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A SINGULAR PERTURBATION OF THE HEAT EQUATION WITH MEMORY

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ABSTRACT: In this paper we consider a hyperbolic equation, with a memory term in time, which can be seen as a singular perturbation of the heat equation with memory. The qualitative properties of the solutions of the initial boundary value problems associated with both equations are studied. We propose numerical methods for the hyperbolic and parabolic models and their stability properties are analysed. Finally, we include numerical experiments illustrating the performance of those methods.

KEYWORDS: Viscoelasticity problem, heat equation with memory, stability, numerical method.

1. Introduction

Let us consider the hyperbolic equation

$$\begin{aligned}\epsilon\frac{\partial^2 u}{\partial t^2}(x,t)+\alpha\frac{\partial u}{\partial t}(x,t)= \; & \gamma\frac{\partial^2 u}{\partial x^2}(x,t)+\int_0^t k(t-s)\frac{\partial^2 u}{\partial x^2}(x,s)\,ds \\ & +f(x,t,u(x,t)),\end{aligned} \tag{1}$$

for $x \in (a,b)$, $t > 0$, where $k(s)$ is a scalar function, smooth enough, and which will be specified later, with initial conditions

$$\begin{cases} u(x,0) = u_0(x),\; x \in (a,b) \\ \dfrac{\partial u}{\partial t}(x,0) = u_1(x), x \in (a,b) \end{cases} \tag{2}$$

and

$$u(a,t) = u_a(t),\; u(b,t) = u_b(t),\; t > 0. \tag{3}$$

Initial boundary value problem (IBVP) (1)-(3) arises from a variety of mathematical models in engineering and physical sciences. We mention, for instance, the theory of linear viscoelasticity. In this case u represents the displacement of a body with density ϵ, viscosity α, tension γ and under external force f.

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For $\epsilon \to 0$ IBVP (1)-(3) can be seen as a singular perturbation of the partial differential equation

$$\alpha \frac{\partial w}{\partial t}(x,t) = \gamma \frac{\partial^2 w}{\partial x^2}(x,t) + \int_0^t k(t-s) \frac{\partial^2 w}{\partial x^2}(x,s)\, ds + f(x,t), \tag{4}$$

for $x \in (a,b)$, $t > 0$, with initial condition

$$w(x,0) = w_0(x),\ x \in (a,b), \tag{5}$$

and

$$w(a,t) = w_a(t),\ w(b,t) = w_b(t),\ t > 0. \tag{6}$$

The behavior of the displacement u when the density ϵ converges to zero was studied in [7], [8]. In those papers it was shown that, under certain assumptions on the initial displacement, initial velocity and boundary conditions, the displacement u, solution of (1)-(2), and its derivatives converge to the solution w (of the heat IBVP (4)-(5)) and its derivatives, respectively, when the density ϵ goes to zero.

Equation (4) is called heat equation with memory and has been considered, for instance, in [4] and more recently in [9] with $k(s) = \dfrac{\sigma}{\tau} e^{-\frac{s}{\tau}}$. This equation is established combining the mass conservation law with the Jeffreys flux

$$q(x,t) = -k_1 \frac{\partial u}{\partial x}(x,t) - \frac{k_2}{\tau} \int_0^t e^{-\frac{t-s}{\tau}} \frac{\partial u}{\partial x}(x,s)\, ds,$$

where k_1 and k_2 are, respectively, the effective thermal and elastic conductivity constants. Motivated by those considerations we consider in the present paper $k(s) = \dfrac{\sigma}{\tau} e^{-\frac{s}{\tau}}$.

Our aim is to study the behavior of the solutions u and w, respectively, of the hyperbolic IBVP (1)-(3) and the parabolic ϵ-limit IBVP (4)-(6) and to present numerical methods which allow us to compute approximations to u and w with the same behavior.

In Section 2 we study the stability of the IBVP (1)-(3) with respect to perturbations of initial conditions. A numerical method for (1)-(3) is proposed in Section 3. In this section, a discrete version of a stability inequality established in Section 2 is proved. As a corollary of such stability result, the convergence of the numerical method is concluded. In Section 4 we establish a stability result for the ϵ-limit IBVP (4)-(6). In Section 5 a numerical method for (4)-(6) is proposed and its stability and convergence properties are studied. In Section 6 the relation between the numerical approximations for the solutions of the hyperbolic problem (1)-(3) and the parabolic problem (1)-(3) is analysed. Finally, in Section 7, several numerical experiments are presented illustrating the theoretical results established in the previous sections.

2. The hyperbolic perturbed IBVP

In this section we study the stability of the hyperbolic IBVP (1)-(3) when the initial conditions are perturbed. By $(.\,,.)$ we represent the L^2 inner product and we denote by $\|.\|_{L^2}$ the corresponding norm. If v is defined in $[a,b]\times[0,T]$ we represent $v(.\,,t)$ by $v(t)$.

We start by establishing an upper bound for the L^2 norm of the solution of (1)-(3) and for the L^2 norm of the spatial and temporal gradients and its past, with initial conditions u_0, u_1 and homogeneous boundary conditions. Nevertheless we assume general Dirichlet boundary conditions when stability results are established.

Theorem 1. *Let u be a solution of (1)-(3) with homogeneous boundary conditions. Let us suppose that*

$$\begin{aligned}&\frac{\partial^\ell u}{\partial t^\ell}(t), \frac{\partial^\ell u}{\partial x^\ell}(t)\,, \int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)ds \in L^2[a,b], \ell=1,2,\\ &\frac{\partial^2 u}{\partial x\partial t}(t)\in L^2[a,b], t\in(0,T].\end{aligned}\tag{7}$$

Then, for each $t\in(0,T]$, holds

$$\begin{aligned}&\epsilon\|\frac{\partial u}{\partial t}(t)\|_{L^2}^2+(\gamma-\sigma)\|\frac{\partial u}{\partial x}(t)\|_{L^2}^2+\sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds+\frac{\partial u}{\partial x}(t)\|_{L^2}^2\\ &\le\frac{1}{2\alpha+\epsilon}\int_0^t e^{\max\{1,\frac{2\sigma}{\tau(\gamma-\sigma)}\}(t-s)}\|f(s)\|_{L^2}^2\,ds\\ &+e^{\max\{1,\frac{2\sigma}{\tau(\gamma-\sigma)}\}t}\Big(\epsilon\|u_1\|_{L^2}^2+\gamma\|u_0'\|_{L^2}^2\Big),\end{aligned}\tag{8}$$

provided that $\sigma\neq\gamma$.

■

Proof: Multiplying each member of (1) by $\dfrac{\partial u}{\partial t}$ with respect to $(.\,,.)$ and integrating by parts we obtain

$$\begin{aligned}\epsilon(\frac{\partial^2 u}{\partial t^2}(t),\frac{\partial u}{\partial t}(t)) + \alpha\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} = -\gamma(\frac{\partial u}{\partial x}(t),\frac{\partial^2 u}{\partial t\partial x}(t))\\ -\frac{\sigma}{\tau}(\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds,\frac{\partial^2 u}{\partial x\partial t}(t)) + (f,\frac{\partial u}{\partial t}(t))\,.\end{aligned} \tag{9}$$

It can be shown that

$$\begin{aligned}(\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds,\frac{\partial^2 u}{\partial x\partial t}(t)) = \frac{1}{2}\frac{d}{dt}\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|^2_{L^2}\\ -\frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial x}(t)\|^2_{L^2} - \frac{1}{\tau}\|\frac{\partial u}{\partial x}(t)\|^2_{L^2} + \frac{1}{\tau}\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds\|^2_{L^2}.\end{aligned} \tag{10}$$

Considering that

$$(f,\frac{\partial u}{\partial t}(t)) \le \frac{1}{4\eta^2}\|f\|^2_{L^2} + \eta^2\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} \tag{11}$$

for some positive constant $\eta \neq 0$, and

$$(\frac{\partial^2 u}{\partial t^2}(t),\frac{\partial u}{\partial t\partial t}(t)) = \frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial t}(t)\|^2_{L^2}$$

and

$$(\frac{\partial u}{\partial x}(t),\frac{\partial^2 u}{\partial t\partial x}(t)) = \frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial x}(t)\|^2_{L^2},$$

from (9), (10) and (11) we deduce the inequality

$$\begin{aligned}&\frac{d}{dt}\Big(\epsilon\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} + (\gamma-\sigma)\|\frac{\partial u}{\partial x}(t)\|^2_{L^2} + \sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|^2_{L^2}\Big)\\ &\le 2(-\alpha+\eta^2)\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} + \frac{2\sigma}{\tau}\|\frac{\partial u}{\partial x}\|^2_{L^2} - \frac{2\sigma}{\tau}\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds\|^2_{L^2}\\ &+\frac{1}{2\eta^2}\|f\|^2_{L^2}\,.\end{aligned} \tag{12}$$

Let η be defined by $\eta^2 = \alpha + \epsilon/2$. Using the Poincaré-Friedrichs inequality $\|u(t)\|_{L^2}^2 \leq (b-a)^2 \|\frac{\partial u}{\partial x}(t)\|_{L^2}^2$ in (12) we obtain the differential inequality

$$
\begin{aligned}
&\frac{d}{dt}\Big(\epsilon\|\frac{\partial u}{\partial t}(t)\|_{L^2}^2 + (\gamma-\sigma)\|\frac{\partial u}{\partial x}(t)\|_{L^2}^2 + \sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|_{L^2}^2\Big) \\
&\quad \leq \max\left\{1, \frac{2\sigma}{(\gamma-\sigma)\tau}\right\}\Big(\epsilon\|\frac{\partial u}{\partial t}\|_{L^2}^2 + (\gamma-\sigma)\|\frac{\partial u}{\partial t}\|_{L^2}^2\Big) + \frac{1}{\epsilon+2\alpha}\|f\|_{L^2}^2
\end{aligned}
\tag{13}
$$

which allows us to conclude inequality (8).

■

The influence of initial conditions u_0 and u_1 on the behavior of $\epsilon\|\frac{\partial u}{\partial t}(t)\|_{L^2}^2$, $\|\frac{\partial u}{\partial x}(t)\|_{L^2}^2$ $\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|_{L^2}^2$ can be established from inequality (8) for $\sigma \neq \gamma$.

For the particular case $\sigma = \gamma$ similar results can be obtained but we do not get an estimate for $\|\frac{\partial u}{\partial x}\|_{L^2}$.

We are in position to establish the stability of (1)-(3) with respect to perturbation of the initial conditions u_0 and u_1.

Corollary 1. *Let u and $\tilde{u}$ be solutions of (1)-(3) with initial conditions u_0, u_1 and $\tilde{u}_0, \tilde{u}_1$, respectively, satisfying the assumptions of Theorem 1. Then, for $v = u - \tilde{u}$ and for each time $t \in (0, T]$, holds*

$$
\begin{aligned}
&\epsilon\|\frac{\partial v}{\partial t}(t)\|_{L^2}^2 + (\gamma-\sigma)\|\frac{\partial v}{\partial x}(t)\|_{L^2}^2 + \sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial v}{\partial x}(s)\,ds + \frac{\partial v}{\partial x}(t)\|_{L^2}^2 \\
&\quad \leq e^{\max\{1, \frac{2\sigma}{\tau(\gamma-\sigma)}\}t}\Big(\epsilon\|u_1 - \tilde{u}_1\|_{L^2}^2 + \gamma\|u_0' - \tilde{u}_0'\|_{L^2}^2\Big).
\end{aligned}
\tag{14}
$$

Proof: The proof follows the proof of Theorem 1 with $f = 0$.

■

As an immediate consequence of Theorem 1, if (1)-(3) has a solution u then u is unique.

Stability results for the solution of (1)-(3) when $\gamma = \sigma$ can be established following the proof of Theorem 1.

3. The parabolic $\epsilon-$ limit IBVP

The hyperbolic problem (1)-(3) can be seen, for ϵ small enough, a singular perturbation of a heat equation with a memory term. In fact, let us suppose that ϵ is a parameter and the boundary conditions are homogeneous. We suppose that u_0 and u_1 are ϵ depending, that is u_0 and u_1 are replaced by $u_{0,\epsilon}$ and $u_{1,\epsilon}$, and f is also ϵ dependent. Let u_ϵ be the solution of the initial boundary value problem correspondent to problem (1)-(3) with $\alpha = 1$. In [7] and [8] was established that if $f_\epsilon \rightarrow f$, $u_{0,\epsilon} \rightarrow w_0, \epsilon u_{1,\epsilon} \rightarrow 0$ (in L^2) when $\epsilon \rightarrow 0$, then $u_\epsilon \rightarrow w$, $\dfrac{\partial u_\epsilon}{\partial x} \rightarrow \dfrac{\partial w}{\partial x}$ and $\dfrac{\partial u_\epsilon}{\partial t} \rightarrow \dfrac{\partial w}{\partial t}$ (in L^2) where w is solution of the heat equation

$$\frac{\partial w}{\partial t}(x,t) = \gamma \frac{\partial^2 w}{\partial x^2}(x,t) + \frac{\sigma}{\tau}\int_0^t e^{-\frac{t-s}{\tau}} \frac{\partial^2 w}{\partial x^2}(x,s)\, ds + f(x,t),\ x \in (a,b),\ t > 0, \tag{15}$$

with initial and boundary conditions

$$\begin{cases} w(x,0) = w_0(x),\ x \in (a,b), \\ \\ w(a,t) = 0, w(b,t) = 0,\ t > 0\,. \end{cases} \tag{16}$$

In this section we establish for w an estimate analogous to estimate (8). Firstly we remark that taking in (8) the limit when $\epsilon \rightarrow 0$ we conclude for w the following estimate

$$\begin{aligned} &(\gamma - \sigma)\|\frac{\partial w}{\partial x}(t)\|^2_{L^2} + \sigma \|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial w}{\partial x}(s)\, ds + \frac{\partial w}{\partial x}(t)\|^2_{L^2} \\ &\quad \leq \frac{1}{2}\int_0^t e^{\max\{1, \frac{2\sigma}{\tau(\gamma-\sigma)}\}(t-s)} \|f(s)\|^2_{L^2}\, ds\ + e^{\max\{1, \frac{2\sigma}{\tau(\gamma-\sigma)}\}t}\gamma \|w_0'\|^2_{L^2}. \end{aligned} \tag{17}$$

The behavior of $\|w(t)\|_{L^2}$ does not follow directly from inequality (17). In the following we establish an estimate to $\|w(t)\|_{L^2}$ using the energy method.

Theorem 2. *Let w be a solution of (15)-(16). Let us suppose that*

$$\frac{\partial w}{\partial t}(t),\ \frac{\partial^\ell w}{\partial x^\ell}(t),\ \int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial w}{\partial x}(s)ds \in L^2[a,b], \ell = 1, 2, t > 0. \tag{18}$$

Then, for each $t > 0$, holds

$$\begin{aligned}\|w(t)\|_{L^2}^2 + \frac{\sigma}{\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial w}{\partial x}(s)\,ds\|_{L^2}^2 \le e^{2\max\{-\frac{\gamma}{(b-a)^2}+\frac{1}{2},-\frac{1}{\tau}\}t}\|w_0\|_{L^2}^2 \\ + \int_0^t e^{2\max\{-\frac{\gamma}{(b-a)^2}+\frac{1}{2},-\frac{1}{\tau}\}(t-s)}\|f(s)\|_{L^2}^2\,ds\,.\end{aligned} \tag{19}$$

The proof differs only in minor details of the proof of Theorem 1 of [3].

As a corollary of Theorem 2 we have the next result:

Corollary 2. *Let w and $\tilde{w}$ be solutions of (15)-(16) with initial conditions w_0 and $\tilde{w}_0$ respectively satisfying the assumptions of Theorem 2. Then, for $v = w - \tilde{w}$ and for each time $t > 0$, holds*

$$\|v(t)\|_{L^2}^2 + \frac{\sigma}{\tau}\|\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial v}{\partial x}(s)\,ds\|_{L^2}^2 \;\le\; e^{2\max\{-\frac{\gamma}{(b-a)^2}+\frac{1}{2},-\frac{1}{\tau}\}t}\|w_0-\tilde{w}_0\|_{L^2}^2. \tag{20}$$

4. A discrete perturbed IBVP

Let us consider in $[a,b]$ a grid $I_h = \{x_j, j = 0,\ldots,N\}$ with $x_0 = a, x_N = b$ and $x_j - x_{j-1} = h$. In $[0,T]$ we consider the grid $\{t_n, n = 0,\ldots,M\}$ with $t_0 = 0, t_M = T$ and $t_{n+1} - t_n = \Delta t$.

We discretize the second partial derivative with respect to x in (1) and (15) using the second-order centered finite-difference operator $D_{2,x}$ defined by

$$D_{2,x}v_h^n(x_i) = \frac{v_h^n(x_{i+1}) - 2v_h^n(x_i) + v_h^n(x_{i-1})}{h^2}.$$

By $D_{2,t}$ we represent the second-order finite difference operator defined by

$$D_{2,t}v_h^n(x_i) = \frac{v_h^{n+1}(x_i) - 2v_h^n(x_i) + v_h^{n-1}(x_i)}{\Delta t^2}.$$

In the stability and convergence analysis of the numerical methods studied in this paper we consider a discrete version of the L^2 norm that we present in what follows.

We denote by $L^2(I_h)$ the space of grid functions v_h defined in I_h such that $v_h(x_0) = v_h(x_N) = 0$. In $L^2(I_h)$ we consider the discrete inner product

$$(v_h, w_h)_h = h\sum_{i=1}^{N-1} v_h(x_i)w_h(x_i),\; v_h, w_h \in L^2(I_h), \tag{21}$$

and by $\|.\|_{L^2(I_h)}$ we denote the norm induced by the above inner product. For grid functions w_h and v_h defined in I_h we introduce the notations

$$(w_h, v_h)_{h,+} = \sum_{i=1}^{N} h w_h(x_i) v_h(x_i),\ \|w_h\|_{L^2(I_h^+)} = \left(\sum_{i=1}^{N} h w_h(x_i)^2\right)^{1/2}.$$

Discretizing the spatial derivatives using $D_{2,x}$ and $D_{2,t}$ and the memory term using a rectangular rule we obtain a fully discrete approximation u_h^n defined by

$$\epsilon D_{2,t} u_h^n(x_i) + \alpha D_{-t} u_h^{n+1}(x_i) = \gamma D_{2,x} u_h^{n+1}(x_i) + \frac{\sigma}{\tau}\Delta t \sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\tau}} D_{2,x} u_h^j(x_i)$$
$$+ f(x_i, t_{n+1}), i = 1, \dots, N-1,\ n = 1, \dots, M-1, \tag{22}$$

where

$$\begin{aligned} &u_h^j(x_0) = u_a(t_j),\ u_h^j(x_N) = u_b(t_j),\ j = 1, \dots, M-1, \\ &u_h^1(x_i) = u_h^0(x_i) = u_0(x_i), i = 1, \dots, N-1. \end{aligned} \tag{23}$$

In what follows we establish for the numerical approximation defined by (22)-(23), a discrete version of Theorem 1 when $\gamma > \sigma$. In this result we characterize the behavior of the discrete L^2 norm of the numerical temporal and spatial gradients as well the past in time of the numerical spatial gradient. The stability of method (22)-(23) is then concluded.

Theorem 3. *Let u_h^j be defined by (22)-(23) with $u_a(t) = u_b(t) = 0, t > 0$. Then*

$$\begin{aligned} &\epsilon \|D_{-t} u_h^{n+1}\|^2_{L^2(I_h)} + \|D_{-x} u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &+\sigma \|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x} u_h^j + D_{-x} u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &\quad \le S_p^n (1 + \sigma(1 + \frac{\Delta t}{\tau})^2) \|D_{-x} u_h^0\|^2_{L^2(I_h^+)} \\ &\quad + \frac{\Delta t}{\max_{\sigma,\gamma,\tau}(2\alpha + \epsilon)} \sum_{j=1}^{n} S_p^{n+1-j} \|f_h(t_{j+1})\|^2_{L^2(I_h)}, \end{aligned} \tag{24}$$

with

$$S_p = \frac{\max_{\sigma,\gamma,\tau}}{1 - \Delta t}, \tag{25}$$

$$\max_{\sigma,\gamma,\tau} = \max\{1, \gamma+\sigma\big(3e^{-\frac{\Delta t}{\tau}}+\frac{\Delta t}{\tau}+2e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\big)\, , \sigma\Big(e^{-\frac{\Delta t}{\tau}}+2e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\Big)\ \},$$

$$\tau - 2\sigma + \sqrt{(\tau-2\sigma)^2+4\sigma(\gamma-\sigma-1)} > 0, \tag{26}$$

and for Δt *such that*

$$\Delta t \leq \frac{\tau}{2\sigma}\Big(\tau(\tau-2\sigma)+\tau\sqrt{(\tau-2\sigma)^2+4\sigma(\gamma-\sigma-1)}\Big). \tag{27}$$

Proof: Multiplying each member of (22) by $D_{-t}u_h^{n+1}$ with respect to the inner product $(.\,,.)_h$ and using summation by parts we obtain

$$\epsilon(D_{2,t}u_h^n, D_{-t}u_h^{n+1})_h + \alpha\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} = \gamma(D_{2,x}u_h^{n+1}, D_{-t}u_h^{n+1})_h$$
$$+(f_h(t_{n+1}), D_{-t}u_h^{n+1})_h - \sigma(\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}D_{-t}u_h^{n+1})_{h,+}\, , \tag{28}$$

where $f_h(t_{n+1})(x_j) = f(x_j, t_{n+1})$.

Considering that we have

$$\begin{aligned}(D_{2,t}u_h^n, D_{-t}u_h^{n+1})_h &= \frac{\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} - (D_{-t}u_h^n, D_{-t}u_h^{n+1})_h}{\Delta t}\\ &\geq \frac{\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} - \|D_{-t}u_h^n\|^2_{L^2(I_h)}}{2\Delta t},\end{aligned} \tag{29}$$

$$\begin{aligned}(D_{2,x}u_h^{n+1}, D_{-t}u_h^{n+1})_h &= \frac{(D_{-x}u_h^{n+1}, D_{-x}u_h^n)_{h,+} - \|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}}{\Delta t}\\ &\leq \frac{\|D_{-x}u_h^n\|^2_{L^2(I_h^+)} - \|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}}{2\Delta t},\end{aligned} \tag{30}$$

and

$$(f_h(t_{n+1}), D_{-t}u_h^{n+1})_h \leq \eta_1^2\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \frac{1}{4\eta_1^2}\|f_h(t_{n+1})\|^2_{L^2(I_h)}, \tag{31}$$

being $\eta_1 \neq 0$ an arbitrary constant, from (28) we obtain

$$\begin{aligned}&\big(\frac{\epsilon}{2}+\Delta t(\alpha-\eta_1^2)\big)\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \frac{\gamma}{2}\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\ &\leq \frac{\epsilon}{2}\|D_{-t}u_h^n\|^2_{L^2(I_h)} + \frac{\gamma}{2}\|D_{-x}u_h^n\|^2_{L^2(I_h^+)} + \frac{\Delta t}{4\eta_1^2}\|f_h(t_{n+1})\|^2_{L^2(I_h)}\\ &-\sigma(\frac{\Delta t^2}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}D_{-t}u_h^{n+1})_{h,+}\, .\end{aligned} \tag{32}$$

We establish in what follows as estimate to

$$(\frac{\Delta t^2}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}D_{-t}u_h^{n+1})_{h,+}.$$

We have

$$\begin{aligned}
&(\frac{\Delta^2 t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}D_{-t}u_h^{n+1})_{h,+}\\
&= \frac{1}{2}\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&-\frac{1}{2}\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} + \frac{1}{2}\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j\|^2_{L^2(I_h^+)}\\
&+(\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}u_h^n)_{h,+}.
\end{aligned}$$

Attending that

$$\begin{aligned}
&(\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}u_h^n)_{h,+}\\
&\leq \frac{1}{2}e^{-\frac{\Delta t}{\tau}}\|\frac{\Delta t}{\tau}\sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^n\|^2_{L^2(I_h^+)}\\
&+\frac{1}{2}\Big(3e^{-\frac{\Delta t}{\tau}} + \frac{\Delta t}{\tau}\Big)\|D_{-x}u_h^n\|^2_{L^2(I_h^+)} + \frac{\Delta t}{2\tau}\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)},
\end{aligned}$$

$$\begin{aligned}
&\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j\|^2_{L^2(I_h^+)} \leq (\frac{\Delta t}{\tau} + (\frac{\Delta t}{\tau})^2)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+2e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\|\frac{\Delta t}{\tau}\sum_{j=1}^{n} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^n\|^2_{L^2(I_h^+)}\\
&+2e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\|D_{-x}u_h^n\|^2_{L^2(I_h^+)}
\end{aligned}$$

and

$$\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \le \|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j\|^2_{L^2(I_h^+)}$$
$$+2\Delta t(\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}D_{-t}u_h^{n+1})_{h,+} + \|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}$$
$$+2(\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-x}u_h^n)_{h,+}$$

we deduce

$$\begin{aligned}
&-\Delta t(\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j, D_{-t}D_{-x}u_h^{n+1})_{h,+}\\
&\le -\frac{1}{2}\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&\ +\Big(\frac{e^{-\frac{\Delta t}{\tau}}}{2} + e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\Big)\|\frac{\Delta t}{\tau}\sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^n\|^2_{L^2(I_h^+)}\\
&\ +\Big(\frac{1}{2}(3e^{-\frac{\Delta t}{\tau}} + \frac{\Delta t}{\tau}) + e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\Big)\|D_{-x}u_h^n\|^2_{L^2(I_h^+)}\\
&+\frac{1}{2}\Big(1 + 2\frac{\Delta t}{\tau} + (\frac{\Delta t}{\tau})^2\Big)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\,.
\end{aligned}\tag{33}$$

Using (33) in (32) with $\eta_1^2 = \alpha + \frac{\epsilon}{2}$ we obtain

$$\begin{aligned}
&(1-\Delta t)\epsilon\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \Big(\gamma - \sigma - \sigma\big(2\frac{\Delta t}{\tau} + (\frac{\Delta t}{\tau})^2\big)\Big)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&\le \epsilon\|D_{-t}u_h^n\|^2_{L^2(I_h)} + \Big(\gamma + \sigma\big(3e^{-\frac{\Delta t}{\tau}} + \frac{\Delta t}{\tau} + 2e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\big)\Big)\|D_{-x}u_h^n\|^2_{L^2(I_h^+)}\\
&+\sigma\Big(e^{-\frac{\Delta t}{\tau}} + 2e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\Big)\|\frac{\Delta t}{\tau}\sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^n\|^2_{L^2(I_h^+)}\\
&+\frac{\Delta t}{2\alpha+\epsilon}\|f_h(t_{n+1})\|^2_{L^2(I_h)}\,.
\end{aligned}\tag{34}$$

Let γ, σ and τ such that (26) holds. Then, for Δt satisfying (27), from (34) we establish

$$
\begin{aligned}
&\Big(\epsilon\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\Big)\\
&\qquad\le S_p\Big(\epsilon\|D_{-t}u_h^{n}\|^2_{L^2(I_h)} + \|D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}\\
&\qquad+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n} e^{-\frac{t_{n}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}\Big)\\
&\qquad\quad+\frac{\Delta t}{(1-\Delta t)(2\alpha+\epsilon)}\|f_h(t_{n+1})\|^2_{L^2(I_h)}\,.
\end{aligned}
\tag{35}
$$

Finally considering inequality (35) and attending that $u_h^1 = u_h^0$ we obtain (24).

■

Theorem 3 can be seen as a discrete version of Theorem 1 for the numerical approximation defined by method (22)-(23). This result allows us to characterize the behavior of the numerical derivatives and the past in discrete time of the spatial gradient of such approximation. As a corollary of Theorem 3 we have:

Corollary 3. *Let u_h^j be defined by method (22)-(23). Under the assumptions of Theorem 3, if*

$$
\max_{\sigma,\gamma,\tau} \le 1 + C\Delta t, \tag{36}
$$

then exists a positive time and space independent constant $\mathbf{C}$ such that

$$
\begin{aligned}
&\epsilon\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&\le \mathbf{C}\Big((1+\sigma(1+\frac{\Delta t}{\tau})^2)\|D_{-x}u_h^0\|^2_{L^2(I_h^+)}\\
&+\frac{\Delta t}{(1-\Delta t)(2\alpha+\epsilon)}\sum_{j=1}^{n}\|f_h(t_{j+1})\|^2_{L^2(I_h)}\Big).
\end{aligned}
\tag{37}
$$

If u_h^j and $\tilde{u}_h^j$ are defined by method (22)-(23) with initial conditions,respectively, u_0 and $\tilde{u}_0$, then, under the assumptions of Theorem 3 and (36), for $v_h^j = u_h^j - \tilde{u}_h^j$, holds

$$\begin{aligned} &\epsilon\|D_{-t}v_h^{n+1}\|^2_{L^2(I_h)} + \|D_{-x}v_h^{n+1}\|^2_{L^2(I_h^+)} \\ &+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}v_h^j + D_{-x}v_h^{n+1}\|^2_{L^2(I_h^+)} \\ &\leq \mathbf{C}(1+\sigma(1+\frac{\Delta t}{\tau})^2)\|D_{-x}(u_h^0-\tilde{u}_h^0)\|^2_{L^2(I_h^+)}\,. \end{aligned} \tag{38}$$

Proof: From Theorem 3, under assumption (36), we conclude

$$\begin{aligned} &\epsilon\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &\leq e^{n\frac{\Delta t(C+1)}{1-\Delta t}}(1+\sigma(\frac{\Delta t}{\tau})^2)\|D_{-x}u_h^0\|^2_{L^2(I_h^+)} \\ &+\frac{\Delta t}{(1-\Delta t)(2\alpha+\epsilon)}\sum_{j=1}^{n} e^{(n-j)\frac{\Delta t(C+1)}{1-\Delta t}}\|f_h(t_{j+1})\|^2_{L^2(I_h)}\,, \end{aligned} \tag{39}$$

and then we get (37) for some positive time and space independent constant $\mathbf{C}$.

Inequality (38) follows from the fact that v_h^{n+1} satisfies inequality (37) with f_h and u_h^0 replaced respectively by the null function and $u_h^0 - \tilde{u}_h^0$.

■

Let us consider Theorem 3 and Corollary 3 with u_h^j replaced by the error $e_{s,h}^j = u_h^j - R_h u(.\,,t_j)$, where R_h denotes the restriction operator. Attending that the discretization (22)-(23) is consistent provided that the solution u is smooth enough (the required smoothness is detailed in Corollary 4), we conclude the following

$$\begin{aligned} &\epsilon\|D_{-t}e_{s,h}^{n+1}\|^2_{L^2(I_h)} \to 0 \\ &\|D_{-x}e_{s,h}^{n+1}\|^2_{L^2(I_h^+)} \to 0 \\ &\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}e_{s,h}^j + D_{-x}e_{s,h}^{n+1}\|^2_{L^2(I_h^+)} \to 0 \end{aligned} \tag{40}$$

when $\Delta t, h \to 0$. Using the discrete Poincaré-Friedrichs inequality

$$\|e_{s,h}^{n+1}\|^2_{L^2(I_h)} \leq (b-a)^2 \|D_{-x} e_{s,h}^{n+1}\|^2_{L^2(I_h^+)}$$

the convergence

$$\|e_{s,h}^{n+1}\|^2_{L^2(I_h)} \to 0 \tag{41}$$

is obtained.

We proved the following convergence result:

Corollary 4. *If the solution of (1)-(2), u, is such that* $\dfrac{\partial^3 u}{\partial t^3} \in C^0[a,b] \times L^2[0,T]$, $\dfrac{\partial^3 u}{\partial x^3} \in L^2[a,b] \times C^0[0,T]$, $\dfrac{\partial^3 u}{\partial t \partial x^2} \in C^0[a,b] \times L^2[0,T]$, *then, for each time t_{n+1}, exists a unique solution u_h^{n+1} defined by (22)-(23) such that (40), (41) hold provided that (27), (26), (36) are satisfied.*

■

5. A discrete ϵ-limit model

In this section we present a numerical method for the computation of an approximation to the solution of the ϵ-limit heat equation with memory (15). The method is established discretizing the memory term of (15) with a rectangular rule. A splitting approach was followed in [2] for the computation of numerical approximations to the solution of the heat equation (15) but this approach do not enables us to observe for the numerical solution a discrete version of (19).

Let w_h^n be the fully discrete approximation to the solution of (15) defined by

$$\begin{aligned} D_{-t} w_h^{n+1}(x_i) = \gamma D_{2,x} w_h^{n+1}(x_i) + \frac{\sigma}{\tau} \Delta t \sum_{\ell=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\tau}} D_{2,x} w_h^\ell(x_i) \\ + f(x_i, t_{n+1}), i = 1, \dots, N-1, \end{aligned} \tag{42}$$

where

$$\begin{aligned} & w_h^j(x_0) = w_a(t_j),\ w_h^j(x_N) = w_b(t_j),\ j = 1, \dots, M-1, \\ & w_h^0(x_i) = w_0(x_i), i = 1, \dots, N-1. \end{aligned} \tag{43}$$

The scheme was obtained integrating numerically the temporal derivative of (19) using the Euler-Implicit method and considering a rectangular rule on the discretization of the memory term.

Theorem 4. *Let w_h^ℓ be defined by (42)-(43) with $w_a(t) = w_b(t) = 0, t > 0$. Then*

$$\|w_h^{n+1}\|^2_{L^2(I_h)} + \frac{\sigma}{\tau}\|\Delta t \sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x} w_h^j\|^2_{L^2(I_h^+)}$$
$$\leq \Delta t \sum_{j=1}^{n} S_{I_1}^{n+1-j} \|f_h(t_{j+1})\|^2_{L^2(I_h)} + S_{I_1}^n S_{I_2}\Big(\Delta t \|f_h(t_1)\|^2_{L^2(I_h)} + \|w_h^0\|^2_{L^2(I_h)}\Big) \tag{44}$$

where

$$S_{I_1} = \frac{1}{\min\{1, 1 - \Delta t\big(1 - \frac{2\gamma + \Delta t \frac{\sigma}{\tau}}{(b-a)^2}\big)\}} \tag{45}$$

and

$$S_{I_2} = \frac{1}{\min\{1, 1 - \Delta t\big(1 - \frac{2\gamma}{(b-a)^2}\big)\}} \tag{46}$$

provided that

$$1 - \Delta t\big(1 - \frac{2\gamma}{(b-a)^2}\big) > 0. \tag{47}$$

Proof:

(1) Let us consider in (42) $n \in \mathbb{N}$. Multiplying each member of (42) by w_h^{n+1} with respect to the inner product $(.,.)_h$ and using summation by parts we obtain

$$\|w_h^{n+1}\|^2_{L^2(I_h)} = (w_h^n, w_h^{n+1})_h - \gamma \Delta t \|D_{-x} w_h^{n+1}\|^2_{L^2(I_h^+)} + \Delta t (f_h(t_{n+1}), w_h^{n+1})_h$$
$$-\frac{\sigma \Delta t^2}{\tau}\Big(\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x} w_h^j, D_{-x} w_h^{n+1}\Big)_{h,+}, \tag{48}$$

where $f_h(t_{n+1})(x_j) = f(x_j, t_{n+1})$.

As

$$\Big(\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x} w_h^j, D_{-x} w_h^{n+1}\Big)_{h,+} = \frac{1}{2}\|\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x} w_h^j\|^2_{L^2(I_h^+)}$$
$$-\frac{1}{2} e^{-2\frac{\Delta t}{\tau}} \|\sum_{j=1}^{n} e^{-\frac{t_n - t_j}{\tau}} D_{-x} w_h^j\|^2_{L^2(I_h^+)} + \frac{1}{2}\|D_{-x} w_h^{n+1}\|^2_{L^2(I_h^+)}, \tag{49}$$

from (48) we have

$$
\begin{aligned}
&\|w_h^{n+1}\|^2_{L^2(I_h)} + \frac{\sigma}{2\tau}\|\Delta t\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j\|^2_{L^2(I_h^+)}\\
&= (w_h^n, w_h^{n+1})_h + \Delta t (f_h(t_{n+1}), w_h^{n+1})_h\\
&+\frac{\sigma}{2\tau}e^{-2\frac{\Delta t}{\tau}}\|\Delta \sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} D_{-x}w_h^j\|^2_{L^2(I_h^+)} - \Delta t\big(\frac{\sigma\Delta t}{2\tau}+\gamma\big)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\,.
\end{aligned}
\tag{50}
$$

Considering in (50) the discrete Poincaré-Friedrichs inequality and the estimates

$$(w_h^n, w_h^{n+1})_h \leq \frac{1}{2}\|w_h^{n+1}\|^2_{L^2(I_h)} + \frac{1}{2}\|w_h^n\|^2_{L^2(I_h)},$$

$$(f_h(t_{n+1}), w_h^{n+1})_h \leq \frac{1}{2}\|f_h(t_{n+1})\|^2_{L^2(I_h)} + \frac{1}{2}\|w_h^{n+1}\|^2_{L^2(I_h)},$$

we conclude

$$
\begin{aligned}
&\left(1-\Delta t + \Delta t\frac{2\gamma+\frac{\sigma\Delta t}{\tau}}{(b-a)^2}\right)\|w_h^{n+1}\|^2_{L^2(I_h)} + \sigma\|\Delta t\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}w_h^j\|^2_{L^2(I_h^+)}\\
&\quad\leq \Delta t\|f_h(t_{n+1})\|^2_{L^2(I_h)} + \|w_h^n\|^2_{L^2(I_h)} + \frac{\sigma}{\tau}e^{-2\frac{\Delta t}{\tau}}\|\Delta t\sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} D_{-x}w_h^j\|^2_{L^2(I_h^+)}\,.
\end{aligned}
\tag{51}
$$

If we assume that Δt satisfies

$$1-\Delta t\big(1-\frac{2\gamma+\Delta t\frac{\sigma}{\tau}}{(b-a)^2}\big) > 0, \tag{52}$$

which is consequence of (45), inequality (51) enables to conclude

$$
\begin{aligned}
&\|w_h^{n+1}\|^2_{L^2(I_h)} + \frac{\sigma}{\tau}\|\Delta t\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}w_h^j\|^2_{L^2(I_h^+)}\\
&\leq \Delta t\sum_{j=1}^{n} S_I^{n+1-j}\|f_h(t_{j+1})\|^2_{L^2(I_h)}\\
&\quad +S_{I_1}^n\Big(\|w_h^1\|^2_{L^2(I_h)} + \frac{\sigma\Delta t^2}{\tau}\|D_{-x}w_h^1\|^2_{L^2(I_h^+)}\Big)
\end{aligned}
\tag{53}
$$

with S_{I_1} defined by (45).

(2) We consider now in (42) $n = 0$. Following the proof of (51) it can shown that

$$\min\{1, 1 - \Delta t\big(1 - \frac{2\gamma}{(b-a)^2}\big)\}\Big(\|w_h^1\|^2_{L^2(I_h)} + \frac{\sigma}{\tau}\|\Delta t D_{-x} w_h^1\|^2_{L^2(I_h^+)}\Big) \le \Delta t\|f(t_1)\|^2_{L^2(I_h)} + \|w_h^0\|^2_{L^2(I_h)}, \tag{54}$$

and then

$$\|w_h^1\|^2_{L^2(I_h)} + \frac{\sigma}{\tau}\|\Delta t D_{-x} w_h^1\|^2_{L^2(I_h^+)} \le \frac{1}{\min\{1, 1 + \Delta t\big(\frac{2\gamma}{(b-a)^2} - 1\big)\}}\Big(\Delta t\|f_h(t_1)\|^2_{L^2(I_h)} + \|w_h^0\|^2_{L^2(I_h)}\Big) \tag{55}$$

provided that (47) holds.

■

Theorem 4 implies the following stability result:

Corollary 5. *Let $w_h^j, \tilde{w}_h^j$ be defined by (42)-(43) with initial conditions w_0 and $\tilde{w}_0$ respectively. Under the assumptions of Theorem 4, $v_h^j = w_h^j - \tilde{w}_h^j$ satisfies*

$$\|v_h^{n+1}\|^2_{L^2(I_h)} + \frac{\sigma}{\tau}\|\Delta t \sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x} v_h^j\|^2_{L^2(I_h^+)} \le S_{I_1}^n S_{I_2}\|w_h^0 - \tilde{w}_h^0\|^2_{L^2(I_h)}. \tag{56}$$

■

Considering the error equation for the global error $e_h^j = w_h^j - R_h w(.,t_j)$ and following the proof Theorem 4, it can be shown that

$$\|e_h^{n+1}\|^2_{L^2(I_h)} + \frac{\sigma}{\tau}\|\Delta t \sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x} e_h^j\|^2_{L^2(I_h^+)} \to 0, \tag{57}$$

when $\Delta t, h \to 0$, provided that w - solution of (4)-(6) - is smooth enough.

In Corollary 6 we summarize the convergence result.

Corollary 6. *If the solution w of the IBVP (4)-(6) is such that $\frac{\partial^2 w}{\partial t^2} \in C^0[a,b] \times L^2[0,T]$, $\frac{\partial^3 w}{\partial x^3} \in L^2[a,b] \times C^0[0,T]$, $\frac{\partial^3 w}{\partial t \partial x^2} \in C^0[a,b] \times L^2[0,T]$, then, for each for each time t_{n+1}, exists a unique solution w_h^{n+1} defined by (42)-(43) such that (57) holds provided that Δt satisfies (47).*

■

6. The two discrete models

In this section we study the behavior of $\|u_h^{n+1}-w_h^{n+1}\|_{L^2(I_h)}$ where u_h^{n+1} and w_h^{n+1} are defined by (22)-(23) and (42)-(43) respectively. We suppose that Corollaries 4 and 6 hold.

As we have

$$\|u_h^{n+1}-w_h^{n+1}\|_{L^2(I_h)} \leq \|e_{s,h}^{n+1}\|_{L^2(I_h)} + \|R_h(u-w)(.\,,t_{n+1})\|_{L^2(I_h)} + \|e_h^{n+1}\|_{L^2(I_h)}\,, \tag{58}$$

and, from (41) and (57),

$$\|e_{s,h}^{n+1}\|_{L^2(I_h)} + \|e_h^{n+1}\|_{L^2(I_h)} \to 0,$$

if we prove

$$\lim_{\epsilon\to 0}\lim_{h,\Delta t\to 0}\|R_h(u-w)(.\,,t_{n+1})\|_{L^2(I_h)} = \lim_{\Delta t,h\to 0}\lim_{\epsilon\to 0}\|R_h(u-w)(.\,,t_{n+1})\|_{L^2(I_h)}, \tag{59}$$

we conclude

$$\lim_{h,\Delta t\to 0}\lim_{\epsilon\to 0}\|u_h^{n+1}-w_h^{n+1}\|_{L^2(I_h)} = \lim_{\epsilon\to 0}\lim_{h,\Delta t\to 0} + \|u_h^{n+1}-w_h^{n+1}\|_{L^2(I_h)} = 0, \tag{60}$$

provided that u is such that

$$\|\frac{\partial^3 u}{\partial t^3}\|_{C^0[a,b]\times L^2[0,T]}\,,\ \|\frac{\partial^3 u}{\partial x^3}\|_{L^2[a,b]\times C^0[0,T]}\,,\ \|\frac{\partial^3 u}{\partial t\partial x^2}\|_{C^0[a,b]\times L^2[0,T]}$$

are ϵ-uniformly bounded.

Convergence (59) is an immediate consequence of

$$\begin{aligned}\|R_h(u-w)(.\,,t_{n+1})\|^2_{L^2(I_h)} &\leq \|(u-w)(.\,,t_{n+1})\|^2_{L^2}\\ &\quad +2h\|(u-w)(.\,,t_{n+1})\|_{L^2(a,b)} + \|\frac{\partial}{\partial x}(u-w)(.\,,t_{n+1})\|_{L^2},\end{aligned}$$

provided that $\|\frac{\partial u}{\partial x}\|_{L^2\times C^0[0,T]}$ is $\epsilon-$ uniformly bounded.

7. Numerical simulation

Let us start by illustrating the performance of method (22)-(23) on the computation of numerical approximations to the solution of (1)-(3) with $a = 0, b = 1$ and homogeneous boundary conditions. The numerical experiments were obtained with

$$u_{0,\epsilon}(x) = \begin{cases} 0, \ \ x \in [0, 0.4-\epsilon) \cup (0.6+\epsilon, 1] \\ 1 + (x - 0.4 - \epsilon)/(2\epsilon), \ \ x \in [0.4-\epsilon, 0.4+\epsilon] \\ 1 - (x - 0.6 + \epsilon)/(2\epsilon) \ \ x \in [0.6-\epsilon, 0.6+\epsilon] \\ 1, \ \ x \in [0.4+\epsilon, 0.6-\epsilon] \end{cases} \tag{61}$$

which converges to

$$w_0(x) = \begin{cases} 0, \ \ x \in [0, 0.4) \cup (0.6, 1] \\ 1, \ \ x \in [0.4, 0.6] \end{cases} \tag{62}$$

when $\epsilon \longrightarrow 0$.

In Figure 1 we plot the results obtained with $\epsilon = 0.05$, $f_\epsilon = 0$, $\gamma = 0.15, \sigma = 0.1$, $h = \Delta t = 0.01$, $\tau = 1$ and $\tau = 0.001$. This figure illustrates the behavior of u when τ increases. In this case, attending that the weight of the second order spatial derivative in the memory term decreases, we observe an increasing of the smoothness of the solution.

The same smoothness behavior is observed when ϵ decreases. In Figures 2 we plot the numerical solutions obtained with $\epsilon = 0.1$ and $\epsilon = 0.0001$. As we expected, when ϵ decreases the hyperbolic character of equation (1) also decreases.

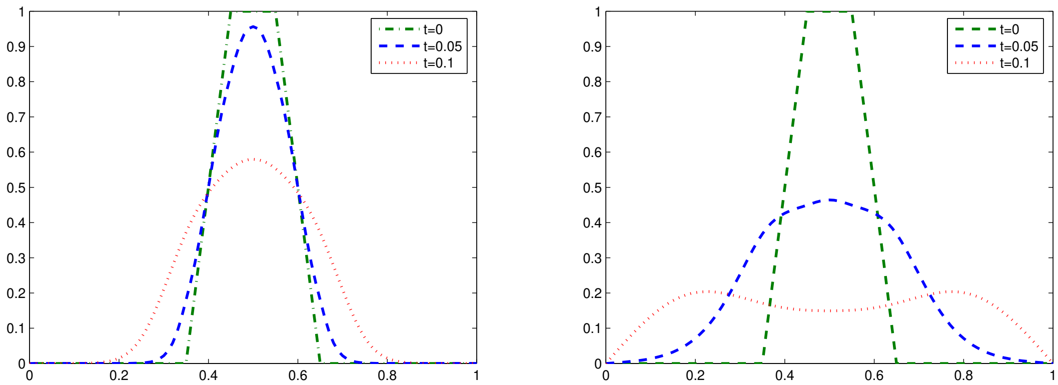

FIGURE 1. Numerical solutions obtained with method (22)-(23), for $\epsilon = 0.05, h = \Delta t = 0.01$, $\tau = 1$ (left) and $\tau = 0.001$ (right).

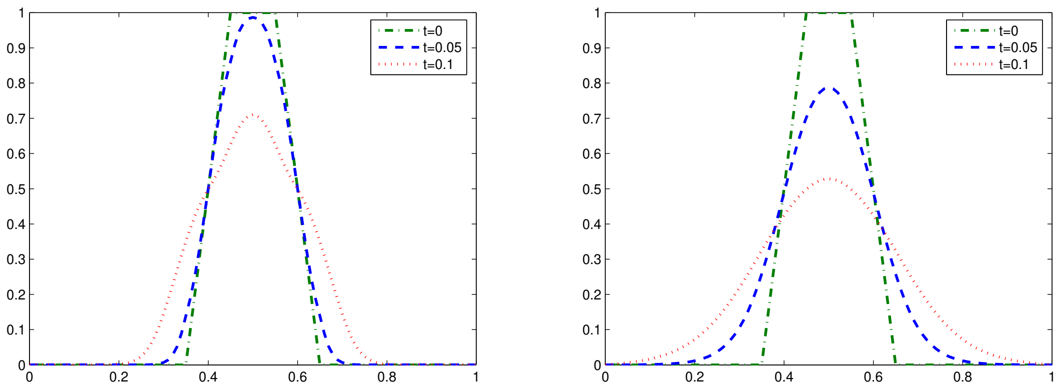

FIGURE 2. Numerical solutions obtained with method (22)-(23), for $h = \Delta t = 0.01$, $\epsilon = 0.1$(left) and $\epsilon = 0.0001$(right).

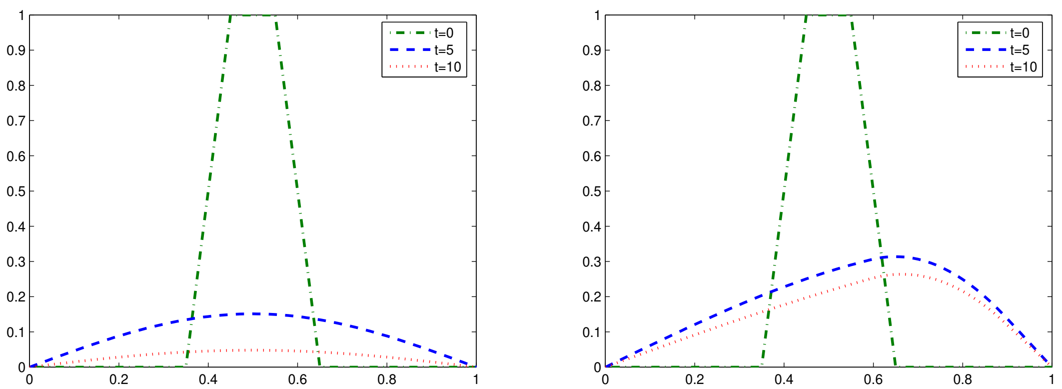

FIGURE 3. Numerical solutions obtained with method (22)-(23), for $h = \Delta t = 0.01$, at $t = 0, t = 5$ and $t = 10$, with $f_\epsilon = 0$(left) and $f_{2\epsilon}$ defined by (63)(right).

In order to capture the behavior of the solution of (1)-(3) when a source function in applied, we took, in the next numerical experiments, $\gamma = 0.015, \sigma = 0.01, \tau = 1$, $T = 10$ and

$$f_\epsilon(x,t) = \begin{cases} 0, & x \in [0, 0.6) \cup (0.9, 1] \\ \epsilon, & x \in [0.6, 0.9]\,. \end{cases} \tag{63}$$

In Figure 3 we plot the numerical results obtained with $h = \Delta t = 0.01$, $\epsilon = 0.05$ and $f_{2\epsilon}$.

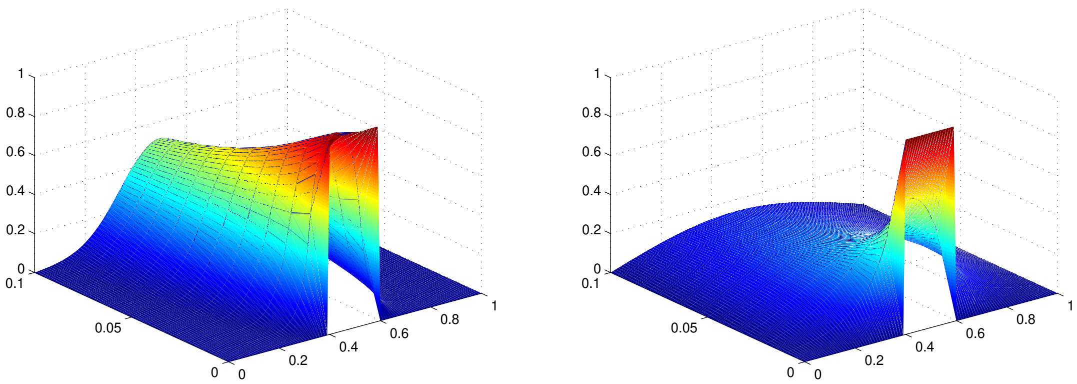

FIGURE 4. Numerical solutions obtained with methods (42)-(43), for $h = \Delta t = 0.01$, with $\tau = 1$(left) and $\tau = 0.001$(right).

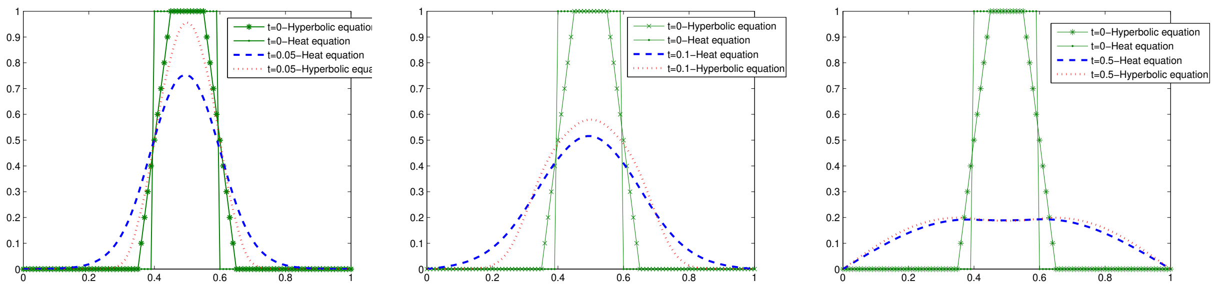

FIGURE 5. Numerical solutions obtained with methods (22)-(23) and (42)-(43), for $h = \Delta t = 0.01$, with $\epsilon = 0.05$ at $t = 0.05$(left), $t = 0.1$ (center) and $t = 0.5$(right).

In what follows we illustrate the behavior of method (42)-(43) with initial condition (62) and $f = 0$. In Figure 4 we plot the numerical results obtained with $\gamma = 0.15, \sigma = 0.1$, $h = \Delta t = 0.01$, and $\tau = 1, 0.001$. The decreasing of τ implies an increasing of the smoothness of the solution the heat equation with memory.

Let us consider now the convergence behavior of the difference between the numerical approximations to the solutions of the IBVPs (1)-(2) , (15)-(16) when $\epsilon \to 0$. In order to observe the previous behavior we start by taking $f_\epsilon = f = 0$, $\gamma = 0.15, \sigma = 0.1, \tau = 1$ and $h = \Delta t = 0.01$. In Figures 5 and 6 we plot the numerical solutions obtained considering method (22)-(23) with $u_{0,\epsilon}$ defined by (61) for $\epsilon = 0.05, 0.0001$ and (42)-(43) at $t = 0.05, 0.1, 0.5$.

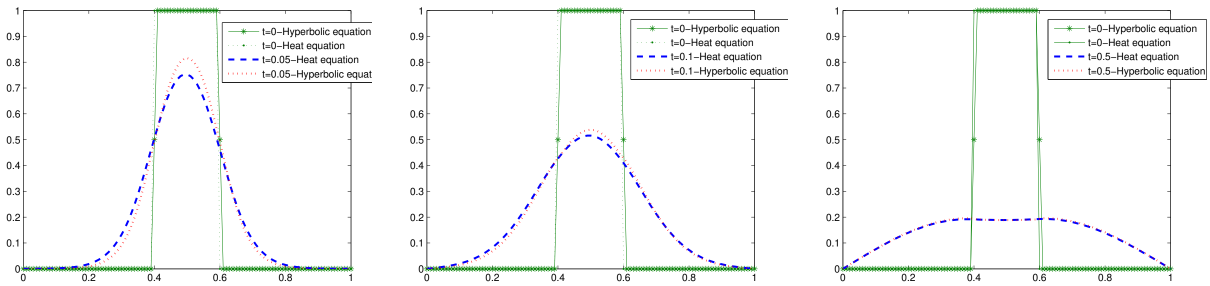

FIGURE 6. Numerical solutions obtained with methods (22)-(23) and (42)-(43), for $h = \Delta t = 0.01$, with $\epsilon = 0.0001$ at $t = 0.05$(left), $t = 0.1$ (center) and $t = 0.5$(right).

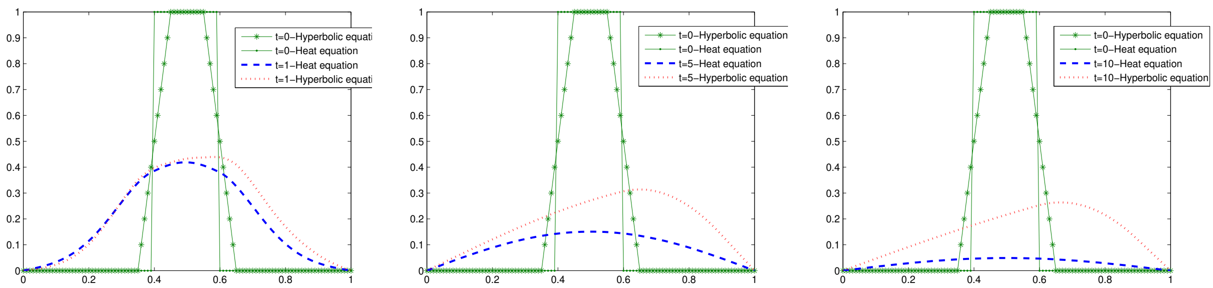

FIGURE 7. Numerical solutions obtained with methods (22)-(23) and (42)-(43), for $h = \Delta t = 0.01$ with $\epsilon = 0.05$ at $t = 1$(left), $t = 5$ (center) and $t = 10$(right).

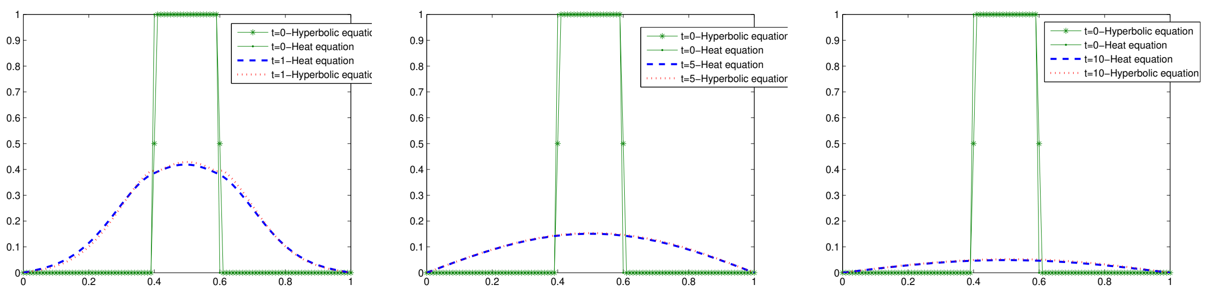

FIGURE 8. Numerical solutions obtained with methods (22)-(23) and (42)-(43), for $h = \Delta t = 0.01$, with $\epsilon = 0.0001$ at $t = 1$(left), $t = 5$ (center) and $t = 10$(right).

Finally in Figures 7 and 8 we consider $f_{2\epsilon}$ defined by (63), $\gamma = 0.015, \sigma = 0.01, \tau = 1$, $\epsilon = 0.05$ and $\epsilon = 0.001$ for $t = 1, 5, 10$.

The numerical results plotted in Figures 5, 6, 7 and 8 illustrate in fact the the convergence of $\|u_h^{n+1} - w_h^{n+1}\|_{L^2(I_h)} \to 0$ when $h, \Delta t, \epsilon \to 0$.

References

[1] A. Araújo, J.A.Ferreira, P. de Oliveira, 2005, Qualitative behavior of numerical traveling wave solutions for reaction-diffusion equations with memory, Applicable Analysis, 84, 1231-1246.

[2] A. Araújo, J.A.Ferreira, P. de Oliveira, 2006, The effect of memory terms in diffusion phenomena, Journal of Computational Mathematics, 24, 91-102.

[3] J. Branco, J.A. Ferreira, P. de Oliveira, On numerical methods for generalized Fisher-Kolmogorov-Piskunov Equation, to appear in Applied Numerical Mathematics.

[4] D.D. Joseph, L.Preziosi, 1989, Heat Waves, Review of Modern Physics, 61, 1, 41-73.

[5] M.He, 2005, A class of integrodifferential equations and applications, Discrete and Continuous Dynamical Systems, Suplement Volume, 386-396.

[6] B.Hu, H-M. Yin, 1997, The DeGiorgi-Nasher-Moser type of estimate for parabolic Volterra integrrodifferential equations, Pacific Journal of Mathematics, 178, 265-277.

[7] J. H. Liu, 1993, A singular perturbation problem in integrodifferential equations, Electronic Journal of Differential Equations, 1993, 1-10.

[8] J. H. Liu, 1994, Singular perturbations of integrodifferential equations in banach space, Proc. Amer. Math. Soc. 122, 791-799.

[9] X. Zhou, K. Tamma, C. Anderson, 2001, On a new C- and F- processes heat conduction constitutive models and the associated generalized theory of dynamic thermoelasticity, Journal of Thermal Stresses, 24, 531-564.

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