Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 06–33

CRITICAL OPERATORS FOR THE DEGREE OF THE MINIMAL POLYNOMIAL OF DERIVATIONS RESTRICTED TO GRASSMANN SPACES

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ABSTRACT: Let V be a finite dimension vector space. For a linear operator on V, f, $D(f)$ denotes the restriction of the derivation associated with f to the mth Grassmann space of V. In [Cyclic Spaces for Grassmann Derivatives and Additive Theory, Bull. London Math. Soc. 26(1994) 140-146] Dias da Silva and Hamidoune obtained a lower bound for the degree of the minimal polynomial of $D(f)$, over an arbitrary field. Over a field of zero characteristic that lower bound is given by

$$\deg\left(P_{D(f)}\right) \geq m(\deg(P_f) - m) + 1 \, .$$

Using additive number theory results, results on the elementary divisors of $D(f)$ and methods presented by Marcus and Ali in [Minimal Polynomials of Additive Commutators and Jordan Products, J. Algebra 22(1972) 12-33] we obtain a characterization of equality cases in the former inequality, over a field of zero characteristic, whenever m does not exceed the number of distinct eigenvalues of f.

KEYWORDS: Grassmann space, derivation, minimal polynomial.
AMS SUBJECT CLASSIFICATION (2000): 15A69.

1. Introduction

Let $\mathbb{F}$ be a field of zero characteristic and let V be a finite dimension vector space over $\mathbb{F}$ such that $\dim V \geq m \geq 2$, where m is an integer. Let S_m be the symmetric group of degree m. For $\sigma \in S_m$, $P(\sigma)$ denotes the unique linear operator on the mth tensor power product of V, $\otimes^m V$, such that

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)} \, ,$$

for all $v_1, v_2, \ldots, v_m \in V$.

Let ε be the alternating character on S_m and consider the symmetrizer defined on $\otimes^m V$ by

$$T_\varepsilon = \frac{1}{m!} \sum_{\sigma \in S_m} \varepsilon(\sigma) P(\sigma) \, .$$

Received July 21, 2006.
This research was done within the activities of "Centro de Matemática da Universidade de Coimbra".

The mth Grassmann space of V is $\wedge^m V = T_\varepsilon(\otimes^m V)$. For $v_1, v_2, \ldots, v_m \in V$, $v_1 \wedge v_2 \wedge \cdots \wedge v_m$ denotes $T_\varepsilon(v_1 \otimes v_2 \otimes \cdots \otimes v_m)$.

For a linear operator, g, on a vector space over $\mathbb{F}$, P_g denotes the minimal polynomial of g and $\deg(P_g)$ denotes its degree. The spectrum of g, i.e., the set of all eigenvalues of g in the algebraic closure of $\mathbb{F}$, is denoted by $\sigma(g)$.

We are going to use the well known fact that, for a simple structure linear operator, the degree of its minimal polynomial is equal to the cardinality of its spectrum.

Let f be a linear operator on V. The derivation associated with f is the linear operator on $\otimes^m V$,

$$f \otimes I_V \otimes \cdots \otimes I_V + I_V \otimes f \otimes \cdots \otimes I_V + \cdots + I_V \otimes I_V \otimes \cdots \otimes f \, .$$

The derivation associated with f commutes with T_ε [2, section 3.2]. Hence, $\wedge^m V$ is an invariant subspace of the derivation associated with f. Let $D(f)$ denote the restriction of the derivation associated with f to $\wedge^m V$. In [1] Dias da Silva and Hamidoune obtained a lower bound for the degree of the minimal polynomial of $D(f)$, over an arbitrary field. Over a field of zero characteristic that lower bound is given by

$$\deg\left(P_{D(f)}\right) \geq m(\deg(P_f) - m) + 1 \, . \tag{1}$$

Using additive number theory results, results on the elementary divisors of $D(f)$ and methods presented in [3] we shall obtain a characterization of equality cases in (1) (for zero characteristic), whenever m does not exceed the number of distinct eigenvalues of f.

2. Additive number theory results

Let r and n be positive integers. By $Q_{r,n}$ we denote the set of all strictly increasing maps from $\{1, \ldots, r\}$ into $\{1, \ldots, n\}$. If $\alpha \in Q_{r,n}$ we use the r-tuple notation for α, that is, $\alpha = (\alpha(1), \ldots, \alpha(r))$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite non-empty subset of $\mathbb{F}$, such that $|A| = n \geq m$, where $|A|$ denotes the cardinality of A.

By $\wedge^m A$ we denote the set of sums of m distinct elements in A, that is,

$$\wedge^m A = \left\{ \sum_{i=1}^{m} a_{\alpha(i)} : \alpha \in Q_{m,n} \right\} .$$

In [1] Dias da Silva and Hamidoune obtained a lower bound for the cardinality of $\wedge^m A$, for A subset of an arbitrary field. In zero characteristic that lower bound is given by

$$| \wedge^m A| \geq m(|A| - m) + 1\,. \tag{2}$$

For subsets of $\mathbb{Q}$ it is well known a characterization of equality cases in (2).

Lemma 1. [6, Theorem 1.10] *Let A be a finite subset of $\mathbb{Q}$ such that $|A| \geq m \geq 2$. Then*

$$| \wedge^m A| = m(|A| - m) + 1$$

if and only if one of the following cases holds:

- **(1):** $|A| \in \{m, m+1\}$*;*
- **(2):** *A is an arithmetic progression;*
- **(3):** *$m = 2$, $|A| = 4$ and there exist $a \in \mathbb{Q}$, $q, q' \in \mathbb{Q} \setminus \{0\}$ such that $q \neq q'$, $q + q' \neq 0$ and*

$$A = a + \{0, q, q', q + q'\}\,.$$

The next two lemmas will be used to prove that Lemma 1 holds in any field of zero characteristic. The first one plays a similar role to the one played by Lemma 2.3 in [3].

Lemma 2. *Let $k, m \in \mathbb{N}$ be such that $k \geq m \geq 2$ and let $\varphi \in S_k$. Then*

$$\left| \left\{ \left(\sum_{i=1}^{m} \alpha(i), \sum_{i=1}^{m} \varphi(\alpha(i)) \right) : \alpha \in Q_{m,k} \right\} \right| = m(k - m) + 1$$

if and only if one of the following cases holds:

- **(1):** $k \in \{m, m+1\}$*;*
- **(2):** $\varphi \in \left\{ \mathrm{id}, \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix} \right\}$*;*
- **(3):** *$m = 2$, $k = 4$ and* $\varphi \in \{\mathrm{id}, (2\,3), (1\,4), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3), (1\,2\,4\,3), (1\,3\,4\,2)\}$.

Proof

Let $\varphi \in S_k$ and $\mathcal{S} = \left\{ \left(\sum_{i=1}^{m} \alpha(i), \sum_{i=1}^{m} \varphi(\alpha(i)) \right) : \alpha \in Q_{m,k} \right\}$. Note that

$$\begin{aligned} |\mathcal{S}| \geq \left| \left\{ \sum_{i=1}^{m} \alpha(i) : \alpha \in Q_{m,k} \right\} \right| &= \left| \left[\frac{m(m+1)}{2}, \frac{m(2k-m+1)}{2} \right] \cap \mathbb{N} \right| \\ &= m(k - m) + 1\,. \end{aligned} \tag{3}$$

Sufficient condition

(1): If $k \in \{m, m+1\}$ then $|Q_{m,k}| = m(k-m)+1$. Since $|\mathcal{S}| \le |Q_{m,k}|$, from (3), we have $|\mathcal{S}| = m(k-m)+1$.

(2): If $\varphi = \mathrm{id}$ then $|\mathcal{S}| = |\{\sum_{i=1}^m \alpha(i) : \alpha \in Q_{m,k}\}| = m(k-m)+1$.

Suppose $\varphi = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix}$.

Then

$$\begin{aligned} &\left|\left\{\left(\sum_{i=1}^m \alpha(i), \sum_{i=1}^m \varphi(\alpha(i))\right) : \alpha \in Q_{m,k}\right\}\right| \\ &\qquad = \left|\left\{\left(\sum_{i=1}^m \alpha(i), m(k+1) - \sum_{i=1}^m \alpha(i)\right) : \alpha \in Q_{m,k}\right\}\right| \\ &\qquad = m(k-m)+1 . \end{aligned}$$

(3): Suppose $m=2$, $k=4$ and $\varphi \in \{\mathrm{id}, (2\,3), (1\,4), (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3), (1\,2\,4\,3), (1\,3\,4\,2)\}$.
Since $Q_{2,4} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$,

$$\begin{aligned} \mathcal{S} &= \{(3, \varphi(1)+\varphi(2)), (4, \varphi(1)+\varphi(3)), (5, \varphi(1)+\varphi(4)), \\ &\qquad (5, \varphi(2)+\varphi(3)), (6, \varphi(2)+\varphi(4)), (7, \varphi(3)+\varphi(4))\} . \end{aligned}$$

For any possible choice of φ we have $\varphi(1)+\varphi(4) = \varphi(2)+\varphi(3)$. Therefore $|\mathcal{S}| = 5$.

Necessary condition

Let $\varphi \in S_k$ and suppose $|\mathcal{S}| = m(k-m)+1$, where $k \ge m+2$.
Consider the sets of positive integers, $B_1, \ldots, B_m$, given by

$$\begin{aligned} B_1 &= \{1 + \cdots + (m-2) + (m-1) + i : i = m, \ldots, k\} \\ &= \tfrac{m(m+1)}{2} + \{0, 1, \ldots, k-m\} ; \\ B_j &= \{\underbrace{1 + \cdots + (m-j)}_{m-j} + i + \underbrace{(k-j+2) + \cdots + k}_{j-1} : i = m-j+2, \ldots, \\ &\qquad\qquad k-j+1\} \\ &= \tfrac{m(m+1)}{2} + (k-m)(j-1) + \{1, 2, \ldots, k-m\} , \qquad j = 2, \ldots m . \end{aligned}$$

Then $\{\sum_{i=1}^m \alpha(i) : \alpha \in Q_{m,k}\} = \left[\frac{m(m+1)}{2}, \frac{m(2k-m+1)}{2}\right] \cap \mathbb{N}$ is the disjoint union of the sets $B_1, B_2, \ldots, B_m$.

For $j = 1, \ldots, m$, let $\mathcal{S}_j = \{(a,b) \in \mathcal{S} : a \in B_j\}$. Then $|\mathcal{S}_j| \ge |B_j|$, for all j, and $\mathcal{S}$ is the disjoint union of $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_m$.

From

$$m(k-m)+1 = |\mathcal{S}| = \sum_{j=1}^{m} |\mathcal{S}_j| \geq \sum_{j=1}^{m} |B_j| = m(k-m)+1\,,$$

it follows that $|\mathcal{S}_j| = |B_j|$, $j = 1, \ldots, m$.

Let $j \in \{1, \ldots, m-1\}$ and $\ell \in \{m-j+2, \ldots, k-j\}$. Define $\alpha_{\ell,j}, \beta_{\ell,j} \in Q_{m,k}$ by

$$\alpha_{\ell,j} = (\underbrace{1, \ldots, m-j-1}_{m-j-1}, m-j+1, \ell, \underbrace{k-j+2, \ldots, k}_{j-1})$$

$$\beta_{\ell,j} = (\underbrace{1, \ldots, m-j-1}_{m-j-1}, m-j, \ell+1, \underbrace{k-j+2, \ldots, k}_{j-1})\,.$$

Since $\sum_{i=1}^{m} \alpha_{\ell,j}(i) = \sum_{i=1}^{m} \beta_{\ell,j}(i) \in B_j$,

$$\left(\sum_{i=1}^{m} \alpha_{\ell,j}(i), \sum_{i=1}^{m} \varphi\left(\alpha_{\ell,j}(i)\right)\right), \left(\sum_{i=1}^{m} \beta_{\ell,j}(i), \sum_{i=1}^{m} \varphi\left(\beta_{\ell,j}(i)\right)\right) \in \mathcal{S}_j\,.$$

From $|\mathcal{S}_j| = |B_j|$ it follows that $\sum_{i=1}^{m} \varphi\left(\alpha_{\ell,j}(i)\right) = \sum_{i=1}^{m} \varphi\left(\beta_{\ell,j}(i)\right)$, that is, $\varphi(m-j+1) + \varphi(\ell) = \varphi(m-j) + \varphi(\ell+1)$.

Hence we have proved that

$$\varphi(m-j+1) - \varphi(m-j) = \varphi(\ell+1) - \varphi(\ell)\,, \tag{4}$$

for $j = 1, \ldots, m-1$ and $\ell = m-j+2, \ldots, k-j$.

(I): $k \geq m+3$

First suppose $m = 2$. From (4) we have $\varphi(\ell+1) - \varphi(\ell) = \varphi(2) - \varphi(1)$, for $\ell = 3, \ldots, k-1$. Since $k \geq 5$, $\alpha = (3, k-1)$ and $\beta = (2, k)$ are in $Q_{2,k}$. From $\alpha(1) + \alpha(2) = \beta(1) + \beta(2) \in B_2$ and $|\mathcal{S}_2| = |B_2|$, it follows that $\varphi(3) + \varphi(k-1) = \varphi(2) + \varphi(k)$ and so, $\varphi(3) - \varphi(2) = \varphi(k) - \varphi(k-1) = \varphi(2) - \varphi(1)$. Hence, for $m = 2$ we have

$$\varphi(i+1) - \varphi(i) = \varphi(2) - \varphi(1)\,, \quad i = 1, 2, \ldots, k-1\,.$$

Next we prove that this is also true for $m \geq 3$. Suppose $m \geq 3$ and let $i \in \{1, \ldots, m-2\}$.

Taking $j = i$ and $\ell = m-i+2$ in (4) we obtain $\varphi(m-i+1) - \varphi(m-i) = \varphi(m-i+3) - \varphi(m-i+2)$.

Taking $j = i+1$ and $\ell = m-(i+1)+3 \leq k-(i+1)$ in (4) we obtain $\varphi(m-i)-\varphi(m-i-1) = \varphi(m-i+3)-\varphi(m-i+2)$.

Then $\varphi(m-i+1)-\varphi(m-i) = \varphi(m-i)-\varphi(m-i-1)$, for $i = 1,\ldots,m-2$.

Hence

$$\varphi(i+1)-\varphi(i) = \varphi(2)-\varphi(1)\,, \quad i = 1,\ldots,m-1\,.$$

Taking $j = 2$ and $\ell = m$ in (4) we get $\varphi(m+1)-\varphi(m) = \varphi(m-1)-\varphi(m-2) = \varphi(2)-\varphi(1)$.

For $i = m+1,\ldots,k-1$, taking $j = 1$ and $\ell = i$ in (4) we have $\varphi(i+1)-\varphi(i) = \varphi(m)-\varphi(m-1) = \varphi(2)-\varphi(1)$.

Thus we have proved that

$$\varphi(i+1)-\varphi(i) = \varphi(2)-\varphi(1)\,, \quad i = 1,\ldots,k-1\,.$$

Let $r = \varphi(2)-\varphi(1) \neq 0$. Then $\varphi(i) = \varphi(1)+(i-1)r$, for $i = 1,\ldots,k$.

If $r > 0$ then $\varphi(1) < \varphi(2) < \cdots < \varphi(k)$ and $\varphi = \mathrm{id}$.

If $r < 0$ then $\varphi(1) > \varphi(2) > \cdots > \varphi(k)$ and

$$\varphi = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix}.$$

(II): $k = m+2$

In this case, from (4), we have

$$\varphi(k-j+1)-\varphi(k-j) = \varphi(k-j-1)-\varphi(k-j-2)\,, \quad j = 1,\ldots,k-3\,. \quad (5)$$

That is,

$$\varphi(k)-\varphi(k-1) = \varphi(k-2)-\varphi(k-3) = \cdots = \begin{cases} \varphi(2)-\varphi(1) & \text{if } k \text{ is even} \\ \varphi(3)-\varphi(2) & \text{if } k \text{ is odd} \end{cases}$$

and

$$\varphi(k-1)-\varphi(k-2) = \varphi(k-3)-\varphi(k-4) = \cdots = \begin{cases} \varphi(3)-\varphi(2) & \text{if } k \text{ is even} \\ \varphi(2)-\varphi(1) & \text{if } k \text{ is odd} \end{cases}.$$

Let

$$r = \begin{cases} \varphi(2)-\varphi(1) & \text{if } k \text{ is even} \\ \varphi(3)-\varphi(2) & \text{if } k \text{ is odd} \end{cases} \quad \text{and} \quad r' = \begin{cases} \varphi(3)-\varphi(2) & \text{if } k \text{ is even} \\ \varphi(2)-\varphi(1) & \text{if } k \text{ is odd} \end{cases}.$$

Suppose $k \geq 5$. Since $m = k-2$, $B_1 = \frac{(k-2)(k-1)}{2} + \{0,1,2\}$ and $B_2 = \frac{(k-2)(k-1)}{2} + \{3,4\}$. Let

$$\alpha = (1,2,\ldots,k-4,k-2,k),\ \beta = (1,2,\ldots,k-5,k-3,k-2,k-1) \in Q_{k-2,k}\,.$$

Since $\sum_{i=1}^{k-2} \alpha(i) = \sum_{i=1}^{k-2} \beta(i) = \frac{(k-2)(k-1)}{2} + 3 \in B_2$ and $|\mathcal{S}_2| = |B_2|$, we have $\sum_{i=1}^{k-2} \varphi(\alpha(i)) = \sum_{i=1}^{k-2} \varphi(\beta(i))$, that is, $\varphi(k-4) + \varphi(k) = \varphi(k-3) + \varphi(k-1)$.

Then $r = r'$ and $\varphi(i+1) - \varphi(i) = \varphi(2) - \varphi(1)$, for $i = 1, \ldots, k-1$. As we have seen in (I),

$$\varphi \in \left\{ \mathrm{id}, \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix} \right\}.$$

For $k = 4$ and $m = 2$, from (5), we have $\varphi(1) + \varphi(4) = \varphi(2) + \varphi(3)$. Then $\{\varphi(1), \varphi(4)\} \in \{\{1, 4\}, \{2, 3\}\}$ and case (3) holds. ∎

Next lemma is a straightforward generalization of Lemma 2.1 from [3].

Lemma 3. *Let $m \geq 2$ and let V be an n-dimensional vector space over a field of zero characteristic, $\mathbb{F}$. Let $k \in \mathbb{N}$ and let $u_1, \ldots, u_k \in V$ be distinct. Then there exists a basis $\{f_1, \ldots, f_n\}$ of V^*, such that, for each $j \in \{1, \ldots, n\}$, $f_j(u_1), \ldots, f_j(u_k)$ are k distinct elements in $\mathbb{F}$ and*

$$\left| \left\{ \sum_{i=1}^{m} u_{\alpha(i)} : \alpha \in Q_{m,k} \right\} \right| \geq |\wedge^m \{f_j(u_1), \ldots, f_j(u_k)\}| \geq m(k-m) + 1\,.$$

Proposition 1. *Let $\mathbb{F}$ be a field of zero characteristic and let A be a finite subset of $\mathbb{F}$ such that $|A| \geq m \geq 2$. Then*

$$|\wedge^m A| = m(|A| - m) + 1$$

if and only if one of the following cases holds:

(1): *$|A| \in \{m, m+1\}$;*

(2): *A is an arithmetic progression;*

(3): *$m = 2$, $|A| = 4$ and there exist $a \in \mathbb{F}$, $q, q' \in \mathbb{F} \setminus \{0\}$ such that $q \neq q'$, $q + q' \neq 0$ and*

$$A = a + \{0, q, q', q + q'\}\,.$$

Proof The sufficient condition's proof is obvious, so we include only the necessary condition's proof. Suppose $A = \{a_1, \ldots, a_k\}$, where $k = |A| \geq m + 2 \geq 4$, and $|\wedge^m A| = m(k-m) + 1$.

Consider the vector space over $\mathbb{Q}$,

$$W = \left\{ \sum_{i=1}^{k} q_i a_i : q_i \in \mathbb{Q} \right\}$$

and let $n = \dim_{\mathbb{Q}} W \leq k$. From Lemma 3 there exists a basis of W^*, $\{f_1, \ldots, f_n\}$, such that, for $t = 1, \ldots, n$,

$$|\{f_t(a_1), \ldots, f_t(a_k)\}| = k$$

and

$$|\wedge^m A| \geq |\wedge^m \{f_t(a_1), \ldots, f_t(a_k)\}| \geq m(k-m) + 1 \,.$$

Since $|\wedge^m A| = m(k-m)+1$, it follows that

$$|\wedge^m \{f_t(a_1), \ldots, f_t(a_k)\}| = m(k-m) + 1 \,, \quad t = 1, \ldots, n \,. \tag{6}$$

From (6) and Lemma 1, for each $t \in \{1, \ldots, n\}$, one of the following cases holds.

(i): $\{f_t(a_1), \ldots, f_t(a_k)\}$ is an arithmetic progression;

(ii): $m = 2$, $k = 4$ and there exist $b'_t \in \mathbb{Q}$, $q_t, q'_t \in \mathbb{Q} \setminus \{0\}$ such that $q_t \neq q'_t$, $q_t + q'_t \neq 0$ and

$$\{f_t(a_1), f_t(a_2), f_t(a_3), f_t(a_4)\} = b'_t + \{0, q_t, q'_t, q_t + q'_t\} \,.$$

First we assume that $k \geq 5$ or $m \geq 3$. Then, for each $t \in \{1, \ldots, n\}$, $\{f_t(a_1), \ldots, f_t(a_k)\}$ is an arithmetic progression. For some $\varphi_t \in S_k$, $r_t \in \mathbb{Q} \setminus \{0\}$ and $b_t \in \mathbb{Q}$ we have

$$f_t(a_i) = b_t + r_t \varphi_t(i) \,, \quad i = 1, \ldots, k \,.$$

Considering the basis $\left\{ \frac{1}{r_1} f_1, \frac{1}{r_1} f_2, \ldots, \frac{1}{r_1} f_n \right\}$ of W^*, for which Lemma 3 is also true, we can assume that $r_1 = 1$.

Let $\{e_1, \ldots, e_n\}$ be the basis of W having $\{f_1, f_2, \ldots, f_n\}$ as its dual. Then

$$a_i = \sum_{j=1}^{n} f_j(a_i) e_j = \sum_{j=1}^{n} \left(b_j + r_j \varphi_j(i)\right) e_j \,, \quad i = 1, \ldots, k \,. \tag{7}$$

Reordering $a_1, \ldots, a_k$ in such way that

$$f_1(a_i) = b_1 + i \,, \quad i = 1, \ldots, k$$

we may suppose that $\varphi_1 = \mathrm{id}$.

From (7) it follows that

$$a_i = (b_1 + i)e_1 + \sum_{j=2}^{n} (b_j + r_j\varphi_j(i))\, e_j\,, \quad i = 1, \ldots, k\,.$$

Let $b = \sum_{j=1}^{n} b_j e_j \in W$ and consider the elements in W given by $d_i = a_i - b$, for $i = 1, \ldots, k$. Since $| \wedge^m \{d_1, \ldots, d_k\}| = | \wedge^m A|$, and A is an arithmetic progression if and only if $\{d_1, \ldots, d_k\}$ is an arithmetic progression, we may assume that $b = 0$, that is, we may assume that $b_j = 0$, for $j = 1, \ldots, n$.

Then

$$a_i = ie_1 + \sum_{j=2}^{n} r_j\varphi_j(i)e_j\,, \quad i = 1, \ldots, k\,.$$

If $n = 1$ then $a_i = ie_1$ for $i = 1, \ldots, k$ and A is an arithmetic progression. Suppose $n > 1$ and let $t \in \{2, \ldots, n\}$. For each $\alpha \in Q_{m,k}$ we have

$$\begin{aligned}
\sum_{\ell=1}^{m} a_{\alpha(\ell)} &= \sum_{\ell=1}^{m} \alpha(\ell)e_1 + \sum_{\ell=1}^{m}\sum_{j=2}^{n} r_j\varphi_j(\alpha(\ell))e_j \\
&= \sum_{\ell=1}^{m} \alpha(\ell)e_1 + r_t\left(\sum_{\ell=1}^{m} \varphi_t(\alpha(\ell))\right) e_t + \sum_{\substack{j=2\\ j\neq t}}^{n}\sum_{\ell=1}^{m} \varphi_j(\alpha(\ell))r_je_j
\end{aligned}$$

and, since $e_1, \ldots, e_n$ are linearly independent,

$$\begin{aligned}
m(k-m)+1 = | \wedge^m A| &\geq \left|\left\{\left(\sum_{\ell=1}^{m} \alpha(\ell), \sum_{\ell=1}^{m} \varphi_t(\alpha(\ell))\right) : \alpha \in Q_{m,k}\right\}\right| \\
&\geq m(k-m)+1\,.
\end{aligned}$$

Then

$$\left|\left\{\left(\sum_{\ell=1}^{m} \alpha(\ell), \sum_{\ell=1}^{m} \varphi_t(\alpha(\ell))\right) : \alpha \in Q_{m,k}\right\}\right| = m(k-m)+1$$

and, from Lemma 2,

$$\varphi_t \in \left\{\mathrm{id}, \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix}\right\}\,, \quad t = 2, \ldots, n\,.$$

Suppose $\varphi_{t_1} = \varphi_{t_2} = \cdots = \varphi_{t_p} = \mathrm{id}$ and

$$\varphi_{s_1} = \varphi_{s_2} = \cdots = \varphi_{s_q} = \begin{pmatrix} 1 & 2 & \cdots & k-1 & k \\ k & k-1 & \cdots & 2 & 1 \end{pmatrix},$$

where $p+q=n$, $t_1 = 1$ and $\{t_2, \ldots, t_p, s_1, s_2, \ldots, s_q\} = \{2, \ldots, n\}$. Then, for $i = 1, \ldots, k$,

$$\begin{aligned} a_i &= ie_1 + \sum_{j=2}^{n} r_j \varphi_j(i) e_j \\ &= ie_1 + \sum_{j=1}^{p} r_{t_j} i e_{t_j} + \sum_{j=1}^{q} r_{s_j}(k-i+1)e_{s_j} \\ &= i\left(e_1 + \sum_{j=1}^{p} r_{t_j} e_{t_j} - \sum_{j=1}^{q} r_{s_j} e_{s_j}\right) + (k+1)\sum_{j=1}^{q} r_{s_j} e_{s_j} \end{aligned}$$

and A is an arithmetic progression.

Next suppose $k=4$ and $m=2$. For $t = 1, \ldots, n \leq 4$ there exist $b'_t \in \mathbb{Q}$, $q_t, q'_t \in \mathbb{Q} \setminus \{0\}$ such that $q_t \neq q'_t$, $q_t + q'_t \neq 0$ and

$$\{f_t(a_1), f_t(a_2), f_t(a_3), f_t(a_4)\} = b'_t + \{0, q_t, q'_t, q_t + q'_t\}.$$

(This includes both cases (i) and (ii)).

For $j \in \{1, \ldots, n\}$ there exists a permutation $\varphi_j \in S_4$ such that

$$f_j(a_i) = b'_j + \left(\left\lfloor \frac{\varphi_j(i)}{2} \right\rfloor - \left\lfloor \frac{\varphi_j(i)-1}{2} \right\rfloor\right) q_j + \left\lfloor \frac{\varphi_j(i)-1}{2} \right\rfloor q'_j, \quad i = 1, 2, 3, 4,$$

where, for $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

As in case $k \geq 5$ or $m \geq 3$ we can assume that $q_1 = 1$, $\varphi_1 = \mathrm{id}$ and $b'_j = 0$, for each j. Then, for $i = 1, 2, 3, 4$,

$$a_i = \sum_{j=1}^{n} f_j(a_i) e_j = \left(\left\lfloor \frac{i}{2} \right\rfloor - \left\lfloor \frac{i-1}{2} \right\rfloor + \left\lfloor \frac{i-1}{2} \right\rfloor q'_1\right) e_1 + \sum_{j=2}^{n} f_j(a_i) e_j$$

and

$$\begin{aligned}
&a_1 + a_2 = e_1 + \textstyle\sum_{j=2}^{n} (f_j(a_1) + f_j(a_2))e_j;\\
&a_1 + a_3 = q_1' e_1 + \textstyle\sum_{j=2}^{n} (f_j(a_1) + f_j(a_3))e_j;\\
&a_1 + a_4 = (1 + q_1')e_1 + \textstyle\sum_{j=2}^{n} (f_j(a_1) + f_j(a_4))e_j;\\
&a_2 + a_3 = (1 + q_1')e_1 + \textstyle\sum_{j=2}^{n} (f_j(a_2) + f_j(a_3))e_j;\\
&a_2 + a_4 = (2 + q_1')e_1 + \textstyle\sum_{j=2}^{n} (f_j(a_2) + f_j(a_4))e_j;\\
&a_3 + a_4 = (1 + 2q_1')e_1 + \textstyle\sum_{j=2}^{n} (f_j(a_3) + f_j(a_4))e_j.
\end{aligned}$$

Since $q_1' \neq 0$, $q_1' \neq 1 = q_1$, $1 + q_1' = q_1 + q_1' \neq 0$ and $|\wedge^2 A| = 5$, it follows that $a_2 + a_3 = a_1 + a_4$ and (3) holds. ∎

3. Elementary divisors

Let $m \geq 2$, let $\mathbb{F}$ be a field of zero characteristic and let V be a finite dimension vector space over $\mathbb{F}$ such that $\dim V \geq m$. Let f be a linear operator on V. The following characterization of the elementary divisors of $D(f)$ is well known ([4, 5]).

Let

$$(X - \mu_i)^{n_i}, \quad i = 1, 2, \ldots, \ell$$

be the elementary divisors of f, where $\mu_1, \ldots, \mu_\ell \in \overline{\mathbb{F}}$ are not necessarily distinct. Let $k_1, k_2, \ldots, k_\ell$ be nonnegative integers such that

$$k_1 + k_2 + \cdots + k_\ell = m \text{ and } k_i \leq n_i, \quad i = 1, 2, \ldots, \ell. \tag{8}$$

Let $r_1, r_2, \ldots, r_\ell$ be nonnegative integers such that

$$2r_i \leq k_i(n_i - k_i), \quad i = 1, 2, \ldots, \ell. \tag{9}$$

For $s \in \{1, 2, \ldots, \ell\}$ define

$$E_s = k_s(n_s - k_s) - 2r_s + 1 \quad \text{and} \quad \mathcal{E}_s = \sum_{i=1}^{s} E_i.$$

For $t_1, t_2, \ldots, t_{\ell-1}$ integers such that

$$1 \leq t_s \leq \min\{\mathcal{E}_s - 2(t_1 + \cdots + t_{s-1}) + s - 1, E_{s+1}\}, \quad s = 1, \ldots, \ell - 1, \tag{10}$$

define

$$\eta(r_1, \ldots, r_\ell, t_1, \ldots, t_{\ell-1}) = \mathcal{E}_\ell - 2(t_1 + t_2 + \cdots + t_{\ell-1}) + \ell - 1.$$

Let $s \in \{1, 2, \ldots, \ell\}$. For each positive integer j we denote by $p_{s,j}$ the number of partitions of j into not more than k_s parts, each part at most $n_s - k_s$ and define $p_{s,0} = 1$.

For each $s \in \{1, 2, \ldots, \ell\}$ let

$$c_s = \begin{cases} 1 \text{ if } r_s = 0 \\ p_{s,r_s} - p_{s,r_s-1}\,, \text{ if } r_s > 0 \end{cases} .$$

Theorem 1. [4, 5] *The elementary divisors of $D(f)$ are*

$$\left(X - \sum_{s=1}^{\ell} k_s \mu_s\right)^{\eta(r_1,\ldots,r_\ell,t_1,\ldots,t_{\ell-1})}, \quad c_1 c_2 \cdots c_\ell \text{ times,}$$

when $k_1, \ldots, k_\ell, r_1, \ldots r_\ell, t_1, \ldots, t_{\ell-1}$ run over the sets of nonnegative integers satisfying (8), (9) and (10).

Remark 1. *For $k_1, \ldots, k_\ell, r_1, \ldots r_\ell, t_1, \ldots, t_{\ell-1}$ satisfying (8), (9) and (10), we have*

$$\eta(r_1, \ldots, r_\ell, t_1, \ldots, t_{\ell-1}) \leq \mathcal{E}_\ell - \ell + 1 \leq \sum_{s=1}^{\ell} k_s(n_s - k_s) + 1 .$$

Remark 2. *If we consider $r_1 = \cdots = r_\ell = 0$ and $t_1 = \cdots = t_{\ell-1} = 1$, we obtain $c_1 = \cdots = c_\ell = 1$ and*

$$\eta(\underbrace{0, \ldots, 0}_{\ell}, \underbrace{1, \ldots, 1}_{\ell-1}) = \sum_{s=1}^{\ell} k_s(n_s - k_s) + 1 .$$

It follows that, if $k_1 + \cdots + k_\ell = m$ and $0 \leq k_i \leq n_i$, $i = 1, \ldots, \ell$, then

$$\left(X - \sum_{s=1}^{\ell} k_s \mu_s\right)^{\sum_{s=1}^{\ell} k_s(n_s - k_s)+1}$$

is an elementary divisor of $D(f)$.

The following well know results can be obtained as corollaries from Theorem 1.

Corollary 1. *If $a_1, \ldots, a_r \in \overline{\mathbb{F}}$ are the distinct eigenvalues of f and*

$$(X - a_i)^{n_{i,j}}, \quad j = 1, 2, \ldots, s_i\,, \quad i = 1, \ldots, r$$

are the elementary divisors of f then

$$\sigma(D(f)) = \left\{ \sum_{i=1}^{r} m_i a_i : \sum_{i=1}^{r} m_i = m\,,\ m_i \in \mathbb{N}_0 \text{ and } m_i \leq \sum_{j=1}^{s_i} n_{i,j}\,, i = 1, \ldots, r \right\} .$$

Corollary 2. *If f is of simple structure then also $D(f)$ is of simple structure.*

Corollary 3.

(1) $\wedge^m \sigma(f) \subseteq \sigma(D(f))$;
(2) *If* $\dim V = |\sigma(f)|$ *then* $\wedge^m \sigma(f) = \sigma(D(f))$.

For $m = 2$ there is a considerably simpler characterization for the elementary divisors of $D(f)$.

Theorem 2. [2, Chapter 7, Theorem 2.6] *Let*

$$(X - \mu_i)^{n_i}, \quad i = 1, 2, \ldots, \ell,$$

be the elementary divisors of f, where $\mu_1, \ldots, \mu_\ell \in \overline{\mathbb{F}}$ are not necessarily distinct. The elementary divisors of the restriction of the derivation associated with f to $\wedge^2 V$ are:

$$(X - 2\mu_i)^k, \quad k = 2n_i - 3, 2n_i - 7, \ldots, \begin{cases} 1 \text{ if } n_i \text{ is even} \\ 3 \text{ if } n_i \text{ is odd} \end{cases}, \quad 1 \le i \le \ell$$

and

$$(X - \mu_i - \mu_j)^{n_i + n_j - 2t + 1}, \quad 1 \le t \le \min\{n_i, n_j\}, \quad 1 \le i < j \le \ell.$$

4. Main Result

Theorem 3. *Let $\mathbb{F}$ be a field of zero characteristic, let V be a vector space over $\mathbb{F}$ with finite dimension $n \ge m$ and let f be a linear operator on V. Suppose $r := |\sigma(f)| \ge m$. Let $D(f)$ be the restriction of the derivation associated with f to $\wedge^m V$. Then*

$$\deg(P_{D(f)}) = m(\deg(P_f) - m) + 1$$

if and only if one of the following cases holds:

(1): $r = m = n$;
(2): $r = m + 1 = n$;
(3): *The elementary divisors of f are*

$$X - b_1, \ldots, X - b_{m-1}, (X - b_m)^2,$$

where $b_1, \ldots, b_m \in \overline{\mathbb{F}}$ are distinct;
(4): $r \ge m + 1$ *and the elementary divisors of f are*

$$X - b_i, \quad s_i \text{ times}, \quad i = 1, \ldots, r,$$

where $b_1, \ldots, b_r$ is an arithmetic progression with first term b_1, $s_1 = \cdots = s_{m-1} = 1$ and $s_{r-m+2} = \cdots = s_r = 1$;

(5): *$m=2$ and the elementary divisors of f are*

$$X-b\,,\quad (X-b-q)^2\,,\quad X-b-2q\,,$$

where $b,q\in\overline{\mathbb{F}}$ and $q\neq 0$;

(6): *$m=2$ and the elementary divisors of f are*

$$X-b\,,\quad X-b-q\,,\quad X-b-q'\,,\quad X-b-q-q'\,,$$

where $b\in\overline{\mathbb{F}}$, $q,q'\in\overline{\mathbb{F}}\setminus\{0\}$, $q\neq q'$ and $q+q'\neq 0$;

(7): *$m=2$ and the elementary divisors of f are*

$$(X-b_1)^2,(X-b_2)^2\,,$$

where $b_1,b_2\in\overline{\mathbb{F}}$ and $b_1\neq b_2$.

Proof

<u>Sufficient condition</u>

(1), (2) and (6): In any of these cases f is of simple structure and $\dim V=|\sigma(f)|$. Then (Corollaries 2, 3 and Proposition 1)

$$\deg(P_{D(f)})=|\sigma(D(f))|=|\wedge^m\sigma(f)|=m(r-m)+1=m(\deg(P_f)-m)+1\,.$$

(3): By Corollary 1, the eigenvalues of $D(f)$ are the m elements

$$z_i=b_m+\sum_{\substack{j=1\\ j\neq i}}^{m}b_j\,,\quad i=1,\dots,m$$

and (Remark 2) $X-z_1,X-z_2,\dots,X-z_{m-1},(X-z_m)^2$ are elementary divisors of $D(f)$. Since $\dim\wedge^mV=\binom{m+1}{m}=m+1$, it follows that

$$P_{D(f)}=(X-z_m)^2\prod_{i=1}^{m-1}(X-z_i)$$

and $\deg(P_{D(f)})=m+1=m(\deg(P_f)-m)+1$.

(4): Suppose $b_i=b_1+(i-1)q$, where $q\in\overline{\mathbb{F}}\setminus\{0\}$. From Corollary 1, $\sigma(D(f))$ is the set

$$\left\{mb_1+q\sum_{i=1}^{r}m_i(i-1):\sum_{i=1}^{r}m_i=m\ \text{ and }\ 0\leq m_i\leq s_i\,,i=1,\dots,r\right\}.$$

Since $s_1=\cdots=s_{m-1}=1$ and $s_{r-m+2}=\cdots=s_r=1$,

$$\left\{\sum_{i=1}^{r} m_i(i-1) : m_1+\cdots+m_r = m \text{ and } 0 \le m_i \le s_i\,, i=1,\ldots,r\right\}$$
$$= \left[\frac{m(m-1)}{2}, mr - \frac{m(m+1)}{2}\right] \cap \mathbb{N}\,.$$

Then

$$\sigma(D(f)) = \left\{mb_1 + qz : z \in \left[\frac{m(m-1)}{2}, mr - \frac{m(m+1)}{2}\right] \cap \mathbb{N}\right\} = \wedge^m \sigma(f)\,.$$

Since f is of simple structure, also $D(f)$ is of simple structure and $\deg(P_{D(f)}) = |\sigma(D(f))| = rm - m^2 + 1 = m\deg(P_f) - m^2 + 1$.

(5): From Theorem 2 the elementary divisors of $D(f)$ are

$$(X-2b-q)^2\,, \quad X-2b-2q\,, \quad X-2b-2q\,, \quad (X-2b-3q)^2\,.$$

Then $P_{D(f)} = (X-2b-2q)(X-2b-q)^2(X-2b-3q)^2$ and $\deg(P_{D(f)}) = 5 = 2\deg(P_f) - 3$.

(7): In this case $P_{D(f)} = (X-2b_1)(X-2b_2)(X-b_1-b_2)^3$ and $\deg(P_{D(f)}) = 5 = 2\deg(P_f) - 3$.

Necessary condition

Suppose $\deg(P_{D(f)}) = m\deg(P_f) - m^2 + 1$. Let $a_1, \ldots, a_r \in \overline{\mathbb{F}}$ (where $r \ge m$) be the distinct eigenvalues of f and let

$$(X-a_i)^{n_{i,j}}\,, \quad j = 1,2,\ldots,t_i\,, \quad i = 1,\ldots,r$$

be the elementary divisors of f, where, for each i, $n_i := n_{i,1} \ge n_{i,2} \ge \cdots \ge n_{i,t_i}$. Then $P_f = (X-a_1)^{n_1}\cdots(X-a_r)^{n_r}$.

Consider the $\mathbb{Q}$-vector space, $W = \{\sum_{i=1}^r q_i a_i : q_i \in \mathbb{Q}\}$. Let d be its dimension and let $\{f_1, \ldots, f_d\}$ be a basis of W^* satisfying the conditions in Lemma 3, for the distinct elements in W, $a_1, a_2, \ldots, a_r$.

From Lemma 3, $f_1(a_1), f_1(a_2), \ldots, f_1(a_r)$ are distinct rational numbers. Without loss of generality we may assume that $a_1, a_2, \ldots, a_r$ are ordered in such way that

$$f_1(a_1) < f_1(a_2) < \cdots < f_1(a_r)\,. \tag{11}$$

We consider two cases: $r = m$ and $r \ge m+1$.

(I): $r = m$:

Suppose that $n_i \ge 2$ for some $i \in \{1, \ldots, r\}$. Let ℓ be the biggest element in $\{1, \ldots, r\}$ such that

$$n_\ell = \max\{n_i : i = 1, \ldots, r\} \ge 2$$

and let k be the biggest element in $\{1,\ldots,r\}\setminus\{\ell\}$ such that

$$n_k = \max\{n_i : i = 1,\ldots,\ell-1,\ell+1\ \ldots,r\}\,.$$

Let

$$z_i = a_\ell + \sum_{\substack{j=1\\ j\neq i}}^{m} a_j\,, \quad i = 1,\ldots,m\,.$$

$z_1,\ldots,z_m$ are distinct eigenvalues of $D(f)$ and, since $n_\ell \geq 2$, $(X-z_\ell)^{\sum_{j=1}^m (n_j-1)+1}$ and

$$(X-z_i)^{\sum_{\substack{j=1\\ j\neq i\\ j\neq \ell}}^m (n_j-1)+2(n_\ell-2)+1}\,, \quad i=1,\ldots,m\,, \quad i\neq\ell$$

are elementary divisors of $D(f)$.

Then, for some monic polynomial $q(X)\in\overline{\mathbb{F}}[X]\setminus\{0\}$,

$$P_{D(f)} = q(X)(X-z_\ell)^{\sum_{j=1}^m (n_j-1)+1}\prod_{\substack{i=1\\ i\neq\ell}}^{m}(X-z_i)^{\sum_{\substack{j=1\\ j\neq i\\ j\neq \ell}}^m (n_j-1)+2(n_\ell-2)+1}\,. \qquad (12)$$

$$\begin{aligned}
\deg(q(X)) &= m\deg(P_f) - m^2 + 1 - \deg(P_f) + m - 1 - \sum_{\substack{i=1\\ i\neq\ell}}^{m}\sum_{\substack{j=1\\ j\neq\ell\\ j\neq i}}^{m}(n_j-1) - \\
&\quad (m-1)(2n_\ell-3)\\
&= (m-1)\deg(P_f) - m^2 + m - \sum_{\substack{i=1\\ i\neq\ell}}^{m}\sum_{\substack{j=1\\ j\neq\ell\\ j\neq i}}^{m} n_j + (m-1)(m-2) - \\
&\quad (m-1)(2n_\ell-3)\\
&= (m-1)\deg(P_f) - m^2 - \sum_{\substack{i=1\\ i\neq\ell}}^{m}(\deg(P_f)-n_i-n_\ell) - 2(m-1)n_\ell + \\
&\quad m^2+m-1\\
&= -(m-1)n_\ell + \sum_{\substack{i=1\\ i\neq\ell}}^{m} n_i + m - 1\\
&= \deg(P_f) - mn_\ell + m - 1\,. \qquad (13)
\end{aligned}$$

We consider two subcases:

(i): $n_k \geq 2$:

From the definition of k, $n_\ell \geq n_k$ and $\deg(P_f) \leq n_\ell + (m-1)n_k$. Then $0 \leq \deg(q(X)) \leq (m-1)(n_k - n_\ell + 1)$ and so $n_k \in \{n_\ell, n_\ell - 1\}$. Suppose $n_k = n_\ell - 1$. Then $\deg(q(X)) = 0$ and

$$\sigma(D(f)) = \{z_1, \ldots, z_m\}\,. \tag{14}$$

If $k < \ell$,

$$w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{\substack{j=1\\ j\neq k}}^{m-1} a_j$$

is an eigenvalue of $D(f)$ such that $f_1(w_1) < f_1(z_m) < \cdots < f_1(z_1)$, which contradicts (14).

If $k > \ell$,

$$w_2 = a_k + a_2 + \cdots + a_m = 2a_k + \sum_{\substack{j=2\\ j\neq k}}^{m} a_j$$

is an eigenvalue of $D(f)$ such that $f_1(w_2) > f_1(z_1) > \cdots > f_1(z_m)$, which contradicts (14).

Then $n_k = n_\ell \geq 2$ and, from the definitions of k and ℓ, we have $k < \ell$. Also in this case

$$w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{\substack{j=1\\ j\neq k}}^{m-1} a_j$$

is an eigenvalue of $D(f)$ not in $\{z_1, \ldots, z_m\}$. Therefore

$$(X - w_1)^{\sum_{\substack{j=1\\ j\neq k}}^{m-1}(n_j-1)+2(n_k-2)+1}$$

divides $q(X)$ and, from (13), it follows that

$$\sum_{\substack{j=1\\ j\neq k}}^{m-1}(n_j - 1) + 2(n_k - 2) + 1 \leq \deg(P_f) - mn_\ell + m - 1\,,$$

that is,

$$\deg(P_f) - n_k - n_m - m + 2 + 2n_k - 3 \leq \deg(P_f) - mn_\ell + m - 1\,.$$

Since $n_k = n_\ell \geq n_m$, we obtain $mn_\ell \leq 2m$ and $n_k = n_\ell = 2$. Then $m+2 = r+2 \leq \deg(P_f) \leq 2m$.

If $m = 2$ then $P_f = (X-a_1)^2(X-a_2)^2$, $\sigma(D(f)) = \{2a_1, 2a_2, a_1 + a_2\}$ and (Theorem 2) $(X - a_1 - a_2)^3$ is an elementary divisor of $D(f)$. Since $\deg(P_{D(f)}) = 5$ we have $P_{D(f)} = (X - 2a_1)(X - 2a_2)(X - a_1 - a_2)^3$. Suppose $t_1 \geq 2$. Then $(X - a_1)^{n_{1,2}}$ is another elementary divisor of f associated with a_1. Hence $(X-2a_1)^{2+n_{1,2}-1}$ is an elementary divisor of $D(f)$, which contradicts $n_{1,2} \geq 1$. Then $t_1 = 1$ and, similarly, $t_2 = 1$. Condition (7) holds.

Assume now that $r = m \geq 3$. Suppose $n_q = 2$ for some $q \in \{1, \ldots, m\} \setminus \{\ell, k\}$. Then $\deg(P_f) \geq r+3 = m+3$. From the definitions of ℓ and k we have $q < k < \ell$. Then

$$w_1 = a_k + a_1 + \cdots + a_{m-1} = 2a_k + \sum_{\substack{j=1 \\ j \neq k}}^{m-1} a_j$$

and

$$w_3 = a_q + a_1 + \cdots + a_{m-1} = 2a_q + \sum_{\substack{j=1 \\ j \neq q}}^{m-1} a_j$$

are eigenvalues of $D(f)$ such that $f_1(w_3) < f_1(w_1) < f_1(z_m) < \cdots < f_1(z_1)$.
Therefore,

$$(X - w_1)^{\sum_{\substack{j=1 \\ j\neq k}}^{m-1}(n_j-1)+2(n_k-2)+1} (X - w_3)^{\sum_{\substack{j=1 \\ j\neq q}}^{m-1}(n_j-1)+2(n_q-2)+1}$$

has degree, at most, equal to the degree of $q(X)$, that is,

$$2\deg(P_f) - 2m - 2n_m + n_k + n_q - 2 \leq \deg(P_f) - m - 1\,,$$

which contradicts $\deg(P_f) \geq m+3$, since $n_k = n_q = 2$ and $n_m \leq 2$. So, for $r = m \geq 3$ and $n_k \geq 2$ it must be $n_k = n_\ell = 2$ and $n_i = 1$ for $i \in \{1, \ldots, m\} \setminus \{\ell, k\}$. Then $\deg(P_f) = m+2$ and $\deg(q(X)) = 1$. Since $w_1 \in \sigma(D(f)) \setminus \{z_1, \ldots, z_m\}$, it follows that $q(X) = X - w_1$. Since

$$(X - w_1)^{\sum_{\substack{j=1 \\ j\neq k}}^{m-1}(n_j-1)+2(n_k-2)+1}$$

is an elementary divisor of $D(f)$ it follows that $\ell = m$ and, from (12), we have

$$P_{D(f)} = (X - z_m)^3 \left(\prod_{\substack{i=1 \\ i \neq k}}^{m-1} (X - z_i)^2 \right) (X - z_k)(X - w_1)\,.$$

If $k \leq m-2$, then

$$w_4 = a_k + a_1 + \cdots + a_{m-2} + a_m = 2a_k + a_m + \sum_{\substack{j=1 \\ j \neq k}}^{m-2} a_j$$

is also an eigenvalue of $D(f)$, and again we have a contradiction, since $f_1(w_1) < f_1(w_4) < f_1(z_m) < \cdots < f_1(z_1)$.
Then $k = m-1$. If $m \geq 4$ then $w_5 = a_3 + \cdots + a_{m-2} + 2a_{m-1} + 2a_m$ is also an eigenvalue of $D(f)$ and, from $f_1(w_1) < f_1(z_m) < \cdots < f_1(z_1) < f_1(w_5)$, we have a contradiction.
Then $m = 3$, $\ell = 3$, $k = 2$, $P_f = (X - a_1)(X - a_2)^2(X - a_3)^2$ and

$$P_{D(f)} = (X - z_3)^3 (X - z_1)^2 (X - z_2)(X - w_1)\,.$$

Since $(X - 2a_2 - a_3)^2$ is an elementary divisor of $D(f)$, $2a_2 + a_3 \in \{z_1, z_3\} = \{a_2 + 2a_3, a_1 + a_2 + a_3\}$, and, once more, we obtain a contradiction.

(ii): $n_k = 1$.
In this case $\deg(P_f) = n_\ell + m - 1$ and, from (13), we obtain $0 \leq \deg(q(X)) = (n_\ell - 2)(1 - m)$. Then $n_\ell = 2$, $\deg(q(X)) = 0$ and

$$P_{D(f)} = (X - z_\ell)^2 \prod_{\substack{i=1 \\ i \neq \ell}}^{m} (X - z_i)\,.$$

Suppose $t_q \geq 2$ for some $q \in \{1, \ldots, m\} \setminus \{\ell\}$. Then, for $i = 1, \ldots, m$, $y_i = a_q + \sum_{\substack{j=1 \\ j \neq i}}^{m} a_j$ is an eigenvalue of $D(f)$ and $f_1(y_1) > f_1(y_2) > \cdots > f_1(y_m)$. Since $\sigma(D(f)) = \{z_1, \ldots, z_m\}$ and $f_1(z_1) > f_1(z_2) > \cdots > f_1(z_m)$, it has to be $z_i = y_i$, for all i, which contradicts $a_q \neq a_\ell$. Then $t_q = 1$, for all $q \in \{1, \ldots, m\} \setminus \{\ell\}$.

Now suppose $t_\ell \geq 2$. Then $(X-a_\ell)^2$ and $(X-a_\ell)^{n_{\ell,2}}$ are elementary divisors of f. If $\ell \geq 2$ then

$$(X-z_1)^{\sum_{\substack{j=2\\ j\neq \ell}}^{m}(n_j-1)+(n_\ell-1)+(n_{\ell,2}-1)+1}$$

is an elementary divisor of $D(f)$, with degree $n_\ell + n_{\ell,2} - 1 \geq 2$ and we obtain a contradiction. Then $\ell = 1$ and

$$(X-z_2)^{\sum_{j=3}^{m}(n_j-1)+(n_1-1)+(n_{1,2}-1)+1}$$

is an elementary divisor of $D(f)$ with degree $n_1 + n_{1,2} - 1 \geq 2$. Once more, we obtain a contradiction. Condition (3) holds.

For $r = m$ it remains to consider the case $n_1 = \cdots = n_m = 1$, that is, the case f is of simple structure.

Suppose $t_\ell \geq 2$ for some $\ell \in \{1, \ldots, m\}$. Then $z_1, \ldots, z_m$, defined as before, are m distinct eigenvalues of $D(f)$, to which $X - z_i\,, \quad i = 1, \ldots, m$ are associated elementary divisors. Then $m \deg(P_f) - m^2 + 1 \geq m$ and this contradicts $\deg(P_f) = r = m$. It follows that $t_1 = \cdots = t_m = 1$ and condition (1) holds.

(II): $r \geq m+1$:

Consider the m subsets of W given by

$$\begin{aligned} L_1 &= \{a_1 + \cdots + a_{m-1} + a_i : i = m, \ldots, r\}\,, \\ L_j &= \{\underbrace{a_1 + \cdots + a_{m-j}}_{m-j} + a_i + \underbrace{a_{r-j+2} + \cdots + a_r}_{j-1} : i = m-j+2, \ldots, \\ &\qquad\qquad r-j+1\}\,, \quad j = 2, \ldots m\,, \end{aligned}$$

and the m subsets of $\mathbb{Q}$ given by

$$\begin{aligned} L'_1 &= f_1(L_1) = \{f_1(a_1) + \cdots + f_1(a_{m-1}) + f_1(a_i) : i = m, \ldots, r\}\,, \\ L'_j &= f_1(L_j) = \{f_1(a_1) + \cdots + f_1(a_{m-j}) + f_1(a_i) + f_1(a_{r-j+2}) + \cdots + \\ &\qquad f_1(a_r) : i = m-j+2, \ldots, r-j+1\}\,, \quad j = 2, \ldots m\,. \end{aligned}$$

For $j = 1, \ldots, m$ let m'_j and M'_j be, respectively, the minimum and the maximum of L'_j. Since $f_1(a_1) < f_1(a_2) < \cdots < f_1(a_r)$, we have $M'_j < m'_{j+1}$, for $j = 1, \ldots, m-1$. Then, for $1 \leq j < k \leq m$, $L'_j \cap L'_k = \emptyset$. Hence, $L_j \cap L_k = \emptyset$, for $1 \leq j < k \leq m$, and the elements in the disjoint union $\bigcup_{j=1}^{m} L_j$ are $m(r-m)+1$ distinct eigenvalues of

$D(f)$, with associated elementary divisors

$$\left(X - a_i - \sum_{k=1}^{m-1} a_k\right)^{\sum_{k=1}^{m-1}(n_k-1)+(n_i-1)+1}, \quad i = m, \ldots, r\,;$$

$$\left(X - a_i - \sum_{k=1}^{m-j} a_k - \sum_{k=r-j+2}^{r} a_k\right)^{\sum_{k=1}^{m-j}(n_k-1)+\sum_{k=r-j+2}^{r}(n_k-1)+(n_i-1)+1},$$

$$i = m-j+2, \ldots, r-j+1\,, \quad j = 2, \ldots, m\,.$$

Let $t(X)$ be the product of these elementary divisors. Then

$$\begin{aligned}
\deg(t(X)) &= (r-m+1)\left(\sum_{k=1}^{m-1} n_k - m + 1\right) + \sum_{i=m}^{r} n_i + \\
&\quad \sum_{j=2}^{m} \sum_{i=m-j+2}^{r-j+1} \left(\sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k + n_i - m + 1\right) \\
&= (r-m)\sum_{k=1}^{m-1} n_k + \deg(P_f) + (-m+1)(rm - m^2 + 1) + \\
&\quad (r-m-1)\sum_{j=2}^{m}\left(\sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k\right) + \sum_{j=2}^{m}(\deg(P_f) - n_{m-j+1}) \\
&= (r-m-1)\sum_{k=1}^{m-1} n_k + m\deg(P_f) + (-m+1)(rm - m^2 + 1) + \\
&\quad (r-m-1)\sum_{j=2}^{m}\left(\sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k\right)
\end{aligned}$$

Let $q(X) \in \overline{\mathbb{F}}[X]$ be such that $P_{D(f)} = q(X)t(X)$. Since $n_i \geq 1$, for all i, we have

$$\begin{aligned}
\deg(q(X)) &= m\deg(P_f) - m^2 + 1 - \deg(t(X)) \\
&= -m^2 + 1 - (r-m-1)\sum_{k=1}^{m-1} n_k + (m-1)(rm - m^2 + 1) - \\
&\quad (r-m-1)\sum_{j=2}^{m}\left(\sum_{k=1}^{m-j} n_k + \sum_{k=r-j+2}^{r} n_k\right) \\
&\leq -m^2 + 1 - (r-m-1)(m-1) + (m-1)(rm - m^2 + 1) - \\
&\quad (r-m-1)(m-1)^2 \\
&= 0\,.
\end{aligned}$$

Then $q(X) = 1$, $P_{D(f)} = t(X)$ and

$$\sigma(D(f)) = \bigcup_{j=1}^{m} L_j\,. \tag{15}$$

Then also

$$f_1(\sigma(D(f))) = \bigcup_{j=1}^{m} L'_j\,. \tag{16}$$

Suppose that $n_\ell \geq 2$ or $t_\ell \geq 2$ for some $\ell \in \{1, \ldots, m-1\}$. Then

$$c = 2a_\ell + \sum_{\substack{j=1 \\ j \neq \ell}}^{m-1} a_j \in \sigma(D(f))\,.$$

Since

$$f_1(c) = f_1(a_\ell) + \sum_{i=1}^{m-1} f_1(a_i) < \sum_{i=1}^{m} f_1(a_i) = m'_1\,,$$

we obtain a contradiction with (16).

Suppose that $n_\ell \geq 2$ or $t_\ell \geq 2$, for some $\ell \in \{r-m+2, \ldots, r\}$. Then

$$d = 2a_\ell + \sum_{\substack{j=r-m+2 \\ j \neq \ell}}^{r} a_j \in \sigma(D(f))$$

and, from

$$f_1(d) = f_1(a_\ell) + \sum_{j=r-m+2}^{r} f_1(a_j) > \sum_{j=r-m+1}^{r} f_1(a_j) = M'_m\,,$$

we obtain a contradiction with (16).

Hence

$$n_i = t_i = 1\,, \quad \text{for } i \in \{1,\dots,m-1\} \cup \{r-m+2,\dots,r\}\,. \tag{17}$$

Suppose f is not of simple structure. Then $r-m+1 \geq m$ and $n_\ell \geq 2$ for some $\ell \in \{m,\dots,r-m+1\}$. Let ℓ be the smallest element in $\{m,\dots,r-m+1\}$ such that $n_\ell \geq 2$. Notice that $\ell \leq r-m+1 \leq r-1$. Suppose $\ell \leq r-2$ and consider

$$\begin{aligned} x_i &= \sum_{j=1}^{m-1} a_j + a_i\,, \quad i = m,\dots,\ell\,; \\ y_i &= \sum_{j=1}^{m-2} a_j + a_i + a_r\,, \quad i = \ell+1,\dots,r-1\,; \\ v_i &= \sum_{j=1}^{m-2} a_j + a_\ell + a_i\,, \quad i = m,\dots,r\,. \end{aligned} \tag{18}$$

Since $f_1(x_m) < f_1(x_{m+1}) < \cdots < f_1(x_\ell) < f_1(v_m) < f_1(v_{m+1}) < \cdots < f_1(v_r) < f_1(y_{\ell+1}) < \cdots < f_1(y_{r-1}) < m'_3$, the elements in (18) are $2r-2m+1$ distinct eigenvalues of $D(f)$, not in $\displaystyle\bigcup_{j=3}^{m} L_j$.

From (17) and $n_1 = \cdots = n_{\ell-1} = 1$, we conclude that

$$\begin{aligned} &(X - x_i)\,, \quad i = m,\dots,\ell-1\,; \\ &(X - x_\ell)^{n_\ell}\,; \\ &(X - y_i)^{n_i}\,, \quad i = \ell+1,\dots,r-1\,; \\ &(X - v_i)^{n_\ell+n_i-1}\,, \quad i = m,\dots,r\,, \quad i \neq \ell\,; \\ &(X - v_\ell)^{2n_\ell-3} \end{aligned}$$

are elementary divisors of $D(f)$. Then

$$\begin{aligned}
m\deg(P_f)-m^2+1 &\geq \ell-m+n_\ell+\sum_{i=\ell+1}^{r-1} n_i+\sum_{\substack{i=m\\ i\neq\ell}}^{r}(n_\ell+n_i-1)+\\
&\qquad 2n_\ell-3+\sum_{j=3}^{m}\sum_{i=m-j+2}^{r-j+1} n_i\\
\Rightarrow m\deg(P_f)-m^2+1 &\geq \ell-m-3+\sum_{i=\ell}^{r-1} n_i+\sum_{i=m}^{r} n_i+(r-m)(n_\ell-1)+\\
&\qquad n_\ell+\sum_{j=3}^{m}\sum_{i=m-j+2}^{r-j+1} n_i\\
\Rightarrow m\deg(P_f)-m^2+1 &\geq \ell-m-3+(\deg(P_f)-\ell)+(\deg(P_f)-m+1)+\\
&\qquad (r-m+1)n_\ell-r+m+\sum_{j=3}^{m}\sum_{i=m-j+2}^{r-j+1} n_i\,.
\end{aligned}$$

For $3\leq j\leq m$ we have $m-j\leq m-3$ and $r-j\geq r-m$. So, if $i\leq m-j+1$ or $i\geq r-j+2$ then $n_i=1$. Hence $\displaystyle\sum_{i=m-j+2}^{r-j+1} n_i=\deg(P_f)-m$ and

$$\begin{aligned}
m\deg(P_f)-m^2+1 &\geq 2\deg(P_f)-m-2-r+(r-m+1)n_\ell+\\
&\qquad (m-2)(\deg(P_f)-m)\\
\Rightarrow\qquad -m^2+1 &\geq (r-m+1)n_\ell-r-m-m^2+2m-2\\
\Rightarrow\qquad (r-m+1)n_\ell &\leq r-m+3\,.
\end{aligned}$$

From the last inequality, since we are assuming that $n_\ell\geq 2$, we have $r=m+1$ and, from $\ell\leq r-2=m-1$, we obtain a contradiction with (17). Then $\ell=r-1$ and, from $\ell\leq r-m+1$, it follows that $m=2$.

So, if f is not of simple structure then $m=2$, $n_{r-1}\geq 2$ and

$$n_i=t_i=1\,,\quad \text{for } i\in\{1,\ldots,r\}\setminus\{r-1\}\,.$$

In this case,

$$x_i = a_1 + a_i\,, \quad i = 2, \ldots, r-1\,;$$
$$v_i = a_{r-1} + a_i\,, \quad i = 2, \ldots, r$$

are $2r-3$ distinct eigenvalues of $D(f)$. Since $n_i = 1$ for $i \neq r-1$ we obtain

$$\begin{aligned} 2\deg(P_f) - 3 &\geq \sum_{i=2}^{r-1} n_i + \sum_{\substack{i=2 \\ i\neq r-1}}^{r} (n_{r-1} + n_i - 1) + 2n_{r-1} - 3 \\ \Rightarrow \quad 2\deg(P_f) - 3 &\geq \deg(P_f) - 2 + (r-2)(n_{r-1} - 1) + \deg(P_f) - n_{r-1} - 1 + \\ & \quad 2n_{r-1} - 3 \\ \Rightarrow \quad 2 &\geq (r-1)(n_{r-1} - 1)\,. \end{aligned}$$

Since $r \geq m+1 = 3$, it follows that $r = 3$ and $n_2 = 2$. From (17) we obtain $t_1 = t_3 = 1$. Suppose $t_2 \geq 2$. Then $(X - a_2)^2$ and $(X - a_2)^{n_{2,2}}$ are elementary divisors of f and

$$(X - a_1 - a_2)^2\,, \quad (X - a_2 - a_3)^2\,, \quad (X - 2a_2)^{n_{2,2}+1}$$

are elementary divisors of $D(f)$ associated to distinct eigenvalues. Hence $5 = \deg(P_{D(f)}) \geq 5 + n_{2,2}$, which leads to a contradiction. Then $t_2 = 1$ and

$$P_{D(f)} = (X - a_1 - a_2)^2 (X - 2a_2)(X - a_2 - a_3)^2\,.$$

Since $a_1 + a_3 \in \sigma(D(f))$ it follows that $a_1 + a_3 = 2a_2$ and $a_1, a_2 = a_1 + (a_3 - a_2), a_3 = a_1 + 2(a_3 - a_2)$ is an arithmetic progression. Condition (5) holds.

Next suppose f is of simple structure. From (15) and (2) it follows that

$$m(r-m) + 1 = |\sigma(D(f))| \geq |\wedge^m \sigma(f)| \geq m(r-m) + 1\,.$$

Hence

$$\sigma(D(f)) = \wedge^m \sigma(f) \tag{19}$$

and

$$|\wedge^m \sigma(f)| = m(r-m) + 1\,.$$

Then (Proposition 1) one of the following conditions holds:

(a): $\sigma(f)$ is an arithmetic progression:
Let b and q be, respectively, the first term and the difference of that arithmetic progression. Since $b, b+q \in W$, then also $q \in W$ and $f_1(b), f_1(b+q), \dots, f_1(b+(r-1)q)$ is an arithmetic progression in $\mathbb{Q}$ with difference $f_1(q) \neq 0$ (from (11)).
If $f_1(q) > 0$ then $f_1(b) < f_1(b+q) < \cdots < f_1(b+(r-1)q)$ and so, from (11), we have $a_i = b + (i-1)q$, for $i = 1, \dots, r$.
If $f_1(q) < 0$ then $a_i = b + (r - i)$, for $i = 1, \dots, r$.
From (17) we have $t_i = 1$ for all $i \in \{1, \dots, m-1\} \cup \{r - m + 2, \dots, m\}$ and condition (4) holds.
(b): $r = m + 1$:
If $m \geq 3$ then $r \geq 4$ and, from (17), we have $t_i = 1$, $i = 1, \dots, r$. Condition (2) holds.
If $m = 2$ then $r = 3$ and, from (17), $t_1 = t_3 = 1$. Suppose $t_2 \geq 2$. Then, from (19) and Corollary 1, we have

$$2a_2 \in \sigma(D(f)) = \{a_1 + a_2, a_1 + a_3, a_2 + a_3\}\,.$$

Therefore $2a_2 = a_1 + a_3$ and $\sigma(f)$ is an arithmetic progression with first term a_1 and difference $a_2 - a_1$. Condition (4) holds.
(c): $m = 2$, $r = 4$ and

$$\sigma(f) = a + \{0, d, d', d + d'\}$$

for some $a \in \overline{\mathbb{F}}$, $d, d' \in \overline{\mathbb{F}} \setminus \{0\}$ such that $d \neq d'$ and $d + d' \neq 0$. Then

$$\begin{aligned} \sigma(D(f)) &= \wedge^2 \sigma(f) \\ &= 2a + \{d, d', d + d', 2d + d', d + 2d'\}\,. \end{aligned}$$

Let

$$\begin{array}{ll} X - a\,, & s_1 \text{ times} \\ X - a - d\,, & s_2 \text{ times} \\ X - a - d'\,, & s_3 \text{ times} \\ X - a - d - d'\,, & s_4 \text{ times} \end{array}$$

be the elementary divisors of f. From (17) we know that, at least, two of the numbers s_1, s_2, s_3, s_4 are equal to 1.
If $s_1 = s_2 = s_3 = s_4 = 1$ then condition (6) holds.
Suppose $s_1 \geq 2$. Then $2a \in \sigma(D(f))$. Hence $2d + d' = 0$ or $d + 2d' = 0$. Then $\sigma(f)$ is an arithmetic progression. Similarly, if $s_i \geq$

2 for some $i \in \{2, 3, 4\}$, then $\sigma(f)$ is an arithmetic progression. As we have seen in (a), condition (4) holds.

■

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