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CATEGORICAL STRUCTURES ENRICHED IN A QUANTALOID: TENSORED AND COTENSORED CATEGORIES

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ABSTRACT: Our subject is that of categories, functors and distributors enriched in a base quantaloid $\mathscr{Q}$. We show how cocomplete $\mathscr{Q}$-categories are precisely those which are tensored and conically cocomplete, or alternatively, those which are tensored, cotensored and order-cocomplete. Bearing this in mind, we analyze how Sup-valued homomorphisms on $\mathscr{Q}$ are related to $\mathscr{Q}$-categories. With an appendix on action, representation and variation.

1. Introduction

The definition of "category enriched in a bicategory $\mathscr{W}$" is as old as the definition of bicategory itself [Bénabou, 1967]. Taking a $\mathscr{W}$ with only one object gives a monoidal category; for symmetric closed monoidal $\mathscr{V}$ the theory of $\mathscr{V}$-categories is well known [Kelly, 1982]. But also categories enriched in a $\mathscr{W}$ with more than one object are interesting. [Walters, 1981] observed that sheaves on a locale give rise to bicategory-enriched categories: "variation" (sheaves on a locale Ω) is related to "enrichment" (categories enriched in $\mathsf{Rel}(\Omega)$). This insight was further developed in [Walters, 1982] and [Betti *et al.*, 1983]. Later [Gordon and Power, 1997, 1999] complemented this work, stressing the important rôle of tensors in bicategory-enriched categories.

Here we wish to discuss "variation and enrichment" in the case of a base quantaloid (a Sup-enriched category). This is, of course, a particular case of the above, but we believe that it is also of particular interest; many examples of bicategory-enriched categories (like Walters') are really quantaloid-enriched. Since in a quantaloid $\mathscr{Q}$ every diagram of 2-cells commutes, many coherence issues disappear, so the theory of $\mathscr{Q}$-enriched categorical structures is very transparent. Moreover, by definition a quantaloid $\mathscr{Q}$ has stable local colimits, hence (by local smallness) it is closed; this is of great help when working with $\mathscr{Q}$-categories. The theory of quantaloids is documented in [Rosenthal, 1996], and [Stubbe, 2005a] provides a reference for all the necessary definitions and basic facts from $\mathscr{Q}$-category theory that will be needed further on.

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Our starting point here is the notion of weighted colimit in a $\mathscr{Q}$-category $\mathbb{C}$ [Kelly, 1982; Street, 1983]. Two particular cases of such weighted colimits are tensors and conical colimits; then $\mathbb{C}$ is cocomplete (i.e. it admits all weighted colimits) if and only if it is tensored and has all conical colimits [Kelly, 1982; Gordon and Power, 1999] (see also 2.6 below). But we may consider the family of ordered sets of objects of the same type in $\mathbb{C}$; we call $\mathbb{C}$ "order-cocomplete" when these ordered sets admit arbitrary suprema. This is a weaker requirement than for $\mathbb{C}$ to have conical colimits, but for cotensored $\mathbb{C}$ they coincide. Now $\mathbb{C}$ is cocomplete if and only if it is tensored, cotensored and order-cocomplete (as in 2.11). Put differently, for a tensored and cotensored $\mathscr{Q}$-category $\mathbb{C}$, order-theoretical content (suprema) can be "lifted" to $\mathscr{Q}$-categorical content (weighted colimits).

Then a section is devoted to adjunctions. We see how, at least for tensored $\mathscr{Q}$-categories, order-adjunctions can be "lifted" to $\mathscr{Q}$-enriched adjunctions, and how (co)tensoredness may be characterized by enriched adjunctions (analogously to $\mathscr{V}$-categories). As a result, for a tensored $\mathbb{C}$, its cotensoredness is equivalent to certain order-adjunctions (cf. 3.6). With this in mind we analyze the basic biequivalence between tensored $\mathscr{Q}$-enriched categories and closed pseudofunctors on $\mathscr{Q}^{\mathsf{op}}$ with values in $\mathsf{Cat}(\mathbf{2})$ (as in 4.2, a particular case of results in [Gordon and Power, 1997]). A finetuned version thereof (in 4.9) says that right $\mathscr{Q}$-modules are the same thing as cocomplete $\mathscr{Q}$-enriched categories.

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2. More on weighted (co)limits

Throughout $\mathscr{Q}$ denotes a small quantaloid, and our $\mathscr{Q}$-categories have a small set of objects. All notations are as in [Stubbe, 2005a].

(Co)tensors. Let $\mathbb{C}$ be a $\mathscr{Q}$-category. For a $\mathscr{Q}$-arrow $f\colon X \longrightarrow Y$ and an object $y \in \mathbb{C}_0$ of type $ty = \mathsf{cod}(f) = Y$, the *tensor* of y and f is by definition the (f)-weighted colimit of Δy; it will be denoted $y \otimes f$. Thus, whenever it exists, $y \otimes f$ is the (necessarily essentially unique) object of $\mathbb{C}$ (necessarily of type $t(y \otimes f) = \mathsf{dom}(f)$) such that

$$\text{for all } z \in \mathbb{C},\ \mathbb{C}(y \otimes f, z) = \Big[f, \mathbb{C}(y,z)\Big] \text{ in } \mathscr{Q}.$$

A *cotensor* in $\mathbb{C}$ is a tensor in the $\mathcal{Q}^{\mathsf{op}}$-category $\mathbb{C}^{\mathsf{op}}$; in elementary terms, for an arrow $f\colon X \to Y$ in $\mathcal{Q}$ and an object $x \in \mathbb{C}$ of type $tx = \mathsf{dom}(f) = X$, the cotensor of f and x, denoted $\langle f, x\rangle$, is – whenever it exists – the object of $\mathbb{C}$ of type $t\langle f, x\rangle = \mathsf{cod}(f)$ with the universal property that

$$\text{for all } z \in \mathbb{C},\ \mathbb{C}(z, \langle f, x\rangle) = \Big\{f, \mathbb{C}(z, x)\Big\} \text{ in } \mathcal{Q}.$$

Thus, $\langle f, x\rangle$ is the (f)-weighted limit of Δx.

A $\mathcal{Q}$-category $\mathbb{C}$ is *tensored* when for all $f \in \mathcal{Q}$ and $y \in \mathbb{C}_0$ with $ty = \mathsf{cod}(f)$, the tensor $y \otimes f$ exists; and $\mathbb{C}$ is *cotensored* when $\mathbb{C}^{\mathsf{op}}$ is tensored.

When making a theory of (small) tensored $\mathcal{Q}$-categories, there are some size issues to address, as the following indicates.

Lemma 2.1. *A tensored $\mathcal{Q}$-category has either no objects at all, or at least one object of type X for each $\mathcal{Q}$-object X.*

Proof: The empty $\mathcal{Q}$-category is trivially tensored. Suppose that $\mathbb{C}$ is non-empty and tensored; say that there is an object y of type $ty = Y$ in $\mathbb{C}$. Then, for any $\mathcal{Q}$-object X the tensor of y with the zero-morphism $0_{X,Y} \in \mathcal{Q}(X, Y)$ must exist, and is an object of type X in $\mathbb{C}$. $\square$

This motivates once more why *we work over a small base quantaloid $\mathcal{Q}$*.

Example 2.2. The two-element Boolean algebra is denoted **2**; we may view it as a one-object quantaloid so that **2**-categories are ordered sets, functors are order-preserving maps, and distributors are ideal relations. A non-empty **2**-category, i.e. a non-empty order, is tensored if and only if it has a bottom element, and cotensored if and only if it has a top element.

Example 2.3. For any object Y in a quantaloid $\mathcal{Q}$, $\mathscr{P}Y$ denotes the $\mathcal{Q}$-category of contravariant presheaves on the one-object $\mathcal{Q}$-category $*_Y$ whose hom-arrow is 1_Y. It is cocomplete, thus complete (because a $\mathcal{Q}$-category is cocomplete if and only if it is complete [Stubbe, 2005a, 5.10]), thus both tensored and cotensored. For an object $f \in \mathscr{P}Y$ of type $tf = X$ (i.e. a $\mathcal{Q}$-arrow $f\colon X \to Y$) and a $\mathcal{Q}$-arrow $g\colon U \to X$, $f \otimes g = f \circ g\colon U \to Z$ seen as object of type U in $\mathscr{P}Y$. For $h\colon X \to V$, $\langle h, f\rangle = \{h, f\}\colon V \to Y$, an object of type V in $\mathscr{P}Y$. Similarly, $\mathscr{P}^\dagger X$ is the $\mathcal{Q}$-category of covariant presheaves on $*_X$; for $f\colon X \to Y$, $k\colon Y \to M$ and $l\colon N \to Y$, $f \otimes l = [l, f]$ and $\langle k, f\rangle = k \circ f$ in $\mathscr{P}^\dagger X$.

Conical (co)limits. A $\mathcal{Q}$-category $\mathbb{C}$ has an underlying order $(\mathbb{C}_0, \leq)$: put $x' \leq x$ whenever both these objects are of the same type, say $tx = tx' = X$, and

$1_X \leq \mathbb{C}(x', x)$. Conversely, on an ordered set $(A, \leq)$ we may consider the free $\mathcal{Q}(X,X)$-category $\mathbb{A}$:

- $\mathbb{A}_0 = A$, all objects are of type X;
- $\mathbb{A}(a', a) = \begin{cases} 1_X & \text{if } a' \leq a, \\ 0_{X,X} & \text{otherwise.} \end{cases}$

To give a functor $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ is to give objects Fa, Fa', ... of type X in $\mathbb{C}$, such that $Fa' \leq Fa$ in the underlying order of $\mathbb{C}$ whenever $a' \leq a$ in $(A, \leq)$. Consider furthermore the weight $\phi\colon\ *_X \looparrowright \mathbb{A}$ whose elements are $\phi(a) = 1_X$ for all $a \in \mathbb{A}_0$. The ϕ-weighted colimit of $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ (which may or may not exist) is the *conical colimit of* F. (Notwithstanding the adjective "conical", this is still a weighted colimit!) A *conically cocomplete* $\mathcal{Q}$-category is one that admits all conical colimits.

Analogously to 2.1, a conically cocomplete $\mathcal{Q}$-category $\mathbb{C}$ has, for each $\mathcal{Q}$-object X, at least one object of type X. Indeed, the conical colimit on the empty functor from the empty free $\mathcal{Q}(X,X)$-category into $\mathbb{C}$ is an object of type X in $\mathbb{C}$.

The dual notions are those of *conical limit* and *conically complete $\mathcal{Q}$-category*. We do not bother spelling them out.

The following will help us calculate conical colimits.

Proposition 2.4. *Consider a free $\mathcal{Q}(X,X)$-category $\mathbb{A}$ and a functor $F\colon \mathbb{A} \longrightarrow \mathbb{C}$. An object $c \in \mathbb{C}_0$, necessarily of type $tc = X$, is the conical colimit of F if and only if $\mathbb{C}(c, -) = \bigwedge_{a \in \mathbb{A}_0} \mathbb{C}(Fa, -)$ in $\mathsf{Dist}(\mathcal{Q})(\mathbb{C}, *_X)$.*

Proof: For the conical colimit weight $\phi\colon\ *_X \looparrowright \mathbb{A}$, $\phi(a) = 1_X$ for all $a \in \mathbb{A}$, thus $c = \mathrm{colim}(\phi, F)$ if and only if

$$\begin{aligned} \mathbb{C}(c,-) &= \Big[\phi, \mathbb{C}(F-,-)\Big] \\ &= \bigwedge_{a \in \mathbb{A}_0} \Big[\phi(a), \mathbb{C}(Fa,-)\Big] \\ &= \bigwedge_{a \in \mathbb{A}_0} \Big[1_X, \mathbb{C}(Fa,-)\Big] \\ &= \bigwedge_{a \in \mathbb{A}_0} \mathbb{C}(Fa,-). \end{aligned}$$

□

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two right modules (M,ϕ) and (N,ψ), a module-morphism $\alpha\colon M\longrightarrow N$ is a sup-morphism that makes

$$\begin{tikzcd} M\otimes K \arrow[r, "\alpha\otimes 1_K"] \arrow[d, "\phi"'] & N\otimes K \arrow[d, "\psi"] \\ M \arrow[r, "\alpha"'] & N \end{tikzcd}$$

commute. In elementary terms, such a sup-morphism $\alpha\colon M\longrightarrow N\colon m\mapsto\alpha(m)$ satisfies

$$\alpha(\phi(m,f))=\psi(\alpha(m),f).$$

By adjunction – and with notations as above – this gives for any $f\in K$ the commutative square

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[d, "\bar{\phi}(f)"'] & N \arrow[d, "\bar{\psi}(f)"] \\ M \arrow[r, "\alpha"'] & N \end{tikzcd}$$

which expresses precisely the naturality of α viewed as (single) component of a natural transformation $\alpha\colon F\Rightarrow G$, where $F,G\colon\mathcal{Q}^{\mathsf{op}}\rightrightarrows\mathsf{Sup}$ denote the homomorphisms corresponding to M and N.

Conclusively, actions, representations and Sup-presheaves are essentially the same thing. The point now is that the latter presentation straightforwardly makes sense for any quantaloid, and not just those with only one object.

Definition 5.1. *A* right $\mathcal{Q}$-module M *is a homomorphism* $M\colon\mathcal{Q}^{\mathsf{op}}\longrightarrow\mathsf{Sup}$. *And a* module-morphism $\alpha\colon M\Rightarrow N$ *between two right $\mathcal{Q}$-modules M and N is an enriched natural transformation between these homomorphisms.*

That is to say, $\mathsf{QUANT}(\mathcal{Q}^{\mathsf{op}},\mathsf{Sup})$ is the quantaloid of right $\mathcal{Q}$-modules.

We have chosen here to work with *right* actions, *reversed* representations, and *contravariant* Sup-presheaves. Clearly *left* actions correspond to *straight* representations and to *covariant* Sup-valued presheaves.

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In the proof above, to pass from the first line to the second in the series of equations, we used the explicit formula for liftings in the quantaloid $\mathsf{Dist}(\mathcal{Q})$: in general, for distributors $\Theta\colon\mathbb{A}\begin{array}{c}\longrightarrow\end{array}\!\!\!\!\!\!\!\!\circ\ \ \mathbb{C}$ and $\Psi\colon\mathbb{B}\begin{array}{c}\longrightarrow\end{array}\!\!\!\!\!\!\!\!\circ\ \ \mathbb{C}$ between $\mathcal{Q}$-categories, the lifting $[\Psi,\Theta]\colon\mathbb{A}\begin{array}{c}\longrightarrow\end{array}\!\!\!\!\!\!\!\!\circ\ \ \mathbb{B}$ has elements, for $a\in\mathbb{A}_0$ and $b\in\mathbb{C}_0$, $[\Psi,\Theta](b,a)=\bigwedge_{c\in\mathbb{C}_0}[\Psi(c,b),\Theta(c,a)]$, where the liftings on the right are calculated in $\mathcal{Q}$.

Proposition 2.5. *A $\mathcal{Q}$-category $\mathbb{C}$ is conically cocomplete if and only if for any family $(c_i)_{i\in I}$ of objects of $\mathbb{C}$, all of the same type, say $tc_i=X$, there exists an object c in $\mathbb{C}$, necessarily also of that type, such that $\mathbb{C}(c,-)=\bigwedge_{i\in I}\mathbb{C}(c_i,-)$ in* $\mathsf{Dist}(\mathcal{Q})(\mathbb{C},*_X)$.

Proof: One direction is a direct consequence of 2.4. For the other, given a family $(c_i)_{i\in I}$ of objects of $\mathbb{C}$, all of type $tc_i=X$, consider the free $\mathcal{Q}(X,X)$-category $\mathbb{I}$ on the ordered set $(I,\leq)$ with $i\leq j\iff c_i\leq c_j$ in $\mathbb{C}$. The conical colimit of the functor $F\colon\mathbb{I}\longrightarrow\mathbb{C}\colon i\mapsto c_i$ is an object $c\in\mathbb{C}_0$ such that $\mathbb{C}(c,-)=\bigwedge_{i\in I}\mathbb{C}(c_i,-)$, precisely what we wanted. $\square$

In what follows we will often speak of "the conical (co)limit of a family of objects with the same type", referring to the construction as in the proof above.

Theorem 2.6. *A $\mathcal{Q}$-category $\mathbb{C}$ is cocomplete if and only if it is tensored and conically cocomplete.*

Proof: For the non-trivial implication, the alternative description of conical cocompleteness in 2.5 is useful. If $\phi\colon *_X\begin{array}{c}\longrightarrow\end{array}\!\!\!\!\!\!\!\!\circ\ \ \mathbb{C}$ is *any* presheaf on $\mathbb{C}$, then the conical colimit of the family $(x\otimes\phi(x))_{x\in\mathbb{C}_0}$ is the ϕ-weighted colimit of $1_{\mathbb{C}}$: for this is an object $c\in\mathbb{C}_0$ such that

$$\begin{aligned}\mathbb{C}(c,-)&=\bigwedge_{x\in\mathbb{C}_0}\mathbb{C}(x\otimes\phi(x),-)\\&=\bigwedge_{x\in\mathbb{C}_0}\Big[\phi(x),\mathbb{C}(x,-)\Big]\\&=\Big[\phi,\mathbb{C}(1_{\mathbb{C}}-,-)\Big].\end{aligned}$$

Hence $\mathbb{C}$ is cocomplete (indeed, it suffices that $\mathbb{C}$ admit presheaf-weighted colimits of $1_{\mathbb{C}}$). $\square$

Tensors and conical colimits allow for a very explicit description of colimits in a cocomplete category.

Corollary 2.7. *If $\mathbb{C}$ is a cocomplete $\mathcal{Q}$-category, then the colimit of*

$$\mathbb{A} \xrightarrow{\Phi} \mathbb{B} \xrightarrow{F} \mathbb{C}$$

is the functor $\mathrm{colim}(\Phi, F)\colon \mathbb{A} \longrightarrow \mathbb{C}$ *sending an object* $a \in \mathbb{A}_0$ *to the conical colimit of the family* $(Fb \otimes \Phi(b,a))_{b \in \mathbb{B}_0}$. *A functor* $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ *between cocomplete $\mathcal{Q}$-categories is cocontinuous if and only if it preserves tensors and conical colimits.*

In 2.13 we will discuss a more user-friendly version of the above: we can indeed avoid the *conical colimits*, and replace them by suitable *suprema*.

A third kind of (co)limit. It makes no sense to ask for the underlying order $(\mathbb{C}_0, \leq)$ of a $\mathcal{Q}$-category $\mathbb{C}$ to admit arbitrary suprema: two objects of different type cannot even have an upper bound! So let us now denote $\mathbb{C}_X$ for the *ordered set of $\mathbb{C}$-objects with type X* (which is thus the empty set when $\mathbb{C}$ has no such objects); in these orders it does make sense to talk about suprema. We will say that $\mathbb{C}$ is *order-cocomplete* when each $\mathbb{C}_X$ admits all suprema.

An order-cocomplete $\mathcal{Q}$-category $\mathbb{C}$ has, for each $\mathcal{Q}$-object X, at least one object of type X. Namely, each $\mathbb{C}_X$ contains the empty supremum, i.e. has a bottom element.

The dual notion is that of *order-complete $\mathcal{Q}$-category*; but of course "order-complete" and "order-cocomplete" are always equivalent since each order $\mathbb{C}_X$ is *small*. Nevertheless we will pedantically use both terms, to indicate whether we take suprema or infima as primitive structure.

Proposition 2.8. *Let $\mathbb{C}$ be a $\mathcal{Q}$-category. The conical colimit of a family $(c_i)_{i \in I} \in \mathbb{C}_X$ is also its supremum in $\mathbb{C}_X$.*

Proof: Use that $\mathbb{C}(c, -) = \bigwedge \mathbb{C}(c_i, -)$ in $\mathsf{Dist}(\mathcal{Q})(\mathbb{C}, *_X)$ for the conical colimit $c \in \mathbb{C}_0$ of the given family to see that $c = \bigvee_i c_i$ in $\mathbb{C}_X$. $\square$

So if $\mathbb{C}$ is a conically cocomplete $\mathcal{Q}$-category, then it is also order-cocomplete. The converse is not true in general without extra assumptions.

Example 2.9. Consider the $\mathcal{Q}$-category $\mathbb{C}$ that has, for each $\mathcal{Q}$-object X, precisely one object of type X; denote this object as 0_X. The hom-arrows in $\mathbb{C}$ are defined as $\mathbb{C}(0_X, 0_X) = 1_X$ (the identity arrow in $\mathcal{Q}(X,X)$) and $\mathbb{C}(0_Y, 0_X) = 0_{X,Y}$ (the bottom element in $\mathcal{Q}(X,Y)$). Then each $\mathbb{C}_X = \{0_X\}$ is a sup-lattice, so $\mathbb{C}$ is order-cocomplete. However the conical colimit of the *empty* family of objects of type X does not exist as soon as the identity arrows in $\mathcal{Q}$ are not the top elements, or as soon as $\mathcal{Q}$ has more than one object.

5. Appendix: action, representation and variation

Let K denote a quantale, i.e. a one-object quantaloid. Now thinking of K as a monoid in Sup, let "unit" and "multiplication" in K (the single identity arrow and the composition in the one-object quantaloid) correspond to sup-morphisms $\varepsilon\colon I \longrightarrow K$ and $\gamma\colon K \otimes K \longrightarrow K$. A *right action* of K on some sup-lattice M is a sup-morphism $\phi\colon M \otimes K \longrightarrow M$ such that the diagrams

$$\begin{tikzcd}
M \otimes K \otimes K \arrow[r, "1 \otimes \gamma"] \arrow[d, "\phi \otimes 1_K"'] & M \otimes K \arrow[d, "\phi"] & M \otimes I \arrow[l, "1 \otimes \varepsilon"'] \arrow[dl, equal] \\
M \otimes K \arrow[r, "\phi"'] & M &
\end{tikzcd}$$

commute (we do not bother writing the associativity and unit isomorphisms in the symmetric monoidal closed category Sup); (M, ϕ) is then said to be a *right K-module*. In elementary terms we have a set-mapping $M \times K \longrightarrow M\colon (m, f) \mapsto \phi(m, f)$, preserving suprema in both variables, and such that (with obvious notations)

$$\phi(m, 1) = m \text{ and } \phi(m, g \circ f) = \phi(\phi(m, g), f).$$

By closedness of Sup, to the sup-morphism $\phi\colon M \otimes K \longrightarrow M$ corresponds a unique sup-morphism $\bar{\phi}\colon K \longrightarrow \mathsf{Sup}(M, M)$. In terms of elements, this $\bar{\phi}$ sends every $f \in K$ to the sup-morphism $\phi(-, f)\colon M \longrightarrow M$; it satisfies

$$\bar{\phi}(1) = 1_M \text{ and } \bar{\phi}(g \circ f) = \bar{\phi}(f) \circ \bar{\phi}(g).$$

That is to say, $\bar{\phi}\colon K \longrightarrow \mathsf{Sup}(M, M)$ is a *reversed representation* of the quantale K by endomorphisms on the sup-lattice M: a homomorphism of quantales that reverses the multiplication (where $\mathsf{Sup}(M, M)$ is endowed with composition as binary operation and the identity morphism 1_M as unit to form a quantale). Recalling that K is a one-object quantaloid $\mathcal{Q}$, such a multiplication-reversing homomorphism $\bar{\phi}\colon K \longrightarrow \mathsf{Sup}(M, M)$ is really a Sup-valued quantaloid homomorphism $F\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Sup}\colon * \mapsto M, f \mapsto \bar{\phi}(f)$.

In the same way it can be seen that morphisms between modules correspond to Sup-enriched natural transformations between Sup-presheaves. Explicitly, for

(1) *The corresponding lax natural transformation* $\varphi^F\colon \mathscr{F}_{\mathbb{C}} \Rightarrow \mathscr{F}_{\mathbb{C}'}$ *has components in* $\mathsf{Cat}_{\otimes}(\mathbf{2})$.
(2) F *is tensor-preserving if and only if* φ^F *is pseudonatural.*
(3) F *is left adjoint if and only if* φ^F *is pseudonatural and its components are in* $\mathsf{Map}(\mathsf{Cat}_{\otimes}(\mathbf{2}))$.
(4) *If* $\mathbb{C}$ *and* $\mathbb{C}'$ *are moreover cotensored, then* F *is left adjoint if and only if* φ^F *is pseudonatural and its components are in* $\mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathbf{2}))$.
(5) *If* $\mathbb{C}$ *and* $\mathbb{C}'$ *are cocomplete, then* F *is left adjoint if and only if* φ^F *is pseudonatural and its components are in* $\mathsf{Cocont}(\mathbf{2})$.
(6) *If* $\mathbb{C}$ *and* $\mathbb{C}'$ *are skeletal and cocomplete, then* F *is left adjoint if and only if* φ^F *is pseudonatural and its components are in* $\mathsf{Cocont}_{\mathsf{skel}}(\mathbf{2})$.

Proof : (1) Is the content of 4.4.

(2) To say that $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ preserves tensors, means that for any $f\colon X \longrightarrow Y$ in $\mathscr{Q}$ and $y \in \mathbb{C}_Y$, $F(y \otimes f) \cong Fy \otimes f$ in $\mathbb{C}_X$. In terms of the transformation φ^F this means that $\varphi^F_X \circ \mathscr{F}_{\mathbb{C}}(f) \cong \mathscr{F}_{\mathbb{C}'} \circ \varphi^F_Y$ instead of merely the inequality "$\geq$"; hence it is pseudonatural instead of merely lax natural.

(3) By 3.1 and the previous point.

(4) Is a variation on the previous point, using 4.7 (2).

(5) Follows from (3), taking into account that all $\mathbb{C}_X$ and $\mathbb{C}'_X$ are cocomplete orders.

(6) Is a variation on (5). $\square$

We may now state our conclusion.

Theorem 4.9. *The equivalence in 4.3 reduces to the following equivalences of locally ordered 2-categories:*

(1) $\mathsf{Cat}_{\otimes}(\mathscr{Q}) \simeq \mathsf{ClPsd}_{\mathsf{lax}}(\mathscr{Q}^{\mathsf{op}}, \mathsf{Cat}_{\otimes}(\mathbf{2}))$,
(2) $\mathsf{Tens}(\mathscr{Q}) \simeq \mathsf{ClPsd}(\mathscr{Q}^{\mathsf{op}}, \mathsf{Tens}(\mathbf{2}))$,
(3) $\mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathscr{Q})) \simeq \mathsf{ClPsd}(\mathscr{Q}^{\mathsf{op}}, \mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathbf{2})))$,
(4) $\mathsf{Cocont}(\mathscr{Q}) \simeq \mathsf{ClPsd}(\mathscr{Q}^{\mathsf{op}}, \mathsf{Cocont}(\mathbf{2}))$,
(5) $\mathsf{Cocont}_{\mathsf{skel}}(\mathscr{Q}) \simeq \mathsf{ClPsd}(\mathscr{Q}^{\mathsf{op}}, \mathsf{Cocont}_{\mathsf{skel}}(\mathbf{2}))$.

Actually, $\mathsf{Cocont}_{\mathsf{skel}}(\mathbf{2}) = \mathsf{Sup}$ and a closed pseudofunctor from $\mathscr{Q}^{\mathsf{op}}$ to Sup is really a quantaloid homomorphism; moreover, $\mathsf{Cocont}_{\mathsf{skel}}(\mathscr{Q})$ is biequivalent to $\mathsf{Cocont}(\mathscr{Q})$. So we may end with the following.

Corollary 4.10. *The quantaloid of right* $\mathscr{Q}$*-modules (cf. 5.1) is biequivalent to the locally cocompletely ordered category of cocomplete* $\mathscr{Q}$*-categories and cocontinuous functors:* $\mathsf{QUANT}(\mathscr{Q}^{\mathsf{op}}, \mathsf{Sup}) \simeq \mathsf{Cocont}(\mathscr{Q})$.

Proposition 2.10. *Let* $\mathbb{C}$ *be a cotensored* $\mathscr{Q}$*-category. The supremum of a family* $(c_i)_{i\in I} \in \mathbb{C}_X$ *is also its conical colimit in* $\mathbb{C}$.

Proof : By hypothesis the supremum $\bigvee_i c_i$ in $\mathbb{C}_X$ exists, and by 2.8 it is the only candidate to be the wanted conical colimit. Thus we must show that $\mathbb{C}(\bigvee_i c_i, -) = \bigwedge_i \mathbb{C}(c_i, -)$. But this follows from the following adjunctions between orders:

$$\text{for any } y \in \mathbb{C}_Y,\ \begin{tikzcd} \mathbb{C}_X \arrow[r, bend left, "\mathbb{C}(-,y)"] \arrow[r, phantom, "\perp"] & \mathscr{Q}(Y,X)^{\mathsf{op}} \arrow[l, bend left, "\langle -,y\rangle"] \end{tikzcd} \text{ in } \mathsf{Cat}(\mathbf{2}).$$

These adjunctions in $\mathsf{Cat}(\mathbf{2})$ follow from adjunctions in $\mathsf{Cat}(\mathscr{Q})$ which are due to the cotensoredness of $\mathbb{C}$—see 3.2; but a direct proof for this adjunction is easy: one uses cotensors in $\mathbb{C}$ to see that, for any $x \in \mathbb{C}_X$,

- $1_X \leq \Big\{\mathbb{C}(x,y), \mathbb{C}(x,y)\Big\} = \mathbb{C}(x, \langle \mathbb{C}(x,y), y\rangle)$ hence $x \leq \langle \mathbb{C}(x,y), y\rangle$ in $\mathbb{C}_X$;
- $1_X \leq \mathbb{C}(\langle f,y\rangle, \langle f,y\rangle) = \Big\{f, \mathbb{C}(\langle f,y\rangle, y)\Big\}$ hence $\mathbb{C}(\langle f,y\rangle, y) \leq^{\mathsf{op}} f$ in $\mathscr{Q}(Y,X)$.

Any left adjoint between orders preserves all suprema that happen to exist, so for any $y \in \mathbb{C}_Y$, $\mathbb{C}(\bigvee_i c_i, y) = \bigwedge_i \mathbb{C}(c_i, y)$ in $\mathscr{Q}(Y,X)$, hence $\mathbb{C}(\bigvee_i c_i, -) = \bigwedge_i \mathbb{C}(c_i, -)$ in $\mathsf{Dist}(\mathscr{Q})(\mathbb{C}, *_X)$, since infima of distributors are calculated elementwise. $\square$

So if $\mathbb{C}$ is cotensored and order-cocomplete, then it is also conically cocomplete. Put differently, a cotensored $\mathscr{Q}$-category is conically cocomplete if and only if it is order-cocomplete. Dually, a tensored category is conically complete if and only if it is order-complete. So...

Theorem 2.11. *For a tensored and cotensored* $\mathscr{Q}$*-category, all notions of completeness and cocompleteness coincide.*

As usual, for orders the situation is much simpler than for general $\mathscr{Q}$-categories.

Example 2.12. For any $\mathbf{2}$-category (be it *a priori* tensored and cotensored or not) all notions of completeness and cocompleteness coincide: an order is order-cocomplete if and only if it is order-complete, but it is then non-empty and has bottom and top element, thus it is tensored and cotensored, thus it is also conically complete and cocomplete, thus also complete and cocomplete *tout court*.

In 2.7 arbitrary colimits in a cocomplete $\mathcal{Q}$-category are reduced to tensors and conical colimits. But a cocomplete $\mathcal{Q}$-category is always complete too; so in particular cotensored. By cotensoredness the conical colimits may be further reduced to suprema.

Corollary 2.13. *If $\mathbb{C}$ is a cocomplete $\mathcal{Q}$-category, then the colimit of the diagram*

$$\mathbb{A} \xrightarrow{\ \Phi\ } \mathbb{B} \xrightarrow{\ F\ } \mathbb{C}$$

is the functor $\mathrm{colim}(\Phi, F)\colon \mathbb{A} \to \mathbb{C}$ sending an object $a \in \mathbb{A}_0$ to the supremum of the family $(Fb \otimes \Phi(b,a))_{b \in \mathbb{B}_0}$. And a functor $F\colon \mathbb{C} \to \mathbb{C}'$ between cocomplete $\mathcal{Q}$-categories is cocontinuous if and only it preserves tensors and suprema in each of the $\mathbb{C}_X$.

3. (Co)tensors and adjunctions

Adjunctions and adjunctions are two. An adjunction of functors between $\mathcal{Q}$-categories, like

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B}, \arrow[l, bend left, "G"] \end{tikzcd}$$

means that $G \circ F \geq 1_{\mathbb{A}}$ and $F \circ G \leq 1_{\mathbb{B}}$ in $\mathsf{Cat}(\mathcal{Q})$. Since functors are type-preserving, this trivially implies adjunctions

$$\text{for any } \mathcal{Q}\text{-object } X,\ \begin{tikzcd} \mathbb{A}_X \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B}_X \arrow[l, bend left, "G"] \end{tikzcd} \text{ in } \mathsf{Cat}(\mathbf{2}).$$

Now we are interested in the converse: how do adjunctions in $\mathsf{Cat}(\mathbf{2})$ determine adjunctions in $\mathsf{Cat}(\mathcal{Q})$? The pertinent result is the following.

Theorem 3.1. *Let $F\colon \mathbb{A} \to \mathbb{B}$ be a functor between $\mathcal{Q}$-categories, with $\mathbb{A}$ tensored. Then the following are equivalent:*

(1) *F is a left adjoint in $\mathsf{Cat}(\mathcal{Q})$;*
(2) *F preserves tensors and, for all $\mathcal{Q}$-objects X, $F\colon \mathbb{A}_X \to \mathbb{B}_X$ is a left adjoint in $\mathsf{Cat}(\mathbf{2})$.*

of a lax natural transformation $\varphi^F\colon \mathscr{F}_{\mathbb{C}}X \to \mathscr{F}_{\mathbb{C}'}X$ live in $\mathsf{Cat}_{\otimes}(\mathbf{2})$ rather than $\mathsf{Cat}(\mathbf{2})$. $\square$

From this proof it follows that, for a closed pseudofunctor $\mathscr{F}\colon \mathcal{Q}^{\mathsf{op}} \to \mathsf{Cat}_{\otimes}(\mathbf{2})$, the bottom element in a non-empty order $\mathscr{F}X$ is $0_X := \mathscr{F}(0_{X,X})(x)$, where x is an arbitrary element in $\mathscr{F}X$. This allows for the following.

Lemma 4.5. *A pseudonatural transformation $\varphi\colon \mathscr{F} \Rightarrow \mathscr{F}'\colon \mathcal{Q}^{\mathsf{op}} \to \mathsf{Cat}_{\otimes}(\mathbf{2})$ between closed pseudofunctors has components in $\mathsf{Tens}(\mathbf{2})$.*

Proof: If $\mathscr{F}X$ is non-empty, take any $x \in \mathscr{F}X$, then by pseudonaturality of φ,

$$\varphi_X(0_X) = \varphi_X(\mathscr{F}(0_{X,X})(x)) \cong \mathscr{F}'(0_{X,X})(\varphi_X(x)) = 0'_X.$$

So each component $\varphi_X\colon \mathscr{F}X \to \mathscr{F}'X$, *a priori* in $\mathsf{Cat}_{\otimes}(\mathbf{2})$, preserves the bottom element if there is one, thus lives in $\mathsf{Tens}(\mathbf{2})$. $\square$

Lemma 4.6. *Any closed pseudofunctor $\mathscr{F}\colon \mathcal{Q}^{\mathsf{op}} \to \mathsf{Map}(\mathsf{Cat}_{\otimes}(\mathbf{2}))$ actually lands in $\mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathbf{2}))$.*

Proof: Taking an arbitrary $x \in \mathscr{F}X$ (presumed non-empty), $\mathscr{F}(0_{X,X})^*(x)$ gives the top element of $\mathscr{F}X$. Here $\mathscr{F}(0_{X,X})^*$ denotes the right adjoint to $\mathscr{F}(0_{X,X})$ in $\mathsf{Cat}_{\otimes}(\mathbf{2})$. So each $\mathscr{F}X$ is an object of $\mathsf{Cat}_{\otimes,\langle\rangle}(\mathbf{2})$ rather than $\mathsf{Cat}_{\otimes}(\mathbf{2})$. $\square$

Now we can apply all this to finetune 4.3.

Proposition 4.7. *Let $\mathbb{C}$ be a tensored $\mathcal{Q}$-category.*

(1) *The associated pseudofunctor $\mathscr{F}_{\mathbb{C}}\colon \mathcal{Q}^{\mathsf{op}} \to \mathsf{Cat}(\mathbf{2})$ factors through $\mathsf{Cat}_{\otimes}(\mathbf{2})$.*
(2) *$\mathbb{C}$ is moreover cotensored if and only if $\mathscr{F}_{\mathbb{C}}$ factors through $\mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathbf{2}))$.*
(3) *$\mathbb{C}$ is cocomplete if and only if $\mathscr{F}_{\mathbb{C}}$ factors through $\mathsf{Cocont}(\mathbf{2})$.*
(4) *$\mathbb{C}$ is skeletal and cocomplete if and only if $\mathscr{F}_{\mathbb{C}}$ factors through $\mathsf{Cocont}_{\mathsf{skel}}(\mathbf{2})$.*

Proof: (1) Is the content of 4.4.

(2) Is a combination of 3.6, 4.3 and 4.6.

(3) By 2.11 a tensored and cotensored $\mathbb{C}$ is cocomplete if and only if it is order-cocomplete, i.e. each $\mathbb{C}_X$ is a cocomplete order. Now apply (2), recalling that $\mathsf{Cocont}(\mathbf{2})$ is precisely the full sub-2-category of $\mathsf{Map}(\mathsf{Cat}_{\otimes,\langle\rangle}(\mathbf{2}))$ determined by the (order-)cocomplete objects.

(4) Is a variation on (3): a $\mathcal{Q}$-category $\mathbb{C}$ is skeletal if and only if each $\mathbb{C}_X$ is an antisymmetric (i.e. skeletal) order. $\square$

Proposition 4.8. *Let $F\colon \mathbb{C} \to \mathbb{C}'$ be a functor between tensored $\mathcal{Q}$-categories.*

- for $x \in \mathbb{C}^{\mathscr{F}}_X$ and $y \in \mathbb{C}^{\mathscr{F}}_Y$, $\mathbb{C}^{\mathscr{F}}(y,x) = \bigvee\{f\colon X \longrightarrow Y \text{ in } \mathcal{Q} \mid \mathscr{F}(f)(y) \leq x \text{ in } \mathbb{C}^{\mathscr{F}}_X\}$.

The supremum involved is really an expression of the closedness of the pseudofunctor: $x \mapsto \mathbb{C}^{\mathscr{F}}(y,x)$ is the right adjoint to $f \mapsto \mathscr{F}(f)(y)$ in $\mathsf{Cat}(\mathbf{2})$. Then $\mathbb{C}^{\mathscr{F}}$ is a tensored $\mathcal{Q}$-category: the tensor of some $f\colon X \longrightarrow Y$ and $y \in \mathscr{F}Y$ is precisely $\mathscr{F}(f)(y)$, by (the dual of) 3.2. It is clear that $\mathscr{F} \cong \mathscr{F}_{\mathbb{C}^{\mathscr{F}}}$. So far for essential surjectivity of (8).

Finally, given tensored $\mathcal{Q}$-categories $\mathbb{C}$ and $\mathbb{C}'$, the ordered sets $\mathsf{Cat}_{\otimes}(\mathcal{Q})(\mathbb{C},\mathbb{C}')$ and $\mathsf{Psd}_{\mathsf{lax}}(\mathcal{Q}^{\mathsf{op}},\mathsf{Cat}(\mathbf{2}))(\mathscr{F}_{\mathbb{C}},\mathscr{F}_{\mathbb{C}'})$ are isomorphic: a functor $F\colon \mathbb{C} \longrightarrow \mathbb{C}'$ between (tensored) $\mathcal{Q}$-categories is completely determined by its action on objects, hence by the family of (order-preserving) mappings $\mathbb{C}_X \longrightarrow \mathbb{C}'_X\colon x \mapsto Fx$, hence by the components of the corresponding transformation $\varphi^F\colon \mathscr{F}_{\mathbb{C}} \Longrightarrow \mathscr{F}_{\mathbb{C}'}$. From the proof of 4.2 it is clear that F is a functor if and only if φ^F is lax natural (thanks to tensoredness of $\mathbb{C}$ and $\mathbb{C}'$). Furthermore, to say that $F \leq G\colon \mathbb{C} \rightrightarrows \mathbb{C}'$ in $\mathsf{Cat}(\mathcal{Q})$ means that, for any $\mathcal{Q}$-object X and any $x \in \mathbb{C}_X$, $Fx \leq Gx$ in $\mathbb{C}'_X$. For the lax natural transformations φ^F, φ^G corresponding to F, G this is really the same thing as saying that $\varphi^F_X \leq \varphi^G_X$ in $\mathsf{Cat}(\mathbf{2})$, in other words, $\varphi^F \leq \varphi^G$ as arrows between (closed) pseudofunctors. $\square$

It follows from 2.1 and 4.3 that a closed pseudofunctor $\mathscr{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Cat}(\mathbf{2})$ either has all of the $\mathscr{F}X$ empty, or none of them. A direct proof is easy too (it is of course a transcription of 2.1 modulo the equivalence in 4.3): if $y \in \mathscr{F}Y$, then $\mathscr{F}(0_{X,Y})(y) \in \mathscr{F}X$, where $0_{X,Y} \in \mathcal{Q}(X,Y)$ is the bottom element. So as soon as one of the $\mathscr{F}X$ is non-empty, all of them are. And the empty pseudofunctor is trivially closed.

Finetuning. Here are some seemingly innocent specifications concerning the 2-functor in 4.3.

Lemma 4.4. *Any closed pseudofunctor $\mathscr{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Cat}(\mathbf{2})$ lands in $\mathsf{Cat}_{\otimes}(\mathbf{2})$. And any lax natural transformation $\varphi\colon \mathscr{F} \Longrightarrow \mathscr{F}'\colon \mathcal{Q} \longrightarrow \mathsf{Cat}(\mathbf{2})$ between closed pseudofunctors has components in $\mathsf{Cat}_{\otimes}(\mathbf{2})$ rather than $\mathsf{Cat}(\mathbf{2})$.*

Proof: For any closed pseudofunctor $\mathscr{F}\colon \mathcal{Q}^{\mathsf{op}} \longrightarrow \mathsf{Cat}(\mathbf{2})$, for every X in $\mathcal{Q}$ and $x \in \mathscr{F}X$, $\mathscr{F}(-)(x)\colon \mathcal{Q}(X,X) \longrightarrow \mathscr{F}X$ preserves all suprema, thus in particular the empty supremum, i.e. the bottom element $0_{X,X} \in \mathcal{Q}(X,X)$. This implies that every non-empty $\mathscr{F}X$ must have a bottom element. Thus $\mathscr{F}$ lands in $\mathsf{Cat}_{\otimes}(\mathbf{2})$ rather than $\mathsf{Cat}(\mathbf{2})$. Precisely because of this, the components $\varphi_X\colon \mathscr{F}X \longrightarrow \mathscr{F}'X$

Proof: One direction is trivial. For the other, denote the assumed adjunctions in $\mathsf{Cat}(\mathbf{2})$ as

$$\begin{tikzcd} \mathbb{A}_X \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B}_X \arrow[l, bend left, "G_X"] \end{tikzcd} \text{, one for each } \mathcal{Q}\text{-object } X.$$

First, for any $a \in \mathbb{A}_X$ and $b \in \mathbb{B}_Y$,

$$\begin{array}{rcl} \mathbb{A}(a, G_Y b) & \leq & \mathbb{B}(Fa, FG_Y b) \\ & = & \mathbb{B}(Fa, FG_Y b) \circ 1_Y \\ & \leq & \mathbb{B}(Fa, FG_Y b) \circ \mathbb{B}(FG_Y b, b) \\ & \leq & \mathbb{B}(Fa, b). \end{array}$$

The first inequality holds by functoriality of F; to pass from the second to the third line, use the pertinent adjunction $F \dashv G_Y$: $FG_Y b \leq b$ in $\mathbb{B}_Y$, so $1_Y \leq \mathbb{B}(FG_Y b, b)$. For the converse inequality, use tensors in $\mathbb{A}$ and the fact that F preserves them, plus the adjunction $F \dashv G_Y$ where appropriate: for $a \in \mathbb{A}_X$ and $b \in \mathbb{B}_Y$,

$$\begin{array}{rcl} \mathbb{B}(Fa, b) \leq \mathbb{A}(a, G_Y b) & \iff & 1_Y \leq \Big[\mathbb{B}(Fa, b), \mathbb{A}(a, G_Y b)\Big] \\ & \iff & 1_Y \leq \mathbb{A}\Big(a \otimes \mathbb{B}(Fa, b), G_Y b\Big) \\ & \iff & 1_Y \leq \mathbb{B}\Big(F(a \otimes \mathbb{B}(Fa, b)), b\Big) \\ & \iff & 1_Y \leq \mathbb{B}\Big(Fa \otimes \mathbb{B}(Fa, b), b\Big) \\ & \iff & 1_Y \leq \Big[\mathbb{B}(Fa, b), \mathbb{B}(Fa, b)\Big] \end{array}$$

which is certainly true. It remains to prove that $G\colon \mathbb{B} \longrightarrow \mathbb{A}\colon b \mapsto G_{tb} b$ is a functor; but for $b \in \mathbb{B}_Y$ and $b' \in \mathbb{B}_{Y'}$,

$$\begin{array}{rcl} \mathbb{B}(b', b) & = & 1_{Y'} \circ \mathbb{B}(b', b) \\ & \leq & \mathbb{B}(FG_{Y'} b', b') \circ \mathbb{B}(b', b) \\ & \leq & \mathbb{B}(FG_{Y'} b', b) \\ & = & \mathbb{A}(G_{Y'} b', G_Y b). \end{array}$$

Here we use once more the suitable $F \dashv G_{Y'}$, and also the composition in $\mathbb{B}$. $\square$

In a way, 3.1 resembles 2.10: in both cases **2**-categorical content is "lifted" to $\mathcal{Q}$-categorical content (suprema are "lifted" to conical colimits, adjunctions between orders are "lifted" to adjunctions between categories), and in both cases the price to pay has to do with (existence and preservation of) (co)tensors.

There is a "weaker" version of 3.1: for two functors $F\colon \mathbb{A}\to\mathbb{B}$ and $G\colon \mathbb{B}\to\mathbb{A}$, $F \dashv G$ in $\mathsf{Cat}(\mathcal{Q})$ if and only if, for each $\mathcal{Q}$-object X, $F_X \dashv G_X$ in $\mathsf{Cat}(\mathbf{2})$. Here one need not ask $\mathbb{A}$ to be tensored nor F to preserve tensors (although it does *a posteriori* for it is a left adjoint). But the point is that for this "weaker" proposition one *assumes the existence* of some functor G and one proves that it is the right adjoint to F, whereas in 3.1 one *proves the existence* of the right adjoint to F.

Were we to prove 3.1 under the hypothesis that $\mathbb{A}$, $\mathbb{B}$ are cocomplete $\mathcal{Q}$-categories, we simply could have applied 2.13: for such categories, $F\colon \mathbb{A}\to\mathbb{B}$ is left adjoint if and only if it is cocontinuous, if and only if preserves tensors and each $\mathbb{A}_X\to\mathbb{B}_X\colon a\mapsto Fa$ preserves suprema, if and only if it preserves tensors and each $\mathbb{A}_X\to\mathbb{B}_X\colon a\mapsto Fa$ is left adjoint in $\mathsf{Cat}(\mathbf{2})$ (for each $\mathbb{A}_X$ is a cocomplete order). The merit of 3.1 is thus to have generalized 2.13 to the case of a tensored $\mathbb{A}$ and an arbitrary $\mathbb{B}$.

Adjunctions from (co)tensors, and *vice versa*.

Proposition 3.2. *For a $\mathcal{Q}$-category $\mathbb{C}$ and an object $x\in\mathbb{C}_X$, all cotensors with x exist if and only if the functor* $\mathbb{C}(-,x)\colon \mathbb{C}\to\mathscr{P}^\dagger X$ is a left adjoint in $\mathsf{Cat}(\mathcal{Q})$. In this case its right adjoint is $\langle -,x\rangle\colon \mathscr{P}^\dagger X\to\mathbb{C}$.*

Proof: If, for any $f\colon X\to Y$ in $\mathcal{Q}$, $\langle f,x\rangle$ exists, then $\langle -,x\rangle\colon \mathscr{P}^\dagger X\to\mathbb{C}$ is a functor: for $f\colon X\to Y$, $f'\colon X\to Y'$, i.e. two objects of $\mathscr{P}^\dagger X$,

$$\begin{aligned}
\mathscr{P}^\dagger X(f',f)\le\mathbb{C}(\langle f',x\rangle,\langle f,x\rangle) \quad &\Longleftrightarrow\quad \Big\{f,f'\Big\}\le\Big\{f,\mathbb{C}(\langle f',x\rangle,x)\Big\}\\
&\Longleftarrow\quad f'\le\mathbb{C}(\langle f',x\rangle,x)\\
&\Longleftrightarrow\quad 1_{Y'}\le\mathbb{C}(\langle f',x\rangle,\langle f',x\rangle)
\end{aligned}$$

which is true. And $\mathbb{C}(-,x)\dashv\langle -,x\rangle$ holds by the universal property of the cotensor itself.

*In principle, $\mathbb{C}(-,x)\colon *_X \relbar\!\!\!\!\circ\!\!\to \mathbb{C}$ is a covariant presheaf on $\mathbb{C}$, i.e. a distributor; but these correspond precisely to functors from $\mathbb{C}$ to the completion of $*_X$, which we denote as $\mathscr{P}^\dagger X$; see section 6 of [Stubbe, 2005a] for details. We do not notationally distinguish between distributor and functor here.

Proof: For a tensored $\mathcal{Q}$-category $\mathbb{C}$, $\mathscr{F}_\mathbb{C}$ as in the statement of the proposition is well-defined: each $\mathbb{C}_X$ is an order and each $-\otimes f\colon \mathbb{C}_Y\to\mathbb{C}_X$ preserves order (by 3.4). Moreover, this action is pseudofunctorial (again by 3.4). And from (the dual of) 3.2 we know that, for each X,Y in $\mathcal{Q}$ and $y\in\mathbb{C}_Y$,

$$y\otimes -\colon \mathcal{Q}(X,Y)\to\mathbb{C}_X\colon f\mapsto y\otimes f$$

is a left adjoint; so $\mathscr{F}_\mathbb{C}$ is a closed pseudofunctor.

A functor $F\colon \mathbb{C}\to\mathbb{C}'$ is a type-preserving mapping $F\colon \mathbb{C}_0\to\mathbb{C}'_0\colon x\mapsto Fx$ of objects such that $\mathbb{C}(y,x)\le\mathbb{C}'(Fy,Fx)$ for all $x,y\in\mathbb{C}_0$. With (4), this functor-inequality may be rewritten as

$$\begin{aligned}
&\mathbb{C}(y,x)\le\mathbb{C}'(Fy,Fx)\\
&\iff \text{for any } f\colon X\to Y \text{ in } \mathcal{Q}, \text{ if } y\otimes f\le x \text{ in } \mathbb{C}_X \text{ then } Fy\otimes f\le Fx \text{ in } \mathbb{C}'_X\\
&\iff \text{for any } f\colon X\to Y \text{ in } \mathcal{Q},\ Fy\otimes f\le F(y\otimes f).
\end{aligned}$$

(For the last equivalence, necessity follows by application of the previous sentence to $y\otimes f\le y\otimes f$, whereas for sufficiency one first notes that $y\otimes f\le x$ implies anyway that $F(y\otimes f)\le Fx$ so combined with the assumption this gives $Fy\otimes f\le Fx$.) Thus, such a functor $F\colon\mathbb{C}\to\mathbb{C}'$ is really just a family of mappings $\mathbb{C}_X\to\mathbb{C}'_X\colon x\mapsto Fx$, one for each $\mathcal{Q}$-object X, which are all order-preserving (by functoriality of F) and satisfy furthermore for any $f\colon X\to Y$ in $\mathcal{Q}$ and $y\in\mathbb{C}_Y$ that $Fy\otimes f\le F(y\otimes f)$. Having defined components φ^F_X as in (7), this says that $\mathscr{F}_{\mathbb{C}'}(f)\circ\varphi^F_Y\le\varphi^F_X\circ\mathscr{F}_\mathbb{C}(f)$, for any $f\colon X\to Y$ in $\mathcal{Q}$. So $\varphi\colon\mathscr{F}_\mathbb{C}\to\mathscr{F}_{\mathbb{C}'}$ is a lax natural transformation. $\square$

Theorem 4.3. *For any quantaloid $\mathcal{Q}$, the action*

$$\mathsf{Cat}_\otimes(\mathcal{Q})\to\mathsf{ClPsd}_{\mathsf{lax}}(\mathcal{Q}^{\mathsf{op}},\mathsf{Cat}(\mathbf{2}))\colon \Big(F\colon\mathbb{C}\to\mathbb{C}'\Big)\mapsto\Big(\varphi^F\colon\mathscr{F}_\mathbb{C}\Rightarrow\mathscr{F}_{\mathbb{C}'}\Big) \qquad (8)$$

is an equivalence of 2-categories.

Proof: Straightforwardly the action in (8) is functorial: the lax natural transformation corresponding to an identity functor is an identity lax natural transformation; the lax natural transformation corresponding to the composition of functors is the composition of the lax natural transformations corresponding to each of the functors involved.

Now let $\mathscr{F}\colon\mathcal{Q}^{\mathsf{op}}\to\mathsf{Cat}(\mathbf{2})$ be any closed pseudofunctor; then define a $\mathcal{Q}$-category $\mathbb{C}^\mathscr{F}$ by:

- for each $\mathcal{Q}$-object X, $\mathbb{C}^\mathscr{F}_X:=\mathscr{F}X$,

Some more notions and notations, now from the realm of "variation": Let $\mathscr{A}$ and $\mathscr{B}$ be locally ordered 2-categories (i.e. $\mathsf{Cat}(\mathbf{2})$-enriched categories). A *pseudofunctor* $\mathscr{F}\colon \mathscr{A}\longrightarrow\mathscr{B}$ is an action on objects and morphisms that respects the local order and such that functoriality holds up to local isomorphism (we need not require any coherence because our 2-categories are locally ordered). For two such pseudofunctors $\mathscr{F},\mathscr{F}'\colon \mathscr{A}\rightrightarrows\mathscr{B}$, a *lax natural transformation* $\varphi\colon \mathscr{F}\Rightarrow\mathscr{F}'$ is a family of $\mathscr{B}$-morphisms $(\varphi_X\colon \mathscr{F}X\longrightarrow\mathscr{F}'X)_{X\in\mathscr{A}_0}$ satisfying, for any $f\colon X\longrightarrow Y$ in $\mathscr{A}$, $\mathscr{F}'f\circ\varphi_X\leq\varphi_Y\circ\mathscr{F}f$ in $\mathscr{B}(\mathscr{F}X,\mathscr{F}'Y)$. Such a transformation is *pseudonatural* when these inequalities are isomorphisms. Lax natural transformations are ordered componentwise. There are locally ordered 2-categories $\mathsf{Psd}_{\mathsf{lax}}(\mathscr{A},\mathscr{B})$, resp. $\mathsf{Psd}(\mathscr{A},\mathscr{B})$, with pseudofunctors as objects and lax natural transformations, resp. pseudonatural transformations, as arrows.

Now consider a pseudofunctor $\mathscr{F}\colon \mathscr{A}\longrightarrow\mathsf{Cat}(\mathbf{2})$; it is *closed* when, for every X,Y in $\mathscr{A}$ and $x\in\mathscr{F}X$,

$$\mathscr{F}(-)(x)\colon \mathscr{A}(X,Y)\longrightarrow\mathscr{F}Y\colon f\mapsto\mathscr{F}(f)(x)$$

is a left adjoint in $\mathsf{Cat}(\mathbf{2})$. We write $\mathsf{ClPsd}_{\mathsf{lax}}(\mathscr{A},\mathsf{Cat}(\mathbf{2}))$ and $\mathsf{ClPsd}(\mathscr{A},\mathsf{Cat}(\mathbf{2}))$ for the full sub-2-categories of $\mathsf{Psd}_{\mathsf{lax}}(\mathscr{A},\mathsf{Cat}(\mathbf{2}))$ and $\mathsf{Psd}(\mathscr{A},\mathsf{Cat}(\mathbf{2}))$ determined by the closed pseudofunctors.

We will be interested in closed pseudofunctors on the opposite of a quantaloid $\mathcal{Q}$; the closedness of a pseudofunctor $\mathscr{F}\colon\mathcal{Q}^{\mathsf{op}}\longrightarrow\mathsf{Cat}(\mathbf{2})$ reduces to the fact that, for each X,Y in $\mathcal{Q}$ and $y\in Y$,

$$\mathscr{F}(-)(y)\colon \mathcal{Q}(X,Y)\longrightarrow\mathscr{F}X\colon f\mapsto\mathscr{F}(f)(y) \qquad (5)$$

preserves arbitrary suprema (for $\mathcal{Q}(X,Y)$ is a sup-lattice). When we replace $\mathsf{Cat}(\mathbf{2})$ by any of its sub-2-categories like $\mathsf{Cat}_{\otimes}(\mathbf{2})$, $\mathsf{Tens}(\mathbf{2})$ and so on, the closedness condition for pseudofunctors still makes sense: we will mean precisely that the order-morphisms in (5) preserve suprema (i.e. are left adjoints in $\mathsf{Cat}(\mathbf{2})$).

The basic biequivalence.

Proposition 4.2. *A tensored $\mathcal{Q}$-category $\mathbb{C}$ determines a closed pseudofunctor*

$$\mathscr{F}_{\mathbb{C}}\colon\mathcal{Q}^{\mathsf{op}}\longrightarrow\mathsf{Cat}(\mathbf{2})\colon\Big(f\colon X\longrightarrow Y\Big)\mapsto\Big(-\otimes f\colon\mathbb{C}_Y\longrightarrow\mathbb{C}_X\Big). \qquad (6)$$

And a functor $F\colon\mathbb{C}\longrightarrow\mathbb{C}'$ between tensored $\mathcal{Q}$-categories determines a lax natural transformation

$$\varphi^F\colon\mathscr{F}_{\mathbb{C}}\Rightarrow\mathscr{F}_{\mathbb{C}'}\text{ with components }\varphi^F_X\colon\mathbb{C}_X\longrightarrow\mathbb{C}'_X\colon x\mapsto Fx. \qquad (7)$$

Conversely, if $\mathbb{C}(-,x)\colon\mathbb{C}\longrightarrow\mathscr{P}^{\dagger}X$ has a right adjoint $R_x\colon\mathscr{P}^{\dagger}X\longrightarrow\mathbb{C}$, then in particular for all $f\colon X\longrightarrow Y$ in $\mathcal{Q}$, $R_x(f)$ is an object of type Y in $\mathbb{C}$, satisfying

$$\text{for all } y\in\mathbb{C},\ \mathbb{C}(y,R_x(f))=\mathscr{P}^{\dagger}X\Big(\mathbb{C}(y,x),f\Big)=\Big\{f,\mathbb{C}(y,x)\Big\}.$$

This says precisely that $R_x(f)$ is the cotensor of x with f. □

In the situation of 3.2 it follows that

$$\text{for each }\mathcal{Q}\text{-object }Z,\ \begin{tikzcd}\mathbb{C}_Z \arrow[r, bend left, "\mathbb{C}(-,x)"] \arrow[r, phantom, "\perp"] & \mathcal{Q}(X,Z)^{\mathsf{op}} \arrow[l, bend left, "\langle -,x\rangle"]\end{tikzcd}\ \text{in }\mathsf{Cat}(\mathbf{2}), \qquad (1)$$

$$\text{for each }z\in\mathbb{C}_Z,\ \mathbb{C}(z,x)=\bigwedge\{f\colon X\longrightarrow Z\text{ in }\mathcal{Q}\mid z\leq\langle f,x\rangle\text{ in }\mathbb{C}_Z\}. \qquad (2)$$

The dual version of the above will be useful too: it says that tensors with $y\in\mathbb{C}_Y$ exist if and only if $\mathbb{C}(y,-)\colon\mathbb{C}\longrightarrow\mathscr{P}Y$ is a right adjoint in $\mathsf{Cat}(\mathcal{Q})$, in which case its left adjoint is $y\otimes-\colon\mathscr{P}Y\longrightarrow\mathbb{C}$. And then moreover

$$\text{for each }\mathcal{Q}\text{-object }Z,\ \begin{tikzcd}\mathbb{C}_Z \arrow[r, bend right, "\mathbb{C}(y,-)"'] \arrow[r, phantom, "\perp"] & \mathcal{Q}(Z,Y) \arrow[l, bend right, "y\otimes -"']\end{tikzcd}\ \text{in }\mathsf{Cat}(\mathbf{2}), \qquad (3)$$

$$\text{for each }z\in\mathbb{C}_Z,\ \mathbb{C}(y,z)=\bigvee\{f\colon Z\longrightarrow Y\text{ in }\mathcal{Q}\mid y\otimes f\leq z\text{ in }\mathbb{C}_Z\}. \qquad (4)$$

Here is a useful application of the previous results. For any $\mathcal{Q}$-category $\mathbb{C}$ the Yoneda embedding $Y^{\dagger}_{\mathbb{C}}\colon\mathbb{C}\longrightarrow\mathscr{P}^{\dagger}\mathbb{C}\colon c\mapsto\mathbb{C}(c,-)$ is a cocontinuous functor; in particular, for any $x\in\mathbb{C}_X$ the functor $\mathbb{C}(-,x)\colon\mathbb{C}\longrightarrow\mathscr{P}^{\dagger}X$ preserves tensors. (A direct proof of this latter fact is easy too: for $f\colon Y\longrightarrow Z$ in $\mathcal{Q}$ and $z\in\mathbb{C}_Z$, suppose that $z\otimes f$ exists in $\mathbb{C}$. Then $\mathbb{C}(z\otimes f,x)=[f,\mathbb{C}(z,x)]=\mathbb{C}(z,x)\otimes f$ in $\mathscr{P}^{\dagger}X$, because this is how tensors are calculated in $\mathscr{P}^{\dagger}X$.)

Corollary 3.3. *If $\mathbb{C}$ is a tensored $\mathcal{Q}$-category, then the following are equivalent:*

(1) *for all $\mathcal{Q}$-objects X and Y and each $x\in\mathbb{C}_X$, $\mathbb{C}(-,x)\colon\mathbb{C}_Y\longrightarrow\mathcal{Q}(X,Y)^{\mathsf{op}}$ is a left adjoint in $\mathsf{Cat}(\mathbf{2})$;*
(2) *for each $x\in\mathbb{C}_X$, $\mathbb{C}(-,x)\colon\mathbb{C}\longrightarrow\mathscr{P}^{\dagger}X$ is a left adjoint in $\mathsf{Cat}(\mathcal{Q})$;*
(3) *$\mathbb{C}$ is cotensored.*

In 3.2 we have results about "(co)tensoring with a fixed object"; now we are interested in studying "tensoring with a fixed arrow". Recall that a tensor is a colimit of which such an arrow is the weight. So we may apply general lemmas on

weighted colimits [Stubbe, 2005a, 5.2 and 5.3] to obtain the following particular results.

Proposition 3.4. *Let $\mathbb{C}$ denote a $\mathcal{Q}$-category.*

(1) *For all $y \in \mathbb{C}_Y$, $y \otimes 1_Y \cong y$.*
(2) *For $g\colon W \to X$ and $f\colon X \to Y$ in $\mathcal{Q}$ and $y \in \mathbb{C}_Y$, if all tensors involved exist then $y \otimes (f \circ g) \cong (y \otimes f) \otimes g$.*
(3) *for $(f_i\colon X \to Y)_{i \in I}$ in $\mathcal{Q}$ and $y \in \mathbb{C}_Y$, if all tensors involved exist then $y \otimes (\bigvee_i f_i) \cong \bigvee_i (y \otimes f_i)$.*
(4) *For $f\colon X \to Y$ in $\mathcal{Q}$ and $y, y' \in \mathbb{C}_Y$, if all tensors involved exist then $y \leq y'$ in $\mathbb{C}_Y$ implies $y \otimes f \leq y' \otimes f$ in $\mathbb{C}_X$.*

Of course there is a dual version about cotensors, but we do not bother spelling it out. However, there is an interesting interplay between tensors and cotensors.

Proposition 3.5. *Let $f\colon X \to Y$ be a $\mathcal{Q}$-arrow and suppose that all tensors and all cotensors with f exist in some $\mathcal{Q}$-category $\mathbb{C}$. Then*

$$\begin{tikzcd} \mathbb{C}_Y \arrow[r, bend left, "- \otimes f"] \arrow[r, phantom, "\perp"] & \mathbb{C}_X \arrow[l, bend left, "\langle f, - \rangle"] \end{tikzcd} \quad \textit{in } \mathsf{Cat}(\mathbf{2}).$$

Proof: It follows from 3.4 (and its dual) that

$$- \otimes f\colon \mathbb{C}_Y \to \mathbb{C}_X \text{ and } \langle f, - \rangle\colon \mathbb{C}_X \to \mathbb{C}_Y$$

are order-preserving morphisms. Furthermore, for $x \in \mathbb{C}_X$ and $y \in \mathbb{C}_Y$,

$$\begin{aligned} y \otimes f \leq x \quad &\iff \quad 1_X \leq \mathbb{C}(y \otimes f, x) = \Big[f, \mathbb{C}(y, x)\Big] \\ &\iff \quad f \leq \mathbb{C}(y, x) \\ &\iff \quad 1_Y \leq \Big\{f, \mathbb{C}(y, x)\Big\} = \mathbb{C}(y, \langle f, x \rangle) \\ &\iff \quad y \leq \langle f, x \rangle. \end{aligned}$$

$\square$

We can push this further.

Proposition 3.6. *A tensored $\mathcal{Q}$-category $\mathbb{C}$ is cotensored if and only if, for every $f\colon X \to Y$ in $\mathcal{Q}$, $- \otimes f\colon \mathbb{C}_Y \to \mathbb{C}_X$ is a left adjoint in $\mathsf{Cat}(\mathbf{2})$. In this case, its right adjoint is $\langle f, - \rangle\colon \mathbb{C}_X \to \mathbb{C}_Y$.*

Proof: Necessity follows from 3.5. As for sufficiency, by 3.3 it suffices to show that for all $\mathcal{Q}$-objects X and Y and every $x \in \mathbb{C}_X$,

$$\mathbb{C}(x, -)\colon \mathbb{C}_Y \to \mathcal{Q}(X, Y)^{\mathsf{op}}\colon y \mapsto \mathbb{C}(x, y)$$

has a right adjoint in $\mathsf{Cat}(\mathbf{2})$. Denoting, for a $\mathcal{Q}$-arrow $f\colon X \to Y$, the right adjoint to $- \otimes f\colon \mathbb{C}_Y \to \mathbb{C}_X$ in $\mathsf{Cat}(\mathbf{2})$ as $R_f\colon \mathbb{C}_X \to \mathbb{C}_Y$, the obvious candidate right adjoint to $y \mapsto \mathbb{C}(x, y)$ is $f \mapsto R_f(x)$. First note that, if $f \leq^{\mathsf{op}} f'$ in $\mathcal{Q}(X, Y)$ then $R_f(x) \otimes f' \leq R_f(x) \otimes f \leq x$ using $- \otimes f \dashv R_f$, which implies by $- \otimes f' \dashv R_{f'}$ that $R_f(x) \leq R_{f'}(x)$: so

$$R_{(-)}(x)\colon \mathcal{Q}(X, Y)^{\mathsf{op}} \to \mathbb{C}_Y\colon f \mapsto R_f(x)$$

preserves order. Further, for $f \in \mathcal{Q}(X, Y)$ and $y \in \mathbb{C}_Y$,

$$\begin{aligned} \mathbb{C}(y, x) \leq^{\mathsf{op}} f \quad &\iff \quad f \leq \mathbb{C}(y, x) \\ &\iff \quad y \otimes f \leq x \\ &\iff \quad y \leq R_f(x), \end{aligned}$$

so indeed $\mathbb{C}(x, -) \dashv R_{(-)}(x)$ in $\mathsf{Cat}(\mathbf{2})$. Now $\mathbb{C}$ is tensored and cotensored, so by 3.5 it follows that $R_f(x)$ must be $\langle f, x \rangle$ (since both are right adjoint to $- \otimes f$). $\square$

4. Enrichment and variation

Terminology and notations. We must introduce some notation. By $\mathsf{Cat}_{\otimes}(\mathcal{Q})$ we denote the full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ whose objects are tensored categories, and $\mathsf{Tens}(\mathcal{Q})$ the sub-2-category whose objects are tensored categories and morphisms are tensor-preserving functors. Similarly we use $\mathsf{Cat}_{\langle\rangle}(\mathcal{Q})$ for the full sub-2-category of $\mathsf{Cat}(\mathcal{Q})$ whose objects are cotensored categories, and moreover the obvious combination $\mathsf{Cat}_{\otimes,\langle\rangle}(\mathcal{Q})$. Recall also that $\mathsf{Cocont}(\mathcal{Q})$ denotes the locally completely ordered 2-category whose objects are cocomplete $\mathcal{Q}$-categories and morphisms are cocontinuous (equivalently, left adjoint) functors; and $\mathsf{Cocont}_{\mathsf{skel}}(\mathcal{Q})$ denotes its biequivalent full sub-quantaloid whose objects are skeletal.

Example 4.1. $\mathsf{Cat}(\mathbf{2})$ is the locally ordered 2-category of orders and order preserving maps. $\mathsf{Cat}_{\otimes}(\mathbf{2})$ has orders with bottom element as objects and all order-preserving maps as morphisms, whereas $\mathsf{Tens}(\mathbf{2})$ has the same objects but the morphisms are required to send bottom onto bottom. $\mathsf{Cocont}(\mathbf{2})$ is biequivalent to the quantaloid of sup-lattices and sup-morphisms; taking only skeletal $\mathbf{2}$-categories (i.e. antisymmetric orders) we have $\mathsf{Cocont}_{\mathsf{skel}}(\mathbf{2}) = \mathsf{Sup}$.