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QUASI–STEADY STOKES FLOW OF MULTIPHASE FLUIDS WITH SHEAR–DEPENDENT VISCOSITY

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ABSTRACT: The quasi–steady power–law Stokes flow of a mixture of incompressible fluids with shear–dependent viscosity is studied. The fluids are immiscible and have constant densities. Existence results are presented for both the no–slip and the no–stick boundary value conditions. Use is made of Schauder's fixed–point theorem, compactness arguments, and DiPerna-Lions renormalized solutions.

KEYWORDS: Multiphase fluid, shear–dependent viscosity, no–slip condition, no–stick condition, renormalized solution.
AMS SUBJECT CLASSIFICATION (2000): 35Q30, 76D03, 35J65, 76T30.

1. Introduction

In chemical engineering, blood rheology, ice mechanics, and geology, one comes across a wide range of incompressible fluids that cannot be adequately described by using the Navier–Stokes theory. Those fluids are generally referred to as non–Newtonian fluids. There are many examples for which the viscosity depends on the modulus of the symmetric part of the velocity gradient; cf. [11, 12, 14]. Such fluids are called generalised Newtonian fluids or fluids with shear–dependent viscosity.

Our purpose is to study the quasi–steady flow of M immiscible fluids with shear–dependent viscosity. We assume that the fluids occupy time–dependent subdomains $\Omega_m(t)$, $1 \leq m \leq M$, of a fixed domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$. In each subdomain the Cauchy stress tensor σ_m has p–structure, i.e.,

$$\sigma_m = 2\mu(\rho_m)\, T(e(\mathbf{u}_m)) - \pi_m \,\mathrm{Id}, \tag{1.1}$$

where $T(e(\mathbf{u}_m)) = \nu(\kappa + |e(\mathbf{u}_m)|)^{p-2} e(\mathbf{u}_m)$, $1 < p < \infty$, $e(\mathbf{u}_m)$ is the symmetric part of the velocity gradient, ρ_m is the density, $\mu(\rho_m)$ is the viscosity, and π_m is the pressure. For different values of p different phenomena are captured. The quantity $|e(\mathbf{u}_m)|$ is called the shear rate and a fluid obeying the constitutive law (1.1) is named shear thinning if $p < 2$, and shear thickening if $p > 2$. The non–miscibility conditions at the interfaces between the fluids are equivalent to a

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transport equation for the viscosities on the whole domain; cf. [16]. Coupling this transport equation with the quasi–steady power–law Stokes equations leads to the system

$$\begin{aligned} -\mathrm{div}\,(2\,\mu(\rho)\,T(e(\mathbf{u}))) + \nabla\pi &= \rho\,\mathbf{f}, \\ \rho_t + \mathrm{div}\,(\rho\,\mathbf{u}) &= 0, \\ \mathrm{div}\,\mathbf{u} &= 0. \end{aligned}$$

We consider this system under no–slip and under no–stick boundary value conditions. The aim of the paper is to derive existence results for weak solutions of the coupled system. This will be obtained by a Galerkin ansatz and a fixed–point argument. We first show how to solve the transport equation by using the concept of renormalized solution introduced by DiPerna and Lions [5]. Next we consider an approximated Stokes problem in a finite dimensional space and solve it using monotonicity techniques and Korn's inequality (incidentally, an extension of Korn's second inequality to general boundary conditions is obtained, cf. Lemma 7.1, which is interesting in its own right). A solution to the coupled approximated problem is then obtained through Schauder's fixed–point theorem. Finally, we pass to the limit in the dimension using rather delicate compactness arguments.

Up to now, mixtures of incompressible viscous fluids have only been studied in the linear case $p = 2$. In [16] an existence theorem for the multi–fluid Stokes problem is given. The full incompressible multi–fluid Navier–Stokes system is treated in [15]. Moreover, two–dimensional flows are studied in [3], and two–phase flows of fluids with surface tension are considered in [17].

The outline of the paper is as follows. The model and the assumptions on the data are stated in the next section. In Section 3 we discuss the continuity equation. Sections 4 and 5 are devoted to some auxiliary Galerkin problems and the fixed–point argument. The main results are stated and proved in Sections 6 and 7.

2. Quasi–steady Stokes flow

We consider M fluids with shear–dependent viscosity flowing in an open domain $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$. Let $\Omega_m(t)$ be the domain occupied by the m-th fluid at time $t \in (0, T]$; thus, for each t, we have

$$\overline{\Omega} = \bigcup_{m=1}^{M} \overline{\Omega}_m(t).$$

Denote by $\mathbf{u}_m$ the velocity field of the m-th fluid, with components $u_m^1, \ldots, u_m^d$, by ρ_m its density, and by $\mu(\rho_m)$ its viscosity. We assume that the density ρ_m of each fluid is constant, and that μ is a C^1–function. Furthermore, each fluid is incompressible, that is,

$$\operatorname{div} \mathbf{u}_m = 0 \qquad \text{in } \Omega_m(t),\ a.e.\ t \in (0, T].$$

We define the velocity $\mathbf{u}$ and the density ρ, globally in $\Omega \times (0, T]$, by setting

$$\mathbf{u}(x, t) = \mathbf{u}_m(x, t) \quad \text{and} \quad \rho(x, t) = \rho_m(x, t),$$

for $x \in \Omega_m(t)$ and $1 \leq m \leq M$.

Now, let us establish the system of equations for $\mathbf{u}$ and ρ. For $1 \leq m \leq M$, let

$$T(e(\mathbf{u}_m)) = \nu(\kappa + |e(\mathbf{u}_m)|)^{p-2} e(\mathbf{u}_m),$$

where $e(\mathbf{u}_m) = \frac{1}{2}\left(\nabla \mathbf{u}_m + (\nabla \mathbf{u}_m)^T\right)$ is the symmetric part of the velocity gradient, $1 < p < \infty$, $\kappa \geq 0$, and $\nu > 0$. We define the stress tensor

$$\sigma_m = 2\,\mu(\rho_m)\,T(e(\mathbf{u}_m)) - \pi_m\,\mathrm{Id}\,,$$

where π_m is the pressure. The balance of momentum for the quasi–steady Stokes flow is given by

$$-\operatorname{div} \sigma_m = \rho_m\,\mathbf{f}_m\,,$$

where $\mathbf{f}_m = (f_m^1, \ldots, f_m^d)^T$ is a given body force. Furthermore, the mathematical formulation of the physical principle of mass conservation is expressed by the continuity equation

$$\partial_t \rho_m + \operatorname{div}(\rho_m \mathbf{u}_m) = 0.$$

We only consider immiscible fluids, *i.e.*, $\mathbf{u}_m \cdot \mathbf{n}_m = 0$ holds on the interfaces between the fluids, where $\mathbf{n}_m$ is the outward normal of $\partial\Omega_m(t)$, $t \in (0, T]$. The immiscibility property is equivalent to the fact that

$$\partial_t \mu(\rho) + \mathbf{u} \cdot \nabla \mu(\rho) = 0 \qquad \text{in } \Omega \times (0, T],$$

if the interfaces are smooth; cf. [16]. This equation is satisfied if $\rho_t + \operatorname{div}(\rho\,\mathbf{u}) = 0$, $\operatorname{div} \mathbf{u} = 0$, and μ is a C^1–function. Thus, we arrive at the following quasi–steady Stokes system:

$$\begin{aligned} -\operatorname{div}\left(2\,\mu(\rho)\,T(e(\mathbf{u}))\right) + \nabla\pi &= \rho\,\mathbf{f} &&\text{in } \Omega \times (0, T], &&(2.1)\\ \rho_t + \operatorname{div}(\rho\,\mathbf{u}) &= 0 &&\text{in } \Omega \times (0, T], &&(2.2)\\ \operatorname{div} \mathbf{u} &= 0 &&\text{in } \Omega \times (0, T], &&(2.3) \end{aligned}$$

where $T(e(\mathbf{u})) = \nu(\kappa + |e(\mathbf{u})|)^{p-2} e(\mathbf{u})$ and $e(\mathbf{u}) = \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right)$ are $d \times d$–matrices and $\mathbf{f}$ is defined globally as before.

We consider the system (2.1)–(2.3) under the initial condition

$$\rho(x,0) = \rho_0 \qquad \text{in } \Omega, \tag{2.4}$$

where $\rho_0(x) \in \{\rho_1, \dots, \rho_M\}$, a.e. in Ω, and the no–slip boundary condition

$$\mathbf{u} = 0 \qquad \text{on } \partial\Omega \times (0,T]. \tag{2.5}$$

We refer to (2.1)–(2.5) as problem $(\mathcal{P}_\mathbf{1})$ and our aim is to show the existence of M fluids, with constant densities $\rho_1, \dots, \rho_M$, satisfying $(\mathcal{P}_\mathbf{1})$ in the sense of the following

Definition 2.1: *A weak solution of problem $(\mathcal{P}_\mathbf{1})$ is a triple $(\mathbf{u}, \rho, \pi)$ such that*

$$\mathbf{u} \in L^\infty\left(0,T; W_0^{1,p}(\Omega;\mathbb{R}^d)\right), \quad \operatorname{div}\mathbf{u} = 0;$$

$$\rho \in L^\infty(\Omega_T), \quad \pi \in L^\infty(0,T;L^{p'}(\Omega));$$

$$\int_\Omega 2\,\mu(\rho)\,T(e(\mathbf{u})) : e(\mathbf{w}) - \int_\Omega \pi\,\nabla\cdot\mathbf{w} = \int_\Omega \rho\,\mathbf{f}\cdot\mathbf{w}, \quad \forall \mathbf{w} \in W_0^{1,p}(\Omega;\mathbb{R}^d),$$

for a.e. $t \in (0,T]$; and ρ is a renormalized solution of the initial-value problem

$$\begin{aligned} \rho_t + \operatorname{div}(\rho\,\mathbf{u}) &= 0 \qquad \text{in } \Omega\times(0,T], \\ \rho(x,0) &= \rho_0 \qquad \text{in } \Omega. \end{aligned}$$

We need the following set of assumptions on the data:

- **(i)** $\Omega \subset \mathbb{R}^d$ ($d \geq 2$) is a smooth bounded domain.
- **(ii)** $1 < p < \infty$, $\kappa \geq 0$, and $\nu > 0$.
- **(iii)** $\rho_i > 0$, $i = 1, \dots, M$.
- **(iv)** $\mu \in C^1(\mathbb{R})$ and $\mu \geq \mu_0 > 0$.
- **(v)** $\mathbf{f} \in L^{p'}(\Omega;\mathbb{R}^d)$.

Theorem 2.1: Under the previous assumptions, there exists a weak solution $(\mathbf{u}, \rho, \pi)$ of problem $(\mathcal{P}_\mathbf{1})$ in the sense of Definition 2.1. Moreover, there is a constant c, depending only on the data, such that

$$\|\mathbf{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\pi\|_{L^\infty(0,T;L^{p'}(\Omega))} \leq c,$$

and $\rho(x,t) \in \{\rho_1, \dots, \rho_M\}$, a.e. in $\Omega \times (0,T]$.

Remark 2.1: If $M = 1$, the system is the well-known p–Stokes system for incompressible fluids with constant densities. Various regularity results are available; see, for instance, [6, 7, 8], where steady flows are treated.

Remark 2.2: Our method of proof can be applied to the full multi–fluid system of incompressible fluids with shear-dependent viscosities. Due to the low regularity of the convective term this leads to restrictions concerning the range of p.

Remark 2.3: In Section 7, we prove similar results for the no–stick boundary condition

$$\begin{aligned} \mathbf{u}\cdot\mathbf{n} &= 0 \qquad \text{on } \partial\Omega, \\ \mathbf{n}\cdot\sigma(\mathbf{u},\pi)\cdot\mathbf{t} &= 0 \qquad \text{on } \partial\Omega, \text{ for all } \mathbf{t}\in M_{\mathbf{n}}\,, \end{aligned}$$

where $\mathbf{n}$ is the outward normal of $\partial\Omega$, $\sigma(\mathbf{u},\pi) = 2\,\mu(\rho)\,T(e(\mathbf{u})) - \pi\,\mathrm{Id}$, and

$$M_{\mathbf{n}} = \{\mathbf{t}\in\mathbb{R}^d : \mathbf{t}\cdot\mathbf{n} = 0, \text{ on } \partial\Omega\}.$$

If $p = 2$ this is the classical slip–condition.

We conclude this section by fixing some notation. The space-time cylinder is denoted by $\Omega_T = \Omega\times(0,T)$, $\mathbf{u} = (u_1,\ldots,u_d)^T$ is a vector field, and $\partial_i = \frac{\partial}{\partial x_i}$. The Euclidean scalar products in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d}$ are denoted by $\mathbf{u}\cdot\mathbf{v}$ and $\nabla\mathbf{u} : \nabla\mathbf{v}$, respectively, and (f,g) is the $L^2(\Omega)$–scalar product. We use the usual notation for Sobolev spaces, and c is a constant which is allowed to vary from equation to equation.

3. The continuity equation

Existence and uniqueness results in the context of the Cauchy problem for the transport equation follow from the method of characteristics in the classical setting of a velocity field $\mathbf{w}\in L^1\left(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)\right)$. For less regular velocity fields, as is the case of the coupled problem we are considering, for which the natural assumption is $\mathbf{w}\in L^\infty\left(0,T;W^{1,p}(\Omega;\mathbb{R}^d)\right)$, we have to resort to the theory of renormalized solutions introduced by DiPerna and Lions in their celebrated paper [5]. They only considered the case of an equation defined in the whole space $\mathbb{R}^d$ but included a remark about the possible extension of the results for bounded smooth domains $\Omega\subset\mathbb{R}^d$ and a velocity field satisfying $\mathbf{w}\in L^1\left(0,T;W^{1,1}(\Omega;\mathbb{R}^d)\right)$ and the condition

$$\mathbf{w}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

which prevents the need to use boundary conditions. For this more complex case, the extension was pursued in [16].

Define the solenoidal vector spaces

$$V = \left\{\mathbf{v}\in W^{1,p}(\Omega;\mathbb{R}^d) \;:\; \operatorname{div}\mathbf{v} = 0 \quad \text{and} \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on } \partial\Omega\right\} \tag{3.1}$$

$$V_0 = \left\{ \mathbf{v} \in W_0^{1,p}(\Omega; \mathbb{R}^d) \ : \ \mathrm{div}\, \mathbf{v} = 0 \right\} \tag{3.2}$$

and consider the initial value problem for the transport equation

$$\rho_t + \mathrm{div}\,(\rho\, \mathbf{w}) = 0 \qquad \text{in } \Omega \times (0, T], \tag{3.3}$$

$$\rho(x, 0) = \rho_0 \qquad \text{in } \Omega. \tag{3.4}$$

Definition 3.1: *Let* $\mathbf{w} \in L^\infty(0, T; V)$. *A weak solution of (3.3)–(3.4) is a function* $\rho \in L^\infty(\Omega_T)$ *such that*

$$\int_0^T \int_\Omega \rho \left(\frac{\partial \varphi}{\partial t} + \mathbf{w} \cdot \nabla \varphi \right) = \int_\Omega \rho_0 \varphi(0), \qquad \forall\, \varphi \in C^\infty(\Omega_T), \ \textit{with } \varphi(T) = 0.$$

A renormalized solution of (3.3)–(3.4) is a function $\rho \in L^\infty(\Omega_T)$ *such that, for any* $\beta \in C^1(\mathbb{R})$, $\beta(\rho)$ *is a weak solution of (3.3)–(3.4) for the initial datum* $\beta(\rho_0)$.

The following existence result is proved in [16, section 4].

Theorem 3.1: For any given vector field $\mathbf{w} \in L^\infty(0, T; V_0)$, and any initial datum $\rho_0 \in L^\infty(\Omega)$ such that $\rho_0 \in \{\rho_1, \ldots, \rho_M\}$, a.e. in Ω, there exists a unique weak solution of (3.3)–(3.4). Moreover, this weak solution is a renormalized solution and satisfies

$$\rho(x, t) \in \{\rho_1, \ldots, \rho_M\}, \qquad \text{a.e. in } \Omega \times (0, T]. \tag{3.5}$$

Remark 3.1: It is a simple matter to obtain (3.5); cf. [16]. Indeed, let β be a $C^1(\mathbb{R})$–function, $\beta(s) = 0$ for $s = \rho_1, \ldots, \rho_M$, and $\beta > 0$ elsewhere. Then, $\beta(\rho)$ is a weak solution of (3.3)–(3.4). Since $\beta(\rho_0) = 0$, it follows from the uniqueness that $\beta(\rho) = 0$, a.e. in Ω_T. This yields (3.5).

We conclude this section by stating the following result, which is a straightforward extension to the L^p setting of Corollary 5.1 in [16].

Lemma 3.1: *Let* $\mathbf{w}_i \rightharpoonup \mathbf{w}$ *weakly–$*$ in* $L^\infty(0, T; W^{1,p}(\Omega; \mathbb{R}^d))$. *If* ρ_i *and* ρ *are the associated renormalized solutions of* (3.3)–(3.4)*, then* $\rho_i \to \rho$ *strongly in* $L^p(\Omega_T)$.

4. The approximated Stokes problem

We make a Galerkin ansatz. The space V_0, defined by (3.2), is separable. Thus, there is a set of divergence free $W^{1,\infty}(\Omega; \mathbb{R}^d)$–functions $\mathbf{y}_k$, $k = 1, 2, \ldots$, that is dense in V_0. We now introduce the space

$$S_n = \left\{ \mathbf{v} \in L^\infty(0,T;V_0) \ : \ \mathbf{v}(x,t) = \sum_{k=1}^{n} \alpha_k(t)\, \mathbf{y}_k(x), \ \alpha_k \in L^\infty(0,T) \right\}.$$

For a given $\rho \in L^\infty(\Omega_T)$, we are looking for a function $\mathbf{w}^n(x,t) \in S_n$ solving the algebraic system

$$(2\,\mu(\rho)\,T(e(\mathbf{w}^n)), e(\mathbf{v}^n)) = (\rho\,\mathbf{f}, \mathbf{v}^n), \qquad \forall \mathbf{v}^n \in S_n, \ \text{a.e. } t \in (0,T]\,. \tag{4.1}$$

Before proving the existence and uniqueness of a solution, let us state some useful inequalities.

Lemma 4.1: *Let* $\mathbf{v}, \mathbf{w} \in W^{1,p}(\Omega;\mathbb{R}^d)$. *For each* $1 < p < 2$*, there exists a constant* $c = c(\|e(\mathbf{v})\|_{L^p(\Omega)}\,, \|e(\mathbf{w})\|_{L^p(\Omega)}) > 0$ *such that*

$$\|e(\mathbf{v}) - e(\mathbf{w})\|^2_{L^p(\Omega)} \le c \int_\Omega (T(e(\mathbf{v})) - T(e(\mathbf{w}))) : (e(\mathbf{v}) - e(\mathbf{w}))\,. \tag{4.2}$$

For each $2 < p < \infty$*, there exists a constant* $c' > 0$ *such that*

$$\|e(\mathbf{v}) - e(\mathbf{w})\|^p_{L^p(\Omega)} \le c' \int_\Omega (T(e(\mathbf{v})) - T(e(\mathbf{w}))) : (e(\mathbf{v}) - e(\mathbf{w}))\,. \tag{4.3}$$

Proof: For $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{d\times d}$, and $1 < p < \infty$, there exists a constant $c > 0$ such that

$$\int_0^1 (\kappa + |t\mathbf{A} + (1-t)\mathbf{B}|)^{p-2} dt \ge c(\kappa + |\mathbf{A}| + |\mathbf{B}|)^{p-2}\,; \tag{4.4}$$

cf. [4]. Using Taylor's expansion and (4.4), it easily follows that there exists a constant $c_p > 0$, depending on p, such that

$$\begin{aligned} &\big((\kappa + |\mathbf{A}|)^{p-2}\mathbf{A} - (\kappa + |\mathbf{B}|)^{p-2}\mathbf{B}\big) : (\mathbf{A} - \mathbf{B}) \\ &\ge \ c_p(\kappa + |\mathbf{A}| + |\mathbf{B}|)^{p-2}|\mathbf{A} - \mathbf{B}|^2. \end{aligned} \tag{4.5}$$

The Hölder inequality yields (with $p_1 = \frac{2}{p}$ and $p_2 = \frac{2}{2-p}$)

$$\begin{aligned}
&\int_\Omega |e(\mathbf{v}) - e(\mathbf{w})|^p \\
&= \int_\Omega (\kappa + |e(\mathbf{v})| + |e(\mathbf{w})|)^{\frac{(p-2)p}{2}} |e(\mathbf{v}) - e(\mathbf{w})|^p (\kappa + |e(\mathbf{v})| + |e(\mathbf{w})|)^{\frac{(2-p)p}{2}} \\
&\leq \Big(\int_\Omega (\kappa + |e(\mathbf{v})| + |e(\mathbf{w})|)^{p-2} |e(\mathbf{v}) - e(\mathbf{w})|^2 \Big)^{\frac{p}{2}} \times \\
&\quad \times \Big(\int_\Omega (\kappa + |e(\mathbf{v})| + |e(\mathbf{w})|)^p \Big)^{\frac{2-p}{2}} .
\end{aligned}$$

Using estimate (4.5), the assertion (4.2) follows. Furthermore, for $2 < p < \infty$, we find

$$|\mathbf{A} - \mathbf{B}|^p = |\mathbf{A} - \mathbf{B}|^{p-2} |\mathbf{A} - \mathbf{B}|^2 \leq (\kappa + |\mathbf{A}| + |\mathbf{B}|)^{p-2} |\mathbf{A} - \mathbf{B}|^2.$$

Taking (4.5) into account we obtain assertion (4.3). ∎

We now solve problem (4.1).

Lemma 4.2: *Given $\rho \in L^\infty(\Omega_T)$, there exists a unique solution $\mathbf{w}^n \in S_n$ of (4.1). Moreover, the following a priori estimate holds:*

$$\|\mathbf{w}^n\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq c, \tag{4.6}$$

where $c = c\,(data, \|\rho\|_\infty)$ is independent of n.

Proof: Note that $\mu \geq \mu_0 > 0$. Using Lemma 4.1, we estimate, for $\mathbf{w}_1^n, \mathbf{w}_2^n \in S_n$,

$$\int_\Omega 2\,\mu(\rho)\;(T(e(\mathbf{w}_1^n)) - T(e(\mathbf{w}_2^n))) : (e(\mathbf{w}_1^n) - e(\mathbf{w}_2^n)) \geq c\,\|e(\mathbf{w}_1^n) - e(\mathbf{w}_2^n)\|_{L^p(\Omega)}^{\alpha(p)},$$

where $\alpha(p) = 2$ if $p < 2$, and $\alpha(p) = p$ if $p > 2$. From Korn's second inequality [9],

$$\exists\, c > 0 \;:\; \|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq c\,\|e(\mathbf{v})\|_{L^p(\Omega)}, \quad \forall\, \mathbf{v} \in W_0^{1,p}(\Omega;\mathbb{R}^d),$$

we obtain the monotonicity, putting $\mathbf{v} = \mathbf{w}_1^n - \mathbf{w}_2^n$. This implies, using classical results concerning monotone operators, the existence and uniqueness of a solution. Testing equation (4.1) with $\mathbf{w}^n$, we deduce, for a.e. $t \in (0, T]$,

$$2\mu_0\nu \int_\Omega (\kappa + |e(\mathbf{w}^n)|)^{p-2} |e(\mathbf{w}^n)|^2 \leq \|\rho\|_\infty \|\mathbf{f}\|_{L^{p'}(\Omega)} \|\mathbf{w}^n\|_{L^p(\Omega)} . \tag{4.7}$$

Due to Poincaré's inequality and Korn's second inequality, estimate (4.6) follows. ∎

The following is a stability result for problem (4.1).

Lemma 4.3: *If* $(\rho_i, \mathbf{w}_i^n) \in L^\infty(\Omega_T) \times S_n$ *solve* (4.1) *and* $\rho_i \to \rho$ *in* $L^p(\Omega_T)$ *and* $\mathbf{w}_i^n \rightharpoonup \mathbf{w}^n$ *weakly–$*$ in* $L^\infty(0,T;V_0)$*, then* $(\rho, \mathbf{w}^n)$ *also solve* (4.1)*.*

Proof: Due to the assumed convergence, the sequence $\mathbf{w}_i^n$ is uniformly bounded in $L^\infty(0,T;V_0)$ and we have

$$\|\mathbf{w}_i^n - \mathbf{w}^n\|_{L^p(\Omega)} \to 0, \quad \text{a.e. } t \in (0,T],$$

as $i \to \infty$. Using the representations $\mathbf{w}_i^n = \sum_{k=1}^n \alpha_k^i \, \mathbf{y}_k$ and $\mathbf{w}^n = \sum_{k=1}^n \alpha_k \, \mathbf{y}_k$ we may conclude that

$$\alpha_k^i(t) \to \alpha_k(t), \quad \text{a.e. } t \in (0,T], \qquad \forall k \in \{1,\ldots,n\}.$$

Therefore,

$$\|T(e(\mathbf{w}_i^n)) - T(e(\mathbf{w}^n))\|_{L^{p'}(\Omega)} \to 0, \quad \text{a.e. } t \in (0,T].$$

Noting that $\mu(\rho_i) \to \mu(\rho)$ in $L^p(\Omega_T)$, it follows that, for any $\mathbf{v}^n \in S_n$,

$$(2\,\mu(\rho_i)\,T(e(\mathbf{w}_i^n)), e(\mathbf{v}^n)) \to (2\,\mu(\rho)\,T(e(\mathbf{w}^n)), e(\mathbf{v}^n)), \quad \text{a.e. } t \in (0,T],$$

and also

$$(\rho_i\,\mathbf{f}, \mathbf{v}^n) \to (\rho\,\mathbf{f}, \mathbf{v}^n), \quad \text{a.e. } t \in (0,T].$$

The conclusion follows. ∎

5. The fixed–point argument

The purpose of this section is to prove the existence, for each $n \in \mathbb{N}$, of a solution to the following approximated problem:

$(\mathcal{P}_\mathbf{1}^\mathbf{n})$ Find $(\mathbf{u}^n, \rho^n) \in S_n \times L^\infty(\Omega_T)$ such that

$$(2\,\mu(\rho^n)\,T(e(\mathbf{u}^n)), e(\mathbf{v}^n)) = (\rho^n \mathbf{f}, \mathbf{v}^n)\,, \quad \forall \mathbf{v}^n \in S_n, \text{ a.e. } t \in (0,T]; \tag{5.1}$$

and ρ^n is a renormalized solution of

$$\rho_t^n + \operatorname{div}(\rho^n\,\mathbf{u}^n) = 0 \quad \text{in } \Omega \times (0,T]; \tag{5.2}$$

$$\rho^n(x,0) = \rho_0 \quad \text{in } \Omega, \tag{5.3}$$

where $\rho_0 \in \{\rho_1, \ldots, \rho_M\}$, a.e. in Ω.

The solution will be obtained, using Schauder's theorem, as a fixed–point of a nonlinear mapping $\mathcal{T}$, defined in the closed and convex set

$$K = \{\varrho \in L^p(\Omega_T) \;:\; \rho_1 \le \varrho(x,t) \le \rho_M, \text{ a.e. in } \Omega_T\}$$

of the Banach space $L^p(\Omega_T)$.

Given $\varrho \in K$, solve the Stokes problem obtained from (5.1) by replacing ρ^n with ϱ. Since $\varrho \in L^\infty(\Omega_T)$, it follows from Lemma 4.2 that there exists a unique solution $\mathbf{u}^n \in S_n$. For this $\mathbf{u}^n$, solve the initial value problem (5.2)–(5.3) for the transport equation using Theorem 3.1. Since $\mathbf{u}^n \in S_n \subset L^\infty(0,T;V_0)$, we obtain a unique $\rho^n \in L^\infty(\Omega_T)$. We finally define $\mathcal{T}(\varrho) := \rho^n$. From (3.5), it is apparent that $\mathcal{T}(K) \subset K$ so it remains to be proved that $\mathcal{T}$ is continuous and that $\mathcal{T}(K)$ is precompact.

$\mathcal{T}$ is continuous: Let $\varrho_i \in K$ be a sequence such that

$$\varrho_i \to \varrho_0 \quad \text{strongly in } L^p(\Omega_T) \tag{5.4}$$

for some function $\varrho_0 \in L^p(\Omega_T)$. Consider the sequence of solutions $(\mathbf{u}_i^n)_i \in S_n$ of the Stokes problem (5.1) corresponding to the choice $\rho^n = \varrho_i$. Since the constant c in (4.6) only depends on ρ through its L^∞ norm, we obtain the uniform bound $\|\mathbf{u}_i^n\|_{L^\infty(0,T;V_0)} \leq c$. We can then extract a subsequence, still denoted by $(\mathbf{u}_i^n)_i$, such that

$$\mathbf{u}_i^n \rightharpoonup \mathbf{u}^n \quad \text{weakly}-* \text{ in } L^\infty(0,T;V_0) \tag{5.5}$$

for some function $\mathbf{u}^n \in L^\infty(0,T;V_0)$. Using Lemma 4.3, and the convergences (5.4) and (5.5), we conclude that $\mathbf{u}^n$ is the solution of the Stokes problem (5.1) corresponding to the choice $\rho^n = \varrho_0$.

Finally, we consider the solutions of the initial value problem for the transport equation corresponding to $\mathbf{u}_i^n$ and $\mathbf{u}^n$, *i.e.*, $\mathcal{T}(\varrho_i)$ and $\mathcal{T}(\varrho_0)$. By Lemma 3.1, we obtain that

$$\mathcal{T}(\varrho_i) \to \mathcal{T}(\varrho_0) \quad \text{strongly in } L^p(\Omega_T)$$

and the conclusion follows.

$\mathcal{T}(K)$ is precompact: Take an arbitrary sequence $\varrho_i \in K$. The corresponding solutions of the Stokes problem form a bounded sequence in $L^\infty(0,T;V_0)$, so we can extract a subsequence weakly converging to some $\mathbf{u} \in L^\infty(0,T;V_0)$. Again by Lemma 3.1, we conclude that, up to a subsequence, $\mathcal{T}(\varrho_i)$ strongly converges in $L^p(\Omega_T)$ to the solution of (5.2)–(5.3) with velocity field $\mathbf{u}$.

We have just proved, as a consequence of Schauder's fixed–point theorem, the following

Theorem 5.1: For each $n \in \mathbb{N}$, there exists a weak solution $(\mathbf{u}^n, \rho^n) \in S_n \times L^\infty(\Omega_T)$ to the approximated problem $(\mathcal{P}_\mathbf{1}^\mathbf{n})$.

Remark 5.1: Weak solutions $(\mathbf{u}^n, \rho^n)$ of problem $(\mathcal{P}_\mathbf{1}^\mathbf{n})$ are uniformly bounded in $L^\infty(0,T;V_0) \times L^\infty(\Omega_T)$. In fact, (4.6) yields a constant c independent of n such that

$$\|\mathbf{u}^n\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq c,$$

and (3.5) implies that $\rho^n(x,t) \in \{\rho_1, \ldots, \rho_M\}$, a.e. in $\Omega \times (0,T]$.

6. Existence for problem $(\mathcal{P}_\mathbf{1})$

The aim of this section is to take the limit as $n \to \infty$ in problems $(\mathcal{P}_\mathbf{1}^\mathbf{n})$ and obtain existence for problem $(\mathcal{P}_\mathbf{1})$, thus proving Theorem 2.1. First, we discuss an analogue of problem $(\mathcal{P}_\mathbf{1})$ in the solenoidal vector space V_0.

Proposition 6.1: *There exists* $(\mathbf{u}, \rho) \in L^\infty(0,T;V_0) \times L^\infty(\Omega_T)$ *such that*

$$\int_\Omega 2\,\mu(\rho)\,T(e(\mathbf{u})) : e(\mathbf{v}) = \int_\Omega \rho\,\mathbf{f}\cdot\mathbf{v}, \qquad \forall \mathbf{v} \in V_0,\ \textit{a.e. } t \in (0,T]\,, \tag{6.1}$$

and ρ *is a renormalized solution of*

$$\begin{aligned} \rho_t + \mathrm{div}\,(\rho\,\mathbf{u}) &= 0 \qquad \text{in } \Omega \times (0,T], && (6.2)\\ \rho(x,0) &= \rho_0 \qquad \text{in } \Omega, && (6.3) \end{aligned}$$

satisfying $\rho(x,t) \in \{\rho_1, \ldots, \rho_M\}$*, a.e. in* $\Omega \times (0,T]$.

Proof: Theorem 5.1 yields a weak solution $(\mathbf{u}^n, \rho^n) \in S_n \times L^\infty(\Omega_T)$ to the approximated problem $(\mathcal{P}_\mathbf{1}^\mathbf{n})$, fulfilling $\rho^n(x,t) \in \{\rho_1, \ldots, \rho_M\}$, a.e. in $\Omega \times (0,T]$. Note that the sequence $(\mathbf{u}^n)_n$ is uniformly bounded in $L^\infty(0,T;V_0)$. Thus, there is a subsequence and a function $\mathbf{u} \in L^\infty(0,T;V_0)$ such that $\mathbf{u}^n \rightharpoonup \mathbf{u}$ weakly$-*$ in $L^\infty(0,T;V_0)$. Let $\mathbf{v}^n \in S_n$ be the best approximation of $\mathbf{u}$, that is,

$$\|\mathbf{v}^n - \mathbf{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq \|\mathbf{w}^n - \mathbf{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))}\,, \qquad \forall \mathbf{w}^n \in S_n.$$

Noting that $\mathbf{u}^n - \mathbf{v}^n$ is an admissible test function in equation (5.1) we get

$$\int_\Omega 2\mu(\rho^n)\,T(e(\mathbf{u}^n)) : (e(\mathbf{u}^n) - e(\mathbf{v}^n)) = \int_\Omega \rho^n\,\mathbf{f}\cdot(\mathbf{u}^n - \mathbf{v}^n)\,.$$

Therefore,

$$\begin{aligned} J_1 &:= \int_\Omega 2\mu(\rho^n)\,(T(e(\mathbf{u}^n)) - T(e(\mathbf{u}))) : (e(\mathbf{u}^n) - e(\mathbf{v}^n)) \\ &= \int_\Omega \rho^n\,\mathbf{f}\cdot(\mathbf{u}^n - \mathbf{v}^n) \\ &\quad - \int_\Omega 2\mu(\rho^n)\,T(e(\mathbf{u})) : (e(\mathbf{u}^n) - e(\mathbf{v}^n)) \\ &=: J_2 + J_3. \end{aligned}$$

Let us note that

$$\|\mathbf{v}^n - \mathbf{u}\|_{L^p(\Omega)} \to 0, \qquad \text{a.e. } t \in (0,T].$$

Moreover, there is a subsequence of $(\mathbf{u}^n)_n$, denoted again by $(\mathbf{u}^n)_n$, such that

$$\|\mathbf{u}^n - \mathbf{u}\|_{L^p(\Omega)} \to 0, \qquad \text{a.e. } t \in (0,T].$$

Therefore, we have

$$\|\mathbf{u}^n - \mathbf{v}^n\|_{L^p(\Omega)} \to 0, \qquad \text{a.e. } t \in (0,T].$$

Noting that $\mathbf{u}^n - \mathbf{v}^n \in S_n$ and recalling the simple form of S_n-functions this implies that

$$\|\mathbf{u}^n - \mathbf{v}^n\|_{W^{1,p}(\Omega)} \to 0, \qquad \text{a.e. } t \in (0,T]$$

(see the proof of Lemma 4.3). Due to the uniform boundedness of the sequences $(\rho^n)_n$ and $(\mu(\rho^n))_n$ in $L^\infty(\Omega_T)$ we deduce

$$\lim_{n\to\infty} J_2 = 0 \qquad \text{and} \qquad \lim_{n\to\infty} J_3 = 0,$$

and thus $\lim_{n\to\infty} J_1 = 0$. Next, we write

$$\begin{aligned} J_1 &= \int_\Omega 2\mu(\rho^n)\,(T(e(\mathbf{u}^n)) - T(e(\mathbf{u}))) : (e(\mathbf{u}^n) - e(\mathbf{u})) \\ &\quad - \int_\Omega 2\mu(\rho^n)\,(T(e(\mathbf{u}^n)) - T(e(\mathbf{u}))) : (e(\mathbf{v}^n) - e(\mathbf{u})) \\ &=: J_{11} + J_{12}\,, \end{aligned}$$

and estimate J_1 from below to show that also $\lim_{n\to\infty} J_{11} = 0$. Using Taylors' expansion we find

$$|T(e(\mathbf{u})) - T(e(\mathbf{u}^n))| \le c\,|e(\mathbf{u}) - e(\mathbf{u}^n)| \int_0^1 (\kappa + |te(\mathbf{u}) + (1-t)e(\mathbf{u}^n)|)^{p-2}\,dt\,.$$

Using the fact that (see [4])

$$\int_0^1 (\kappa + |t\mathbf{A} + (1-t)\mathbf{B}|)^{p-2} \leq c(\kappa + |\mathbf{A}| + |\mathbf{B}|)^{p-2},$$

and noting that $|\mathbf{A}| + |\mathbf{A} - \mathbf{B}| \leq 2(|\mathbf{A}| + |\mathbf{B}|) \leq 4(|\mathbf{A}| + |\mathbf{A} - \mathbf{B}|)$, we obtain

$$\begin{aligned} &|T(e(\mathbf{u})) - T(e(\mathbf{u}^n))|\,|e(\mathbf{u}) - e(\mathbf{v}^n)| \\ &\quad \leq\ c\,(\kappa + |e(\mathbf{u})| + |e(\mathbf{u}) - e(\mathbf{u}^n)|)^{p-2}\,|e(\mathbf{u}) - e(\mathbf{u}^n)|\,|e(\mathbf{u}) - e(\mathbf{v}^n)|\,. \end{aligned}$$

Now we apply the following Young–like inequality that can be found in [2]: for any $\varepsilon > 0$, there is a constant $c_\varepsilon > 0$ such that, for all $\lambda, a, b \geq 0$,

$$(\lambda + a)^{p-2} a\, b \leq \varepsilon(\lambda + a)^{p-2} a^2 + c_\varepsilon(\lambda + b)^{p-2} b^2.$$

Putting $\lambda = \kappa + |e(\mathbf{u})|$, $a = |e(\mathbf{u}) - e(\mathbf{u}^n)|$, and $b = |e(\mathbf{u}) - e(\mathbf{v}^n)|$ we get

$$\begin{aligned} &(\kappa + |e(\mathbf{u})| + |e(\mathbf{u}) - e(\mathbf{u}^n)|)^{p-2}\,|e(\mathbf{u}) - e(\mathbf{u}^n)|\,|e(\mathbf{u}) - e(\mathbf{v}^n)| \\ &\quad \leq\ \varepsilon(\kappa + |e(\mathbf{u})| + |e(\mathbf{u}) - e(\mathbf{u}^n)|)^{p-2}\,|e(\mathbf{u}) - e(\mathbf{u}^n)|^2 \\ &\qquad + c_\varepsilon(\kappa + |e(\mathbf{u})| + |e(\mathbf{u}) - e(\mathbf{v}^n)|)^{p-2}\,|e(\mathbf{u}) - e(\mathbf{v}^n)|^2\,. \end{aligned}$$

Using estimate (4.5), we have

$$\begin{aligned} &(\kappa + |\mathbf{A}| + |\mathbf{A} - \mathbf{B}|)^{p-2}|\mathbf{A} - \mathbf{B}|^2 \\ &\quad \leq\ c\,(\kappa + |\mathbf{A}| + |\mathbf{B}|)^{p-2}|\mathbf{A} - \mathbf{B}|^2 \\ &\quad \leq\ c\,((\kappa + |\mathbf{B}|)^{p-2}\mathbf{B} - (\kappa + |\mathbf{A}|)^{p-2}\mathbf{A}) : (\mathbf{B} - \mathbf{A})\,. \end{aligned}$$

Altogether, we conclude that, for any $\varepsilon > 0$,

$$\begin{aligned} |J_{12}| \ \leq\ & \varepsilon 2\bar{\mu} \int_\Omega (T(e(\mathbf{u}^n)) - T(e(\mathbf{u}))) : (e(\mathbf{u}^n) - e(\mathbf{u})) \\ & + c_\varepsilon 2\bar{\mu} \int_\Omega (T(e(\mathbf{v}^n)) - T(e(\mathbf{u}))) : (e(\mathbf{v}^n) - e(\mathbf{u}))\,, \end{aligned}$$

where $\bar{\mu} = \max_{1\leq k\leq M} \mu(\rho_k)$. Putting $\underline{\mu} = \min_{1\leq k\leq M} \mu(\rho_k)$ and $\varepsilon = \underline{\mu}(2\bar{\mu})^{-1}$ we deduce

$$\frac{1}{2} J_{11} \leq J_1 + c\int_\Omega (T(e(\mathbf{v}^n)) - T(e(\mathbf{u}))) : (e(\mathbf{v}^n) - e(\mathbf{u}))$$

and, in fact, $\lim_{n\to\infty} J_{11} = 0$. Estimating $\mu(\rho^n) \geq \underline{\mu}$ we infer

$$\lim_{n\to\infty} \int_\Omega (T(e(\mathbf{u}^n)) - T(e(\mathbf{u}))) : (e(\mathbf{u}^n) - e(\mathbf{u})) = 0. \tag{6.4}$$

Due to Lemma 4.1 we conclude that

$$\lim_{n\to\infty} \|e(\mathbf{u}^n) - e(\mathbf{u})\|_{L^p(\Omega)} = 0, \qquad \text{a.e. } t \in (0,T]\,. \tag{6.5}$$

Applying Korn's second inequality it follows that

$$\nabla \mathbf{u}^n \to \nabla \mathbf{u}, \qquad \text{a.e. in } \Omega_T\,.$$

Now let ρ be the renormalized solution of the continuity equation associated with $\mathbf{u}$. Lemma 3.1 implies that

$$\rho^n \to \rho \qquad \text{strongly in } L^p(\Omega_T)\,.$$

Utilizing Vitali's convergence theorem we obtain

$$\lim_{n\to\infty} (2\,\mu(\rho^n)\,T(e(\mathbf{u}^n)), e(\mathbf{v}^n)) = (2\,\mu(\rho)\,T(e(\mathbf{u})), e(\mathbf{v})) \quad \forall \mathbf{v} \in L^\infty(0,T;V_0)\,,$$

as well as $(\rho^n\,\mathbf{f}, \mathbf{v}^n) \to (\rho\,\mathbf{f}, \mathbf{v})$. This yields the assertion. ■

We finally prove our main result.

Proof of Theorem 2.1: Due to Proposition 6.1, there exists a weak solution $(\mathbf{u}, \rho)$ of (6.1)–(6.3). Moreover, $\rho(x,t) \in \{\rho_1, \ldots, \rho_M\}$, a.e. in $\Omega \times (0,T]$, and there is a constant c, depending only on the data, such that

$$\|\mathbf{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))} \leq c. \tag{6.6}$$

For $t \in (0,T]$, let us introduce the functional $F \in W^{-1,p'}(\Omega;\mathbb{R}^d)$ by setting

$$\langle F, \mathbf{w}\rangle := -(\text{div}\,(2\,\mu(\rho)\,T(e(\mathbf{u}))), \mathbf{w}) - (\rho\,\mathbf{f}, \mathbf{w}).$$

We have

$$\langle F, \mathbf{w}\rangle = 0, \qquad \forall \mathbf{w} \in V_0,\ \text{a.e. } t \in (0,T].$$

Applying a simplification of De Rahm's theorem [1, Lemma. 2.7], we obtain a function $\pi \in L^{p'}(\Omega)$ such that

$$F = -\nabla\pi, \qquad \text{a.e. } t \in (0,T].$$

Moreover, noting that $\|F\|_{W^{-1,p'}(\Omega)} \leq c$, uniformly in t, due to (6.6), we get

$$\|\nabla\pi\|_{W^{-1,p'}(\Omega)} \leq c', \qquad \text{a.e. } t \in (0,T].$$

In view of the following estimate of Nečas [13],

$$\|v\|_{L^q(\Omega)} \leq c_0 \left(\sum_{i=1}^d \|\partial_i v\|_{W^{-1,q}(\Omega)} + \|v\|_{W^{-1,q}(\Omega)} \right),$$

valid for any $1 < q < \infty$, and any distribution v such that $\partial_i v \in W^{-1,q}(\Omega)$, $i = 0, 1, \ldots, d$, we conclude that there is a generic constant c, depending only on the data, such that

$$\|\pi\|_{L^{p'}(\Omega)} \le c, \qquad \text{a.e. } t \in (0, T].$$

This completes the proof. ∎

7. No-stick boundary condition

In this final section, we extend our results to the case of the no–stick boundary condition

$$\begin{aligned} \mathbf{u} \cdot \mathbf{n} = 0 \qquad & \text{on } \partial\Omega, \\ \mathbf{n} \cdot \sigma(\mathbf{u}, \pi) \cdot \mathbf{t} = 0 \qquad & \text{on } \partial\Omega, \text{ for all } \mathbf{t} \in M_{\mathbf{n}}, \end{aligned} \tag{7.1}$$

where $\mathbf{n}$ is the outward normal of $\partial\Omega$, $\sigma(\mathbf{u}, \pi) = 2\,\mu(\rho)\,T(e(\mathbf{u})) - \pi\,\text{Id}$, and

$$M_{\mathbf{n}} = \{\mathbf{t} \in \mathbb{R}^d : \mathbf{t} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\}.$$

We refer to (2.1)–(2.4), (7.1) as problem $(\mathcal{P}_2)$.

Recall definition (3.1) of the solenoidal vector space V and let us introduce

$$W = \left\{\mathbf{w} \in W^{1,p}(\Omega; \mathbb{R}^d) : \mathbf{w} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\right\}.$$

Since the Green formula

$$\int_\Omega (-\text{div}\,(2\,\mu(\rho)\,T(e(\mathbf{u}))) + \nabla\pi) \cdot \mathbf{w} = \int_\Omega 2\,\mu(\rho)\,T(e(\mathbf{u})) : e(\mathbf{w})$$
$$- \int_\Omega \pi\,\nabla \cdot \mathbf{w} - \int_{\partial\Omega} \mathbf{n} \cdot (2\,\mu(\rho)\,T(e(\mathbf{u})) - \pi\,\text{Id}) \cdot \mathbf{w}$$

holds for all $\mathbf{w} \in W^{1,p}(\Omega; \mathbb{R}^d)$, we obtain the following

Definition 7.1: *A weak solution of problem $(\mathcal{P}_2)$ is a triple $(\mathbf{u}, \rho, \pi)$ such that*

$$\mathbf{u} \in L^\infty(0, T; V), \quad \rho \in L^\infty(\Omega_T), \quad \pi \in L^\infty(0, T; L^{p'}(\Omega));$$

$$\int_\Omega 2\,\mu(\rho)\,T(e(\mathbf{u})) : e(\mathbf{w}) - \int_\Omega \pi\,\nabla \cdot \mathbf{w} = \int_\Omega \rho\,\mathbf{f} \cdot \mathbf{w}, \quad \forall \mathbf{w} \in W, \text{ a.e. } t \in (0, T];$$

and ρ is a renormalized solution of

$$\begin{aligned} \rho_t + \text{div}\,(\rho\,\mathbf{u}) &= 0 \qquad \text{in } \Omega \times (0, T], \\ \rho(x, 0) &= \rho_0 \qquad \text{in } \Omega. \end{aligned}$$

In order to prove existence for this problem we need a version of Korn's second inequality for the type of boundary conditions we are considering. Since a proof seems to be missing in the literature, we include it here for the sake of completeness. We feel the result is interesting in its own right.

Lemma 7.1: *For any $1 < p < \infty$, there exists a constant c such that*

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq c \, \|e(\mathbf{v})\|_{L^p(\Omega)} \, , \tag{7.2}$$

for all $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$, and $\mathbf{n}\cdot T(e(\mathbf{v}))\cdot\mathbf{t} = 0$ on $\partial\Omega$, $\forall \mathbf{t} \in M_{\mathbf{n}}$.

Proof: It is known that

$$\|\mathbf{v}\|_{W^{1,p}(\Omega)} \leq c \left(\|\mathbf{v}\|_{L^p(\Omega)} + \|e(\mathbf{v})\|_{L^p(\Omega)} \right); \tag{7.3}$$

cf. [9]. It remains to show that

$$\|\mathbf{v}\|_{L^p(\Omega)} \leq c' \, \|e(\mathbf{v})\|_{L^p(\Omega)} \, . \tag{7.4}$$

We argue by contradiction and assume that estimate (7.4) is false. Thus, there exists a sequence $(\mathbf{v}^k)_k$ of functions $\mathbf{v}^k \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that

$$\forall k \geq 1, \quad \left\|\mathbf{v}^k\right\|_{L^p(\Omega)} = 1, \quad \text{and} \lim_{k\to\infty} \left\|e(\mathbf{v}^k)\right\|_{L^p(\Omega)} = 0.$$

Using (7.3), we find a subsequence and a limit function $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^d)$ such that $\mathbf{v}^n \rightharpoonup \mathbf{v}$ weakly in $W^{1,p}(\Omega;\mathbb{R}^d)$, $\mathbf{v}^n \to \mathbf{v}$ strongly in $L^p(\Omega;\mathbb{R}^d)$, and $e(\mathbf{v}) = 0$. Moreover, it is known that a function $\mathbf{v} \in W^{1,p}(\Omega;\mathbb{R}^d)$ satisfies $e(\mathbf{v}) = 0$ if and only if it is of the form $\mathbf{v}(x) = \mathbf{A}x + \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{d\times d}$ is skew–symmetric and $\mathbf{b} \in \mathbb{R}^d$; cf. [10]. Now let us consider a point $x \in \partial\Omega$ where $\mathbf{n}(x)$ is equal to the i-th unit vector in $\mathbb{R}^d$. Let v_i be the i-th component of $\mathbf{v}$. Then $\mathbf{n}\cdot T(e(\mathbf{v}))\cdot\mathbf{t} = 0$ implies that $\partial_j v_i = 0$, for all $1 \leq j \leq d$, $j \neq i$. Hence, it follows that $a_{ij} = 0$, for all $j \neq i$. Moreover, $\mathbf{A}$ is skew–symmetric; thus, we have $a_{ii} = 0$. We conclude that $\mathbf{A} = 0$. Furthermore, $\mathbf{v}\cdot\mathbf{n} = 0$ on $\partial\Omega$ yields $v_i = 0$ on $\partial\Omega$. Therefore, we also deduce $\mathbf{b} = 0$. Hence, it holds that $\mathbf{v} = 0$. This contradicts $\left\|\mathbf{v}^k\right\|_{L^p(\Omega)} = 1$ $\forall k \geq 1$. ■

We can then establish the following existence result.

Theorem 7.1: Under assumptions **(i)**–**(v)**, there exists a weak solution $(\mathbf{u}, \rho, \pi)$ of problem $(\mathcal{P}_\mathbf{2})$ in the sense of Definition 7.1. Moreover, there is a constant c, depending only on the data, such that

$$\|\mathbf{u}\|_{L^\infty(0,T;W^{1,p}(\Omega))} + \|\pi\|_{L^\infty(0,T;L^{p'}(\Omega))} \leq c,$$

and $\rho(x,t) \in \{\rho_1, \ldots, \rho_M\}$, a.e. in $\Omega \times (0,T]$.

Proof: We argue as in the proof of Theorem 2.1, but instead of V_0 we use the space V. We now consider a finite dimensional space $\widetilde{S}_n \subset L^\infty(0,T;V)$. Proceeding as in Sections 4 and 5, and making use of (7.2), we obtain, for each $n \in \mathbb{N}$, the existence of a weak solution $(\mathbf{u}^n, \rho^n)$ of the following approximated problem:

$(\mathcal{P}_\mathbf{2}^\mathbf{n})$ Find $(\mathbf{u}^n, \rho^n) \in \widetilde{S}_n \times L^\infty(\Omega_T)$ such that

$$(2\,\mu(\rho^n)\,T(e(\mathbf{u}^n)), e(\mathbf{v}^n)) = (\rho^n \mathbf{f}, \mathbf{v}^n)\,, \quad \forall \mathbf{v}^n \in \widetilde{S}_n, \text{ a.e. } t \in (0,T];$$

and ρ^n is a renormalized solution of

$$\begin{aligned} \rho_t^n + \operatorname{div}(\rho^n\,\mathbf{u}^n) &= 0 \quad \text{in } \Omega \times (0,T]; \\ \rho^n(x,0) &= \rho_0 \quad \text{in } \Omega, \end{aligned}$$

where $\rho_0 \in \{\rho_1, \ldots, \rho_M\}$, a.e. in Ω.

Taking the limit as $n \to \infty$, we obtain the analogue of Proposition 6.1. Applying again De Rahm's theorem, the assertion follows. ■

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