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ON POINT-FINITENESS IN POINTFREE TOPOLOGY

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Dedicated to Bernhard Banaschewski on the occasion of his 80th birthday

ABSTRACT: In pointfree topology, the point-finite covers introduced by Dowker and Strauss do not behave similarly to their classical counterparts with respect to transitive quasi-uniformities, contrarily to what happens with other familiar types of interior-preserving covers. The purpose of this paper is to remedy this by modifying the definition of Dowker and Strauss. We present arguments to justify that this modification turns out to be the right pointfree definition of point-finiteness. Along the way we place point-finite covers among the classes of interior-preserving and closure-preserving families of covers that are relevant for the theory of (transitive) quasi-uniformities, completing the study initiated with [6].

KEYWORDS: frame, locale, sublocale lattice, congruence lattice, cover, interior-preserving cover, closure-preserving cover, point-finite cover, locally finite cover.

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1. Introduction

Recall that pointfree topology deals with the category Frm of frames and frame homomorphisms where a *frame* is a complete lattice satisfying the distributive law

$$a \wedge \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \wedge b_i) \tag{1.1}$$

and a *frame homomorphism* is a map : $L \to M$ between frames preserving finitary meets and arbitrary joins.

In the theory of functorial quasi-uniformities on topological spaces, interior-preserving covers play a central role [9]. Indeed each family of interior-preserving open covers of a space $(X, \mathcal{T})$ generates, in a canonical way (usually referred to as the *Fletcher construction* [8]) a transitive quasi-uniformity on X

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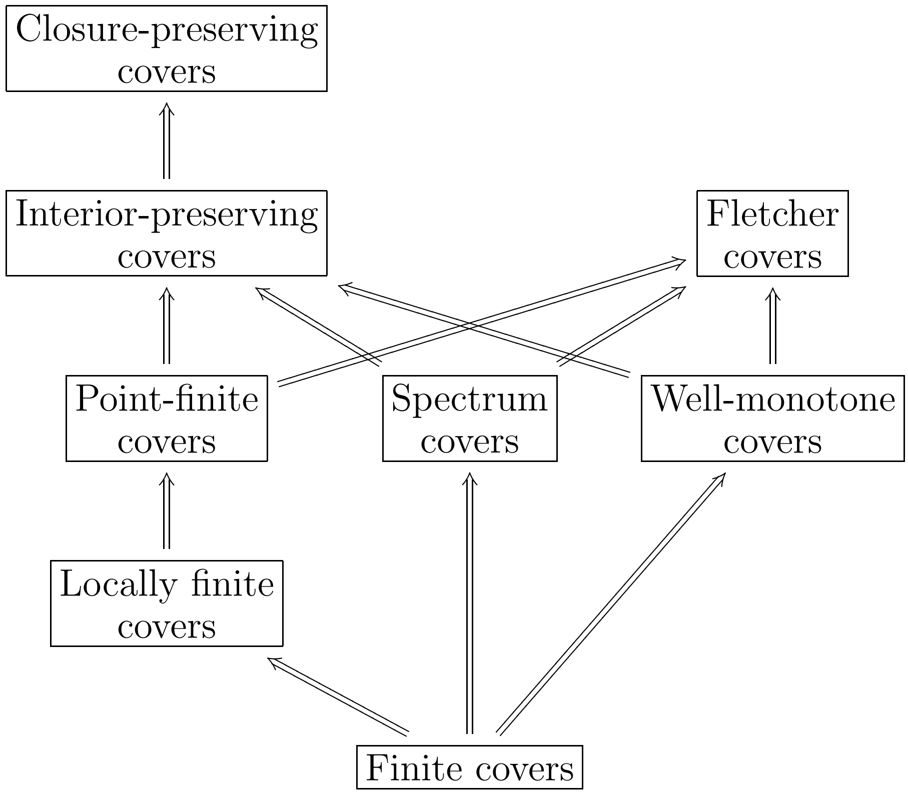
compatible with the topology $\mathcal{T}$ (that is, which induces as its first topology the given topology $\mathcal{T}$). For instance, with the set of all finite covers one gets the Pervin quasi-uniformity and with the family of all point-finite (resp. locally finite, spectrum, well-monotone) covers one gets the point-finite (resp. locally finite, spectrum, well-monotone) quasi-uniformity [9]. Moreover, this construction gives all transitive quasi-uniformities in the sense that, for any transitive quasi-uniformity μ on X, compatible with $\mathcal{T}$, there exists a family of interior-preserving covers of $(X, \mathcal{T})$ that generates μ [2].

In the setting of frames, however, matters are a bit more technical: interior-preserving (and, more generally, closure-preserving) covers induce transitive quasi-uniformities in case they satisfy a certain condition which we refer to as the *Fletcher condition* ([6], [7]). That is the case with finite, locally finite, spectrum or well-monotone covers but not with the point-finite covers of Dowker and Strauss [4]. Moreover, the notion of a point-finite family (which is a notion weaker than local finiteness) is not strong enough to imply the interior-preserving condition.

By formulating the classical definition inside the sublocale lattice, we propose a stronger notion of point-finiteness, still weaker than local finiteness, but stronger enough to imply both interior-preserving and Fletcher conditions. This makes the point-finite quasi-uniformity on frames behaving closer to its classical role and keeps the pointfree version of the theorem of Lefschetz that characterizes normality in terms of point-finite covers (due to Dowker and Strauss [4]) true.

With this refinement of the point-finiteness definition one may redraw the general picture in ([7], Section 4), making it very similar to the corresponding classical picture:



2. The sublocale lattice

The dual category of Frm is the category Loc of *locales*. We adopt the usual distinction [10], in notation and terminology, between frames and locales, which are the same thing, but where morphisms go in the opposite direction. Given a locale X, we write $\mathcal{O}(X)$ for the corresponding frame. Given a morphism $f : Y \to X$ in Loc, we write $f^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ for the corresponding frame homomorphism. For general notions and results concerning frames and locales we refer to [11] or [14].

A *sublocale* $j : S \rightarrowtail X$ of a locale X is a regular subobject of X in Loc, or, in frame terms, a quotient $j^* : \mathcal{O}(X) \twoheadrightarrow \mathcal{O}(S)$ of the frame $\mathcal{O}(X)$. On the class $\mathfrak{S}(X)$ of sublocales of X we have the natural preorder $j_1 \leq j_2$ if and only if there is a j such that $j_2 \cdot j = j_1$ (note that this j is necessarily again a sublocale). Sublocales $j_i : S_i \rightarrowtail X$ $(i = 1, 2)$ are *equivalent* if $j_1 \leq j_2$ and $j_2 \leq j_1$ or, equivalently, if there is an isomorphism $j : S_1 \to S_2$ such that $j_2 \cdot j = j_1$.

The partially ordered set $\mathfrak{S}(X)$ is a complete lattice, much more complicated than its topological counterpart. Indeed, in the latter every element has a complement (which makes it a complete Boolean algebra) whilst in the former most elements are not complemented: in general, for each locale X, $\mathfrak{S}(X)$ is a co-frame (that is, it satisfies the dual law of (1.1)) but it is not a frame unless X is scattered [16]; indeed, $S \wedge \bigvee_{i\in I} S_i = \bigvee_{i\in I}(S \wedge S_i)$ for all $\{S_i \mid i \in I\} \subseteq \mathfrak{S}(X)$ if and only if S is complemented [10]. However, there are sufficient complemented elements in order for $\mathfrak{S}(X)$ to be generated by them (i.e. $\mathfrak{S}(X)$ is zero-dimensional). Specifically:

There are, for every $a \in \mathcal{O}X$, the *open sublocales* $X_a \rightarrowtail X$ given by the frame homomorphisms

$$\begin{array}{rccl} \hat{a}: & \mathcal{O}X & \longrightarrow & \downarrow a := \{x \in \mathcal{O}X \mid x \leq a\} \\ & x & \longmapsto & x \wedge a, \end{array}$$

and the *closed sublocales* $X\text{-}X_a \rightarrowtail X$ given by the frame homomorphisms

$$\begin{array}{rccl} \check{a}: & \mathcal{O}X & \longrightarrow & \uparrow a := \{x \in \mathcal{O}X \mid x \geq a\} \\ & x & \longmapsto & x \vee a. \end{array}$$

Open and closed sublocales are complements in $\mathfrak{S}(X)$ and every sublocale $j : Y \rightarrowtail X$ satisfies

$$j = \bigwedge\{X_a \vee X\text{-}X_b \mid j^*(a) = j^*(b)\}.$$

Further, families $\{X_a \mid a \in A\}$ of open sublocales are *distributive* [15], that is, satisfy $S \wedge \bigvee_{a\in A} X_a = \bigvee_{a\in A}(S \wedge X_a)$ for all $S \in \mathfrak{S}(X)$.

Every sublocale $j : S \rightarrowtail X$ has a *closure* $\mathrm{cl}(j) : \mathrm{cl}(S) \rightarrowtail X$, which is the smallest closed sublocale that contains j and an *interior* $\mathrm{int}(j) : \mathrm{int}(S) \rightarrowtail X$ (the largest open sublocale contained in j).

A family $\mathcal{S} = \{S_i \mid i \in I\}$ of sublocales of X is *closure-preserving* if for all $J \subseteq I$,

$$\mathrm{cl}(\bigvee_{j\in J} S_j) = \bigvee_{j\in J} \mathrm{cl}(S_j)$$

and it is a *cover* of X if $\bigvee_{i\in I} S_i = X$ [16]. Similarly, we say that $\mathcal{S}$ is *interior-preserving* if for all $J \subseteq I$,

$$\mathrm{int}(\bigwedge_{j\in J} S_j) = \bigwedge_{j\in J} \mathrm{int}(S_j).$$

Further, $\mathcal{S}$ is *locally finite* [16] if there exists an open cover $\mathcal{C}$ of X such that for all $C \in \mathcal{C}$ there exists a finite set I_C for which $C \wedge \bigvee_{j\in J} S_j = C \wedge \bigvee_{j\in J\cap I_C} S_j$ for all $J \subseteq I$. The cover $\mathcal{C}$ is said to *witness* the local finiteness of $\mathcal{S}$.

3. Interior-preserving covers

Sublocales of X correspond to quotients on the frame $\mathcal{O}(X)$, thus to frame congruences on $\mathcal{O}(X)$, that is, equivalence relations on $\mathcal{O}(X)$ closed under finite meets and arbitrary joins. With the inclusion ordering, the set of all congruences on $L = \mathcal{O}(X)$ forms a frame $\mathfrak{C}(L)$, called the *congruence frame* of L, and $\mathfrak{S}(X)^{op} \cong \mathfrak{C}(L)$. Open sublocales X_a correspond to congruences

$$\Delta_a = \{(x,y) \mid x \wedge a = y \wedge b\}$$

and closed sublocales X-X_a correspond to congruences

$$\nabla_a = \{(x,y) \mid x \vee a = y \vee b\}.$$

Therefore, each ∇_a is complemented in $\mathfrak{C}(L)$ with complement Δ_a.

Further, $\nabla_L := \{\nabla_a \mid a \in L\}$ is a subframe of $\mathfrak{C}(L)$ and the correspondence $a \mapsto \nabla_a$ defines an epimorphism and a monomorphism $\nabla_L : L \to \mathfrak{C}(L)$ which gives an isomorphism $L \to \nabla_L$. On the other hand, the map $a \mapsto \Delta_a$ is a dual poset embedding $L \to \Delta_L$ taking finitary meets to finitary joins and arbitrary joins to arbitrary meets, where Δ_L denotes the subframe of $\mathfrak{C}(L)$ generated by $\{\Delta_a \mid a \in L\}$.

Therefore, open covers $\{X_a \mid a \in A \subseteq \mathcal{O}X\}$ of X correspond to families $\{\Delta_a \mid a \in A\}$ of open congruences satisfying $\bigwedge_{a\in A} \Delta_a = 0$, that is, $\Delta_{\bigvee A} = 0$; this means that $\bigvee A = 1$ i.e. A is a *cover* of the frame $L = \mathcal{O}X$. Note that $\mathrm{cl}(\Delta_a) = \nabla_{a^*}$ and $\mathrm{int}(\nabla_a) = \Delta_{a^*}$ (where a^* denotes the pseudocomplement of a).

For each cover A of L let $\mathcal{O}_A$ and $\mathcal{C}_A$ denote respectively, the corresponding open cover $\{\Delta_a \mid a \in A\}$ and closed co-cover $\{\nabla_a \mid a \in A\}$.

Proposition 3.1.

(1) $\mathcal{O}_A$ *is interior-preserving if and only if* $\bigvee_B \Delta_b = \Delta_{\bigwedge_B b}$, *for all* $B \subseteq A$.

(2) $\mathcal{O}_A$ *is closure-preserving if and only if* $\bigwedge_B \nabla_{b^*} = \nabla_{\bigwedge_B b^*}$, *for all* $B \subseteq A$.

(3) $\mathcal{C}_A$ *is interior-preserving if and only if* $\bigvee_B \Delta_{b^*} = \Delta_{\bigwedge_B b^*}$, *for all* $B \subseteq A$.

(4) $\mathcal{C}_A$ *is closure-preserving if and only if* $\bigwedge\limits_B \nabla_b = \nabla_{\bigwedge_B b}$, *for all* $B \subseteq A$.

Proof. (1) By definition, $\mathcal{O}_A$ is interior-preserving if and only if, for every $B \subseteq A$, $\mathrm{int}(\bigvee_{b\in B} \Delta_b) = \bigvee_{b\in B} \mathrm{int}(\Delta_b)$. Since $\mathrm{int}(\Delta_b) = \Delta_b$, we only have to show that $\mathrm{int}(\bigvee_{b\in B} \Delta_b) = \Delta_{\bigwedge_{b\in B} b}$, that is, that $\Delta_{\bigwedge_{b\in B} b}$ is the smallest open congruence that contains $\bigvee_{b\in B} \Delta_b$. Obviously $\Delta_{\bigwedge_{b\in B} b}$ is open and contains $\bigvee_{b\in B} \Delta_b$. Let Δ_x be an open congruence of L such that $\bigvee_{b\in B} \Delta_b \leq \Delta_x$. Then $\Delta_b \leq \Delta_x$, for every $b \in B$, that is, $x \leq b$, thus $x \leq \bigwedge_{b\in B} b$ and $\Delta_{\bigwedge_{b\in B} b} \leq \Delta_x$.

(2) By definition, $\mathcal{O}_A$ is closure-preserving if and only if, for every $B \subseteq A$, $\mathrm{cl}(\bigwedge_{b\in B} \Delta_b) = \bigwedge_{b\in B} \mathrm{cl}(\Delta_b)$, that is, $\mathrm{cl}(\Delta_{\bigvee_{b\in B} b}) = \bigwedge_{b\in B} \nabla_{b^*}$. The congruence $\nabla_{\bigwedge_{b\in B} b^*}$ is closed and it is contained in $\Delta_{\bigvee_{b\in B} b}$, since

$$\nabla_{\bigwedge\limits_{b\in B} b^*} \wedge \nabla_{\bigvee\limits_{b\in B} b} = \nabla_{\bigwedge\limits_{b\in B} b^* \wedge \bigvee\limits_{b\in B} b} = \nabla_0 = 0.$$

It remains to check that this is the largest closed congruence of L satisfying those conditions. Let ∇_x such that $\nabla_x \leq \Delta_{\bigvee_{b\in B} b}$. Then $\nabla_x \wedge \nabla_{\bigvee_{b\in B} b} = 0$, that is, $\nabla_{x \wedge \bigvee_{b\in B} b} = 0$. Therefore $x \wedge \bigvee_{b\in B} b = 0$ and $x \leq \bigwedge_{b\in B} b^*$, which implies $\nabla_x \leq \nabla_{\bigwedge_{b\in B} b^*}$.

Assertions (3) and (4) may be proved similarly. ☐

Since the necessary and sufficient condition in 3.1(4) is a consequence of the one in 3.1(1) by pseudocomplementation, we have:

Corollary 3.2. *For any cover A of L, if $\mathcal{O}_A$ is interior-preserving then $\mathcal{C}_A$ is closure-preserving.* ☐

Contrarily to what happens in the classical case, the converse to Corollary 3.2 does not hold, in general, as the following example shows.

Example 3.3. Let A be the cover $\{n \mid n \in \mathbb{N}\}$ of the frame

$$L = (\omega + 1)^{op} = \{\infty < \cdots < 2 < 1\}.$$

We show that $\mathcal{C}_A$ is closure-preserving. Let B be a subset of A. If B is finite then, trivially, $\bigwedge_{n\in B} \nabla_n = \nabla_{\bigwedge_{n\in B} n}$. In case B is infinite then $\bigwedge_{n\in B} n = \infty$ and $\nabla_{\bigwedge_{n\in B} n} = \nabla_\infty = 0$. On the other hand, if $(x, y) \in \bigwedge_{n\in B} \nabla_n$ then $x \vee n = y \vee n$, for all $n \in B$, which implies $x = y$ and, consequently, $\bigwedge_{n\in B} \nabla_n = 0$.

In order to prove that $\mathcal{O}_A$ is not interior-preserving it suffices to show that $\Delta_{\bigwedge_{n\in\mathbb{N}} n} > \bigvee_{n\in\mathbb{N}} \Delta_n$. Of course $\Delta_{\bigwedge_{n\in\mathbb{N}} n} = \Delta_\infty = 1$. On the other hand, it is easy to check that

$$\{(a,b)\in L\times L \mid a=\infty \text{ iff } b=\infty\}$$

is a congruence of L that contains $\bigcup_{n\in\mathbb{N}} \Delta_n$. Further, it is even the least congruence of L that satisfies this condition, so

$$\bigvee_{n\in\mathbb{N}} \Delta_n = \{(a,b)\in L\times L \mid a=\infty \text{ iff } b=\infty\}$$

which shows that $\bigvee_{n\in\mathbb{N}} \Delta_n < 1$.

However, if L is *scattered*, that is, if every congruence of L is complemented [15] or, equivalently,

$$\theta \vee \bigwedge_{i\in I} \theta_i = \bigwedge_{i\in I} (\theta\vee\theta_i), \text{ for every } \theta, \theta_i \in \mathfrak{C}L,\ i\in I,\ [16]$$

the conditions in 3.1(4) and 3.1(1) are equivalent, as for spaces:

Proposition 3.4. *For any cover A of a scattered frame L, $\mathcal{O}_A$ is interior-preserving if and only if $\mathcal{C}_A$ is closure-preserving.*

Proof. By Corollary 3.2, if $\mathcal{O}_A$ is interior-preserving then $\mathcal{C}_A$ is closure-preserving. Conversely, if $\mathcal{C}_A$ is closure-preserving then, by Proposition 3.1(4) $\bigwedge_{b\in B} \nabla_b = \nabla_{\bigwedge_{b\in B} b}$, for all $B\subseteq A$, and then, by pseudocomplementation, $(\bigwedge_{b\in B}\nabla_b)^* = \Delta_{\bigwedge_{b\in B} b}$. Since L is scattered, the second De Morgan law is valid in $\mathfrak{C}L$ thus $\bigvee_{b\in B}\Delta_b = \Delta_{\bigwedge_{b\in B} b}$ and $\mathcal{O}_A$ is interior-preserving. □

The preceding results justify to say that a cover A of a frame L is *interior-preserving* if it satisfies the condition in 3.1(1). If A satisfies only condition 3.1(4) we say that A is *closure-preserving.*

Therefore, a cover A of L is interior-preserving if and only if $\bigvee_{b\in B}\Delta_b$ is open for every $B\subseteq A$; A is closure-preserving if and only if the congruence $\bigwedge_{b\in B}\nabla_b$ is closed for all $B\subseteq A$.

Remark 3.5. The closure-preserving subsets of a frame L are precisely the conservative subsets introduced by Dowker and Strauss in [4] and studied in detail by Chen in [3]. They are characterized [3, Lemma 2.3] as the subsets A of L for which

$$x \vee (\bigwedge_{b\in B} b) = \bigwedge\{x \vee b \mid b \in B\}, \text{ for every } B \subseteq A \text{ and every } x \in L.$$

4. Point-finite families of Dowker and Strauss

We recall from [4] that a subset A of a frame L is *point-finite* if

$$x = \bigwedge_{F\in\mathcal{P}_f(A)} (x \vee c_F) \quad \text{for every } x \in L, \tag{4.1}$$

where $\mathcal{P}_f(A)$ denotes the set of all finite subsets of A and $c_F = \bigvee(A \setminus F)$. This notion is stronger than local finiteness. Indeed, if A is locally finite, that is, if there exists a cover C of L for which $A_c := \{a \in A \mid a \wedge c \neq 0\}$ is finite for all $c \in C$, then, for every $c \in C$ and $x \in L$, we have

$$c \wedge \bigwedge_{F\in\mathcal{P}_f(A)} (x \vee c_F) \leq c \wedge (x \vee c_{A_c}) = c \wedge x;$$

since C is a cover, this implies $\bigwedge_{F\in\mathcal{P}_f(A)}(x \vee c_F) \leq x$, which is the non-trivial part of the point-finiteness condition.

Furthermore, point-finite covers are closure-preserving [6] but they are not, in general, interior-preserving [6, Example 3.2].

For any cover A of L let

$$\delta_A = \bigvee\Big\{(\bigwedge_{a\in A_1} \nabla_a) \wedge (\bigwedge_{a\in A_2} \Delta_a) \mid A_1 \cup A_2 = A\Big\}.$$

A cover A is said to be a *Fletcher cover* [6] whenever $\delta_A = 1$. These covers are crucial for the construction of transitive compatible quasi-uniformities on frames: besides being closure-preserving, they must satisfy this condition [6].

Examples of Fletcher covers are finite covers, locally finite covers, spectrum covers and well-monotone covers (see [6] for the details). However, that does not happen with the point-finite covers of Dowker and Strauss, as the following example shows.

Consider the *frame of reals* $\mathfrak{L}(\mathbb{R})$ [1], generated by pairs of rationals (p, q), and its subframe

$$\mathfrak{L}[0,1] = {\uparrow}((-,0) \vee (1,-))$$

(the *frame of reals in the closed unit interval*), which is generated by elements

$$\rrbracket p,q\llbracket = (p,q) \vee (-,0) \vee (1,-), \quad p,q \in \mathbb{Q}.$$

Let A be the cover $\left\{1, \left\rrbracket \frac{1}{n+1}, \frac{1}{n} \right\llbracket : n \in \mathbb{N}\right\}$ of $\mathfrak{L}[0,1]$.

Lemma 4.1. *A is interior-preserving.*

Proof. Note that any cover in which every infinite subset B contains b_1 and b_2 satisfying $b_1 \wedge b_2 = 0$ is interior-preserving:

$$\bigvee_{b \in B} \Delta_b \geq \Delta b_1 \vee \Delta_{b_2} = \Delta_{b_1 \wedge b_2} = \Delta_0 \geq \Delta_{\bigwedge B}.$$

This happens precisely with cover A: for any $n \neq m$, $\rrbracket \frac{1}{n+1}, \frac{1}{n} \llbracket \wedge \rrbracket \frac{1}{m+1}, \frac{1}{m} \llbracket = 0$. □

The cover A is not Fletcher, that is, $\delta_A \neq 1$, as we prove in the sequel.
Since A is interior-preserving, then

$$\delta_A = \bigvee_{A_1 \cup A_2 = A} (\nabla_{\bigwedge\limits_{a \in A_1} a} \wedge \Delta_{\bigvee\limits_{a \in A_2} a}).$$

In this join it suffices to consider the partitions $A_1 \cup A_2$ of A in which $1 \in A_1$, since $1 \in A_2$ implies $\Delta_{\bigvee_{a \in A_2} a} = \Delta_1 = 0$. On the other hand, if A_1 contains two distinct elements $\left\rrbracket \frac{1}{n+1}, \frac{1}{n} \right\llbracket$ and $\left\rrbracket \frac{1}{m+1}, \frac{1}{m} \right\llbracket$ then $\bigwedge_{a \in A_1} a = 0$. Therefore the remaining partitions are

$$A_1 = \left\{1, \left\rrbracket \frac{1}{n+1}, \frac{1}{n} \right\llbracket \right\} \quad \text{and} \quad A_2 = A \setminus A_1.$$

Thus,

$$\delta_A = \bigvee_{n \in \mathbb{N}} \big(\nabla_{\rrbracket \frac{1}{n+1}, \frac{1}{n} \llbracket} \wedge \Delta_{\bigvee\limits_{m \in \mathbb{N} \setminus \{n\}} \rrbracket \frac{1}{m+1}, \frac{1}{m} \llbracket}\big).$$

But, as we show in the following lemma,

$$\nabla_{\rrbracket \frac{1}{n+1}, \frac{1}{n} \llbracket} \leq \Delta_{\bigvee\limits_{m \in \mathbb{N} \setminus \{n\}} \rrbracket \frac{1}{m+1}, \frac{1}{m} \llbracket}, \quad \text{for every } n \in \mathbb{N},$$

hence

$$\delta_A = \bigvee_{n \in \mathbb{N}} \nabla_{\rrbracket \frac{1}{n+1}, \frac{1}{n} \llbracket} = \nabla_{\bigvee\limits_{n \in \mathbb{N}} \rrbracket \frac{1}{n+1}, \frac{1}{n} \llbracket},$$

which implies that $\delta_A < \nabla_1 = 1$.

Lemma 4.2. *For any* $n \in \mathbb{N}$,

$$\nabla_{\rrbracket\frac{1}{n+1},\frac{1}{n}\llbracket} \leq \Delta_{\bigvee\limits_{m\in\mathbb{N}\setminus\{n\}} \rrbracket\frac{1}{m+1},\frac{1}{m}\llbracket}.$$

Proof. We have, for each $n \in \mathbb{N}$:

$$\nabla_{\rrbracket\frac{1}{n+1},\frac{1}{n}\llbracket} \wedge \nabla_{\bigvee\limits_{m\in\mathbb{N}\setminus\{n\}} \rrbracket\frac{1}{m+1},\frac{1}{m}\llbracket} = \nabla_{\rrbracket\frac{1}{n+1},\frac{1}{n}\llbracket \ \wedge (\bigvee\limits_{m\in\mathbb{N}\setminus\{n\}} \rrbracket\frac{1}{m+1},\frac{1}{m}\llbracket)} = \nabla_0 = 0.$$

Thus

$$\nabla_{\rrbracket\frac{1}{n+1},\frac{1}{n}\llbracket} \leq \Big(\nabla_{\bigvee\limits_{m\in\mathbb{N}\setminus\{n\}} \rrbracket\frac{1}{m+1},\frac{1}{m}\llbracket}\Big)^* = \Delta_{\bigvee\limits_{m\in\mathbb{N}\setminus\{n\}} \rrbracket\frac{1}{m+1},\frac{1}{m}\llbracket}.$$

□

We are finally ready to show that A is point-finite, that is, for every $x \in \mathfrak{L}[0,1]$,

$$\bigwedge_{F\in\mathcal{P}_f(\mathbb{N})} \Big(x \vee \bigvee_{m\notin F} \Big\rrbracket \frac{1}{m+1}, \frac{1}{m} \Big\llbracket \Big) \leq x. \tag{4.2}$$

We begin by showing that (4.2) holds for $x = 0$, that is,

$$\bigwedge_{F\in\mathcal{P}_f(\mathbb{N})} \bigvee_{m\notin F} \Big\rrbracket \frac{1}{m+1}, \frac{1}{m} \Big\llbracket = 0.$$

Consider $n \in \mathbb{N}$. The set $F = \{1, 2, \dots, n\}$ belongs to $\mathcal{P}_f(\mathbb{N})$ and

$$\bigvee_{m\notin F} \Big\rrbracket \frac{1}{m+1}, \frac{1}{m} \Big\llbracket = \bigvee_{m\geq n+1} \Big\rrbracket \frac{1}{m+1}, \frac{1}{m} \Big\llbracket,$$

which implies

$$\bigwedge_{F\in\mathcal{P}_f(\mathbb{N})} \bigvee_{m\notin F} \Big\rrbracket \frac{1}{m+1}, \frac{1}{m} \Big\llbracket \leq \bigwedge_{n\in\mathbb{N}} \bigvee_{m\geq n+1} \Big\rrbracket \frac{1}{m+1}, \frac{1}{m} \Big\llbracket = \bigwedge_{n\in\mathbb{N}} \Big\rrbracket \frac{1}{n+1}, \frac{1}{n} \Big\llbracket = 0.$$

The proof of (4.2) for a non-zero element of $\mathfrak{L}[0,1]$ is an immediate consequence of the following result:

Lemma 4.3. *Let* $x, y \in \mathfrak{L}[0,1]$. *If* $y \leq x \vee \bigvee_{m\notin F} \Big\rrbracket \frac{1}{m+1}, \frac{1}{m} \Big\llbracket$ *for every* $F \in \mathcal{P}_f(\mathbb{N})$, *then* $y \leq x$.

Proof. The result is obviously true for $y = 0$. Assume that $y \nleq x$ is a nonzero element of $\mathfrak{L}[0,1]$ which satisfies the hypothesis. This means that there exists $]\!]p,q[\![\leq y$, with $q > 0$, $p < 1$ and $p < q$, such that $]\!]p,q[\![\nleq x$. Then, there exists a rational $r \in [0,1]$ that satisfies

$$p < r < q \text{ and }]\!]p',q'[\![\nleq x, \text{ for every }]\!]p',q'[\![\text{ such that } p' < r < q'. \tag{4.3}$$

There are two possibilities:

Case 1: $r \in]0,1]$. Then there exists $n \in \mathbb{N}$ such that $\frac{1}{n+1} < r \leq \frac{1}{n}$. Therefore $p \vee \frac{1}{n+1} < r < q$ and

$$\left]\!\!\right]p \vee \frac{1}{n+1}, q\left[\!\!\right[=]\!]p,q[\![\wedge \left]\!\!\right]\frac{1}{n+1}, q\left[\!\!\right[\leq y \leq x \vee \bigvee_{m \notin F} \left]\!\!\right]\frac{1}{m+1}, \frac{1}{m}\left[\!\!\right[$$

for $F = \{1,2,\ldots,n\}$. On the other hand,

$$\left]\!\!\right]\frac{1}{n+1}, q\left[\!\!\right[\wedge \left(\bigvee_{m \notin F} \left]\!\!\right]\frac{1}{m+1}, \frac{1}{m}\left[\!\!\right[\right) = 0.$$

Thus, $\left]\!\!\right]p \vee \frac{1}{n+1}, q\left[\!\!\right[\leq x$, which contradicts (4.3).

Case 2: $r = 0$ is the only rational that satisfies (4.3). Then $]\!]0,q[\![\leq x$. Consider $n \in \mathbb{N}$ such that $\frac{1}{n+1} < q \leq \frac{1}{n}$ (we may assume, without loss of generality, that $q \leq 1$) and take $F = \{1,2,\ldots,n\}$. By hypothesis,

$$y \leq x \vee \bigvee_{m \notin F} \left]\!\!\right]\frac{1}{m+1}, \frac{1}{m}\left[\!\!\right[= x,$$

which contradicts (4.3). $\square$

In conclusion we have:

Proposition 4.4. *The cover A is an interior-preserving and point-finite cover of $\mathfrak{L}[0,1]$ which is not Fletcher.* $\square$

5. Point-finite families

A family $\mathcal{S} = \{S_i \mid i \in I\}$ of subsets of a set X is called *point-finite* [5] if for every $x \in X$ the set $\{i \in I \mid x \in S_i\}$ is finite. Denoting $\bigcup\{S_i \mid i \in I \setminus F\}$, for any finite $F \subseteq I$, by $\mathcal{S}_F$, it is obvious that $\mathcal{S}$ is point-finite if and only if $\bigcap_{F\in\mathcal{P}_f(I)} \mathcal{S}_F = \emptyset$ or, equivalently, $T = \bigcap_{F\in\mathcal{P}_f(I)}(T \cup \mathcal{S}_F)$ for every $T \subseteq X$. Clearly, any locally finite family is point-finite and any point-finite open cover is interior-preserving [5].

In view of this definition, it is natural to define a family $\mathcal{S} = \{S_i \mid i \in I\}$ of sublocales of a locale X *point-finite* whenever

$$T = \bigwedge_{F\in\mathcal{P}_f(I)} (T \vee \mathcal{S}_F) \text{ for every } T \in \mathfrak{S}(X), \tag{5.1}$$

where $\mathcal{S}_F$ is the sublocale $\bigvee\{S_i \mid i \in I \setminus F\}$. Note that, by the co-frame distributive law of $\mathfrak{S}(X)$, (5.1) is equivalent to

$$\bigwedge_{F\in\mathcal{P}_f(I)} \mathcal{S}_F = 0.$$

Proposition 5.1. *Each locally finite family of sublocales is point-finite.*

Proof. Let $\mathcal{S} = \{S_i \mid i \in I\}$ be a locally finite family of sublocales of X and consider the corresponding witnessing open cover $\mathcal{C}$. Since open families are distributive we have

$$\bigwedge_{F\in\mathcal{P}_f(I)} \mathcal{S}_F = \Big(\bigvee_{C\in\mathcal{C}} C\Big) \wedge \bigwedge_{F\in\mathcal{P}_f(I)} \mathcal{S}_F = \bigvee_{C\in\mathcal{C}} (C \wedge \bigwedge_{F\in\mathcal{P}_f(I)} \mathcal{S}_F) \leq \bigvee_{C\in\mathcal{C}} (C \wedge \mathcal{S}_{I_C}) = 0.$$

□

Proposition 5.2. *Each point-finite cover of a locale is interior-preserving.*

Proof. We only need to show that $\mathrm{int}(\bigwedge_{j\in J} S_j) \leq \bigwedge_{j\in J} \mathrm{int}(S_j)$ for every infinite $J \subseteq I$. For each $F \in \mathcal{P}_f(I)$ there exists $j_F \in J \setminus F$. Consequently, $\mathcal{S}_F \geq \mathcal{S}_{j_F}$ and

$$0 = \bigwedge_{F\in\mathcal{P}_f(I)} \mathcal{S}_F \geq \bigwedge_{F\in\mathcal{P}_f(I)} \mathcal{S}_{j_F} \geq \mathrm{int}(\bigwedge_{F\in\mathcal{P}_f(I)} \mathcal{S}_{j_F}) \geq \mathrm{int}(\bigwedge_{j\in J} \mathcal{S}_j).$$

□

Now, let $\mathcal{S} = \{X_a \mid a \in A\}$ be a point-finite open cover of X and consider the corresponding open cover $\{\Delta_a \mid a \in A\}$ of congruences. The point-finiteness of $\mathcal{S}$ means that

$$\bigvee_{F \in \mathcal{P}_f(A)} \Big(\bigwedge_{a \in A \setminus F} \Delta_a \Big) = 1.$$

Since $\bigwedge_{a \in A \setminus F} \Delta_a = \Delta_{c_F}$, this is equivalent to $\bigvee_{F \in \mathcal{P}_f(I)} \Delta_{c_F} = 1$.

This motivates us to define a *point-finite* cover A of a frame L as a cover satisfying

$$\bigvee_{F \in \mathcal{P}_f(A)} \Delta_{c_F} = 1. \tag{5.2}$$

Remarks 5.3. (1) Condition (5.2) is stronger than condition (4.1) of Dowker and Strauss. Indeed, by De Morgan law, $\bigvee_{F \in \mathcal{P}_f(A)} \Delta_{c_F} = 1$ implies $\bigwedge_{F \in \mathcal{P}_f(A)} \nabla_{c_F} = 0$ thus, for every $x \in L$,

$$\nabla_x = \nabla_x \vee \bigwedge_{F \in \mathcal{P}_f(A)} \nabla_{c_F} = \bigwedge_{F \in \mathcal{P}_f(A)} (\nabla_x \vee \nabla_{c_F}) = \bigwedge_{F \in \mathcal{P}_f(A)} \nabla_{x \vee c_F} \geq \nabla_{\bigwedge_{F \in \mathcal{P}_f(A)} (x \vee c_F)}$$

from which it follows that $x \geq \bigwedge_{F \in \mathcal{P}_f(A)} (x \vee c_F)$. Hence (4.1) holds.

The reverse implication is not true as the example of Proposition 4.4, combined with Proposition 5.5 below, shows.

(2) By the properties of open congruences, $\bigvee_{F \in \mathcal{P}_f(A)} \Delta_{c_F} \leq \Delta_{\bigvee_{F \in \mathcal{P}_f(A)} c_F}$. Therefore condition (5.2) implies, in particular, that $\bigwedge_{F \in \mathcal{P}_f(A)} c_F = 0$. Condition (4.1) of Dowker and Strauss lies precisely between those two conditions:

$$(5.2) \Rightarrow (4.1) \Rightarrow \bigwedge_{F \in \mathcal{P}_f(A)} c_F = 0.$$

Lemma 5.4. *For every point-finite family A of L,*

$$\bigvee_{F \in \mathcal{P}_f(A)} (\nabla_{\bigwedge F} \wedge \Delta_{c_F}) = \Big(\bigvee_{F \in \mathcal{P}_f(A)} \nabla_{\bigwedge F} \Big) \wedge \Big(\bigvee_{F \in \mathcal{P}_f(A)} \Delta_{c_F} \Big).$$

Proof. Trivially

$$\bigvee_{F \in \mathcal{P}_f(A)} (\nabla_{\bigwedge F} \wedge \Delta_{c_F}) \leq \Big(\bigvee_{F \in \mathcal{P}_f(A)} \nabla_{\bigwedge F} \Big) \wedge \Big(\bigvee_{F \in \mathcal{P}_f(A)} \Delta_{c_F} \Big),$$

so it suffices to check that

$$\nabla_{\bigwedge F} \wedge \Delta_{c_G} \leq \bigvee_{H \in \mathcal{P}_f(A)} (\nabla_{\bigwedge H} \wedge \Delta_{c_H})$$

for all $F, G \in \mathcal{P}_f(A)$. Let $F, G \in \mathcal{P}_f(A)$, $F \neq G$.

Case 1: $\exists a \in F : a \notin G$. This case is obvious: $\nabla_{\bigwedge F} \wedge \Delta_{c_G} \leq \nabla_a \wedge \Delta_a = 0$.

Case 2: $F \subset G$**.** Let $G = F \cup \{a_1, a_2, \ldots, a_n\}$,

$$F_0 = F, F_1 = F_0 \cup \{a_1\}, \ldots, F_n = F_{n-1} \cup \{a_n\} = G$$

and

$$G_0 = G, G_1 = G_0 \setminus \{a_1\}, \ldots, G_n = G_{n-1} \setminus \{a_n\} = F.$$

It is easy to prove, by induction over $n \in \mathbb{N}$, that

$$\nabla_{\bigwedge F} \wedge \Delta_{c_G} = \bigvee_{i=0}^{n} (\nabla_{\bigwedge F_i} \wedge \Delta_{c_{F_i}}).$$

Indeed, the case $n = 1$ is straightforward: $F_0 = F, F_1 = G, G_0 = G, G_1 = F$ and

$$\begin{aligned}
\nabla_{\bigwedge F} \wedge \Delta_{c_G} &= (\nabla_{\bigwedge F} \wedge \Delta_{c_G}) \wedge (\nabla_{a_1} \vee \Delta_{a_1}) \\
&= (\nabla_{\bigwedge F \wedge a_1} \wedge \Delta_{c_G}) \vee (\nabla_{\bigwedge F} \wedge \Delta_{c_G \vee a_1}) \\
&= (\nabla_{\bigwedge F_1} \wedge \Delta_{c_{F_1}}) \vee (\nabla_{\bigwedge F_0} \wedge \Delta_{c_{F_0}}).
\end{aligned}$$

Now, let $n = k + 1$ and assume the result holds for k. Then

$$\begin{aligned}
\nabla_{\bigwedge F} \wedge \Delta_{c_G} &= (\nabla_{\bigwedge F} \wedge \Delta_{c_G}) \wedge (\nabla_{a_1} \vee \Delta_{a_1}) \\
&= (\nabla_{\bigwedge F \wedge a_1} \wedge \Delta_{c_G}) \vee (\nabla_{\bigwedge F} \wedge \Delta_{c_G \vee a_1}) \\
&= (\nabla_{\bigwedge F_1} \wedge \Delta_{c_G}) \vee (\nabla_{\bigwedge F} \wedge \Delta_{c_{G_1}}),
\end{aligned}$$

which, by inductive hypothesis, is equal to

$$\bigvee_{i=1}^{k+1} (\nabla_{\bigwedge F_i} \wedge \Delta_{c_{F_i}}) \vee \bigvee_{i=0}^{k} (\nabla_{\bigwedge F_i} \wedge \Delta_{c_{F_i}}) = \bigvee_{i=0}^{k+1} (\nabla_{\bigwedge F_i} \wedge \Delta_{c_{F_i}}).$$

□

Proposition 5.5. *Each point-finite cover is a Fletcher cover.*

Proof. Let

$$\delta_A = \bigvee\{ \bigwedge_{a\in A_1} \nabla_a \wedge \bigwedge_{a\in A_2} \Delta_a \mid A_1 \cup A_2 = A\}.$$

Then

$$\begin{aligned}\delta_A &\geq \bigvee\{ \bigwedge_{a\in F} \nabla_a \wedge \bigwedge_{a\in A\setminus F} \Delta_a \mid F \in \mathcal{P}_f(A)\} \\ &= \bigvee_{F\in\mathcal{P}_f(A)} (\nabla_{\bigwedge F} \wedge \Delta_{c_F}),\end{aligned}$$

so, using Lemma 5.4, we get

$$\delta_A \geq \Big(\bigvee_{F\in\mathcal{P}_f(A)} \nabla_{\bigwedge F}\Big) \wedge \Big(\bigvee_{F\in\mathcal{P}_f(A)} \Delta_{c_F}\Big) = \bigvee_{F\in\mathcal{P}_f(A)} \nabla_{\bigwedge F} = \nabla_{\bigvee_{F\in\mathcal{P}_f(A)} \bigwedge F} = 1,$$

since A is a cover. □

From Propositions 5.2 and 5.5 it follows that the family $\mathcal{A}$ of all point-finite covers of a frame L induces, by the method introduced in [6], a transitive quasi-uniformity $\mathcal{PF}$ on $\mathfrak{C}L$ compatible with L. From the following result it follows that $\mathcal{A}$ is an adequate kind of covers [7], which, in particular, implies that $\mathcal{PF}$ is functorial (and so we may add it to our table of examples in [7]). In order to prove the result we need to recall the well-known fact that, for any frame homomorphism $h : L \to M$, there exists a (unique) frame homomorphism $\overline{h} : \mathfrak{C}L \to \mathfrak{C}M$, given by $\overline{h}(\nabla_x) = \nabla_{h(x)}$ and $\overline{h}(\Delta_x) = \Delta_{h(x)}$, making the diagram

$$\begin{tikzcd} L \arrow[r, "\nabla_L"] \arrow[d, "h"'] & \mathfrak{C}L \arrow[d, dashed, "\overline{h}"] \\ M \arrow[r, "\nabla_M"'] & \mathfrak{C}M \end{tikzcd}$$

commute [12].

Proposition 5.6. *Let $h : L \to M$ be a frame homomorphism. If A is a point-finite family of L then $h[A]$ is a point-finite family of M.*

Proof. For $F \in \mathcal{P}_f(A)$ and $G \in \mathcal{P}_f(h[A])$ let c_F^A and $c_G^{h[A]}$ denote respectively $\bigvee(A \setminus F)$ and $\bigvee(h[A] \setminus G)$. From the hypothesis $\bigvee_{F\in\mathcal{P}_f(A)} \Delta_{c_F^A} = 1$ it readily follows that

$$1 = \overline{h}\Big(\bigvee_{F\in\mathcal{P}_f(A)} \Delta_{c_F^A}\Big) = \bigvee_{F\in\mathcal{P}_f(A)} \overline{h}(\Delta_{c_F^A}) = \bigvee_{F\in\mathcal{P}_f(A)} \Delta_{h(c_F^A)}.$$

So it suffices to show that

$$\bigvee_{F\in\mathcal{P}_f(A)} \Delta_{h(c_F^A)} \leq \bigvee_{G\in\mathcal{P}_f(h[A])} \Delta_{c_G^{h[A]}},$$

which is easy, since $h[F] \in \mathcal{P}_f(h[A])$ for any $F \in \mathcal{P}_f(A)$ and

$$h(c_F^A) = h\Big(\bigvee_{a\in A\setminus F} a\Big) = \bigvee_{a\in A\setminus F} h(a) \geq c_{h[F]}^{h[A]}.$$

□

Finally, recall from [4] that a cover $A = \{a_i \mid i \in I\}$ of L is *shrinkable* if there exists a cover $B = \{b_i \mid i \in I\}$ such that $a_i \vee b_i^* = 1$ for every $i \in I$. Note that a frame L is *normal* if for any $a, b \in L$ with $a \vee b = 1$ there exist $u, v \in L$ with $u \wedge v = 0$, $a \vee u = 1$ and $b \vee v = 1$, which is equivalent to saying that any binary cover $\{a, b\}$ is shrinkable.

The pointfree version of Lefschetz Theorem [13] presented in [4, Proposition 1] improves this by asserting that a frame is normal if and only if each point-finite cover (in the sense of Dowker and Strauss) is shrinkable. Now, since our condition of point-finiteness is stronger than the one by Dowker and Strauss but still weaker than finiteness, the pointfree version of Lefschetz Theorem also holds for our notion:

Proposition 5.7. *A frame L is normal if and only if each point-finite cover is shrinkable.* □

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