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ON PROPERTIES OF HYPERGEOMETRIC TYPE-FUNCTIONS

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ABSTRACT: The functions of hypergeometric type are the solutions $y = y_\nu(z)$ of the differential equation $\sigma(z)y'' + \tau(z)y' + \lambda y = 0$, where σ and τ are polynomials of degrees not higher than 2 and 1, respectively, and λ is a constant. Here we consider a class of functions of hypergeometric type: those that satisfy the condition $\lambda + \nu\tau' + \frac{1}{2}\nu(\nu-1)\sigma'' = 0$, where ν is an arbitrary complex (fixed) number. We also assume that the coefficients of the polynomials σ and τ do not depend on ν. To this class of functions belong Gauss, Kummer and Hermite functions, and also the classical orthogonal polynomials. In this work, using the constructive approach introduced by Nikiforov and Uvarov, several structural properties of the hypergeometric type functions $y = y_\nu(z)$ are obtained. Applications to hypergeometric functions and classical orthogonal polynomials are also given.

KEYWORDS: Hypergeometric Type Functions, Recurrence Relations, Classical Orthogonal Polynomials.
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1. Introduction

When solving numerous theoretical and applied quantum mechanical problems, one is led to potentials that can be solved analytically (see e.g. [2, 10, 14, 16, 17, 18]). In most cases the Schrödinger equation for such potentials can be transformed into the *generalized hypergeometric type* differential equation [12] that has the form

$$u''(z) + \frac{\tilde{\tau}(z)}{\sigma(z)}u'(z) + \frac{\tilde{\sigma}(z)}{\sigma^2(z)}u(z) = 0,$$

where σ, $\tilde{\sigma}$ and $\tilde{\tau}$ are polynomials, $\deg[\sigma] \leq 2$, $\deg[\tilde{\sigma}] \leq 2$ and $\deg[\tilde{\tau}] \leq 1$. By a certain change of variable (see [12]) it can be transformed into the hypergeometric type equation

$$\sigma(z)y''(z) + \tau(z)y'(z) + \lambda y(z) = 0\,, \tag{1.1}$$

where σ and τ are polynomials with degrees not higher than two and one, respectively, and λ is a constant. Their solutions are known as *hypergeometric type functions* and to this class belong the Bessel, Airy, Weber, Whittaker,

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Gauss, Kummer and Hermite functions, the classical orthogonal polynomials, among others.

The class of functions $y = y_\nu(z)$ we are dealing with in this work, corresponds to the solutions of the hypergeometric equation (1.1) under the condition

$$\lambda + \nu\tau' + \frac{\nu(\nu-1)}{2}\sigma'' = 0,$$

where ν is a complex number. One basic important property of this class of functions is that *their derivatives are again a hypergeometric type functions.* The converse is also true when $\deg[\sigma(s)] = 2 \ \vee \ \deg[\tau(s)] = 1$: *any hypergeometric type function is the derivative of a hypergeometric type function.* More precisely [12]:

(1) if $y = y(z)$ is a solution of (1.1) then, the n-th derivative of $y(z)$, $v_n(z) := y^{(n)}(z)$, is a solution of

$$\sigma v_n''(z) + \tau_n(z)v_n'(z) + \mu_n v_n(z) = 0, \tag{1.2}$$

where

$$\tau_n(z) = \tau(z) + n\sigma'(z); \tag{1.3}$$

(2) if $v_n(z)$ is a solution of (1.2) and $\mu_k \neq 0$ for $k = 1, \ldots, n-1$, then $v_n = y^{(n)}(z)$ where $y = y(z)$ is a solution of (1.1).

Joining these two properties it is possible to derive many other ones [12]. Numerous structural properties of this class of functions has been under attention in the last two decades [4, 5, 6, 7, 8, 19].

The main aim of this work is to complete the study of the hypergeometric type functions, i.e., the solutions of the differential equation (refE1.1) obtaining, in a unified way, several of their algebraic characteristics. The structure of the paper is as follows: In section 2 the preliminar results are presented. The main results of the paper are in Section 3, where three four therm recurrence are obtained and from them, several three term recurrence relations are explicitly written down. Finally, in Section 4, applications to the hypergeometric, confluent hypergeometric and Hermite functions, as well as to the classical orthogonal polynomials are given.

2. Preliminaries

Here we will follow the notation and results of [12]. The above properties (1) and (2) allow us to construct a family of particular solutions of (1.1) for a given λ. In fact, when $\mu_n = 0$, equation (1.2) has the particular solution

$v_n(z) = C$, (constant). By (2), $v_n(z) = y^{(n)}$ where $y = y(z)$ is a solution of (1.1). This means that when

$$\lambda = \lambda_n := -n\tau' - \frac{n(n-1)}{2}\sigma'' \tag{2.1}$$

the equation (1.1) has a (particular) polynomial solution $y(z) = y_n(z)$, with $\deg[y_n(z)] = n$. Such polynomials are known as *polynomials of hypergeometric type* and correspond to the case when $\lambda = \lambda_n$ is given by (2.1). In particular, for them we have the *Rodrigues formula*

$$y_n(z) = \frac{B_n}{\rho(z)}\big[\sigma^n(z)\rho(z)\big]^{(n)}, \tag{2.2}$$

where $\rho_n(z) = \sigma^n(z)\rho(z)$, $n = 0, 1, 2, \ldots$, and $\rho(z)$ is a solution of the Pearson equation

$$\big[\sigma(z)\rho(z)\big]' = \tau(z)\rho(z)\,. \tag{2.3}$$

Assuming that ρ is an analytic function on and inside a closed contour $\mathcal{C}$ surrounding the point $s = z$ and making use of the Cauchy's integral theorem (see e.g. [9]) we may write

$$y_n(z) = \frac{C_n}{\rho(z)}\int_{\mathcal{C}} \frac{\sigma^n(s)\rho(s)}{(s-z)^{(n+1)}}ds\,, \tag{2.4}$$

where the $C_n = n!B_n/(2\pi i)$ is a normalizing constant and $\rho(z)$ satisfies (2.3). This suggests [12] to look for a particular solution of (1.1) in the form

$$y_\nu(z) = \frac{C_\nu}{\rho(z)}\int_{\mathcal{C}} \frac{\sigma^\nu(s)\rho(s)}{(s-z)^{(\nu+1)}}ds\,, \tag{2.5}$$

where C_ν is a normalizing constant and ν is an arbitrary complex parameter connected with λ by

$$\lambda = \lambda_\nu = -\nu\tau' - \frac{\nu(\nu-1)}{2}\sigma''\,. \tag{2.6}$$

The following theorem asserts that the above suggestion is true.

Theorem A([12, page 10]) *Let $\rho(z)$ satisfy the Pearson equation (2.3) where ν is a solution of (2.6) and let D be a region of the complex plane that contain the piecewise smooth curve $\mathcal{C}$ of finite length. Then, equation (1.1) has a particular solution of the form (2.5) provided that the functions $\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{(\nu+k)}}$, for $k = 1, 2,$*

- *are continuous as functions of the variables $s \in \mathcal{C}$, $z \in D$;*
- *for each fixed $s \in \mathcal{C}$, they are analytic as functions of $z \in D$;*

and $\mathcal{C}$ is such that $\left.\frac{\sigma^{\nu+1}(s)\rho(s)}{(s-z)^{\nu+2}}\right|_{s_1}^{s_2} = 0$, where s_1 and s_2 are the endpoints of $\mathcal{C}$.

If the integral in (2.5) is an improper one, then the result remains valid if the convergence of the integral is uniform [9, page 188].

In next sections, generalizing (1.3) to complex ν, we will use the notation

$$\tau_\nu(z) = \tau(z) + \nu\sigma'(z) = \tau_\nu' \, z + \tau_\nu(0), \tag{2.7}$$

and, in order to keep the above mentioned property (2), we will restrict ourselves to the condition $\deg[\sigma(s)] = 2 \;\vee\; \deg[\tau(s)] = 1$.

3. Recurrence Relations for the hypergeometric type functions

Now we are ready to establish the main results of this paper.

3.1. Four-Term Recurrence Relations.

Theorem 1. *Consider the hypergeometric type functions $y_{\nu-1}^{(k)}(z)$, $y_\nu^{(k)}(z)$, $y_\nu^{(k+1)}(z)$ and $y_{\nu+1}^{(k+1)}(z)$ defined by (2.5). Suppose that $\rho(z)$ is a solution of (2.3) and*

$$\left.\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k-1}} s^m\right|_{s_1}^{s_2} = 0, \quad m = 0, 1, 2, ..., \tag{3.1}$$

where s_1 and s_2 are the end points of $\mathcal{C}$. Then, there exist polynomial coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, not all identically zero, such that

$$A_{1k}(z)y_{\nu-1}^{(k)}(z) + A_{2k}(z)y_\nu^{(k)}(z) + A_{3k}(z)y_\nu^{(k+1)}(z) + A_{4k}(z)y_{\nu+1}^{(k+1)}(z) = 0. \tag{3.2}$$

Moreover, the functions A_{ik}, $i = 1,2,3,4$ are given by

$$
\begin{cases}
A_{1k}(z) = -\tau'_{\nu}\tau'_{\frac{\nu+k-1}{2}}\tau'_{\frac{\nu+k-2}{2}}\left[\tau^2_{\nu-1}(0)\dfrac{\sigma''}{2} - \tau'_{\nu-1}\Big(\tau_{\nu-1}(0)\sigma'(0) - \tau'_{\nu-1}\sigma(0)\Big)\right] \\
\qquad\qquad \times\dfrac{C_\nu}{C_{\nu-1}}\Big(R(z) - 2\sigma'(z)\Big), \\
A_{2k}(z) = (\nu-k)\tau'_{\frac{\nu}{2}-1}\tau'_{\nu}\tau'_{\frac{\nu+k-1}{2}}\left[\Big(R(z)-2\sigma'(z)\Big)\left(\tau_{\nu-1}(z)\dfrac{\sigma''}{2}-\sigma'(z)\tau'_{\nu-1}\right)-\tau'_{\nu-1}\sigma''\sigma(z)\right], \\
A_{3k}(z) = \left[\tau'_{\nu}\tau'_{\frac{\nu+k-1}{2}}R(z) + (\nu-k)\dfrac{(\sigma'')^2}{2}\tau_\nu(z) - 2\tau'_{\nu-\frac{1}{2}}\tau'_{\nu}\sigma'(z)\right]\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\sigma(z), \\
A_{4k}(z) = (\nu-k)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau'_{\nu-1}\sigma''\sigma(z),
\end{cases}
\tag{3.3}
$$

where $R(z)$ is an arbitrary polynomial of z.

Proof: From [12, Eq. (9), page 17] we have

$$y_\nu^{(k)}(z) = \frac{C_\nu^{(k)}}{\sigma^k(z)\rho(z)}\int_{\mathcal{C}}\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k+1}}ds, \qquad C_\nu^{(n)} = \left(\prod_{j=0}^{n-1}\tau'_{\frac{\nu+j-1}{2}}\right)C_\nu. \tag{3.4}$$

Now, using [12, Eqs. (4), page 16, and (9), page 17]

$$y_\nu^{(k)}(z) = \frac{C_\nu^{(k)}}{\sigma^k(z)\rho(z)}\frac{1}{\nu-k}\int_{\mathcal{C}}\frac{\tau_{\nu-1}(s)\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}}ds. \tag{3.5}$$

Substituting the above expressions (3.4) and (3.5) in

$$S(z) = A_{1k}(z)y_{\nu-1}^{(k)}(z) + A_{2k}(z)y_\nu^{(k)}(z) + A_{3k}(z)y_\nu^{(k+1)}(z) + A_{4k}(z)y_{\nu+1}^{(k+1)}(z),$$

we obtain

$$S(z) = \frac{1}{\sigma^{k+1}(z)\rho(z)}\int_{\mathcal{C}}\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}}P(s)ds\,, \tag{3.6}$$

where

$$P(s) = A_{1k}(z)C_{\nu-1}^{(k)}\sigma(z) + A_{2k}(z)\frac{C_\nu^{(k)}}{\nu-k}\sigma(z)\tau_{\nu-1}(s) + A_{3k}(z)C_\nu^{(k+1)}\sigma(s) +$$
$$A_{4k}(z)\frac{C_{\nu+1}^{(k+1)}}{\nu-k}\tau_\nu(s)\sigma(s)\,. \tag{3.7}$$

Let us define a function $Q(z,s)$ which is, for every fixed z, a polynomial in s such that

$$\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}}P(s) = \frac{\partial}{\partial s}\left[\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k-1}}Q(z,s)\right]\,.$$

If such function Q exists, then the integral (3.6) vanish by the boundary conditions (3.1) and therefore (3.2) holds. Let us show that the aforementioned function Q always exists.

Taking the derivative of the right hand side of the last equality, one gets

$$P(s) = [\tau_{\nu-1}(s)(s-z) - (\nu-k-1)\sigma(s)]\, Q(z,s) + \sigma(s)(s-z)\frac{\partial Q}{\partial s}(z,s)\,. \tag{3.8}$$

Comparing the expressions (3.7) and (3.8) we may conclude that, with respect to the variable s, $\deg\,[Q(z,s)] = \deg_s\,[Q(z,s)] \leq 1$. So, using the expansions

$$Q(z,s) = Q(z,z) + \frac{\partial Q}{\partial s}(z,z)(s-z),$$

$$\tau_\nu(s) = \tau_\nu(z) + \tau_\nu'(s-z), \quad \sigma(s) = \sigma(z) + \sigma'(z)(s-z) + \frac{\sigma''}{2}(s-z)^2\,, \tag{3.9}$$

as well as (2.7), we have

$$\begin{cases} A_{1k}(z)C_{\nu-1}^{(k)}\sigma(z) + A_{2k}(z)\dfrac{C_\nu^{(k)}}{\nu-k}\tau_{\nu-1}(z)\sigma(z) + A_{3k}(z)C_\nu^{(k+1)}\sigma(z) + \\ \qquad + A_{4k}(z)\dfrac{C_{\nu+1}^{(k+1)}}{\nu-k}\tau_\nu(z)\sigma(z) = -(\nu-k-1)\sigma(z)Q(z,z), \\ A_{2k}(z)\dfrac{C_\nu^{(k)}}{\nu-k}\tau_{\nu-1}'\sigma(z) + A_{3k}(z)C_\nu^{(k+1)}\sigma'(z) + A_{4k}(z)\dfrac{C_{\nu+1}^{(k+1)}}{\nu-k}\left[\tau_\nu(z)\sigma'(z) + \tau_\nu'\sigma(z)\right] \\ \qquad = \tau_k(z)Q(z,z) - (\nu-k-2)\sigma(z)\dfrac{\partial Q}{\partial s}(z,z), \\ A_{3k}(z)C_\nu^{(k+1)}\dfrac{\sigma''}{2} + A_{4k}(z)\dfrac{C_{\nu+1}^{(k+1)}}{\nu-k}\left[\tau_\nu(z)\dfrac{\sigma''}{2} + \tau_\nu'\sigma'(z)\right] = \tau'_{\frac{\nu+k-1}{2}}Q(z,z) + \\ \qquad \tau_{k+1}(z)\dfrac{\partial Q}{\partial s}(z,z), \\ A_{4k}(z)\dfrac{C_{\nu+1}^{(k+1)}}{\nu-k}\tau_\nu'\sigma'' = 2\tau'_{\frac{\nu+k}{2}}\dfrac{\partial Q}{\partial s}(z,z)\,. \end{cases} \tag{3.10}$$

Therefore, we have an indeterminate linear system of four equations with six unknown: the functions $A_{ik}(z)$, $i=1,2,3,4$, and the coefficients in the variable z of the polynomial (on s) $Q(z,s)$. This guarantees not only the existence of the functions $A_{ik}(z)$, $i=1,2,3,4$, but also the polynomial $Q(z,s)$

introduced above. Assuming that $\sigma'' \neq 0$, the above system can be written as

$$\begin{cases} A_{1k}(z)C_{\nu-1}^{(k)}\sigma(z) = -\dfrac{\tau_{\nu-1}(z)}{\tau'_{\nu-1}}\left(Q(z,z) - \dfrac{2}{\sigma''}\sigma'(z)\dfrac{\partial Q}{\partial s}(z,z)\right) \times \\ \qquad\qquad \left[\tau_{\nu-1}(z) + \dfrac{2}{\sigma''}\tau'_{\nu-1} \times \left(\dfrac{\tau'_{\nu-1}}{\tau_{\nu-1}(z)}\sigma(z) - \sigma'(z)\right)\right], \\ A_{2k}(z)\dfrac{C_{\nu}^{(k)}}{\nu-k} = \dfrac{1}{\sigma(z)\tau'_{\nu-1}}\left(\tau_{\nu-1}(z) - \dfrac{2}{\sigma''}\sigma'(z)\tau'_{\nu-1}\right)\left(Q(z,z) - \dfrac{2}{\sigma''}\sigma'(z)\dfrac{\partial Q}{\partial s}(z,z)\right) \\ \qquad\qquad -\dfrac{2}{\sigma''}\dfrac{\partial Q}{\partial s}(z,z), \\ A_{3k}(z)C_{\nu}^{(k+1)} = \dfrac{2}{\sigma''}\left[\tau'_{\frac{\nu+k-1}{2}}Q(z,z) + \left(\dfrac{\tau_{\nu}(z)}{\tau'_{\nu}}(\nu-k)\dfrac{\sigma''}{2} - \dfrac{2}{\sigma''}\sigma'(z)\tau'_{\nu-\frac{1}{2}}\right)\dfrac{\partial Q}{\partial s}(z,z)\right], \\ A_{4k}(z)\dfrac{C_{\nu+1}^{(k+1)}}{\nu-k} = \dfrac{2}{\sigma''}\dfrac{\tau'_{\frac{\nu+k}{2}}}{\tau'_{\nu}}\dfrac{\partial Q}{\partial s}(z,z). \end{cases}$$

Substituting the values A_{ik}, $i = 1, 2, 3, 4$, from above in the equation (3.2), which is an homogeneous linear equation, choosing $Q(z,z) = \frac{\partial Q}{\partial s}(z,z)R(z)/\sigma''$, where $R(z)$ is an arbitrary function of z, and simplifying the common factors, we get the non-trivial solution (3.3). ■

Notice that formulae (3.3) are still valid for $\sigma'' = 0$. This is a consequence of (3.2) and the principle of analytic continuation. Moreover, if one chooses $R(z)$ to be a polynomial in z then the corresponding expressions for the coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, in (3.3) are polynomials in z. Finally, let us mention that this method enables one to construct other type of solutions, not necessarily polynomials, since $R(z)$ is an arbitrary function of z.

Remark 1. *Let us make a short analysis of the cases when $\sigma'' = 0$. In this case, from (3.3), we have two possibilities*

(1) $\deg[\sigma(s)] = 1 \wedge \deg[\tau(s)] = 1$. *Then* $A_{4k} = 0$
(2) $\deg[\sigma(s)] = 0 \wedge \deg[\tau(s)] = 1$. *Then* $A_{2k} = 0 = A_{4k}$.

Since we are looking for solutions with non vanishing at the same time coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, we need to compare the expressions (3.7) and (3.8). From this analysis follows that $deg_s[Q(z,s)] = 0$. In the first case $Q(z,s)$ is a constant $\neq 0$ while in the second one $Q(z,s)$ is identically zero. Notice that in both cases the resulting systems are not equivalent to (3.10). The coefficients in these two cases are given

- *when* $\deg\sigma = 1$, $\sigma(s) = \sigma(z) + \sigma'(z)(s-z)$ *and* $\tau(s) = \tau(z) + \tau'(s-z)$, *by*

$$\begin{cases} A_{1k}(z) = \dfrac{2C_\nu}{C_{\nu-1}}\tau'\Big(\tau'\sigma(z) - \sigma'(z)\tau_{\nu-1}(z)\Big), \\ A_{2k}(z) = (\nu-k)\tau'\sigma'(z), \\ A_{3k}(z) = -(\nu-k)\big(\sigma'(z)\big)^2 - 2\tau'\sigma(z), \\ A_{4k}(z) = \dfrac{C_\nu}{C_{\nu+1}}(\nu-k)\sigma'(z)\,; \end{cases}$$

- *when* $\deg\sigma = 1$, $\sigma(s) = \sigma(z)$ *and* $\tau(s) = \tau(z) + \tau'(s-z)$, *by*

$$\begin{cases} A_{1k}(z) = -\dfrac{C_\nu}{C_{\nu-1}}\tau' A_{3k}(z), \\ A_{2k}(z) = -\dfrac{C_{\nu+1}}{C_\nu}\tau' A_{4k}(z), \end{cases}$$

where $A_{3k}(z)$ *and* $A_{4k}(z)$ *are arbitrary polynomials in* z.

In a similar fashion we can prove the following theorem:

Theorem 2. *Consider the functions of hypergeometric type* $y_{\nu-1}^{(k+1)}(z)$, $y_\nu^{(k)}(z)$, $y_\nu^{(k+1)}(z)$ *and* $y_{\nu+1}^{(k+1)}(z)$. *Suppose that* $\rho(z)$ *is a solution of (2.3),*

$$\left.\frac{\sigma^\nu(s)\rho(s)}{(s-z)^{\nu-k-1}}s^m\right|_{s_1}^{s_2} = 0\,, \quad m = 0, 1, 2, \ldots,$$

where s_1 *and* s_2 *are the end points of* $\mathcal{C}$. *Then, there exist polynomial coefficients* $A_{ik}(z)$, $i = 1, 2, 3, 4$, *not all identically zero, such that*

$$A_{1k}(z)y_{\nu-1}^{(k+1)}(z) + A_{2k}(z)y_\nu^{(k)}(z) + A_{3k}(z)y_\nu^{(k+1)}(z) + A_{4k}(z)y_{\nu+1}^{(k+1)}(z) = 0\,. \tag{3.11}$$

Moreover, the functions A_{ik}, $i=1,2,3,4$ are given by

$$\begin{cases} A_{1k}(z)=\dfrac{C_\nu}{C_{\nu-1}}\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}\Big[\tau^2_{\nu-1}(0)\dfrac{\sigma''}{2}-\tau'_{\nu-1}\Big(\tau_{\nu-1}(0)\sigma'(0)-\tau'_{\nu-1}\sigma(0)\Big)\Big]\times \\ \qquad\qquad \Big(R(z)-2\sigma'(z)\Big), \\ A_{2k}(z)=-(\nu-k)\tau'_\nu\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\Big[R(z)\tau'_{\nu-1}+\Big(\tau_{\nu-1}(z)\sigma''-2\sigma'(z)\tau'_{\nu-1}\Big)\Big], \\ A_{3k}(z)=\tau'_\nu\tau'_{\frac{\nu}{2}-1}\left[\tau'_{\frac{\nu+k-1}{2}}R(z)+(\nu-k)\dfrac{(\sigma'')^2}{2\tau'_\nu}\tau_\nu(z)-2\tau'_{\nu-\frac{1}{2}}\sigma'(z)\right]\tau_{\nu-1}(z), \\ A_{4k}(z)=(\nu-k)\sigma''\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau_{\nu-1}(z), \end{cases} \tag{3.12}$$

where $R(z)$ is an arbitrary polynomial of z.

Remark 2. *As in Remark 1, when $\sigma''=0$ from (3.12) the following two cases follow*

(1) $\deg[\sigma(s)]=1 \wedge \deg[\tau(s)]=1$, *then* $A_{4k}=0$,
(2) $\deg[\sigma(s)]=0 \wedge \deg[\tau(s)]=1$, *then* $A_{4k}=0$.

Solving the corresponding systems we find

- $\deg\sigma=1$ *and* $\deg\tau=1$, $\sigma(s)=\sigma(z)+\sigma'(z)(s-z)$ *and* $\tau(s)=\tau(z)+\tau'(s-z)$

$$\begin{cases} A_{1k}(z)=\dfrac{2C_\nu}{C_{\nu-1}}\Big(\sigma'(z)\tau_{\nu-1}(z)-\tau'\sigma(z)\Big), \\ A_{2k}(z)=(\nu-k)\tau', \\ A_{3k}(z)=-2\tau_{\frac{3\nu-k-2}{2}}, \\ A_{4k}(z)=\dfrac{C_\nu}{C_{\nu+1}}(\nu-k)\,; \end{cases}$$

- $\deg\sigma=0$ *and* $\deg\tau=1$, $\sigma(s)=\sigma(z)$ *and* $\tau(s)=\tau(z)+\tau'(s-z)$

$$\begin{cases} A_{3k}(z)=\dfrac{C_{\nu-1}}{\tau' C_\nu}\tau(z)A_{1k}(z), \\ A_{4k}(z)=-\dfrac{C_{\nu-1}}{\tau' C_{\nu+1}}(\nu-k)A_{1k}(z)-\dfrac{C_\nu}{\tau' C_{\nu+1}}A_{2k}(z)\,, \end{cases}$$

where $A_{1k}(z)$ and $A_{2k}(z)$ are arbitrary polynomials of z.

Theorem 3. *Consider the functions of hypergeometric type $y^{(k)}_{\nu-1}(z)$, $y^{(k)}_{\nu}(z)$, $y^{(k+1)}_{\nu}(z)$ and $y^{(k)}_{\nu+1}(z)$. Suppose that*

$$\left.\frac{\sigma^{\nu}(s)\rho(s)}{(s-z)^{\nu-k-1}}s^m\right|_{s_1}^{s_2}=0,\quad m=0,1,2,\dots, \tag{3.13}$$

where s_1 and s_2 are the end points of $\mathcal{C}$. Then, there exist polynomial coefficients $A_{ik}(z)$, $i=1,2,3,4$, not all identically zero, such that

$$A_{1k}(z)y^{(k)}_{\nu-1}(z)+A_{2k}(z)y^{(k)}_{\nu}(z)+A_{3k}(z)y^{(k+1)}_{\nu}(z)+A_{4k}(z)y^{(k)}_{\nu+1}(z)=0. \tag{3.14}$$

Moreover, the functions A_{ik}, $i=1,2,3,4$ are given by

$$\begin{cases} A_{1k}=\dfrac{C_\nu}{C_{\nu-1}}\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}\tau'_{\frac{\nu+k-2}{2}}\left(H(z)-\tau'_{\nu-\frac12}\right)\times \\ \qquad\qquad\left[\tau^2_{\nu-1}(0)\frac{\sigma''}{2}+\tau'_{\nu-1}\Big(\sigma(0)\tau'_{\nu-1}-\sigma'(0)\tau_{\nu-1}(0)\Big)\right], \\ A_{2k}=(\nu-k)\tau'_{\nu-\frac12}\tau'_\nu\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\times \\ \qquad\qquad\left[\Big(\tau(0)\sigma''-\sigma'(0)\tau'\Big)-\dfrac{\sigma''}{2}\tau_{\nu-1}(z)+H(z)\dfrac{\sigma'(0)\tau'_\nu-\tau_\nu(0)\sigma''}{\tau'_\nu}\right], \\ A_{3k}=-\tau'_{\nu-1}\tau'_{\nu-\frac12}\tau'_{\frac{\nu}{2}-1}\tau'_\nu\left(H(z)-\tau'_{\frac{\nu+k-1}{2}}\right)\sigma(z), \\ A_{4k}=H(z)(\nu-k)(\nu-k+1)\dfrac{\sigma''}{2}\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\nu-1}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}, \end{cases} \tag{3.15}$$

where $H(z)$ is an arbitrary polynomial of z.

Proof: Substituting (3.4) and (3.5) in the equation

$$S(z)=A_{1k}(z)y^{(k)}_{\nu-1}(z)+A_{2k}(z)y^{(k)}_{\nu}(z)+A_{3k}(z)y^{(k+1)}_{\nu}(z)+A_{4k}(z)y^{(k)}_{\nu+1}(z),$$

we obtain

$$S(z)=\frac{1}{\sigma^{k+1}(z)\rho(z)}\int_C\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}}P(s)ds\,,$$

where $P(s)$ is given by

$$P(s) = A_{1k} C_{\nu-1}^{(k)} \sigma(z) + A_{2k} \frac{C_{\nu}^{(k)}}{\nu - k} \sigma(z) \tau_{\nu-1}(s) + A_{3k} C_{\nu}^{(k+1)} \sigma(s) + \\ A_{4k} \frac{C_{\nu+1}^{(k)}}{(\nu-k-1)(\nu-k)} \sigma(z) \left(\tau_{\nu}' \sigma(s) + \tau_{\nu}(s) \tau_{\nu-1}(s)\right) . \tag{3.16}$$

Reasoning as in the proof of Theorem 1, we define a polynomial $Q(z,s)$ in the variable s such that

$$\frac{\sigma^{\nu-1}(s)\rho(s)}{(s-z)^{\nu-k}} P(s) = \frac{\partial}{\partial s} \left[\frac{\sigma^{\nu}(s)\rho(s)}{(s-z)^{\nu-k-1}} Q(z,s) \right] .$$

Therefore, if the boundary conditions (3.13) hold, $S(z) = 0$ and (3.14) follows. Taking the derivative of the right hand side we find

$$P(s) = \left[\tau_{\nu-1}(s)(s-z) - (\nu-k-1)\sigma(s)\right] Q(z,s) + \sigma(s)(s-z) \frac{\partial Q}{\partial s}(z,s) . \tag{3.17}$$

Hence, by comparing (3.16) with (3.17), we conclude that $\deg_s[Q(z,s)] = 0$, i.e., $Q(z,s) = f(z)$ that we choose, without loss of generality, equal to 1.

Substituting the expansions (3.9) of $\tau_{\nu-1}(s)$, $\tau_{\nu}(s)$ and $\sigma(s)$ in powers of $s-z$ in (3.16) and (3.17) we obtain

$$\begin{cases} A_{1k} C_{\nu-1}^{(k)} + A_{2k} \frac{C_{\nu}^{(k)}}{\nu-k} \tau_{\nu-1}(z) + A_{3k}(z) C_{\nu}^{(k+1)} + \\ \qquad A_{4k} \frac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)} \left[\tau_{\nu}' \sigma(z) + \tau_{\nu}(z) \tau_{\nu-1}(z)\right] = -(\nu-k-1), \\ A_{2k} \frac{C_{\nu}^{(k)}}{\nu-k} \sigma(z) \tau_{\nu-1}' + A_{3k} C_{\nu}^{(k+1)} \sigma'(z) + A_{4k} \frac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)} \sigma(z) 2\tau_{\nu}(z) \tau_{\nu-\frac{1}{2}}' = \tau_k(z), \\ A_{3k} C_{\nu}^{(k+1)} \frac{\sigma''}{2} + A_{4k} \frac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)} \sigma(z) \tau_{\nu}' \tau_{\nu-\frac{1}{2}}' = \tau_{\frac{\nu+k-1}{2}}' . \end{cases}$$

Assuming $\sigma'' \neq 0$, from last equation we get

$$A_{3k}(z) C_{\nu}^{(k+1)} = \frac{2}{\sigma''} \left[\tau_{\frac{\nu+k-1}{2}}' - A_{4k}(z) \frac{C_{\nu+1}^{(k)}}{(\nu-k)(\nu-k+1)} \sigma(z) \tau_{\nu}' \tau_{\nu-\frac{1}{2}}' \right] .$$

Choosing now

$$A_{4k}(z) = R(z) \tau_{\frac{\nu+k-1}{2}}' \frac{(\nu-k)(\nu-k+1)}{C_{\nu+1}^{(k)} \sigma(z) \tau_{\nu}' \tau_{\nu-\frac{1}{2}}'},$$

where $R(z)$ is an arbitrary function of z, we obtain

$$A_{3k}(z)C_\nu^{(k+1)} = \frac{2}{\sigma''}\tau'_{\frac{\nu+k-1}{2}}\left(1 - R(z)\right)\,,$$

and therefore

$$\begin{cases} A_{4k}(z) = R(z)\tau'_{\frac{\nu+k-1}{2}}\frac{(\nu-k)(\nu-k+1)}{C_{\nu+1}^{(k)}\tau'_\nu\tau'_{\nu-\frac{1}{2}}} \\ A_{3k}(z)C_\nu^{(k+1)} = \dfrac{2}{\sigma''}\tau'_{\frac{\nu+k-1}{2}}\Big(1 - R(z)\Big)\sigma(z) \\ A_{2k}(z)C_\nu^{(k)} = \frac{\nu-k}{\tau'_{\nu-1}}\left[2\Big(\tau(0)-\frac{\sigma'(0)}{\sigma''}\tau'\Big)-\tau_{\nu-1}(z)+2R(z)\tau'_{\frac{\nu+k-1}{2}}\left(\frac{\sigma'(0)}{\sigma''}-\frac{\tau_\nu(0)}{\tau'_\nu}\right)\right] \\ A_{1k}(z)C_{\nu-1}^{(k)}=-\dfrac{\tau_{\nu-1}(z)}{\tau'_{\nu-1}}\left[\tau_{\nu-1}(z)-\dfrac{2}{\sigma''}\tau'_{\nu-1}+2R(z)\tau'_{\frac{\nu+k-1}{2}}\left(\dfrac{\sigma'(0)}{\sigma''}-\dfrac{\tau_\nu(0)}{\tau'_\nu}\right)\right]- \\ \quad (\nu-k-1)\sigma(z) - R(z)\tau'_{\frac{\nu+k-1}{2}}\frac{\tau'_\nu\sigma(z)+\tau_\nu(z)\tau_{\nu-1}(z)}{\tau'_\nu\tau'_{\nu-\frac{1}{2}}} - \frac{2}{\sigma''}\sigma(z)\tau'_{\frac{\nu+k-1}{2}}(1-R(z)) \end{cases}$$

If we now substitute the above values A_{ik}, $i = 1, 2, 3, 4$, in (3.14), put $R(z) = H(z)/\tau'_{\frac{\nu+k-1}{2}}$, where $H(z)$ is a polynomial in z, and simplify the resulting expressions we obtain the values (3.15). ■

Notice that if we choose $H(z)$ a polynomial in z, then the corresponding coefficients A_{ik}, $i = 1, 2, 3, 4$, will be polynomials in z too. Formulae (3.15) are still valid, by analytic continuation, when $\sigma'' = 0$.

Remark 3. *In the case when $\sigma'' = 0$, from (3.15), the following two cases follow*

(1) *if* $\deg[\sigma(s)] = 1 \wedge \deg[\tau(s)] = 1$ *then* $A_{4k}(z) = 0$,
(2) *if* $\deg[\sigma(s)] = 0 \wedge \deg[\tau(s)] = 1$ *then* $A_{2k}(z) = 0 = A_{4k}(z)$.

Thus, a similar analysis yields

- *in the first case,* $\sigma(s) = \sigma(z) + \sigma'(z)(s - z)$ *and* $\tau(s) = \tau(z) + \tau'(s - z)$

$$\begin{cases} A_{1k}(z) = \dfrac{2C_\nu}{C_{\nu-1}}\tau_\nu(z)\Big(\tau'\sigma(z) - \sigma'(z)\tau_{\nu-1}(z)\Big), \\ A_{2k}(z) = (\nu - k)\sigma'\tau_\nu(z), \\ A_{3k}(z) = -2\tau_{\frac{3\nu-k}{2}}\sigma(z), \\ A_{4k}(z) = \dfrac{C_\nu}{C_{\nu+1}}(\nu - k - 1)(\nu - k)\sigma'(z)\,; \end{cases}$$

- *in the second case* $\sigma(s) = \sigma(z)$ *and* $\tau(s) = \tau(z) + \tau'(s-z)$

$$\begin{cases} A_{1k}(z) = -\dfrac{C_\nu}{C_{\nu-1}}(\nu - k - 1)\tau'\sigma(z), \\ A_{2k}(z) = -(\nu - k)\tau(z), \\ A_{3k}(z) = \sigma(z), \\ A_{4k}(z) = \dfrac{C_\nu}{C_{\nu+1}}(\nu - k - 1)(\nu - k). \end{cases}$$

3.2. Three-Term Recurrence Relations. In general, in order to obtain three-term recurrence relations involving functions of hypergeometric type and its derivatives of any order, one could follow the technique described in the previous section (see e.g. [19]). Here we will obtain several three-term recurrence relations that follows from Theorems 1 (Corollaries 1–3), 2 (Corollaries 4) and 3 (Corollaries 5–8), when one of the coefficients $A_{ik}(z)$, $i = 1, 2, 3, 4$, is chosen to be identically zero. Since the proofs of all Corollaries are quite similar we will include here only the first one.

Corollary 1.

$$B_{1k}(z)y_\nu^{(k)}(z) + B_{2k}(z)y_\nu^{(k+1)}(z) + B_{3k}(z)y_{\nu+1}^{(k+1)}(z) = 0, \tag{3.18}$$

$$\begin{cases} B_{1k}(z) = -\tau'_{\frac{\nu+k-1}{2}}\tau'_\nu, \\ B_{2k}(z) = -\tau'_\nu\sigma'(0) + \frac{\sigma''}{2}\left(\tau_\nu(0) - \tau'_\nu z\right), \\ B_{3k}(z) = \dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu-1}{2}}. \end{cases} \tag{3.19}$$

Proof: Using the fact that $R(z)$ in Theorem 1 is an arbitrary polynomial of z, and putting $R(z) = 2\sigma'(z)$ we get $A_{1k} = 0$. Thus relation (3.2) becomes into

$$B_{1k}(z)y_\nu^{(k)}(z) + B_{2k}(z)y_\nu^{(k+1)}(z) + B_{3k}(z)y_{\nu+1}^{(k+1)}(z) = 0\,,$$

where the coefficients $B_{1k} = A_{2k}$, $B_{2k} = A_{3k}$ and $B_{3k} = A_{4k}$ are given by

$$\begin{cases} B_{1k}(z) = -(\nu - k)\tau'_{\frac{\nu+k-1}{2}}\sigma''\sigma(z), \\ B_{2k}(z) = \dfrac{\nu - k}{\tau'_\nu}\sigma''\sigma(z)\left[-\dfrac{1}{2}\left(2\tau'_\nu\sigma'(0) + \tau'_\nu\sigma'' z - \tau_\nu(0)\sigma''\right)\right], \\ B_{3k}(z) = (\nu - k)\dfrac{C_\nu}{C_{\nu+1}}\dfrac{\tau'_{\frac{\nu-1}{2}}}{\tau'_\nu}\sigma''\sigma(z). \end{cases}$$

Hence, after some simplifications, we obtain (3.19). ■

The previous three-term recurrence relation was published in [8].

Corollary 2.

$$B_{1k}(z)y^{(k)}_{\nu-1}(z) + B_{2k}(z)y^{(k)}_\nu(z) + B_{3k}(z)y^{(k+1)}_{\nu+1}(z) = 0, \tag{3.20}$$

$$\begin{cases} B_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\left[\tau_{\nu-1}(0)\left(\tau'_{\nu-1}\sigma'(0) - \dfrac{\sigma''}{2}\tau_{\nu-1}(0)\right) - (\tau'_{\nu-1})^2\sigma(0)\right]\times \\ \qquad\qquad \left(\dfrac{\sigma''}{2}\tau'_\nu z + \sigma'(0)\tau'_\nu - \dfrac{\sigma''}{2}\tau_\nu(0)\right), \\ B_{2k}(z) = (\nu - k)\tau'_{\frac{\nu}{2}-1}\left(\dfrac{\sigma''}{2}\tau'_\nu z + \sigma'(0)\tau'_\nu - \dfrac{\sigma''}{2}\tau_\nu(0)\right)\times \\ \qquad \left(\dfrac{\sigma''}{2}\tau_{\nu-1}(0) - \sigma'(0)\tau'_{\nu-1} - \dfrac{\sigma''}{2}\tau'_{\nu-1}z\right) - \tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\tau'_{\nu-1}\sigma(z)\tau'_\nu, \\ B_{3k}(z) = \dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau'_{\nu-1}\sigma(z). \end{cases} \tag{3.21}$$

Corollary 3.

$$B_{1k}(z)y^{(k)}_{\nu-1}(z) + B_{2k}(z)y^{(k+1)}_\nu(z) + B_{3k}(z)y^{(k+1)}_{\nu+1}(z) = 0,$$

$$\begin{cases} B_{1k}(z) = -\dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-1}{2}}\tau'_{\frac{\nu+k-2}{2}}\tau'_\nu\Big[\tau^2_{\nu-1}(0)\sigma'' - 2\tau'_{\nu-1}\Big(\tau_{\nu-1}(0)\sigma'(0) - \tau'_{\nu-1}\sigma(0)\Big)\Big], \\ B_{2k}(z) = \dfrac{\nu-k}{2}\tau'_{\frac{\nu}{2}-1}\Big(\tau_\nu(0)\sigma'' - 2\sigma'(0)\tau'_\nu - \sigma''\tau'_\nu z\Big)\times \\ \qquad\qquad \Big(\tau_{\nu-1}(0)\sigma'' - 2\sigma'(0)\tau'_{\nu-1} - \sigma''\tau'_{\nu-1}z\Big) + 2\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k-1}{2}}\tau'_{\nu-1}\tau'_\nu\sigma(z), \\ B_{3k}(z) = (\nu-k)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\Big(\tau_{\nu-1}(0)\sigma'' - 2\sigma'(0)\tau'_{\nu-1} - \sigma''\tau'_{\nu-1}z\Big). \end{cases}$$

Corollary 4.

$$B_{1k}(z)y^{(k+1)}_{\nu-1}(z) + B_{2k}(z)y^{(k)}_\nu(z) + B_{3k}(z)y^{(k+1)}_{\nu+1}(z) = 0,$$

$$\begin{cases} B_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\Big[\tau_{\nu-1}(0)\Big(\sigma'(0)\tau'_{\nu-1} - \dfrac{\sigma''}{2}\tau_{\nu-1}(0)\Big) - \left(\tau'_{\nu-1}\right)^2\sigma(0)\Big] \\ \qquad\qquad \times\Big[\sigma'(0)\tau'_\nu - \frac{\sigma''}{2}\Big(\tau_\nu(0) - \tau'_\nu z\Big)\Big], \\ B_{2k}(z) = (\nu-k-1)\tau'_{\frac{\nu}{2}-1}\tau'_\nu\Big(\sigma'(0)\tau'_{\nu-1} - \frac{\sigma''}{2}\tau_{\nu-1}(0)\Big) + \tau_\nu(0)\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu+k}{2}}\tau'_{\nu-1} \\ \qquad\qquad +\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\tau'_{\nu-\frac{1}{2}}\tau'_\nu z, \\ B_{3k}(z) = -\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau_{\nu-1}(z)\,. \end{cases}$$

Corollary 5.

$$B_{1k}(z)y^{(k)}_{\nu-1}(z) + B_{2k}(z)y^{(k)}_\nu(z) + B_{3k}(z)y^{(k)}_{\nu+1}(z) = 0,$$

$$\begin{cases} B_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\tau'_\nu\left[\tau'_{\nu-1}\left(\sigma'(0)\tau_{\nu-1}(0) - \sigma(0)\tau'_{\nu-1}\right) - \tau^2_{\nu-1}(0)\dfrac{\sigma''}{2}\right], \\ B_{2k}(z) = -\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-\frac{1}{2}}\left[\tau'_\nu\tau'_{\nu-1}z + \tau'\tau_{2\nu-k}(0) + \sigma''\left(k\tau(0) - \tau_{\nu(1-\nu)}(0)\right)\right], \\ B_{3k}(z) = (\nu-k+1)\dfrac{C_\nu}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\tau'_{\nu-1}. \end{cases}$$

This Corollary was first published in [19] and it is nothing else that the *standard* three-term recurrence relation for the derivative of any order of hypergeometric functions.

Corollary 6.

$$B_{1k}(z)y_{\nu}^{(k)}(z)+B_{2k}(z)y_{\nu}^{(k+1)}(z)+B_{3k}(z)y_{\nu+1}^{(k)}(z)=0,$$

$$\begin{cases} B_{1k}(z)=\tau'_{\frac{\nu+k-1}{2}}\tau_{\nu}(z), \\ B_{2k}(z)=\tau'_{\nu}\sigma(z), \\ B_{3k}(z)=-(\nu-k+1)\dfrac{C_{\nu}}{C_{\nu+1}}\tau'_{\frac{\nu-1}{2}}\,. \end{cases}$$

Corollary 7.

$$B_{1k}(z)y_{\nu-1}^{(k)}(z)+B_{2k}(z)y_{\nu}^{(k)}(z)+B_{3k}(z)y_{\nu}^{(k+1)}(z)=0,$$

$$\begin{cases} B_{1k}(z)=-\dfrac{C_{\nu}}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\left[\tau_{\nu-1}^{2}(0)\dfrac{\sigma''}{2}+\tau'_{\nu-1}\Big(\sigma(0)\tau'_{\nu-1}-\sigma'(0)\tau_{\nu-1}(0)\Big)\right], \\ B_{2k}(z)=-\dfrac{\nu-k}{2}\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\left(\sigma''z+2\sigma'(0)-\sigma''\dfrac{\tau_{\nu-1}(0)}{\tau'_{\nu-1}}\right), \\ B_{3k}(z)=\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-1}\sigma(z). \end{cases}$$

The above relation was firstly obtained in [6].

Corollary 8.

$$B_{1k}(z)y_{\nu-1}^{(k)}(z)+B_{2k}(z)y_{\nu}^{(k+1)}(z)+B_{3k}(z)y_{\nu+1}^{(k)}(z)=0,$$

$$\begin{cases} B_{1k}(z)=\frac{C_{\nu}}{C_{\nu-1}}\tau'_{\frac{\nu+k-2}{2}}\tau'_{\frac{\nu+k-1}{2}} \\ \qquad\qquad\times\Big[\tau_{\nu-1}(0)\left(2\sigma'(0)\tau'_{\frac{\nu-1}{2}}-\tau(0)\sigma''\right)-2\left(\tau'_{\nu-1}\right)^{2}\sigma(0)\Big]\tau_{\nu}(z), \\ B_{2k}(z)=2\tau'_{\frac{\nu}{2}-1}\tau'_{\nu-\frac{1}{2}}\Big[\tau'_{\nu}\tau'_{\nu-1}z+\tau_{\nu}(0)\tau'_{k-1}+\tau'_{\nu}\sigma'(0)(\nu-k)\Big]\sigma(z), \\ B_{3k}(z)=-(\nu-k)(\nu-k+1)\frac{C_{\nu}}{C_{\nu+1}}\tau'_{\frac{\nu}{2}-1}\tau'_{\frac{\nu-1}{2}}\left[\sigma''\tau'_{\nu-1}z+2\sigma'(0)\tau'_{\frac{\nu-1}{2}}-\sigma''\tau(0)\right]. \end{cases}$$

4. Applications

4.1. Recurrence relations for Hypergeometric type Functions. We can reduce equation (1.1) to a canonical form by a linear change of independent variable. According to [12, 13], there exists three different cases, corresponding to the different possibilities for the degrees of σ:

- $\deg\big(\sigma(z)\big) = 2$

$$z(1-z)\,u'' + \big[\gamma - \big(\alpha+\beta+1\big)z\big]\,u' - \alpha\beta\,u = 0\,; \tag{4.1}$$

It corresponds to equation 1.1 with

$$\sigma(z) = z(1-z), \qquad \tau(z) = \gamma - \big(\alpha+\beta+1\big)z, \qquad \lambda = -\alpha\beta\,. \tag{4.2}$$

- $\deg\big(\sigma(z)\big) = 1$

$$z\,u'' + \big(\gamma - z\big)\,u' - \alpha\,u = 0\,; \tag{4.3}$$

This is equation (1.1) with

$$\sigma(z) = z, \qquad \tau(z) = \gamma - z, \qquad \lambda = -\alpha\,. \tag{4.4}$$

- $\deg\big(\sigma(z)\big) = 0$

$$u'' - 2z\,u' + 2\nu\,u = 0\,, \tag{4.5}$$

i.e., it corresponds to the equation (1.1) with

$$\sigma(z) = 1, \qquad \tau(z) = -2z, \qquad \lambda = 2\nu\,. \tag{4.6}$$

Equations (4.1), (4.3) and (4.5) are known as the *hypergeometric*, *confluent hypergeometric* and *Hermite* equations, respectively. Explicit solutions for the different three above equations are well known (see e.g. [12, 13]). In [12, §20, section 2. page 255] particular solutions were found using the corresponding integral representations: the hypergeometric $F(\alpha,\beta,\gamma,z)$, confluent hypergeometric $F(\alpha,\gamma,z)$ and Hermite $H_\nu(z)$ functions. In terms of the generalized hypergeometric notation [3], the first two correspond to ${}_2F_1(\alpha,\beta;\gamma;z)$ and ${}_1F_1(\alpha;\gamma;z)$, respectively. Here we will present recurrence relations that follows from the Theorems 1, 2, and 3, and some particular examples from the corresponding Corollaries.

In what follows, $R(z)$ represents an arbitrary function of z.

4.1.1. *Relations derived from Theorem 1.*

- *Hypergeometric Equation* $\big($see (4.1) and (4.2)$\big)$. From (4.2) and (2.6) it follows that $\nu = -\alpha$ or $\nu = -\beta$. Choosing $\nu = -\alpha$, relation (3.2) is fulfilled with

$$\begin{cases} A_{1k}(z) = -\alpha(\alpha-1)(\beta+k)(\beta+k-1)(\beta-\gamma)(\beta-\alpha+1)\big(R(z)-1+2z\big), \\ A_{2k}(z) = (\alpha-1)(\alpha+k)(\beta-1)(\beta+k)(\beta-\alpha+1)\times \\ \qquad \Big\{\Big(R(z)-1+2z\Big)\Big[(1-2z)(\beta-\alpha-1)-\Big((\gamma-\alpha-1)-(\beta-\alpha+1)z\Big)\Big]- \\ \qquad (\beta-\alpha-1)z(1-z)\Big\}, \\ A_{3k}(z) = -(\alpha-1)(\beta-1)(\beta-\alpha-1)z(1-z)\times \\ \quad \Big[(\beta-\alpha+1)(\beta+k)R(z)-(\alpha+k)\Big((\gamma-\alpha)-(\beta-\alpha+1)z\Big)-(\beta-\alpha)(\beta-\alpha+1)(1-2z)\Big], \\ A_{4k}(z)=(\alpha+k)\beta(\beta-1)(\alpha-\gamma)(\beta-\alpha-1)z(1-z). \end{cases}$$

- *Confluent Hypergeometric Equation* $\big($see (4.3) and (4.4)$\big)$. Using (4.4) and (2.6) we find $\nu=-\alpha$, being relation (3.2) fulfilled with

$$A_{1k}(z)=-\alpha, \quad A_{2k}(z)=\alpha+k, \quad A_{3k}(z)=z, \quad A_{4k}(z)=0.$$

- *Hermite Equation* $\big($see (4.5) and (4.6)$\big)$. Using now (4.6) and (2.6) we conclude that ν may be an arbitrary complex number and relation (3.2) is fulfilled with

$$A_{1k}(z)=-2\nu, \quad A_{2k}(z)=0, \quad A_{3k}(z)=1, \quad A_{4k}(z)=0.$$

4.1.2. *Relations derived from Theorem 2.*

- *Hypergeometric Equation* $\big($see (4.1) and (4.2)$\big)$. This corresponds to $\nu=-\alpha$ (or $\nu=-\beta$) and relation (3.11) is fulfilled with

$$\begin{cases} A_{1k}(z) = \alpha(\alpha-1)\beta(\beta-\alpha+1)(\beta-\alpha-2)\big(R(z)-1+2z\big) \\ A_{2k}(z) = -(\alpha-1)(\alpha+k)(\beta-1)(\beta+k-1)(\beta-\alpha+1)\Big[(\beta-\gamma)-(\beta-\alpha-1)\Big(R(z)+z\Big)\Big] \\ A_{3k}(z) = (\alpha-1)(\beta-1)\Big[(\gamma-\alpha-1)-(\beta-\alpha-1)z\Big]\times \\ \quad \Big[(\beta-\alpha)(\beta-\alpha+1)(1-2z)-R(z)(\beta+k-1)(\beta-\alpha+1)-(\alpha+k)\Big((\gamma-\alpha)-(\beta-\alpha+1)z\Big)\Big] \\ A_{4k}(z)=-(\alpha+k)\beta(\beta-1)(\gamma-\alpha)\Big[(\gamma-\alpha-1)-(\beta-\alpha-1)z\Big] \end{cases}$$

- *Confluent Hypergeometric Equation* $\big($see (4.3) and (4.4)$\big)$. This corresponds to $\nu=-\alpha$ and relation (3.11) is fulfilled with

$$A_{1k}(z)=\alpha, \quad A_{2k}(z)=-(\alpha+k), \quad A_{3k}(z)=(\gamma-\alpha-1)-z, \quad A_{4k}(z)=0.$$

- *Hermite Equation* $\big($see (4.5) and (4.6)$\big)$. Relation (3.11) is fulfilled, for an arbitrary complex number ν, with

$$A_{1k}(z)=\nu, \quad A_{2k}(z)=\nu-k, \quad A_{3k}(z)=-z, \quad A_{4k}(z)=0.$$

4.1.3. *Relations derived from Theorem 3.*

- *Hypergeometric Equation* $\big($see (4.1) and (4.2)$\big)$. This corresponds to $\nu = -\alpha$ (or $\nu = -\beta$) and relation (3.14) is fulfilled with

$$\begin{cases} A_{1k}(z) = -\alpha(\alpha-1)(\beta+k)(\beta+k-1)(\beta-\gamma)(\beta-\alpha+1)\big[R(z)-(\beta-\alpha)\big] \\ A_{2k}(z) = (\alpha-1)(\alpha+k)(\beta-1)(\beta+k)(\beta-\alpha)\times \\ \qquad\qquad \Big\{\big[(\beta-\gamma)-(\beta-\alpha-1)z\big](\beta-\alpha+1)+R(z)(\beta-3\alpha+2\gamma+1\Big)\Big\} \\ A_{3k}(z) = (\alpha-1)(\beta-1)(\beta-\alpha-1)(\beta-\alpha)(\beta-\alpha+1)\Big(R(z)-(\beta+k)\Big)z(1-z) \\ A_{4k}(z) = R(z)(\alpha+k)(\alpha+k-1)\beta(\beta-1)(\gamma-\alpha)(\beta-\alpha-1) \end{cases}$$

- *Confluent Hypergeometric Equation* $\big($see (4.3) and (4.4)$\big)$. This corresponds to $\nu = -\alpha$ and relation (3.14) is fulfilled with

$$A_{1k}(z) = -\alpha, \quad A_{2k}(z) = \alpha + k, \quad A_{3k}(z) = z, \quad A_{4k}(z) = 0.$$

- *Hermite Equation* $\big($see (4.5) and (4.6)$\big)$. This corresponds to an arbitrary complex number ν and relation (3.14) is fulfilled with

$$A_{1k}(z) = 2\nu, \quad A_{2k}(z) = 0, \quad A_{3k}(z) = -1, \quad A_{4k}(z) = 0.$$

In the following we will put $k = 1$ and use the identities [12, page 261]

$${}_2F_1{}'(\alpha,\beta;\gamma;z) = \frac{\alpha\beta}{\gamma}\,{}_2F_1(\alpha{+}1,\beta{+}1;\gamma{+}1;z), \;\; {}_1F_1{}'(\alpha;\gamma;z) = \frac{\alpha}{\gamma}\,{}_1F_1(\alpha{+}1;\gamma{+}1;z).$$

4.1.4. *Relations derived from Corollary 1.*

- Hypergeometric function. This corresponds to $\nu = -\alpha$ (or $\nu = -\beta$) and substituting the quantities (4.2) in (3.18)–(3.19) we find the following recurrence relation for the hypergeometric function:

$$\gamma\big(\alpha-\beta-1\big)\,{}_2F_1(\alpha,\beta;\gamma;z) + \beta\big(\alpha-\gamma\big)\,{}_2F_1(\alpha,\beta+1;\gamma+1;z) + \\ \alpha\big[\big(\beta-\gamma+1\big)+\big(\beta-\alpha+1\big)z\big]\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z) = 0\,.$$

- Confluent hypergeometric function. This corresponds to $\nu = -\alpha$. Therefore, by (4.4), (3.18)–(3.19) yield, for the hypergeometric confluent function, the following recurrence relation

$$\gamma\,{}_1F_1(\alpha;\gamma;z) + \big(\gamma-\alpha\big)\,{}_1F_1(\alpha;\gamma+1;z) + \alpha\,{}_1F_1(\alpha+1;\gamma+1;z) = 0.$$

- Hermite function. Being ν an arbitrary complex number then, substituting (4.6) in (3.18)–(3.19), we find the following very well known relation for the Hermite function:

$$H'_{\nu+1}(z) = 2\big(\nu+1\big)H_\nu(z).$$

Other recurrences relations can be obtained from the other Corollaries 2–8. Since the technique is similar we just present here the resulting relations.

4.1.5. *Relations derived from Corollary 2.*

- Hypergeometric function

$$\begin{aligned}\gamma\Big\{\alpha\big[\big(\beta-\gamma+1\big)-\big(\beta-\alpha+1\big)z\big]\big[\big(\beta-\gamma\big)-\big(\beta-\alpha-1\big)z\big]-\\ \beta\big[(\beta-\alpha)^2-1\big]z(1-z)\Big\}\,{}_2F_1(\alpha,\beta;\gamma;z)+\\ \alpha\gamma\big(\beta-\gamma\big)\big[\big(\beta-\alpha+1\big)z-\big(\beta-\gamma+1\big)\big]\,{}_2F_1(\alpha+1,\beta;\gamma;z)+\\ \beta^2\big(\alpha-\gamma\big)\big(\beta-\alpha-1\big)z(1-z)\,{}_2F_1(\alpha,\beta+1;\gamma+1;z)=0\,.\end{aligned}$$

- Confluent hypergeometric function

$$-\gamma\big(\alpha+z\big)\,{}_1F_1(\alpha;\gamma;z)+\big(\gamma-\alpha\big)z\,{}_1F_1(\alpha;\gamma+1;z)+\alpha\gamma\,{}_1F_1(\alpha+1;\gamma;z)=0.$$

- Hermite function

$$H'_{\nu+1}(z)=2\big(\nu+1\big)H_\nu(z).$$

4.1.6. *Relations derived from Corollary 3.*

- Hypergeometric function

$$\begin{aligned}\beta\Big\{\alpha\big[\big(\beta-\gamma+1\big)-\big(\beta-\alpha+1\big)z\big]\big[\big(\beta-\gamma\big)-\big(\beta-\alpha-1\big)z\big]+\\ \beta\big[(\beta-\alpha)^2-1\big]z(1-z)\Big\}\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)+\\ +\beta\gamma\big(\beta-\gamma\big)\big(\beta-\alpha+1\big)\,{}_2F_1(\alpha+1,\beta;\gamma;z)-\\ \beta^2\big(\alpha-\gamma\big)\Big[\big(\beta-\gamma\big)-\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha,\beta+1;\gamma+1;z)=0\,.\end{aligned}$$

- Confluent hypergeometric function

$$-\big(\alpha+z\big)\,{}_1F_1(\alpha+1;\gamma+1;z)+\big(\alpha-\gamma\big)z\,{}_1F_1(\alpha;\gamma+1;z)+\gamma\,{}_1F_1(\alpha+1;\gamma;z)=0.$$

- Hermite function

$$H'_\nu(z)=2\nu\,H_{\nu-1}(z).$$

4.1.7. *Relations derived from Corollary 4.*

- Hypergeometric function

$$\gamma\big(\beta-1\big)\Big[\big(\alpha{+}1\big)\big(\beta{-}\alpha{+}1\big)\big(\gamma{-}\beta\big)+\big(\gamma{-}\alpha\big)\big(\beta{-}\alpha{-}1\big)\big(\beta{+}1\big)-$$
$$\big(\beta{-}\alpha{+}1\big)\big(\beta{-}\alpha\big)\big(\beta{-}\alpha{-}1\big)z\Big]\,{}_2F_1(\alpha,\beta;\gamma;z)+$$
$$\beta^2\big(\alpha{-}\gamma\big)\Big[\big(\gamma{-}\alpha{-}1\big)-\big(\beta{-}\alpha{-}1\big)z\Big]\,{}_2F_1(\alpha,\beta{+}1;\gamma{+}1;z)+$$
$$\alpha\beta\big(\alpha{+}1\big)\big(\gamma{-}\beta\big)\Big[\big(\gamma{-}\beta{-}1\big)+\big(\beta{-}\alpha{+}1\big)z\Big]\,{}_2F_1(\alpha{+}2,\beta{+}1;\gamma{+}1;z)=0\,.$$

- Confluent hypergeometric function

$$\gamma\Big[\big(\gamma{-}2\alpha{-}1\big)-z\Big]\,{}_1F_1(\alpha;\gamma;z)+\big(\alpha{-}\gamma\big)\Big[\big(\gamma{-}\alpha{-}1\big)-z\Big]\,{}_1F_1(\alpha;\gamma{+}1;z)+$$
$$\alpha\big(\alpha{+}1\big)\,{}_1F_1(\alpha{+}2;\gamma{+}1;z)=0\,.$$

- Hermite function

$$H'_{\nu+1}(z)=2\big(\nu+1\big)\,H_\nu(z).$$

4.1.8. *Relations derived from Corollary 5.*

- Hypergeometric function

$$\big(\beta-1\big)\big(\gamma-\alpha\big)\big(\beta-\alpha-1\big)\,{}_2F_1(\alpha-1,\beta;\gamma;z)+$$
$$\big(\beta{-}\alpha\big)\Big\{\big[\big(\beta{-}\alpha\big)^2{-}1\big]z-\big(\alpha{+}\beta{+}1\big)\big(\gamma{-}2\alpha\big)+2\big(\gamma{-}\alpha(\alpha{+}1)\big)\Big\}\,{}_2F_1(\alpha,\beta;\gamma;z)$$
$$+\alpha\big(\beta-\alpha+1\big)\big(\gamma-\beta\big)\,{}_2F_1(\alpha+1,\beta;\gamma;z)=0\,.$$

- Confluent hypergeometric function

$$\big(\gamma-\alpha\big)\,{}_1F_1(\alpha-1;\gamma;z)+\Big[z-\big(\gamma-2\alpha\big)\Big]\,{}_1F_1(\alpha;\gamma;z)-\alpha\,{}_1F_1(\alpha+1;\gamma;z)=0.$$

- Hermite function

$$H_{\nu+1}(z)-2z\,H_\nu(z)+2\nu\,H_{\nu-1}(z)=0.$$

4.1.9. *Relations derived from Corollary 6.*

- Hypergeometric function

$$\beta\gamma\big(\alpha{-}\gamma\big)\,{}_2F_1(\alpha{-}1,\beta;\gamma;z)+\gamma\big(\beta{-}1\big)\Big[\big(\gamma{-}\alpha\big)-\big(\beta{-}\alpha{+}1\big)z\Big]\,{}_2F_1(\alpha,\beta;\gamma;z)$$
$$+\alpha\beta\big(\beta{-}\alpha{+}1\big)z(1{-}z)\,{}_2F_1(\alpha{+}1,\beta{+}1;\gamma{+}1;z)=0\,.$$

- Confluent hypergeometric function

$$\gamma\big(\alpha-\gamma\big)\,{}_1F_1(\alpha-1;\gamma;z)+\gamma\Big[\big(\gamma-\alpha\big)-z\Big]\,{}_1F_1(\alpha;\gamma;z)+\alpha z\,{}_1F_1(\alpha+1;\gamma+1;z)=0.$$

- Hermite function

$$H_{\nu+1}(z)-2z\,H_\nu(z)+H'_\nu(z)=0.$$

4.1.10. *Relations derived from Corollary 7.*

- Hypergeometric function

$$\gamma\big[\big(\gamma-\beta\big)+\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha,\beta;\gamma;z)+\gamma\big(\beta-\gamma\big)\,{}_2F_1(\alpha+1,\beta;\gamma;z)-$$
$$\beta\big(\beta-\alpha-1\big)z(1-z)\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)=0\,.$$

- Confluent hypergeometric function

$$\gamma\,{}_1F_1(\alpha;\gamma;z)-\gamma\,{}_1F_1(\alpha;\gamma;z)+z\,{}_1F_1(\alpha+1;\gamma+1;z)=0.$$

- Hermite function

$$H'_\nu(z)=2\nu\,H_{\nu-1}(z).$$

4.1.11. *Relations derived from Corollary 8.*

- Hypergeometric function

$$\beta\big(\alpha-\gamma\big)\big[\big(\gamma-\beta\big)+\big(\beta-\alpha-1\big)z\Big]\,{}_2F_1(\alpha-1,\beta;\gamma;z)+$$
$$\gamma\big(\gamma-\beta\big)\big(\beta-\alpha-1\big)\Big[\big(\gamma-\alpha\big)-\big(\beta-\alpha+1\big)z\Big]\,{}_2F_1(\alpha+1,\beta;\gamma;z)+$$
$$\beta\big(\beta-\alpha\big)\Big\{\Big[\big(\beta-\alpha\big)^2-1\Big]z+2\alpha\beta-\gamma\big(\alpha+\beta-1\big)\Big\}\,{}_2F_1(\alpha+1,\beta+1;\gamma+1;z)=0\,.$$

- Confluent hypergeometric function

$$\gamma\big(\gamma-\alpha\big)\,{}_1F_1(\alpha-1;\gamma;z)-\gamma\Big(\big(\gamma-\alpha\big)-z\Big)\,{}_1F_1(\alpha+1;\gamma;z)+$$
$$\beta z\big(\gamma-z\big)\,{}_1F_1(\alpha+1;\gamma+1;z)=0\,.$$

- Hermite function

$$H'_\nu(z)=2\nu\,H_{\nu-1}(z).$$

Remark 4. *Notice that we can interchange α and β in equation (4.1). Therefore, several other recurrence relations can be obtained by interchanging α and β in all relations obtained from the corollaries (1–8) corresponding to the hypergeometric equation (4.1).*

4.2. Recurrences for Polynomials of Hypergeometric Type. The polynomials of hypergeometric type $p_n(z) := y_n(z)$ are particular cases of the functions of hypergeometric type $y_\nu(z)$ when the parameter $\nu = n$ is a non-negative integer, being $p_n(z) := y_n(z)$ a particular solution of the equation (1.1) where λ is given by (2.1).
They can be represented by the Rodrigues formula (2.2), where B_n are normalizing constants and $\rho(z)$ satisfies the Pearson equation (2.3), or by their integral representation (2.4), where

$$C_n = \frac{n! B_n}{2\pi i}\,. \tag{4.7}$$

If a_n denotes the leading coefficient of the polynomial $y_n(z)$ then, see [12],

$$a_n = B_n \prod_{m=0}^{n-1} \tau'_{\frac{n+m-1}{2}}\,, \quad a_0 = B_0\,. \tag{4.8}$$

If $a_n = 1$ then $y_n(z)$ is said to be a *monic* polynomial.

The polynomials of hypergeometric type are the classical polynomials, i.e., the Hermite $H_n(z)$, Laguerre $L_n^\alpha(z)$ and Jacobi $P_n^{\alpha,\beta}(z)$ polynomials.

TABLE 1. The classical orthogonal polynomials

$P_n(z)$	$H_n(z)$	$L_n^\alpha(z)$	$P_n^{\alpha,\beta}(z)$
$\sigma(z)$	1	z	$1 - z^2$
$\tau(z)$	$-2z$	$-z+\alpha+1$	$-(\alpha+\beta+2)z+\beta-\alpha$
λ_n	$2n$	n	$n(n+\alpha+\beta+1)$
$\rho(z)$	e^{-z^2}	$z^\alpha e^{-z}$ $\alpha > -1$	$(1-z)^\alpha(1+z)^\beta$ $\alpha,\beta > -1$
B_n	$\frac{(-1)^n}{2^n}$	$(-1)^n$	$\frac{(-1)^n}{(n+\alpha+\beta+1)_n}$

A very important property of the orthogonal polynomials is the three-term recurrence relation

$$zp_n(z) = \alpha_n p_{n+1}(z) + \beta_n p_n(z) + \gamma_n p_{n-1}(z).$$

For computing the coefficients α_n, β_n, and γ_n we can use Corollary 5 with $k=0$ as it has been done in [19]. Other important properties of these polynomials are the so-called raising and lowering operators (see e.g. [1]) that can be obtained from Corollaries 6 and 7, respectively. Since they where studied using this method in [15] we will omit it here.

Here we will study another recurrence relation. Namely the so-called structure relation by Marcellán et. al [11]

$$P_n(z) = \frac{P'_{n+1}(z)}{n+1} + r_n \frac{P'_n(z)}{n} + s_n \frac{P'_{n-1}(z)}{n-1}\,, \quad n \geq 2\,, \tag{4.9}$$

where r_n and s_n are some constants. This relation constitutes another characterization theorem for the classical orthogonal polynomials. A complete study of such structure relation was done in [1].

Corollary 9. *For the monic hypergeometric type polynomials the following recurrence relation holds*

$$\hat{y}_n(z) = \hat{A}_1(z)\hat{y}'_{n+1}(z) + \hat{A}_2(z)\hat{y}'_n(z) + \hat{A}_3(z)\hat{y}'_{n-1}(z)\,. \tag{4.10}$$

where the coefficients $\hat{A}_i$, $i=1,2,3$ *are given by*

$$\begin{cases} \hat{A}_1(z) = \dfrac{1}{n+1}, \\ \hat{A}_2(z) = \dfrac{1}{2}\dfrac{\left(\tau'_{n-1}\tau_n(0)+\tau_{n-1}(0)\tau'_n\right)\sigma''-2\sigma'(0)\tau'_n\tau'_{n-1}}{\tau'_n\tau'_{n-\frac{1}{2}}\tau'_{n-1}}, \\ \hat{A}_3(z) = \left(1-\dfrac{\tau'_{\frac{n-1}{2}}}{\tau'_{n-\frac{1}{2}}}\right)\dfrac{2\tau'_{n-1}\left(\tau'_{n-1}\sigma(0)-\tau_{n-1}(0)\sigma'(0)\right)+\tau^2_{n-1}(0)\sigma''}{2\left(\tau'_{n-1}\right)^2\tau'_{n-\frac{3}{2}}}. \end{cases} \tag{4.11}$$

Proof: Since $R(z)$ in Theorem 2 is an arbitrary polynomial of z, we will define the function $Q(z)$ such that

$$R(z) = 2\sigma'(z) - \sigma''\frac{\tau_{\nu-1}(z)}{\tau'_{\nu-1}}\frac{2+Q(z)}{2}\,.$$

Then (3.11) holds and the corresponding coefficients $A_{ik}(z)$, $i = 1,2,3,4$ becomes into

$$\begin{cases} A_{1k}(z) = \dfrac{C_\nu}{C_{\nu-1}} \dfrac{2+Q(z)}{Q(z)} \dfrac{2\tau'_{\nu-1}\left(\tau_{\nu-1}(0)\sigma'(0) - \tau'_{\nu-1}\sigma(0)\right) - \tau^2_{\nu-1}(0)\sigma''}{2(\nu-k)\tau'_{\nu-1}\tau'_{\frac{\nu}{2}-1}} \\ A_{2k}(z) = 1 \\ A_{3k}(z) = \dfrac{1}{(\nu-k)Q(z)} \dfrac{(\nu-k)\tau'_{\nu-1}\left(\tau_\nu(z)\sigma'' - 2\sigma'(z)\tau'_\nu\right) - \tau_{\nu-1}(z)(2+Q(z))\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}}{\tau'_\nu\tau'_{\nu-1}\tau'_{\frac{\nu+k-1}{2}}} \\ A_{4k}(z) = \dfrac{1}{Q(z)} \dfrac{C_\nu}{C_{\nu+1}} \dfrac{2\tau'_{\frac{\nu-1}{2}}}{\tau'_\nu\tau'_{\frac{\nu+k-1}{2}}}\,. \end{cases}$$

Since the polynomials are monic, by (4.8), $B_n = \left(\prod_{m=0}^{n-1} \tau'_{\frac{n+m-1}{2}}\right)^{-1}$. Then choosing $Q(z) = -2\tau'_{n-\frac{1}{2}}/\tau'_{\frac{n-1}{2}}$ and setting $k=0$, the equations (3.11) transforms into (4.10) whereas (3.12) leads to (4.11). Notice that in (4.9) $r_n = n\hat{A}_2$ and $s_n = (n-1)\hat{A}_3$. ∎

From formulas (4.10)–(4.11) of Corollary 9 follows the identities

$$\begin{aligned} H_n(z) &= \frac{1}{n+1}H'_{n+1}(z), \qquad L_n^\alpha(z) = \frac{1}{n+1}(L_{n+1}^\alpha)'(z) + (L_n^\alpha)'(z), \\ P_n^{\alpha,\beta}(z) &= \frac{1}{n+1}(P_{n+1}^{\alpha,\beta})'(z) + \frac{2(\alpha-\beta)}{(2n+\alpha+\beta)(2n+2+\alpha+\beta)}(P_n^{\alpha,\beta})'(z) \\ &\quad - \frac{4n(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}(P_{n-1}^{\alpha,\beta})'(z), \end{aligned}$$

for the Hermite, Laguerre, and Jacobi polynomials, respectively.

4.3. Further examples. In this section we will present several relations for the classical polynomials that follows from the Theorems 1, 2, and 3. In order to obtain the following relations, where $R(z)$ represents an arbitrary function of z, see subsection 4.2 and table 1.

4.3.1. *Relations derived from Theorem 1.*

- *Jacobi Polynomials*

$$\begin{cases} A_{1k}(z) = 4n(n+1)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+n+k)(\alpha+\beta+n+k+1) \\ \qquad\qquad (\alpha+\beta+2n+2)\big(R(z)+4z\big) \\ A_{2k}(z) = (n+1)(n-k)(\alpha+\beta+n)(\alpha+\beta+n+k+1)(\alpha+\beta+2n-1)(\alpha+\beta+2n)(\alpha+\beta+2n+2))\times \\ \qquad\qquad \Big[\Big(R(z)+4z\Big)\Big((\beta-\alpha)+(\alpha+\beta+2n)z\Big)+(\alpha+\beta+2n)\big(1-z^2\big)\Big] \\ A_{3k}(z) = (n+1)\Big[(\alpha+\beta+n+k+1)(\alpha+\beta+2n+2)R(z)+2(n-k)\Big((\beta-\alpha)-(\alpha+\beta+2n+2)z\Big)+ \\ \qquad\qquad 4(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)z\Big](\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2\big(1-z^2\big) \\ A_{4k}(z) = -2(n-k)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)\big(1-z^2\big) \end{cases}$$

- *Laguerre Polynomials*

$$A_{1k}(z) = -n(\alpha+n), \quad A_{2k}(z) = -(n-k), \quad A_{3k}(z) = z, \quad A_{4k}(z) = 0.$$

- *Hermite Polynomials*

$$A_{1k}(z) = -n, \quad A_{2k}(z) = 0, \quad A_{3k}(z) = 1, \quad A_{4k}(z) = 0.$$

4.3.2. *Relations derived from Theorem 2.*

- *Jacobi Polynomials*

$$\begin{cases} A_{1k}(z) = 4n(n+1)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+n+k+1)(\alpha+\beta+2n+2)\big(R(z)+4z\big) \\ A_{2k}(z) = -(n-k)(\alpha+\beta+n)(\alpha+\beta+n+k+1)(\alpha+\beta+2n-1)(\alpha+\beta+2n)(\alpha+\beta+2n+2))\times \\ \qquad\qquad \Big[R(z)(\alpha+\beta+2n)+2\Big((\beta-\alpha)+(\alpha+\beta+2n-2)z\Big)\Big] \\ A_{3k}(z) = (\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)\Big((\beta-\alpha)-(\alpha+\beta+2n)z\Big)\times \\ \quad \Big[2(n-k)\Big((\beta-\alpha)-(\alpha+\beta+2n+2)z\Big)-(\alpha+\beta+2n+2)\Big((\alpha+\beta+n+k+1)R(z)-4(\alpha+\beta+2n)z\Big)\Big] \\ A_{4k}(z) = -(n-k)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)\times \\ \qquad\qquad \Big((\beta-\alpha)-(\alpha+\beta+2n)z\Big) \end{cases}$$

- *Laguerre Polynomials*

$$A_{1k}(z) = -n(\alpha+n), \quad A_{2k}(z) = n-k, \quad A_{3k}(z) = -(\alpha+n-z), A_{4k}(z) = 0.$$

- *Hermite Polynomials*

$$A_{1k}(z) = -n, \quad A_{2k}(z) = -2(n-k), \quad A_{3k}(z) = 2, \quad A_{4k}(z) = 0.$$

4.3.3. *Relations derived from Theorem 3.*

- *Jacobi Polynomials*

$$\begin{cases} A_{1k}(z) = 4n(n+1)(\alpha+n)(\beta+n)(\alpha+\beta+n)(\alpha+\beta+n+k)(\alpha+\beta+n+k+1)(\alpha+\beta+2n+2)\times \\ \qquad\qquad \Big(R(z) + \big(\alpha+\beta+2n+1\big)\Big) \\ A_{2k}(z) = (n-k)(n+1)(\alpha+\beta+n)(\alpha+\beta+n+k+1)(\alpha+\beta+2n-1)(\alpha+\beta+2n)(\alpha+\beta+2n+1))\times \\ \qquad\qquad \Big[\Big((\beta-\alpha) + (\alpha+\beta+2n)z\Big)(\alpha+\beta+2n+2) - 2(\beta-\alpha)R(z)\Big] \\ A_{3k}(z) = -(n+1)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)(\alpha+\beta+2n+2)\times \\ \qquad\qquad \Big(R(z) + (\alpha+\beta+n+k+1)\Big)(1-z^2) \\ A_{4k}(z) = -2R(z)(n-k)(n-k+1)(\alpha+\beta+n)(\alpha+\beta+2n-1)(\alpha+\beta+2n)^2(\alpha+\beta+2n+1)\times \\ \qquad\qquad (\alpha+\beta+2n+2) \end{cases}$$

- *Laguerre Polynomials*

$$A_{1k}(z) = n(\alpha+n), \quad A_{2k}(z) = n-k, \quad A_{3k}(z) = -z, \quad A_{4k}(z) = 0.$$

- *Hermite Polynomials*

$$A_{1k}(z) = n, \quad A_{2k}(z) = 0, \quad A_{3k}(z) = -1, A_{4k}(z) = 0.$$

4.3.4. *A known identity for the Laguerre polynomials.* To conclude this paper let us obtain a very well known formula for the Laguerre polynomials using the method described here.

Putting $k = 0$ in relation (3.20)–(3.21), for the Laguerre Polynomials it becomes

$$A_1(x)L_{n-1}^{(\alpha)}(x) + A_2(x)L_n^{(\alpha)}(x) + A_3(x)\left(L_{n+1}^{(\alpha)}(x)\right)' = 0\,, \tag{4.12}$$

where, by (4.7), the coefficients A_i, $i = 1, 2, 3$, are given by

$$A_1(x) = \alpha + n, \quad A_2(x) = x - n, \quad A_3(x) = x.$$

Then, (4.12) leads to the well known formula for the Laguerre polynomials

$$x\left(L_{n+1}^{(\alpha)}(x)\right)' = (n-x)L_n^{(\alpha)}(x) - (\alpha+n)L_{n-1}^{(\alpha)}(x)\,.$$

Let us also point out that for Jacobi polynomials, if one considers, in Theorem 3, $k = 0$, $\nu = n$ and $C_n = \frac{n!B_n}{2\pi i}$ then, the corresponding coefficients solution (3.14) gives the four-term recurrence relation stated in Corollary 1.1 of [20, page 729].

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