Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 06–24

LARGE DEVIATIONS FOR THE EMPIRICAL MEAN OF ASSOCIATED RANDOM VARIABLES

CARLA HENRIQUES AND PAULO EDUARDO OLIVEIRA

ABSTRACT: Let $\{X_n,\, n \geq 1\}$ be a stationary associated sequence of random variables. We obtain a large deviations upper bound for the empirical mean $\overline{X}_n$, from which we derive convergence rates for the weak and strong law of large numbers. The strong convergence rate that follows from our result is faster than the ones known in the literature for associated random variables. We also find conditions under which the sequence of empirical means $\{\overline{X}_n,\, n \geq 1\}$ satisfies the large deviation principle.

KEYWORDS: association, large deviations, stationarity, weak and strong convergence rates.
AMS SUBJECT CLASSIFICATION (2000): 60E15, 60F10.

1. Introduction

Consider a sequence $\{X_n,\, n \geq 1\}$ of associated real valued random variables. In this paper, we first prove a Bernstein-Hoeffding type exponential inequality, from which, supposing convenient decrease rates of the covariances $\mathrm{Cov}(X_1, X_n)$, we derive convergence rates both for the weak and the strong law of large numbers. Trying to be more precise on the behavior of the tail of the distribution we also prove a large deviation principle (LDP) for the sequence of partial means.

The concept of association was introduced in statistics by Esary, Proschan and Walkup [7] and has found applicability in diverse fields, such as reliability theory, statistical mechanics, stochastic processes, among others. Just to give some insight into the potential applications of this concept, we recall that association is preserved under monotone transformations and that independent random variables are associated, so that association actually occurs in each situation where we have monotone transformations of independent observations. Thus, it is not surprising to notice the interest of statisticians

Received June 1, 2006.
First author supported by Centro de Matemática da Universidade de Coimbra and PRODEP (Action 5.3).
Second author supported by Centro de Matemática da Universidade de Coimbra, Fundação para a Ciência e Tecnologia (FCT) and POCTI..

on this dependence notion, well reflected in the amount of literature that has been published recently. Excellent reviews of this subject can be found in Roussas [17] and Dewan and Prakasa Rao [6], which include some relevant results, numerous references and also several applications of this concept.

Large deviations results of the type of the classical Bernstein-Hoeffding inequalities have been proved for associated random variables, also yielding some convergence rates for the strong law of large numbers, under suitable conditions on the covariance structure (cf. Ioannides and Roussas [10] and Oliveira [15]). In Ioannides and Roussas [10] the authors establish an exponential inequality for uniformly bounded associated random variables. An extension replacing the boundedness assumption by the existence of Laplace transforms was proved in Oliveira [15]. Assuming a geometrical decrease rate of the covariances $\mathrm{Cov}(X_1, X_n)$, it follows from the inequality proved in [10] a convergence rate for the strong law of large numbers of order $\frac{\log^{2/3} n}{n^{1/3}}$. The extension of the exponential inequality to nonbounded variables provides a slower convergence rate as it multiplies the previous one by $\log n$. This is due to a truncation argument used in the course of the proof. We notice that, under the weaker assumption of polinomial decrease of the covariances, neither of the above mentioned exponential inequalities is strong enough to provide a convergence rate for the strong law of large numbers.

As already mentioned, in the present paper we will establish an exponential inequality for associated random variables assuming the variables to be uniformly bounded. Our exponential inequality applies, thus, to the same framework as the one by Ioannides and Roussas [10]. Under the assumption of geometrical decrease of the covariances, our inequality yields a convergence rate of order $\frac{\log n}{n^{1/2}}$ for the strong law of the large numbers. This is much closer to the best possible strong convergence rate for the empirical mean in the independent setting, which is $O\left(\frac{(\log\log n)^{1/2}}{n^{1/2}}\right)$. These results are stated and proved in Section 3 of the present paper. Before that, in Section 2, we present some definitions and auxiliary lemmas needed for the proof of the main results of this paper.

Finally, in Section 4, we will find conditions under which the large deviation principle (LDP) holds for the sequence of empirical means of associated random variables. For an account of the relevant results on LDPs, see, for example, Dembo and Zeitouni [4] and the references therein. We will prove a LDP assuming a hyper-geometric decrease rate on the covariances $\mathrm{Cov}(X_1, X_n)$.

This is in accordance with analogous results obtained under mixing assumptions, where deviations from independence were conveniently controlled (see, for example, Nummelin [13], Bryc [2] and Bryc and Dembo [3]). In fact, as is well known, the covariance structure of a collection of associated randomvariables highly determines its approximate independence (cf. Newman [12]). So, the referred condition on the decrease rate of the covariances is, for association, the counterpart of the hyper-geometric mixing rates assumed to establish the LDP under ϕ-mixing and α-mixing in Bryc [2] and Bryc and Dembo [3] (see Theorem 1 of Bryc [2] and Proposition 2 of Bryc and Dembo [3]).

2. Definitions, assumptions and auxiliary results

A sequence of random variables X_n, $n \geq 1$, is said to be associated if for any $m \in \mathbb{N}$ and any two real-valued coordinatewise nondecreasing functions f and g it holds

$$\mathrm{Cov}\Big(f\left(X_1, \ldots, X_m\right), g\left(X_1, \ldots, X_m\right)\Big) \geq 0\,,$$

whenever this covariance exists.

Throughout the paper we will always assume that $\{X_n,\, n \geq 1\}$ is an associated sequence. This will not be explicitly stated elsewhere in this paper to avoid unnecessary repetitions. Other assumptions to be considered in the sequel are gathered together below.

(A1) The sequence $\{X_n,\, n \geq 1\}$ is strictly stationary.
(A2) The sequence $\{X_n,\, n \geq 1\}$ is covariance stationary.
(A3) The variables of the sequence $\{X_n,\, n \geq 1\}$ are uniformly bounded, that is, $|X_n| \leq M$, $n \geq 1$.
(A4) For each $n \geq 1$, $\overline{X}_n$ has density function bounded by $a_1 (B_1)^n$, for some $a_1 > 0$ and some $B_1 > 1$.
(A5) $\sum_{n=1}^{\infty} \mathrm{Cov}(X_1, X_n) < +\infty$.

In addition, we will consider the following more precise conditions on the decay rate of the covariances

(P) $\mathrm{Cov}(X_1, X_n) = a_0 n^{-a}$, with $a_0 > 0$ and $a > 1$.
(G) $\mathrm{Cov}(X_1, X_n) = a_0 a^{-n}$, with $a_0 > 0$ and $a > 1$.
(H) $\mathrm{Cov}(X_1, X_n) = a_0 \exp(-n \log^{1+a} n)$, with $a_0 > 0$ and $a > 0$.

In the framework of association, it is usual to state conditions on the covariance structure in terms of the sequence

$$u(n)=\sum_{j=n+1}^{\infty}\operatorname{Cov}(X_1,X_j)\,.$$

The previous assumptions on $\operatorname{Cov}(X_1,X_n)$ imply the following decrease rates for the sequence $u(n)$.

Lemma 2.1. (a) *Under (P), there exists $a_1>0$ such that*

$$u(n)\leq a_1 n^{-(a-1)}\,,\ n\geq 1\,.$$

(b) *Under (G), there exists $a_1>0$ such that*

$$u(n)\leq a_1 a^{-n}\,,\ n\geq 1\,.$$

(c) *Under (H), there exists $a_1>0$ such that*

$$u(n)\leq a_1 e^{-n\log^{1+a}n}\,,\ n\geq 1\,.$$

Proof: To prove (a), just use the inequality

$$u(n)\leq\int_n^{\infty}f(x)\,dx\,, \tag{1}$$

where $f(x)=a_0x^{-a}$, $x\in[1,+\infty)$.

The proof of (b) is immediate.

For the proof of (c), let $f(x)=e^{-x\log^{1+a}x}$, $x\in[1,+\infty)$, and define $v(x)=\int_x^{\infty}f(t)\,dt$, $x\in(1,+\infty)$. Taking into account (1), we have $u(n)\leq a_0v(n)$, so that it is enough to prove that $\frac{v(n)}{f(n)}\longrightarrow 0$. This follows from the l'Hospital rule, since, by some elementary calculation, $\lim_{x\to+\infty}\frac{v'(x)}{f'(x)}=0$. ■

The following two lemmas will be needed for the proof of the main result of Section 4. We first present a result contained in Newman [11, 12], which generalizes the classical Hoeffding identity (see relation (2.2) in [11] or (4.10) in [12]). Also, this results is a special case of Theorem 2.3 of Yu [19].

Lemma 2.2. *Let f and g be two absolutely continuous functions. If X_1 and X_2 are random variables such that* $\mathrm{E}\left(f(X_1)^2\right)<+\infty$ *and* $\mathrm{E}\left(g(X_2)^2\right)<+\infty$, *then*

$$\operatorname{Cov}\left(f(X_1),g(X_2)\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f'(x_1)g'(x_2)H_{1,2}(x_1,x_2)\,dx_1\,dx_2\,,$$

where $H_{1,2}(x_1, x_2) = \mathrm{Cov}\left(I_{(x_1,+\infty)}(X_1), I_{(x_2,+\infty)}(X_2)\right)$.

The next result follows from relation (21) in Newman [11] and Corollary to Theorem 1 in Sadikova [18]. The detailed proof is given in Roussas [16] (see Lemma 2.6).

Lemma 2.3. (Roussas [16]) *Let* X_1 *and* X_2 *be two associated random variables having density functions bounded by* B_0*. Then, for all* $x_1, x_2 \in I\!R$*,*

$$Cov\left(I_{(-\infty,x_1]}(X_1), I_{(-\infty,x_2]}(X_2)\right) \leq B_1 \mathrm{Cov}^{1/3}\left(X_1, X_2\right),$$

where $B_1 = 2\max(2/\pi^2,\, 45B_0)$.

3. Exponential inequality and convergence rates

In this section we will obtain a large deviation upper bound for the empirical mean, $\overline{X}_n = n^{-1}\sum_{i=1}^n X_i$, of the kind of Bernstein and Hoeffding inequalities. To do this we use the classical technique, frequently employed to establish exponential type inequalities for dependent variables, of decomposing the sum $X_1 + \ldots + X_n$ into blocks, and then treating the sum of the odd blocks, say $B_{1,n}$, and of the even blocks, say $B_{2,n}$, separately, establishing for each one an exponential bound for the probability $\mathrm{P}(B_{i,n} \geq \varepsilon)$. If the sequence $\{X_n,\, n \geq 1\}$ is asymptotically independent, as each $B_{i,n}$ is a sum of non adjacent blocks, this technique enables an approximation to independence when the blocks are sufficiently far-apart.

Let us define $\overline{S}_n = n^{-1}\sum_{i=1}^n (X_i - \mathrm{E}(X_i))$. In order to decompose this sum into blocks, we will consider a sequence of positive integers p_n, such that $p_n \longrightarrow +\infty$ and, for each $n \in I\!N$, $p_n < n/2$. Also, let r_n be the greatest integer such that $r_n \leq \frac{n}{2p_n}$ and assume that $r_n \longrightarrow +\infty$. We now define the variables which consist of the sum in each block,

$$U_{n,i} = \sum_{j=2(i-1)p_n+1}^{(2i-1)p_n} (X_j - \mathrm{E}(X_j)), \;\; V_{n,i} = \sum_{j=(2i-1)p_n+1}^{2ip_n} (X_j - \mathrm{E}(X_j)), \;\; i = 1, \ldots, r_n,$$

and

$$Z_n = \sum_{j=2r_np_n+1}^{n} (X_j - \mathrm{E}(X_j))\,.$$

Note that these variables are associated, as they are nondecreasing functions of the variables $X_1, X_2, \ldots$, which are assumed throughout to be associated.

Setting

$$\overline{U}_n = \frac{1}{n}\sum_{i=1}^{r_n} U_{n,i}\,, \quad \overline{V}_n = \frac{1}{n}\sum_{i=1}^{r_n} V_{n,i} \quad \text{and} \quad \overline{Z}_n = \frac{1}{n} Z_n\,,$$

we then obtain

$$\overline{S}_n = \overline{U}_n + \overline{V}_n + \overline{Z}_n\,. \tag{2}$$

We will first establish an exponential bound for $\overline{U}_n$, which is also valid for $\overline{V}_n$, and then we will see that the remainder term, $\overline{Z}_n$, is negligible. For the first task we will have to obtain some control over the deviation from independence, that is, over the difference between what we really have and what we would have if the blocks were independent. This control is achieved through the following lemma, which is a version for moment generating functions of a result by Newman [12] for characteristic functions.

Lemma 3.1 (Dewan and Prakasa Rao [5]). *Let $Y_1, Y_2, ..., Y_n$ be associated random variables that are bounded by a constant M. Then, for any $\theta > 0$,*

$$\left| \mathrm{E}\left(e^{\theta \sum_{i=1}^n Y_i}\right) - \prod_{i=1}^n \mathrm{E}\left(e^{\theta Y_i}\right) \right| \le \theta^2 e^{n\theta M} \sum_{1\le i<j\le n} \mathrm{Cov}(Y_i, Y_j)\,.$$

To obtain the exponential bound for $\overline{U}_n$, we will also need to control the independent like term, that is, the product of the moment generating functions of the $U_{n,i}$ blocks. First, an adaptation of the proof of Lemma 2.2 of Oliveira [14] gives control over the asymptotics of the block variances.

Lemma 3.2. *Let $\{X_n,\, n \ge 1\}$ be a sequence of random variables satisfying* (A2) *and* (A5)*, then*

$$\frac{1}{p_n}\mathrm{Var}(U_{n,1}) \longrightarrow \nu = \mathrm{Var}(X_1) + 2\sum_{i=1}^{\infty} \mathrm{Cov}(X_1, X_{i+1})\,.$$

Proof: By (A2), we may write

$$\mathrm{Var}(U_{n,1}) = p_n Var(X_1) + 2\sum_{i=1}^{p_n-1}(p_n - i)\mathrm{Cov}(X_1, X_{i+1})\,.$$

Defining $v_n = \sum_{i=1}^{n-1}(n-i)\mathrm{Cov}(X_1, X_{i+1})$, the result follows from

$$\frac{1}{n} v_n \longrightarrow \sum_{i=1}^{\infty} \mathrm{Cov}(X_1, X_{i+1})\,,$$

which is a direct consequence of the classical result concerning Cesàro means, since $v_{n+1} - v_n = \sum_{i=1}^{n} \mathrm{Cov}(X_1, X_{i+1}) \longrightarrow \sum_{i=1}^{\infty} \mathrm{Cov}(X_1, X_{i+1})$. ■

The proof of the following auxiliary lemma closely follows the arguments used in the proof of Lemma 6 in Henriques and Oliveira [8], to which we refer the reader for any details.

Lemma 3.3. *Suppose* (A2) *is satisfied. We then have, for the variables* $U_{n,i}$, $i = 1, \ldots, r_n$, *defined earlier,*

$$\sum_{1 \le i < j \le r_n} \mathrm{Cov}(U_{n,i}, U_{n,j}) \le r_n p_n \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k)\,.$$

The same result holds for the variables $V_{n,i}$, $i = 1, \ldots, r_n$.

The following lemma provides control over the independent like terms.

Lemma 3.4. *Suppose* (A2), (A3) *and* (A5) *are satisfied and let* $\{d_n,\ n \ge 1\}$ *be a sequence of positive reals such that* $d_n > 1$ *for every sufficiently large* n. *If* $0 < \lambda < \frac{(d_n - 1)\,n}{d_n p_n 2M}$ *then*

$$\prod_{i=1}^{r_n} \mathrm{E}\left(e^{\frac{\lambda}{n} U_{n,i}}\right) \le \exp\left(\frac{\lambda^2}{n^2} r_n d_n \mathrm{Var}(U_{n,1})\right),$$

and

$$\prod_{i=1}^{r_n} \mathrm{E}\left(e^{\frac{\lambda}{n} V_i}\right) \le \exp\left(\frac{\lambda^2}{n^2} r_n d_n \mathrm{Var}(U_{n,1})\right).$$

Proof: By (A3), we have $|U_{n,i}| \le p_n 2M$, for each $i = 1, \ldots, r_n$. Remembering that $\mathrm{E}(U_{n,i}) = 0$, $i = 1, \ldots, r_n$, we get, using a Taylor expansion, and the stationarity assumption (S2),

$$\begin{aligned}
\mathrm{E}\left(e^{\frac{\lambda}{n} U_{n,i}}\right) &= 1 + \sum_{k=2}^{\infty} \left(\frac{\lambda}{n}\right)^k \mathrm{E}(U_{n,i}^k) \frac{1}{k!} \\
&\le 1 + \left(\frac{\lambda}{n}\right)^2 \mathrm{Var}(U_{n,1}) \sum_{k=2}^{\infty} \left(\frac{\lambda}{n} p_n 2M\right)^{k-2} \\
&= 1 + \left(\frac{\lambda}{n}\right)^2 \mathrm{Var}(U_{n,1}) \frac{1}{1 - \lambda p_n 2M/n}\,,
\end{aligned}$$

noticing that $\frac{\lambda}{n}p_n 2M < 1$, as $\lambda < \frac{(d_n-1)\,n}{d_n p_n 2M} < \frac{n}{p_n 2M}$. Finally, as $\frac{1}{1-\lambda p_n 2M/n} < d_n$, we have

$$\mathrm{E}\left(e^{\frac{\lambda}{n}U_{n,i}}\right) \leq 1 + \left(\frac{\lambda}{n}\right)^2 \mathrm{Var}(U_{n,1})d_n \leq \exp\left(\frac{\lambda^2}{n^2}\,d_n \mathrm{Var}(U_{n,1})\right),$$

which completes the proof. ■

The exponential inequality for $\overline{U}_n$ and $\overline{V}_n$ is proved in the next lemma.

Lemma 3.5. *Suppose* (A2), (A3) *and* (A5) *are satisfied. Let* $\{d_n,\, n \geq 1\}$ *and* $\{\varepsilon_n,\, n \geq 1\}$ *be two sequences of positive reals such that* $d_n > 1$ *and* $\varepsilon_n < \frac{\nu}{4M}\frac{d_n-1}{p_n}$ *, for every sufficiently large* n*. Further suppose that*

$$\frac{\varepsilon_n^2\, n}{\nu^2\, d_n^2} \exp\left(\frac{2M\varepsilon_n\, n}{\nu\, d_n}\right) \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k) \leq C_0. \tag{3}$$

Then, for every sufficiently large n*,*

$$\mathrm{P}(\left|\overline{U}_n\right| \geq \varepsilon_n) \leq 2(1+C_0)\exp\left(-\frac{\varepsilon_n^2\, n}{4\nu\, d_n}\right),$$

and the same for $\overline{V}_n$*.*

Proof: Given $\lambda > 0$, using Lemmas 3.1 and 3.3 we obtain

$$\begin{aligned}
&\mathrm{E}\left(e^{\lambda\overline{U}_n}\right) \leq \\
&\quad \leq \prod_{i=1}^{r_n} \mathrm{E}\left(e^{\frac{\lambda}{n}U_{n,i}}\right) + \frac{\lambda^2}{n^2}\exp\left(\frac{\lambda\, r_n\, p_n 2M}{n}\right) r_n\, p_n \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k) \\
&\quad \leq \prod_{i=1}^{r_n} \mathrm{E}\left(e^{\frac{\lambda}{n}U_{n,i}}\right) + \frac{\lambda^2}{2n}\exp\left(\lambda M\right) \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k),
\end{aligned}$$

remembering that $2r_n p_n \leq n$.

Suppose we choose λ such that the requirements of Lemma 3.4 are verified. Thus, applying the Markov inequality and Lemma 3.4 we get

$$\begin{aligned}
\mathrm{P}(\overline{U}_n \geq \varepsilon_n) \leq{}& \exp\left(-\lambda\varepsilon_n + \frac{\lambda^2}{n^2} r_n d_n \mathrm{Var}(U_{n,1})\right) \\
&+ e^{-\lambda\varepsilon_n}\frac{\lambda^2}{2n}\exp\left(\lambda M\right) \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k).
\end{aligned} \tag{4}$$

Now choose $\lambda = \frac{\varepsilon_n n^2}{2r_n d_n \mathrm{Var}(U_{n,1})}$, the minimizer of the first term on the right-hand side above. We will check that this choice for λ fulfils the requirement of Lemma 3.4. In fact, for every sufficiently large n, $\varepsilon_n < \frac{\nu}{4M}\frac{d_n-1}{p_n}$. So, as $\frac{2r_n p_n}{n} \longrightarrow 1$ and, by Lema 3.2, $\frac{\mathrm{Var}(U_{n,1})}{p_n} \longrightarrow \nu$, we have, for every sufficiently large n,

$$\frac{2r_n \mathrm{Var}(U_{n,1})(d_n-1)}{2Mnp_n} > \frac{\nu}{4M}\frac{d_n-1}{p_n} > \varepsilon_n \, ,$$

proving that $\lambda = \frac{\varepsilon_n n^2}{2r_n d_n \mathrm{Var}(U_{n,1})} < \frac{(d_n-1)\, n}{d_n p_n 2M}$.

Inserting this choice for λ in (4), we obtain

$$\begin{aligned} \mathrm{P}(\overline{U}_n \geq \varepsilon_n) \leq \exp\left(-\frac{\varepsilon_n^2 n^2}{4r_n d_n \mathrm{Var}(U_{n,1})}\right) \\ \times \left[1 + \frac{\varepsilon_n^2 n^3}{2[2r_n d_n \mathrm{Var}(U_{n,1})]^2} \exp\left(\frac{M\varepsilon_n n^2}{2r_n d_n \mathrm{Var}(U_{n,1})}\right) \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k)\right]. \end{aligned}$$

Again, by the fact that $\frac{2r_n p_n}{n} \longrightarrow 1$ and that $\frac{\mathrm{Var}(U_{n,1})}{p_n} \longrightarrow \nu$, we have, for every sufficiently large n, $\frac{n^3}{2[2\, r_n d_n \mathrm{Var}(U_{n,1})]^2} < \frac{n}{\nu^2 d_n^2}$, and $\frac{n}{2\nu\, d_n} < \frac{n^2}{2r_n d_n \mathrm{Var}(U_{n,1})} < \frac{2n}{\nu\, d_n}$, so that,

$$\mathrm{P}(\overline{U}_n \geq \varepsilon_n) \leq \exp\left(-\frac{\varepsilon_n^2 n}{4\nu\, d_n}\right)\left[1 + \frac{\varepsilon_n^2 n}{\nu^2 d_n^2} \exp\left(\frac{2M\varepsilon_n n}{\nu\, d_n}\right) \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k)\right],$$

which yields, due to (3),

$$\mathrm{P}(\overline{U}_n \geq \varepsilon_n) \leq (1 + C_0) \exp\left(-\frac{\varepsilon_n^2 n}{4\nu\, d_n}\right).$$

To conclude the proof just notice that the previous inequality also holds for $-\overline{U}_n$. ■

For the term $\overline{Z}_n$ we have the following result.

Lemma 3.6. *Suppose* (A3) *is satisfied. Let* $\{\varepsilon_n,\, n \geq 1\}$ *be a sequence of positive reals such that* $\frac{n\varepsilon_n}{p_n} \longrightarrow +\infty$. *Then, for every sufficiently large* n, $\mathrm{P}(\left|\overline{Z}_n\right| \geq \varepsilon_n) = 0$.

Proof: Since $\left|\overline{Z}_n\right| \le 2M\frac{n-2p_nr_n}{n} \le \frac{4p_n}{n}M$, $\mathrm{P}\left(\left|\overline{Z}_n\right| \ge \varepsilon_n\right) \le \mathrm{P}\left(\frac{4p_n}{n}M \ge \varepsilon_n\right) = 0$, for every sufficiently large n. ■

Taking decomposition (2) into account, the exponential inequality for $\overline{S}_n$ is now a direct consequence of the two previous lemmas.

Theorem 3.7. *Suppose* (A2), (A3) *and* (A5) *are satisfied. Let* $\{d_n,\, n \ge 1\}$ *and* $\{\varepsilon_n,\, n \ge 1\}$ *be two sequences of positive reals such that* $d_n > 1$ *and* $\varepsilon_n < \frac{\nu}{4M}\frac{d_n-1}{p_n}$, *for every sufficiently large* n, *and also* $\frac{n}{p_n}\varepsilon_n \longrightarrow +\infty$. *Further suppose that (3) holds true. Then, for every sufficiently large* n,

$$\mathrm{P}(\left|\overline{S}_n\right| \ge \varepsilon_n) \le 4(1+C_0)\exp\left(-\frac{\varepsilon_n^2\, n}{36\nu\, d_n}\right).$$

Condition (3) is not very explicit, as it depends on the covariances $\mathrm{Cov}(X_1, X_n)$ and also on the sequences p_n, d_n and ε_n. For a more convenient exploration of this condition and in order to determine explicit convergence rates for $\overline{S}_n$, we will have to assume some particular behavior on the covariance structure. We will then suppose (P), (G) or (H) to hold, and, in each case, we will identify convergence rates for the convergence in probability and for the almost sure convergence of $\overline{S}_n$.

We will first investigate the convergence rate for the convergence in probability. For this, consider in the previous theorem ε_n independent of n. Obviously, the best possible convergence rate for $\overline{S}_n$ obtains for d_n such that n/d_n tends to $+\infty$ with maximal rate. For the sake of verification of $\varepsilon < \frac{\nu}{4M}\frac{d_n-1}{p_n}$, for every sufficiently large n, d_n must be chosen such that $d_n \longrightarrow +\infty$ with a rate at least equal to the growth rate of p_n. With these considerations in mind we now prove the following corollary.

Corollary 3.8. *Suppose* (A2), (A3) *and* (A5) *are satisfied. If*

$$\frac{n}{p_n^2}\exp\left(\frac{n}{4\,p_n}\right)\sum_{k=p_n+1}^{\infty}\mathrm{Cov}(X_1, X_k) \le C_1\,, \tag{5}$$

Then, for $\varepsilon \in (0,1)$, *we have, for every sufficiently large* n,

$$\mathrm{P}(\left|\overline{S}_n\right| \ge \varepsilon) \le C_2\exp\left(-\frac{\varepsilon^2\, n}{288\, Mp_n}\right),$$

where $C_2 = 4\left(1+\frac{C_1}{64M^2}\right)$.

Proof: In Theorem 3.7 choose, for each $n \geq 1$, $\varepsilon_n = \varepsilon$ and $d_n = \delta p_n$ where $\delta = \frac{8M}{\nu}$. ■

The preceding corollary implies that we must choose p_n with a minimal growth rate to get the maximal rate of convergence for $\overline{S}_n$. We will now look for the optimal p_n such that (5) holds true under each of the assumptions (P), (G), or (H). In the sequel, c_{onst} stands for a generic positive constant, which may take different values in each appearance.

If we assume (G) to hold, by Lemma 2.1, (5) follows from

$$\log\left(\frac{n}{p_n^2}\right) + \frac{n}{4\,p_n} - p_n \log a \leq c_{onst}.$$

So, the optimal choice for p_n corresponds to $p_n \sim n^{1/2}$. We will then choose $p_n = [b\,n^{1/2}]$, with $b > \frac{1}{(4\log a)^{1/2}}$, where $[x]$ denotes the integer part of x. In fact, with this choice, the sequence on the left-hand side of the previous inequality is easily verified to converge to $-\infty$, being thus bounded above.

Suppose now that (P) holds true. Then, by Lemma 2.1, condition (5) can be rewritten as

$$\log\left(\frac{n}{p_n^2}\right) + \frac{n}{4\,p_n} - (a-1)\log p_n \leq c_{onst}. \tag{6}$$

It is easy to verify that, if p_n has a slower growth rate than $\frac{n}{\log n}$, then the sequence on the left-hand side of (6) tends to $+\infty$, and is consequently not bounded above. If we take $p_n = [b\,\frac{n}{\log n}]$, with $b > \frac{1}{4(a-1)}$, then the sequence on the left-hand side of (6) will converge to $-\infty$.

A similar analysis will lead to the choice $p_n = \left[b\frac{n^{1/2}}{\log^{(1+a)/2} n}\right]$, with $b > 2^{a-1}$, for the case of hyper-geometric decreasing covariances, (H).

The next result, stating explicit convergence rates for the weak law of large numbers under (P), (G) or (H), is now an immediate consequence of Corollary 3.8.

Corollary 3.9. *Suppose* (A2), (A3) *and* (A5) *are satisfied.*

(a) *Under* (G) *we have, for each* $\varepsilon \in (0,1)$ *and for every sufficiently large* n, $\mathrm{P}(\left|\overline{S}_n\right| \geq \varepsilon) \leq c_{onst}\exp\left(-\frac{\varepsilon^2}{288\,Mb}n^{1/2}\right)$, *with* $b > \frac{1}{(4\log a)^{1/2}}$.

(b) *Under* (P) *we have, for each* $\varepsilon \in (0,1)$ *and for every sufficiently large* n, $\mathrm{P}(\left|\overline{S}_n\right| \geq \varepsilon) \leq c_{onst}\, n^{-\frac{\varepsilon^2}{288\,Mb}}$, *with* $b > \frac{1}{4(a-1)}$.

(c) *Under* (H) *we have, for each* $\varepsilon \in (0,1)$ *and for every sufficiently large* n*,* $\mathrm{P}(|\overline{S}_n| \geq \varepsilon) \leq c_{onst} \exp\left(-\frac{\varepsilon^2}{288\,Mb} n^{1/2} \log^{(1+a)/2} n\right)$*, with* $b > 2^{a-1}$*.*

We now turn to the problem of identifying a convergence rate for the strong law of large numbers. For this, we will take

$$\varepsilon_n = \left(36\,\nu\,\alpha \frac{d_n \log n}{n}\right)^{1/2}, \tag{7}$$

for some $\alpha > 1$, in order to obtain a convergent series on the right-hand side of the exponential inequality of Theorem 3.7. Therefore, by the Borel-Cantelli Lemma, $\overline{S}_n = O(\varepsilon_n)$ almost surely if $\varepsilon_n \longrightarrow 0$. We now have to find the sequences d_n and p_n, such that ε_n tends to zero at a maximal rate and, simultaneously, all the assumptions of Theorem 3.7 are verified. We first note that, in order to get $\varepsilon_n \longrightarrow 0$, d_n must satisfy

$$\frac{d_n \log n}{n} \longrightarrow 0\,. \tag{8}$$

Inserting (7) in (3) this assumption becomes equivalent to

$$\frac{36\alpha\ \log n}{\nu\, d_n} \exp\left\{\left(\frac{144\,M^2\,\alpha}{\nu}\,\frac{n \log n}{d_n}\right)^{1/2}\right\} \sum_{k=p_n+1}^{\infty} \mathrm{Cov}(X_1, X_k) \leq c_{onst},$$

which follows from

$$\log\left(\frac{\log n}{d_n}\right) + \left(\frac{144\,M^2\,\alpha}{\nu}\,\frac{n \log n}{d_n}\right)^{1/2} + \log u(p_n) \leq c_{onst}. \tag{9}$$

Assuming (G), this inequality may be replaced by

$$\log\left(\frac{\log n}{d_n}\right) + \left(\frac{144\,M^2\,\alpha}{\nu}\,\frac{n \log n}{d_n}\right)^{1/2} - p_n \log a \leq c_{onst}. \tag{10}$$

By (8), the second term on the left-hand side above tends to $+\infty$. So, in order to have (10) satisfied, we must choose p_n with growth rate at least equal to $\left(\frac{n \log n}{d_n}\right)^{1/2}$. In addition, we must have $\varepsilon_n < \frac{\nu}{4M}\frac{d_n - 1}{p_n}$, for every sufficiently large n, which means that, inserting ε_n, $\frac{d_n p_n^2 \log n}{n(d_n-1)^2}$ must be bounded above. This and the minimal rate of $\left(\frac{n \log n}{d_n}\right)^{1/2}$ for p_n, imply that d_n must grow at least as $\log n$. The above considerations lead to the following choices

$$d_n = b_0 \log n, \quad \text{with } b_0 > \frac{288\, M^2 \alpha}{\nu \log a}\,, \tag{11}$$

and

$$p_n = [b_1 n^{1/2}], \ \text{ with } \frac{12\, M\, \alpha^{1/2}}{\log a\, (\nu\, b_0)^{1/2}} < b_1 < \frac{\nu^{1/2}\, b_0^{1/2}}{24 M \alpha^{1/2}}\,. \tag{12}$$

It is easy to verify that, choosing ε_n, d_n and p_n according to (7), (11) and (12), all the conditions of Theorem 3.7 hold true under (G), from which we obtain part (a) of Theorem 3.10 below. To prove part (b), just follow the same arguments, which lead to the choices

$$d_n = b_0, \quad \text{with } b_0 > 1,$$

and

$$p_n = \left[b_1 \frac{n^{1/2}}{\log^{1/2+a} n} \right], \ \text{ with } b_1 > \left(\frac{144 M^2 4^{1+a} \alpha}{\nu b_0} \right)^{1/2}.$$

Note also that under (P), condition (9) follows from

$$\log\left(\frac{\log n}{d_n}\right) + \left(\frac{144\, M^2\, \alpha}{\nu}\, \frac{n \log n}{d_n}\right)^{1/2} - (a-1) \log p_n.$$

For this condition to hold we would have to choose p_n such that $\log p_n$ has a growth rate at least equal to $\left(\frac{n \log n}{d_n}\right)^{1/2}$. But, this means that ε_n does not converge to zero. We cannot, therefore, identify a convergence rate for the strong law of large numbers, under the weaker assumption of polinomial decreasing covariances, (P).

Theorem 3.10. *Suppose* (A2), (A3) *and* (A5) *are satisfied.*

(a) *Under* (G), *we have*

$$\overline{S}_n = O\left(\frac{\log n}{n^{1/2}}\right) \qquad \text{a.s.}$$

(b) *Under* (H), *we have*

$$\overline{S}_n = O\left(\frac{\log^{1/2} n}{n^{1/2}}\right) \qquad \text{a.s.}$$

4. Large deviation principle

This section is devoted to proving the large deviation principle for the sequence of empirical means of associated variables.

For the proof of the main theorem in this section, we follow the methodology of proof of Theorem 6.4.4 of Dembo and Zeitouni [4], which deals with the large deviation principle of the empirical mean, under a certain mixing assumption.

For $n, m \geq 1$, define

$$\overline{X}_n^m = \frac{1}{n-m} \sum_{i=m+1}^{n} X_i \,.$$

For sake of simplicity, we write $\overline{X}_n$ instead of $\overline{X}_n^0$. The following two results are key tools to prove the LDP for the empirical mean $\overline{X}_n$. These results are the analogues for our framework of Lemmas 6.4.6 and 6.4.7 of Dembo and Zeitouni [4],

Lemma 4.1. *Suppose that* (A1), (A3) *and* (H) *are satisfied. Then, for each function $g : I\!\!R \longrightarrow I\!\!R$ concave, continuous and bounded above, the following limit exists*

$$\Lambda_g = \lim_{n \to +\infty} \frac{1}{n} \log \mathrm{E}\left(e^{n\, g(\overline{X}_n)}\right).$$

Proof: Let $g : I\!\!R \longrightarrow I\!\!R$ be a concave, continuous and bounded above function. Being concave and continuous, g is also Lipschitz continuous on $[-M, M]$, that is, there exists $L > 0$ such that, for all $x, y \in [-M, M]$, $|g(x) - g(y)| \leq L\,|x - y|$. Without loss of generality, we assume that $-\infty < -B \leq g(x) \leq 0$, for all $x \in [-M, M]$.

Defining $h(n) = -\log \mathrm{E}\left(e^{ng(\overline{X}_n)}\right)$, we have

$$h(n+m) \leq 2lLM - \log \mathrm{E}\left(e^{ng(\overline{X}_n)} e^{mg(\overline{X}_{n+m+l}^{n+l})}\right), \tag{13}$$

(see the proof of Lemma 6.4.6 of Dembo and Zeitouni [4] for details).

For each $n \in I\!\!N$, define $f_n(x) = e^{ng(x)}$, $x \in [-M, M]$. Since g is Lipschitz continuous and non-negative on $[-M, M]$, we have, for each $n \in I\!\!N$,

$$|f_n(x) - f_n(y)| \leq |ng(x) - ng(y)| \leq Ln\,|x - y|\,, \quad x, y \in [-M, M]\,.$$

So, the functions f_n are Lipschitz continuous, being then almost everywhere differentiable, with derivative satisfying $|f_n'(x)| \leq nL$. Applying Lemma 2.2, we obtain

$$\begin{aligned}
&\left|\mathrm{Cov}\left(e^{ng(\overline{X}_n)}, e^{mg(\overline{X}^{n+l}_{n+m+l})}\right)\right| \\
&\quad = \left|\int_{[-M,M]^2} f_n'(x) f_m'(y) \mathrm{Cov}\left(I_{(-\infty,x]}(\overline{X}_n), I_{(-\infty,y]}(\overline{X}^{n+l}_{n+m+l})\right)\, dx\, dy\right| \\
&\quad \le nmL^2 \int_{[-M,M]^2} \mathrm{Cov}\left(I_{(-\infty,x]}(\overline{X}_n), I_{(-\infty,y]}(\overline{X}^{n+l}_{n+m+l})\right)\, dx\, dy \\
&\quad = nmL^2 \mathrm{Cov}\left(\overline{X}_n, \overline{X}^{n+l}_{n+m+l}\right),
\end{aligned}$$

remembering that all the covariances above are non-negative by association. Using the stationarity assumption (A2), we get

$$\left|\mathrm{Cov}\left(e^{ng(\overline{X}_n)}, e^{mg(\overline{X}^{n+l}_{n+m+l})}\right)\right| \le L^2 n \sum_{i=l+2}^{\infty} \mathrm{Cov}(X_1, X_i) = L^2(n+m)u(l)\,,$$

and then

$$\begin{aligned}
\frac{\mathrm{E}\left(e^{ng(\overline{X}_n)} e^{mg(\overline{X}^{n+l}_{n+m+l})}\right)}{\mathrm{E}\left(e^{ng(\overline{X}_n)}\right)\mathrm{E}\left(e^{mg(\overline{X}^{n+l}_{n+m+l})}\right)} &\ge 1 - \frac{L^2(n+m)u(l)}{\mathrm{E}\left(e^{ng(\overline{X}_n)}\right)\mathrm{E}\left(e^{mg(\overline{X}^{n+l}_{n+m+l})}\right)} \\
&\ge 1 - L^2(n+m)u(l)e^{(n+m)B}\,,
\end{aligned}$$

as $g(x) \ge -B$, for all $x \in [-M, M]$.

Now, define $\Theta(l,n) = 1 - L^2 n u(l) e^{nB}$, $l, n \in I\!N$. From the preceding inequality we obtain

$$\log \mathrm{E}\left(e^{ng(\overline{X}_n)} e^{mg(\overline{X}^{n+l}_{n+m+l})}\right) \ge -h(n) - h(m) + \log\left(\Theta(l, n+m) \vee 0\right),$$

so that, from (13) we get

$$h(n+m) \le 2lLM + h(n) + h(m) - \log\left(\Theta(l, n+m) \vee 0\right). \tag{14}$$

Under (H), we have, for each $\kappa < a$ and for each $c \in I\!R$,

$$\lim_{n\to+\infty} n\, u\left(\frac{n}{\log^{1+\kappa} n}\right) e^{cn} = 0\,. \tag{15}$$

In fact, as in the proof of Lemma 2.1, part (c), we have $u(n) \le a_0 v(n)$, with $v(n)$ defined there. Using the l'Hospital rule, it is easy to check that $\lim_{x\to+\infty} x\, v\left(\frac{x}{\log^{1+\kappa} x}\right) e^{cx} = 0$, which yields (15).

Let $0 < \delta < a$, where a is given in (H). From (15) it is obvious that

$$\Theta\left(\frac{n}{\log^{1+\delta} n}, n\right) = 1 - L^2 nu\left(\frac{n}{\log^{1+\delta} n}\right) e^{nB} \overset{n\to+\infty}{\longrightarrow} 1.$$

Now, take $l = \left[\frac{n+m}{\log^{1+\delta}(n+m)}\right]$, then, for every sufficiently large $n+m$,

$$-\log\left(\Theta(l, n+m) \vee 0\right) \leq l\,. \tag{16}$$

Therefore, from (14),

$$h(n+m) \leq h(n) + h(m) + (2LM+1)\,\frac{n+m}{\log^{1+\delta}(n+m)}\,.$$

Finally, using Lemma 6.4.10 in Dembo and Zeitouni [4], it follows that

$$\lim_{n\to+\infty} \frac{h(n)}{n} = \lim_{n\to+\infty} \frac{-\log \mathrm{E}\left(e^{ng(\overline{X}_n)}\right)}{n}$$

exists. ■

In what follows we will use the notation $S_x^\delta =]x-\delta, x+\delta[$.

Lemma 4.2. *Suppose that* (A1), (A3), (A4) *and* (H) *are satisfied. If* $x_1, x_2 \in \mathbb{R}$ *are such that, for each* $\delta > 0$,

$$\liminf_{n\to+\infty} \frac{1}{n} \log \mathrm{P}\left(\overline{X}_n \in S_{x_i}^\delta\right) > -\infty,\ i = 1, 2\,,$$

then

$$\inf_{\delta>0} \liminf_{n\to+\infty} \frac{1}{n} \log \frac{\mathrm{P}\left(\overline{X}_{2n} \in S^\delta_{\frac{x_1+x_2}{2}}\right)}{\mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}\right) \mathrm{P}\left(\overline{X}_n \in S_{x_2}^{\delta/2}\right)} \geq 0.$$

Proof: Fix $\delta > 0$. From the hypothesis of the lemma, there exists $c_1 > 0$ such that, for every sufficiently large n,

$$\mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}\right) \mathrm{P}\left(\overline{X}_n \in S_{x_2}^{\delta/2}\right) \geq \exp(-nc_1). \tag{17}$$

We have

$$
\begin{aligned}
&\left|\mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}, \overline{X}_{2n+l}^{n+l} \in S_{x_2}^{\delta/2}\right) - \mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}\right)\mathrm{P}\left(\overline{X}_{2n+l}^{n+l} \in S_{x_2}^{\delta/2}\right)\right| \\
&\quad \le \left|\mathrm{P}\left(\overline{X}_n < x_1 + \tfrac{\delta}{2}, \overline{X}_{2n+l}^{n+l} < x_2 + \tfrac{\delta}{2}\right) - \mathrm{P}\left(\overline{X}_n < x_1 + \tfrac{\delta}{2}\right)\mathrm{P}\left(\overline{X}_{2n+l}^{n+l} < x_2 + \tfrac{\delta}{2}\right)\right| \\
&\quad + \left|\mathrm{P}\left(\overline{X}_n < x_1 + \tfrac{\delta}{2}, \overline{X}_{2n+l}^{n+l} \le x_2 - \tfrac{\delta}{2}\right) - \mathrm{P}\left(\overline{X}_n < x_1 + \tfrac{\delta}{2}\right)\mathrm{P}\left(\overline{X}_{2n+l}^{n+l} \le x_2 - \tfrac{\delta}{2}\right)\right| \\
&\quad + \left|\mathrm{P}\left(\overline{X}_n \le x_1 - \tfrac{\delta}{2}, \overline{X}_{2n+l}^{n+l} < x_2 + \tfrac{\delta}{2}\right) - \mathrm{P}\left(\overline{X}_n \le x_1 - \tfrac{\delta}{2}\right)\mathrm{P}\left(\overline{X}_{2n+l}^{n+l} < x_2 + \tfrac{\delta}{2}\right)\right| \\
&\quad + \left|\mathrm{P}\left(\overline{X}_n \le x_1 - \tfrac{\delta}{2}, \overline{X}_{2n+l}^{n+l} \le x_2 - \tfrac{\delta}{2}\right) - \mathrm{P}\left(\overline{X}_n \le x_1 - \tfrac{\delta}{2}\right)\mathrm{P}\left(\overline{X}_{2n+l}^{n+l} \le x_2 - \tfrac{\delta}{2}\right)\right|
\end{aligned}
$$

By (A4), we may apply Lemma 2.3 to obtain

$$
\begin{aligned}
&\left|\mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}, \overline{X}_{2n+l}^{n+l} \in S_{x_2}^{\delta/2}\right) - \mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}\right)\mathrm{P}\left(\overline{X}_{2n+l}^{n+l} \in S_{x_2}^{\delta/2}\right)\right| \\
&\qquad\qquad \le 4B_n \mathrm{Cov}^{1/3}\left(\overline{X}_n, \overline{X}_{2n+l}^{n+l}\right),
\end{aligned}
$$

where $B_n = 2\max(2/\pi^2,\, 45\, a_1\, B_1^n)$. Therefore, by the stationarity assumption (A1),

$$
\begin{aligned}
&\left|\mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}, \overline{X}_{2n+l}^{n+l} \in S_{x_2}^{\delta/2}\right) - \mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}\right)\mathrm{P}\left(\overline{X}_n \in S_{x_2}^{\delta/2}\right)\right| \\
&\qquad\qquad \le 4B_n\left(\frac{1}{n^2}\, n \sum_{i=l+1}^{\infty} \mathrm{Cov}(X_1, X_i)\right)^{1/3} = 4B_n\left(\frac{u(l)}{n}\right)^{1/3},
\end{aligned}
$$

which yields

$$
\frac{\mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}, \overline{X}_{2n+l}^{n+l} \in S_{x_2}^{\delta/2}\right)}{\mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}\right)\mathrm{P}\left(\overline{X}_n \in S_{x_2}^{\delta/2}\right)} \ge 1 - 4B_n\left(\frac{u(l)}{n}\right)^{1/3}\exp(c_1\, n), \tag{18}
$$

for each $l \in I\!N$ and n large enough, attending to (17).

Following the arguments used in the proof of Lemma 6.4.7 in Dembo and Zeitouni [4], we get, for $l = \frac{\delta\, n}{2M}$

$$
\mathrm{P}\left(\overline{X}_{2n} \in S_{\frac{x_1+x_2}{2}}^{\delta}\right) \ge \mathrm{P}\left(\overline{X}_n \in S_{x_1}^{\delta/2}, \overline{X}_{2n+l}^{n+l} \in S_{x_2}^{\delta/2}\right),
$$

so that, from (18), for every sufficiently large n,

$$\frac{\mathrm{P}\left(\overline{X}_{2n} \in S^{\delta}_{\frac{x_1+x_2}{2}}\right)}{\mathrm{P}\left(\overline{X}_n \in S^{\delta/2}_{x_1}\right)\mathrm{P}\left(\overline{X}_n \in S^{\delta/2}_{x_2}\right)} \geq 1 - 4B_n \left(\frac{u(\frac{\delta\, n}{2M})}{n}\right)^{1/3} \exp(c_1\, n).$$

We then have

$$\liminf_{n\to+\infty} \frac{1}{n} \log \frac{\mathrm{P}\left(\overline{X}_{2n} \in S^{\delta}_{\frac{x_1+x_2}{2}}\right)}{\mathrm{P}\left(\overline{X}_n \in S^{\delta/2}_{x_1}\right)\mathrm{P}\left(\overline{X}_n \in S^{\delta/2}_{x_2}\right)}$$
$$\geq \liminf_{n\to+\infty} \frac{1}{n} \log \left(\left\{1 - 4B_n \left(\frac{u\left(\frac{\delta\, n}{2M}\right)}{n}\right)^{1/3} \exp(c_1\, n)\right\} \vee 0\right).$$

By (15), which is valid under (H), the right-hand side above is equal to zero, from which the desired result follows. ■

We may now formulate the main result of this section.

Theorem 4.3. *Under* (A1), (A3), (A4) *and* (H), *the sequence* $\{\overline{X}_n,\, n \geq 1\}$ *satisfies the large deviation principle with rate function given by*

$$\Lambda^*(x) = \sup_{t\in \mathbb{R}}\{tx - \Lambda(t)\},\ x \in \mathbb{R},$$

which is the Fenchel-Legendre transform of

$$\Lambda(t) = \lim_{n\to+\infty} \frac{1}{n} \log \mathrm{E}\left(e^{n\,t\,\overline{X}_n}\right).$$

Proof: The proof goes along the same lines as that of Theorem 6.4.4 of Dembo and Zeitouni [4], using our Lemma 4.1, in conjunction with Lemma 4.4.8 and Theorem 4.4.10 of [4], to ensure that $\{\overline{X}_n,\, n \geq 1\}$ satisfies the large deviation principle with good rate function I. The convexity of I follows in the same way as in Dembo and Zeitouni [4], applying Lemma 4.2 to obtain, for $x_1, x_2 \in I\!R$ such that $I(x_1) < +\infty$ and $I(x_2) < +\infty$,

$$
\begin{aligned}
-I\left(\frac{x_1+x_2}{2}\right) &= \inf_{\delta>0}\left\{\limsup_{n\to+\infty}\frac{1}{n}\log \mathrm{P}(\overline{X}_n \in S^{\delta}_{\frac{x_1+x_2}{2}})\right\} \\
&\geq \inf_{\delta>0}\left\{\liminf_{n\to+\infty}\frac{1}{2n}\log \mathrm{P}(\overline{X}_{2n} \in S^{\delta}_{\frac{x_1+x_2}{2}})\right\} \\
&\geq \inf_{\delta>0}\left\{\liminf_{n\to+\infty}\frac{1}{2n}\log\left(\frac{\mathrm{P}\left(\overline{X}_{2n}\in S^{\delta}_{\frac{x_1+x_2}{2}}\right)}{\mathrm{P}\left(\overline{X}_n\in S^{\delta/2}_{x_1}\right)\mathrm{P}\left(\overline{X}_n\in S^{\delta/2}_{x_2}\right)}\right)\right\} \\
&\qquad + \inf_{\delta>0}\left\{\liminf_{n\to+\infty}\frac{1}{2n}\log\left(\mathrm{P}\left(\overline{X}_n\in S^{\delta/2}_{x_1}\right)\right)\right\} \\
&\qquad\qquad + \inf_{\delta>0}\left\{\liminf_{n\to+\infty}\frac{1}{2n}\log\left(\mathrm{P}\left(\overline{X}_n\in S^{\delta/2}_{x_2}\right)\right)\right\} \\
&\geq -\frac{1}{2}I(x_1) - \frac{1}{2}I(x_2)\,.
\end{aligned}
$$

The rest of the proof proceeds exactly as in Theorem 6.4.4 of Dembo and Zeitouni [4]. ■

It is worth noticing that the assumptions of the previous theorem are met, for example, when we consider a sequence of independent and identically distributed random variables, say $Y_1, Y_2, \ldots$, and then take $X_n = c\,(Y_n + \ldots + Y_{n+m})$, for some fixed $m \in I\!N$ and $c \in I\!R$. Actually, this is a commonly used method to generate associated sequences. If the distribution of the variables Y_i is taken to be concentrated in a compact set of $I\!R$ and admits a bounded density function, then, as is easily verified, the sequence $X_1, X_2, \ldots$ satisfies the assumptions of the last theorem.

References

[1] Birkel, T. (1988), On the convergence rate in the central limit theorem for associated processes, *Ann. Probab.* 16, 1685–1698.

[2] Bryc, W. (1992), On large deviations for uniformly strong mixing sequences, *Stochastic Processes Appl.* 41, 191–202.

[3] Bryc, W. and Dembo, A. (1996), Large deviatuons and strong mixing, *Ann. Inst. Henri Poincaré* 32, 549-569.

[4] Dembo, A. and Zeitouni, O. (1998), *Large Deviations Techniques and Applications*, Springer-Verlag, New York.

[5] Dewan, I. and Prakasa Rao, B.L.S. (1999), A general method of density estimation for associated random variables, *J. Nonparametr. Statist.* 10, 405–420.

[6] Dewan, I. and Prakasa Rao, B.L.S. (2001), Associated sequences and related inference problems, In: *Stochastic Processes: Theory and Methods*, eds. D.N. Shanbhag and C.R. Rao, Handbook of Statistics 19, North-Holland, Amsterdam, 693–731.
[7] Esary, J.D., Proschan, F. and Walkup, D.W. (1967), Association of random variables, with applications, *Ann. Math. Statist.* 38, 1466–1474.
[8] Henriques, C. and Oliveira, P.E. (2005), Exponential rates for kernel density estimation under association, *Statistica Neerlandica* 59, 448–466.
[9] Henriques, C. and Oliveira, P.E. (2006), Convergence rates for the estimation of two-dimensional distribution functions under association and estimation of the covariance of the limit empirical process, *J. Nonparametric Statistics* 18, 119–128.
[10] Ioannides, D.A. and Roussas, G.G. (1998), Exponential inequality for associated random variables, *Statist. Probab. Lett.* 42, 423–431.
[11] Newman, C.M. (1980), Normal fluctuations and the FKG inequalities, *Comm. Math Phys.* 74, 119–128.
[12] Newman, C.M. (1984), Asymptotic independence and limit theorems for positively and negatively dependent random variables, in: *Inequalities in Statistics and Probability*, Y.L. Tong, ed, IMS Lecture Notes - Monograph Series, Vol. 5, 127–140.
[13] Nummelin, E. (1990), Large deviations for funccionals of stationary processes, *Probab. Th. Rel. Fields* 86, 387–401.
[14] Oliveira, P.E. (2002), Density estimation for associated sampling: a point process influenced approach, *J. Nonparametric Statist.* 14, 495–509.
[15] Oliveira, P.E. (2005), An exponential inequality for associated variables, *Statist. Probab. Letters* 73, 189–197. 2005.
[16] Roussas, G.G. (1995), Asymptotic normality of a smooth estimate of a random field distribution function under association, *Statist. Probab. Letters* 24, 77–90.
[17] Roussas, G.G. (1999), Positive and negative dependence with some statistical applications, In: *Asymptotics, Nonparametrics and Time Series*, ed. S. Ghosh, Statist. Textbooks Monogr. 158, Dekker, New York, 757–788.
[18] Sadikova, S.M. (1996), Two-dimensional analogies of an inequality of Esseen with applications to the central limit theorem, *Theory Probab. Appl.* 11, 325–335.
[19] Yu, H. (1993), A Glivenko-Cantelli lemma and weak convergence for empirical processes of associated sequences, *Probability Theory and Related Fields* 95, 357–370.

Carla Henriques
Escola Superior de Tecnologia de Viseu, 3504-510 Viseu, Portugal

Paulo Eduardo Oliveira
Dep. Matemática, Univ. Coimbra, Apartado 3008, 3001 - 454 Coimbra, Portugal

E-mail address: paulo@mat.uc.pt
URL: http://www.mat.uc.pt