Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 06–13

MORE ON $\mathscr{Q}$-MODULES

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ABSTRACT: A. Joyal and M. Tierney showed that the internal suplattices in the topos of sheaves on a locale are precisely the modules on that locale. Using a totally different technique, I shall show a generalization of this result to the case of (ordered) sheaves on a (small) quantaloid. Then I make a comment on module-equivalence versus sheaf-equivalence, using a recent observation of B. Mesablishvili and the notion of 'centre' of a quantaloid.

KEYWORDS: quantaloid, quantale, locale, ordered sheaf, module, centre.

1. $\mathscr{Q}$-modules are $\mathscr{Q}$-suplattices

Given any quantaloid $\mathscr{Q}$, a new quantaloid $\mathsf{Idm}(\mathscr{Q})$ is built as follows: its objects are the idempotent arrows of $\mathscr{Q}$, and its arrows are "regular bimodules". Clearly there is a full embedding $i\colon \mathscr{Q}\longrightarrow\mathsf{Idm}(\mathscr{Q})$, sending an arrow $f\colon A\longrightarrow B$ to $f\colon 1_A \rightarrow\!\!\!\!\!\shortmid\;\, 1_B$. Note that $\mathsf{Idm}(\mathscr{Q})$ is small whenever $\mathscr{Q}$ is.

Lemma 1.1. *If $\mathscr{R}$ is a quantaloid in which idempotents split, then, for any quantaloid $\mathscr{Q}$,*

$$-\circ i\colon \mathsf{QUANT}(\mathsf{Idm}(\mathscr{Q}),\mathscr{R})\longrightarrow\mathsf{QUANT}(\mathscr{Q},\mathscr{R})$$

is an equivalence of quantaloids.

Sketch of proof : Given $F\colon \mathscr{Q}\longrightarrow\mathscr{R}$, we must define $\overline{F}\colon \mathsf{Idm}(\mathscr{Q})\longrightarrow\mathscr{R}$. But an arrow $b\colon e \rightarrow\!\!\!\!\!\shortmid\;\, f$ in $\mathsf{Idm}(\mathscr{Q})$ is a diagram

$$\begin{tikzcd} A \arrow[loop left, "e"] \arrow[r, "b"] & B \arrow[loop right, "f"] \end{tikzcd}$$

Received March 10, 2006; first written in April 2005, minor corrections made on the occasion of my presentation of this note at the 'Category Theory Seminar' in Coimbra (November 19, 2005) and the 'Q-day II' in Paris (December 7–8, 2005).

in $\mathscr{Q}$, satisfying $e \circ e = e$, $f \circ f = f$, $b \circ e = b = f \circ b$. Applying F we have a similar diagram in $\mathscr{R}$, in which we can thus split the idempotents:

$$
\begin{tikzcd}
FA \arrow[loop above, "Fe"] \arrow[r, "Fb"] \arrow[d, shift right, "p_e"'] & FB \arrow[loop above, "Ff"] \arrow[d, shift right, "p_f"'] \\
FA_{Fe} \arrow[u, shift right, "s_e"'] & FB_{Ff} \arrow[u, shift right, "s_f"']
\end{tikzcd}
$$

Now put

$$\overline{F}\Big(b\colon e \mathrel{-\!\!\circ\!\!\to} f\Big) = \Big(p_f \circ Fb \circ s_e\colon FA_{Fe} \longrightarrow FB_{Ff}\Big)$$

and verify that

$$\overline{(-)}\colon \mathsf{QUANT}(\mathscr{Q}, \mathscr{R}) \longrightarrow \mathsf{QUANT}(\mathsf{Idm}(\mathscr{Q}), \mathscr{R})$$

gives the required inverse to $- \circ i$. $\square$

Since idempotents split in the quantaloid Sup, we have an important special case of the above; recall that $\mathsf{Mod}(\mathscr{Q}) = \mathsf{QUANT}(\mathscr{Q}^{\mathsf{op}}, \mathsf{Sup})$ is the quantaloid of so-called $\mathscr{Q}$-modules.

Proposition 1.2. *For any quantaloid $\mathscr{Q}$,*

$$- \circ i\colon \mathsf{Mod}(\mathsf{Idm}(\mathscr{Q})) \longrightarrow \mathsf{Mod}(\mathscr{Q})$$

is an equivalence of quantaloids.

With the work previously done in [Stubbe, 2004] we can record a corollary; recall that for a small quantaloid $\mathscr{Q}$, $\mathsf{Cocont}(\mathscr{Q})$ denotes the (locally cocompletely ordered) category of cocomplete $\mathscr{Q}$-categories and cocontinuous functors [Stubbe, 2005a].

Corollary 1.3. *For a small quantaloid $\mathscr{Q}$,*

$$\mathsf{Cocont}(\mathscr{Q}) \simeq \mathsf{Mod}(\mathscr{Q}) \simeq \mathsf{Mod}(\mathsf{Idm}(\mathscr{Q})) \simeq \mathsf{Cocont}(\mathsf{Idm}(\mathscr{Q}))$$

are (bi)equivalent locally ordered categories.

We will now study the monadicity of $\mathsf{Mod}(\mathscr{Q})$. Recall that a Kock–Zöberlein doctrine on a locally ordered 2-category $\mathscr{C}$ is a monad

$$(T\colon \mathscr{C} \longrightarrow \mathscr{C}, \eta\colon \mathsf{Id}_{\mathscr{C}} \Rightarrow T, \mu\colon T \circ T \Rightarrow T)$$

for which $T(\eta_C) \leq \eta_{TC}$ for any $C \in \mathscr{C}$. This precisely means that "T-structures are adjoint to units" [Kock, 1995]. Further on we will encounter an instance of the following abstract lemma.

Lemma 1.4. *For locally ordered 2-categories and 2-functors as in*

$$\begin{tikzcd} & \mathscr{B} \arrow[dr, hook, "W"] & \\ \mathscr{A} \arrow[ur, "V"] \arrow[rr, bend left, "U"] \arrow[rr, phantom, "\top"] & & \mathscr{C} \arrow[ll, bend left, "F"] \end{tikzcd}$$

with W a local equivalence, $W \circ V = U$, and $\eta\colon \mathsf{id}_{\mathscr{C}} \Rightarrow U \circ F$ the unit of the involved adjunction, we get that

(1) *$F \circ W \dashv V$ and its unit $\xi\colon \mathsf{Id}_{\mathscr{B}} \Rightarrow V \circ (F \circ W)$ satisfies $\eta * \mathsf{id}_W = \mathsf{id}_V * \xi$, that is, $W(\xi_B) = \eta_{WB}$ for every $B \in \mathscr{B}$.*

Writing $T = U \circ F\colon \mathscr{C} \to \mathscr{C}$ and $S = V \circ (F \circ W)\colon \mathscr{B} \to \mathscr{B}$, these monads satisfy

(2) *$T \circ W = W \circ S$,*
(3) *if T is a KZ doctrine then*
   (a) *also S is a KZ doctrine,*
   (b) *$B \in \mathscr{B}$ is an S-algebra if and only if WB is a T-algebra,*
   (c) *for $A \in \mathscr{A}$, UA is a T-algebra if and only if VA is an S-algebra,*
   (d) *if $\mathscr{A} \simeq \mathscr{C}^T$ then $\mathscr{A} \simeq \mathscr{B}^S$.*

Proof: To prove that $F \circ W \dashv V$, observe that for $B \in \mathscr{B}$ and $C \in \mathscr{C}$,

$$\begin{tikzcd} \mathscr{B}(B, VC) \arrow[d, "\text{apply } W"] \\ \mathscr{A}(WB, WVC) \arrow[d, equal, "\text{use that } U = WV"] \\ \mathscr{A}(WB, UC) \arrow[d, "\text{use that } F \dashv U"] \\ \mathscr{C}(FWB, C) \end{tikzcd}$$

are all equivalences (recall that W is supposed to be a local equivalence). Putting $C = FWB$ in the above, and tracing the element 1_{FWB} through the equivalences, it results indeed that $W(\xi_B) = \eta_{WB}$.

The second part of the lemma is trivial.

For the third part, suppose that $T(\eta_C) \leq \eta_{TC}$ for any $C \in \mathscr{C}$, then also

$$WS(\xi_B) = TW(\xi_B) = T(\eta_{WB}) \leq \eta_{TWB} = \eta_{WSB} = W(\xi_{SB})$$

for every $B \in \mathscr{B}$; but W is locally an equivalence, so $S(\xi_B) \leq \xi_{SB}$ as required to prove (a). Now, by the very nature of the algebras of KZ doctrines, $B \in \mathscr{B}$ is an S-algebra if and only if ξ_B is a right adjoint in $\mathscr{B}$, which is the same as $W(\xi_B) = \eta_{WB}$ being a right adjoint in $\mathscr{C}$ because W is locally an equivalence, and this in turn is just saying that WB is a T-algebra. This proves (b), and (c) readily follows by putting $B = VA$ for an $A \in \mathscr{A}$, and using that $W \circ V = U$; so (d) becomes obvious. $\square$

It is a result from $\mathscr{Q}$-enriched category theory [Stubbe, 2005a] that $\mathsf{Cocont}(\mathscr{Q})$ is monadic over $\mathsf{Cat}(\mathscr{Q})$: the forgetful $\mathsf{Cocont}(\mathscr{Q}) \longrightarrow \mathsf{Cat}(\mathscr{Q})$ admits the presheaf contruction as left adjoint,

$$\begin{tikzcd} \mathsf{Cocont}(\mathscr{Q}) \arrow[r, bend right, "\mathscr{U}"'] \arrow[r, phantom, "\perp"] & \mathsf{Cat}(\mathscr{Q}), \arrow[l, bend right, "\mathscr{P}"'] \end{tikzcd}$$

and moreover the structure map of an algebra for the monad is left adjoint to the unit of the adjunction (i.e. $\mathbb{A} \in \mathsf{Cocont}(\mathscr{Q})$ if and only if $Y_{\mathbb{A}} \colon \mathbb{A} \longrightarrow \mathscr{P}\mathbb{A}$ admits a left adjoint in $\mathsf{Cat}(\mathscr{Q})$, which is then the structure map of the algebra $\mathbb{A}$). Since there is the *fully faithful* forgetful $\mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q}) \longrightarrow \mathsf{Cat}(\mathscr{Q})$, the same thing can be said about the forgetful $\mathsf{Cocont}(\mathscr{Q}) \longrightarrow \mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q})$ (as recalled in the lemma above): the presheaf contruction thus provides a left adjoint, and $\mathsf{Cocont}(\mathscr{Q})$ is precisely the category of algebras for the induced monad on $\mathsf{Cat}_{\mathsf{cc}}(\mathscr{Q})$. We can apply this to the quantaloid $\mathsf{Idm}(\mathscr{Q})$, of course.

Proposition 1.5. *For any small quantaloid $\mathscr{Q}$, $\mathsf{Cocont}(\mathsf{Idm}(\mathscr{Q}))$ is the category of algebras for the presheaf monad $\mathscr{P} \colon \mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathscr{Q})) \longrightarrow \mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathscr{Q}))$.*

In combination with the above remarks on modules, we can now justify the slogan that "$\mathscr{Q}$-modules are $\mathscr{Q}$-suplattices". Recall that $\mathsf{Ord}(\mathscr{Q})$, the (locally ordered) category of ordered sheaves on a small quantaloid $\mathscr{Q}$, is equivalent to the category $\mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathscr{Q}))$ of Cauchy complete categories enriched in $\mathsf{Idm}(\mathscr{Q})$ [Stubbe, 2005c].

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algebras for the presheaf monad on the ordered sheaves. For the third implication, see 3.2. □

It is an interesting problem to study the converse implications in the above proposition. These converse implications do not hold in general, as the following counterexample shows.

Counterexample 4.2. Let $\mathcal{Q}$ be a quantaloid which can not be equivalent to a quantale, for example $\mathcal{Q} = \mathbf{2} + \mathbf{2}$ (coproduct in QUANT):

$$\begin{tikzcd} X \arrow[loop above, "1"] \arrow[loop below, "0"] \arrow[r, bend right, "0"'] & Y \arrow[l, bend right, "0"'] \arrow[loop above, "1"] \arrow[loop below, "0"] \end{tikzcd}$$

Then still, by 2.1 and 3.2, there exists a quantale with the same centre as $\mathcal{Q}$. So, in general, $\mathscr{Z}(\mathcal{Q}) \simeq \mathscr{Z}(\mathcal{Q}')$ does not imply $\mathcal{Q} \simeq \mathcal{Q}'$.

We must thus study extra conditions on $\mathcal{Q}$ and $\mathcal{Q}'$ that allow for the converse implications in 4.1. At least one such special case is that of *commutative quantales*.

Proposition 4.3. *For commutative quantales $\mathcal{Q}$ and $\mathcal{Q}'$,*

$$\mathcal{Q} \simeq \mathcal{Q}' \Leftrightarrow \mathsf{Ord}(\mathcal{Q}) \simeq \mathsf{Ord}(\mathcal{Q}') \Leftrightarrow \mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}').$$

Proof : A quantale is commutative if and only if it equals its centre. □

A locale is in particular a commutative quantale, so the above applies. Moreover – and this in contrast with the case of quantaloids or even quantales – apart from ordered sheaves (“Ord”) and completely ordered sheaves (“Mod”), we may also consider sheaves (“Sh”) on a locale.

Proposition 4.4. *For locales L and L',*

$$L \simeq L' \Leftrightarrow \mathsf{Sh}(L) \simeq \mathsf{Sh}(L') \Leftrightarrow \mathsf{Ord}(L) \simeq \mathsf{Ord}(L') \Leftrightarrow \mathsf{Mod}(L) \simeq \mathsf{Mod}(L').$$

Sketch of proof : The first equivalence follows from the fact that a locale L is (isomorphic to) the locale of subobjects of the terminal object in $\mathsf{Sh}(L)$ (see [Borceux, 1994, vol. 3, 2.2.16] for example). The other equivalences are instances of 4.3. □

References

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Theorem 1.6. *For a small quantaloid $\mathcal{Q}$, the diagram*

$$\begin{tikzcd} \mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Cocont}(\mathsf{Idm}(\mathcal{Q})) \arrow[r, bend right, "\mathscr{U}"'] \arrow[r, phantom, "\perp"] & \mathsf{Cat}_{\mathsf{cc}}(\mathsf{Idm}(\mathcal{Q})) \simeq \mathsf{Ord}(\mathcal{Q}) \arrow[l, bend right, "\mathscr{P}"'] \end{tikzcd}$$

exhibits the quantaloid $\mathsf{Mod}(\mathcal{Q})$ as being (biequivalent to) the (locally completely ordered) category of algebras for the presheaf construction.

It would thus make sense to write $\mathsf{Sup}(\mathcal{Q})$ for any of the equivalent expressions $\mathsf{Cocont}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}) \simeq ...$, and to speak of “$\mathcal{Q}$-suplattices”. It is then the case that $\mathsf{Sup}(\mathbf{2}) \simeq \mathsf{Sup}$ is just the “ordinary” quantaloid of suplattices; and for a locale L, $\mathsf{Sup}(L)$ gives indeed the suplattices in the topos $\mathsf{Sh}(L)$ (which means that the above theorem is an alternative to Joyal and Tierney’s [1984] proof for the fact that L-modules are the suplattices in $\mathsf{Sh}(L)$).

2. Every small quantaloid is Morita-equivalent to a quantale

Bachuki Mesablishvili [2004] observes that every small quantaloid is Morita equivalent to a quantale; in fact he uses Max Kelly’s [1982] powerful but rather abstract $\mathscr{V}$-category theory to prove this result. I will sketch an elementary proof.

Let $\mathcal{Q}$ be a small quantaloid; we may view its object set $\mathcal{Q}_0$ as a $\mathcal{Q}_0$-typed set in the obvious way. Then $\mathsf{Matr}(\mathcal{Q})(\mathcal{Q}_0, \mathcal{Q}_0)$ is certainly a quantale, for it is an endo-hom object in the quantaloid $\mathsf{Matr}(\mathcal{Q})$ of matrices with elements in $\mathcal{Q}$ (see [Stubbe, 2005a]). One can indeed picture the elements of this quantale as gigantic square matrices: an $\mathbb{M} \in \mathsf{Matr}(\mathcal{Q})(\mathcal{Q}_0, \mathcal{Q}_0)$ is a collection of $\mathcal{Q}$-arrows

$$\Big(\mathbb{M}(B, A) \colon A \longrightarrow B \;\Big|\; (A, B) \in \mathcal{Q}_0 \times \mathcal{Q}_0\Big);$$

such matrices are ordered elementwise:

$$\mathbb{M} \leq \mathbb{N} \iff \forall (A, B) \in \mathcal{Q}_0 \times \mathcal{Q}_0 : \mathbb{M}(B, A) \leq \mathbb{N}(B, A)$$

(so supremum of matrices is calculated elementwise); and multiplication is done with the linear algebra formula:

$$(\mathbb{N} \circ \mathbb{M})(B, A) = \bigvee_{X \in \mathcal{Q}_0} \mathbb{N}(B, X) \circ \mathbb{M}(X, A).$$

Theorem 2.1. *Given a small quantaloid $\mathcal{Q}$, put $\mathscr{M} = \mathsf{Matr}(\mathcal{Q})(\mathcal{Q}_0, \mathcal{Q}_0)$; then*

$$\mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathscr{M}).$$

Sketch of proof: We must first introduce some notation: for a $\mathcal{Q}$-arrow $f\colon A \longrightarrow B$, let $\mathbb{M}_f$ denote the square matrix whose elements are

$$\mathbb{M}_f(Y,X) = \begin{cases} f & \text{if } X = A \text{ and } Y = B, \\ 0_{X,Y} & \text{otherwise} \end{cases}$$

Here, $0_{X,Y}$ denotes the bottom element of the suplattice $\mathcal{Q}(X,Y)$.

Given a $\mathcal{Q}$-module $F\colon \mathcal{Q} \longrightarrow \mathsf{Sup}$, regard the elements of the direct sum $\mathscr{L} = \oplus_{A \in \mathcal{Q}} FA$ in Sup as "column vectors" $x = (x_A)_{A \in \mathcal{Q}}$ with $x_A \in FA$. Then F determines an action $\alpha_F\colon \mathscr{M} \times \mathscr{L} \longrightarrow \mathscr{L}\colon (\mathbb{M}, x) \mapsto \alpha_F(\mathbb{M}, x)$ where the A^{th} component of the column vector $\alpha(\mathbb{M}, x)$ is, by definition,

$$\Big(\alpha_F(\mathbb{M}, x)\Big)_A = \bigvee_{X \in \mathcal{Q}} F\Big(\mathbb{M}(A,X)\Big)(x_A).$$

That is to say, we take the image by F of the matrix $\mathbb{M}$ and then perform a matrix multiplication.

Conversely, let $\alpha\colon \mathscr{M} \times \mathscr{L} \longrightarrow \mathscr{L}$ be an action in Sup. Since it is clear that, for $A, B \in \mathcal{Q}$,

$$\mathbb{M}_{1_B} \circ \mathbb{M}_{1_A} = \begin{cases} \mathbb{M}_{1_A} & \text{if } A = B, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that, for any $A \in \mathcal{Q}$, $\alpha(\mathbb{M}_{1_A}, -)\colon \mathscr{L} \longrightarrow \mathscr{L}$ is an idempotent in Sup, and therefore splits over some suplattice $\mathscr{L}_A$:

$$\begin{tikzcd} \mathscr{L} \arrow[loop above, "\alpha(\mathbb{M}_{1_A}, -)"] \arrow[r, shift left, "p_A"] & \mathscr{L}_A. \arrow[l, shift left, "s_A"] \end{tikzcd}$$

(It is easily verified that $\mathscr{L} = \oplus_{A \in \mathcal{Q}} \mathscr{L}_A$.) Now we can define a $\mathcal{Q}$-module $F_\alpha\colon \mathcal{Q} \longrightarrow \mathsf{Sup}$ by putting

$$F_\alpha\Big(f\colon A \longrightarrow B\Big) = p_B \circ \alpha(\mathbb{M}_f, -) \circ s_A\colon \mathscr{L}_A \longrightarrow \mathscr{L}_B.$$

The definitions for $F \mapsto \alpha_F$ and $\alpha \mapsto F_\alpha$ extend to quantaloid homomorphisms $\mathsf{Mod}(\mathcal{Q}) \longrightarrow \mathsf{Mod}(\mathscr{M})$ and $\mathsf{Mod}(\mathscr{M}) \longrightarrow \mathsf{Mod}(\mathcal{Q})$, which prove to be inverse equivalences. $\square$

3. The centre of a quantaloid

The aim of this section is to discuss a notion, namely the centre of a quantaloid, which is invariant under Morita equivalence.

For any quantaloid $\mathcal{Q}$, let $\mathscr{Z}(\mathcal{Q})$ be shorthand for $\mathsf{QUANT}(\mathcal{Q},\mathcal{Q})(\mathsf{Id}_\mathcal{Q}, \mathsf{Id}_\mathcal{Q})$, and call it the centre of $\mathcal{Q}$. This $\mathscr{Z}(\mathcal{Q})$ is by definition a commutative quantale: that $\mathscr{Z}(\mathcal{Q})$ is a quantale, is because it is an endo-hom-object of the quantaloid $\mathsf{QUANT}(\mathcal{Q},\mathcal{Q})$; that it is moreover commutative, is because $\mathsf{QUANT}(\mathcal{Q},\mathcal{Q})$ is in fact monoidal – with tensor given by composition – and that $\mathsf{Id}_\mathcal{Q}$ is the unit object for the tensor. Unraveling the definition, an element $\alpha \in \mathscr{Z}(\mathcal{Q})$ is a collection of endo-arrows

$$\Big(\begin{tikzcd} A \arrow[loop above, "\alpha_A"] \end{tikzcd} \ \Big|\ A \in \mathcal{Q}_0 \Big)$$

such that for every $f\colon A \longrightarrow B$ in $\mathcal{Q}$, $\alpha_B \circ f = f \circ \alpha_A$.

The following proposition was inspired by [Bass, 1968, p. 56]; I have never seen the version below in print, but I suppose that it belongs to folklore.

Proposition 3.1. *For any quantaloid $\mathcal{Q}$, $\mathscr{Z}(\mathcal{Q}) \cong \mathscr{Z}(\mathsf{Mod}(\mathcal{Q}))$.*

Sketch of proof: Given a natural transformation $\alpha\colon \mathsf{Id}_\mathcal{Q} \longrightarrow \mathsf{Id}_\mathcal{Q}$, build the natural transformation $\widehat{\alpha}\colon \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})} \longrightarrow \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})}$ whose component at $M \in \mathsf{Mod}(\mathcal{Q})$ is the natural transformation $\widehat{\alpha}_M\colon M \longrightarrow M$, whose component at $A \in \mathcal{Q}$ is the Sup-arrow

$$\widehat{\alpha}_M^A = M(\alpha_A)\colon M(A) \longrightarrow M(A).$$

Conversely, given a natural transformation $\beta\colon \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})} \longrightarrow \mathsf{Id}_{\mathsf{Mod}(\mathcal{Q})}$, build the natural transformation $\overline{\beta}\colon \mathsf{Id}_\mathcal{Q} \longrightarrow \mathsf{Id}_\mathcal{Q}$ whose component at $A \in \mathcal{Q}$ is the $\mathcal{Q}$-arrow

$$\overline{\beta}_A = \beta^A_{\mathcal{Q}(A,-)}(1_A)\colon A \longrightarrow A.$$

The mappings $\alpha \mapsto \widehat{\alpha}$ and $\beta \mapsto \overline{\beta}$ thus defined are quantale homomorphisms $\mathscr{Z}(\mathcal{Q}) \longrightarrow \mathscr{Z}(\mathsf{Mod}(\mathcal{Q}))$ and $\mathscr{Z}(\mathsf{Mod}(\mathcal{Q})) \longrightarrow \mathscr{Z}(\mathcal{Q})$ which are each other's inverse. $\square$

As an obvious corollary we may record the following.

Corollary 3.2. *Morita-equivalent quantaloids have isomorphic centres.*

4. Module equivalence compared with sheaf equivalence

Proposition 4.1. *For small quantaloids $\mathcal{Q}$ and $\mathcal{Q}'$,*

$$\mathcal{Q} \simeq \mathcal{Q}' \Rightarrow \mathsf{Ord}(\mathcal{Q}) \simeq \mathsf{Ord}(\mathcal{Q}') \Rightarrow \mathsf{Mod}(\mathcal{Q}) \simeq \mathsf{Mod}(\mathcal{Q}') \Rightarrow \mathscr{Z}(\mathcal{Q}) \cong \mathscr{Z}(\mathcal{Q}').$$

Sketch of proof: The first implication is obvious ("equivalent bases give equivalent enriched structures"). The second implication is due to 1.6: modules are precisely