horizontal distribution.* This distribution is a complementary distribution of $T^\pi\mathcal{F}$ that projects onto TM, that is, $H_q \simeq T_{\pi(q)}M$ for all $q \in \mathcal{F}$. Then, the tangente bundle $T\mathcal{F}$ can be decomposed as a direct sum $T\mathcal{F} = H \oplus T^\pi\mathcal{F}$. There are other (equivalent) ways of defining a connection of π (see [2, 8, 16]). For example, we can define a connection as a global section of the first jet bundle $J^1\pi \to \mathcal{F}$ or as a splitting of the exact sequence

$$0 \longrightarrow T^\pi\mathcal{F} \longrightarrow T\mathcal{F} \xrightarrow{\widetilde{\pi}} \mathcal{F} \bowtie TM \longrightarrow 0,$$

where $\mathcal{F} \bowtie TM = \pi^!(TM) = \{(f, X) \in \mathcal{F} \times TM \mid \pi(f) = \kappa_0(X)\}$ is the pull-back of the tangent bundle $\kappa_0 : TM \to M$ by the map $\pi : \mathcal{F} \to M$,

$$\begin{tikzcd} \mathcal{F} \bowtie TM \arrow[r, "\kappa_2"] \arrow[d, "\kappa_1"'] & TM \arrow[d, "\kappa_0"] \\ \mathcal{F} \arrow[r, "\pi"'] & M \end{tikzcd}$$

and $\widetilde{\pi}$ represents the projection of the tangent bundle $T\mathcal{F}$ onto the bundle $\mathcal{F} \bowtie TM$, i.e., a splitting of the exact sequence is a section for $\tilde{\pi}$: a differentiable map $h : \mathcal{F} \bowtie TM \to T\mathcal{F}$ such that $\widetilde{\pi} \circ h$ is the identity map on the vector bundle $\mathcal{F} \bowtie TM$. Examples of these connections and their applications in Physics can be found in [11, 12, 13, 14].

From the definition of ρ-connection of π given by Cantrijn *et. al.* [2], the map $h : \mathcal{F} \bowtie TM \to T\mathcal{F}$ is a id_{TM}-connection of π, i.e. it is a morphism of vector bundles over the identity map on $\mathcal{F}$ such that the following diagram is commutative:

$$\begin{tikzcd} \mathcal{F} \bowtie TM \arrow[r, "h"] \arrow[d, "\kappa_2"'] & T\mathcal{F} \arrow[d, "T\pi"] \\ TM \arrow[r, "\mathrm{id}_{TM}"'] & TM \end{tikzcd}$$

where $\kappa_2 : \mathcal{F} \bowtie TM \to TM$ is a projection given in local coordinates by

$$\kappa_2(x^i, \xi^a, v^j) = (x^i, v^j),$$

where (x^i, v^j) represents the local coordinates on the tangent bundle TM and (x^i, ξ^a) represents the local coordinates on the fibre bundle $\mathcal{F}$. In local coordinates, h is given by

$$h(x^i, \xi^a, v^j) = (x^i, \xi^a, v^j, \Gamma^a_j v^j) \equiv v^j \partial_{x^j} + \Gamma^a_j v^j \partial_{\xi^a},$$

where the symbols Γ^a_j are called the "coefficients" of the connection h.

Let X be a vector field on M that locally is written as $X = X^j \partial_{x^j}$. The horizontal lift $\widetilde{X}$ of the vector field X associated with the connection h is given locally by

$$\widetilde{X} = h(x^i, \xi^a, X^j) = X^j \partial_{x^j} + \Gamma^a_j X^j \partial_{\xi^a}.$$

There is a map $C : \mathfrak{X}^2(M) \to T^\pi\mathcal{F}$ associated with the connection h of $\pi : \mathcal{F} \to M$ called *curvature form* (see [18]), defined by setting

$$C(X, Y) = \widetilde{[X, Y]} - [\widetilde{X}, \widetilde{Y}].$$

The curvature form mesures the lack of integrability of the horizontal distribution H associated with the connection h. The distribution H is integrable if the connection is *flat*, i.e. if the curvature form is zero. Given $X, Y \in \mathfrak{X}(M)$, the horizontal lift of the vector fields X and Y are locally given by $\widetilde{X} = X^i \partial_{x^i} + \Gamma^a_k X^k \partial_{\xi^a}$ and $\widetilde{Y} = Y^j \partial_{x^j} + \Gamma^b_r Y^r \partial_{\xi^b}$, respectively. In local coordinates, the connection is flat if:

$$X^i Y^j \left(\frac{\partial \Gamma^b_j}{\partial x^i} - \frac{\partial \Gamma^b_i}{\partial x^j} + \Gamma^a_i \frac{\partial \Gamma^b_j}{\partial \xi^a} - \Gamma^a_j \frac{\partial \Gamma^b_i}{\partial \xi^a} \right) = 0,$$

that is,

$$\left(\frac{\partial \Gamma^b_j}{\partial x^i} - \frac{\partial \Gamma^b_i}{\partial x^j} + \Gamma^a_i \frac{\partial \Gamma^b_j}{\partial \xi^a} - \Gamma^a_j \frac{\partial \Gamma^b_i}{\partial \xi^a} \right) = 0.$$

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