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WAVELET FRAMES, BERGMAN SPACES AND FOURIER TRANSFORMS OF LAGUERRE FUNCTIONS

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ABSTRACT: The Fourier transforms of Laguerre functions play the same canonical role in Wavelet analysis as do the Hermite functions in Gabor analysis. We will use them as mother wavelets in a similar way as the Hermite functions were recently used as windows in Gabor frames by Gröchenig and Lyubarskii. Using results due to K. Seip concerning lattice sampling sequences on weighted Bergman spaces, we find a sufficient condition for the discretization of the resulting wavelet transform to be a frame. As in Gröchenig-Lyubarskii theorem, the density increases with n, when considering frames generated by translations and dilations of the Fourier transform of the *nth* Laguerre function.

KEYWORDS: Wavelets, Frames, Laguerre functions, Bergman spaces.
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1. Introduction

One of the fundamental questions of applied harmonic analysis is to obtain density conditions on the sequences required to discretize a continuous integral transform in a way that the resulting discretization is a frame in a certain Hilbert space. This question was sharply solved for the frames in the Bargmann-Fock [7], [19], [20] and in the Bergmann space [23]. However, in the cases of the Wavelet and Gabor transforms, very little is known and only a few very special windows and analysing wavelets are understood.

The short time Fourier (Gabor) transform with respect to a gaussian window can be written in terms of the Bargmann transform, mapping isometrically the space $L^2(\mathbf{R})$ onto the Bargmann-Fock space of entire functions. This is the reason why everything is known about the geometry of sequences that generate frames by sampling the Gabor transform with gaussian windows $g(t) = e^{-\pi t^2}$. Apart from this example, the only cases where a description is known of the lattice sequences that generate frames are the hyperbolic secant $g(t) = (\cosh at)^{-1}$ [14] and the characteristic function of an interval

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[15], which turned out to be a nontrivial problem. There is also a necessary condition for Gabor frames due to Ramanathan and Steger [18].

If we consider the wavelet case, things appear to be even more mysterious. There is no known counterpart of Ramanathan and Steger theorem and the only information available so far concerns a special family of analyzing wavelets that maps the problem into Bergmann spaces: The wavelet transform, with positive dilation parameter, with respect to a wavelet of the form (Paul´s Wavelet in some literature)

$$\psi_\alpha(t) = \left(\frac{1}{t+i}\right)^{\alpha+1} \tag{1}$$

can be rescaled as an isometrical integral transform between spaces of analytical functions, namely, between the Hardy space on the upper half plane $H^2(\mathbf{U})$ and the Bergman space in the upper half plane $A^{2\alpha+1}(\mathbf{U})$. It is natural to refer to this isomorphism as the Bergman transform (the designation analytic wavelet is also frequently used). Since the upper half plane can be mapped isomorphically onto the unit disc by using linear fractional transformations, we can construct a transform mapping the Hardy space onto the Bergman space in the unit disc.

The approaches used to deal with the special situations mentioned in the above paragraphs are based in techniques which differ from case to case. It seems highly desirable to follow a more structured approach. The natural place to look for this structure is within the context of Hilbert spaces of analytic functions, where the powerful methods from complex analysis may answer questions that seem hopeless otherwise. Following this line of reasoning implies using windows that allow to carry the problem to such spaces. In this direction a major step was taken recently by Göchenig and Lyubarskii [10], by considering Gabor systems with Hermite functions of order n as windows. They have proved that if the size of the lattice Λ is $< (n+1)^{-1}$ then the referred Gabor system is a frame and provided an example supporting their conjecture that the result is sharp.

In the Bargmann-Fock setting the Hermite functions play a very special role [9, pag. 57]. They are, up to normalization constants, pre-images, under the Bargmann transform, of the monomials $\{z^n\}$ and since the latter constitute an orthogonal basis of the Bargmann-Fock space, their pre-images also constitute an orthogonal basis of $L^2(\mathbf{R})$. That is, to the isomorphism

$$L^2(\mathbf{R}) \overset{\mathcal{B}}{\to} F^2(\mathbf{C}) \tag{2}$$

corresponds

$$h_n \overset{\mathcal{B}}{\to} c_n z^n \tag{3}$$

where $\mathcal{B}$ is the Bargmann transform, $F^2(\mathbf{C})$ is the Bargmann-Fock space, h_n are the Hermite functions and c_n some constants dependent on n. They are canonical to time-frequency analysis in an additional sense, since they constitute the eigenfunctions of the time-frequency-localization operator with Gaussian window [2]. The Hermite functions are eigenfunctions of the Fourier transform and they can be used [16] to describe Feichtinger´s algebra S_o.

Wavelet and Gabor analysis share many similarities and many of their structural aspects can be bound together in a more general theory using representations of locally compact abelian groups [11], [8]. In looking for a Wavelet-analogue of Gröchenig-Lyubarskii structured approach to the density problem, we must first clarify what functions should be used instead of the Hermite functions. In analogy to the above paragraph it is natural to consider the pre-images, under the Bergman transform, of the monomials $\{z^n\}$ in the unit disc (up to an isomorphism with the upper half plane). Since the $\{z^n\}$ form a basis of the weighted Bergman spaces in the unit disc (independently of the weight), their pre-images must be an orthogonal basis of the Hardy space $H^2(\mathbf{U})$ and it is reasonable to expect that such functions will play a similar role in Wavelet analysis as do the Hermite functions in Gabor analysis. Such functions are the Fourier transforms of the Laguerre functions and they will be ad-hoc denoted by S_n^α.

Additional structural evidence that the functions S_n^α are canonical in Wavelet analysis comes from the work of Daubechies and Paul [3], where it is shown that they are the eigenfunctions of a differential operator that commutes with a time–scale localization operator, once windows of the form (1) are chosen. This completely parallels the situation with the time-frequency-localization operators with gaussian windows, which commute with the harmonic oscillator and therefore have as eigenfunctions the Hermite functions [2]. It was observed by Seip [21] that these problems have a more natural formulation when mapped into the convenient spaces of analytical functions.

2. Description of the results

There exists an analogue of the correspondence between (2) and (3), but involving four different functional spaces and the corresponding bases: To the sequence of isomorphisms between the Hilbert spaces

$$L^2(\mathbf{0},\infty) \stackrel{\mathcal{F}}{\longleftrightarrow} H^2(\mathbf{U}) \stackrel{Ber^\alpha}{\rightarrow} A_{2\alpha+1}(\mathbf{U}) \stackrel{T_\alpha}{\rightarrow} A_{2\alpha+1}(\mathbf{D}), \tag{4}$$

where $A_{2\alpha+1}(\mathbf{U})$ and $A_{2\alpha+1}(\mathbf{D})$ are the weighted Bergmann spaces in the upper-half plane and in the unit disc, respectively, corresponds the relations between the basis of the respective spaces:

$$l_n^\alpha \stackrel{\mathcal{F}}{\longleftrightarrow} S_n^\alpha \stackrel{Ber^\alpha}{\rightarrow} c_n^\alpha \Psi_n^\alpha \stackrel{T_\alpha}{\rightarrow} c_n^\alpha z^n. \tag{5}$$

for some constants c_n^α, where l_n^α is the *nth* Laguerre function of order n and parameter α, S_n^α is its Fourier transform and Ψ_n^α is a basis of $A_{2\alpha+1}(\mathbf{U})$ to be defined in section 4. This correspondence was implicit in the conection between papers [3] and [21] but since it was not stated explicitly, we devote section 4 to clarify how exactly does it work.

The functions S_n^α were computed recently in closed form [5], but the connection to the Wavelet transform seems to have been unnoticed. It is also shown in [5] that S_n^α are, up to a fractional transformation, defined in terms of a certain system of orthogonal polynomials on the unit circle. We will show that this family of polynomials is nothing more but the circular Jacobi orthogonal polynomials for which there is, for computational purposes, a very convenient three term recurrence formula. This connection will make our case to call the functions S_n^α the *rational Jacobi orthogonal functions.*

With a computational method available for evaluating the functions S_n^α and graphic evidence of their good localization properties (see for example the plots of their real versions in page 44 of [1]), it is natural to investigate how to use such functions as analysing wavelets and to obtain frames from the resulting discretization. We will obtain the following sufficient condition on the density of the parameters of the hyperbolic lattice $\{(a^j bk, a^j)\}_{j,k\in\mathbf{Z}}$:

$$b \log a < \frac{4\pi}{2n+\alpha}$$

that is, as in Gröchenig and Lyubarskii result [10], this density increases with n. The proof of this sufficiently condition will be dramatically simplified with the observation of the following ocurrence: There exist sequences of numbers

$\{C^{\alpha,n}\}$ and $\{a_k^{\alpha,n}\}$ such that

$$S_n^{\alpha}(\frac{t}{2}) = C^{\alpha,n}\sum_{k=0}^{n} a_k^{\alpha,n}\psi_{k+\frac{\alpha}{2}}(t). \tag{6}$$

This is a lucky coincidence, to say the least. It will allow the use of Seip´s results from [22] and [23] in a relatively strightforward way.

We organize our ideas as follows. The next section contains the main definitions and facts concerning Wavelet transforms, Bergman spaces and Laguerre functions. Section four contains the evaluation of the pre-images required to build up the correspondences (4) and (5). The fifth section contains our main results on Wavelet frames with Fourier transforms of Laguerre functions. We conclude the paper collecting some further properties of the functions S_n^{α} and clarifying their classification within known families of orthogonal polynomials.

3. Preparation

3.1. The Bergman transform. Now we present a sinthesis of ideas that appeared in the section 3.2 of [11] and in [3] (see also [1, pag. 31]) Here they will be exposed in such a way that the role of the Bergman spaces is emphasized.

Consider the dilation and translation operators

$$\begin{aligned} D_s f(x) &= |s|^{-\frac{1}{2}} f(s^{-1}x) \\ T_x f(t) &= f(t-x) \end{aligned}$$

and define

$$\psi^{x,s}(t) = T_x D_s \psi(t) = |s|^{-\frac{1}{2}}\psi(\frac{t-x}{s}).$$

The wavelet transform of a function f, with respect to the wavelet ψ is

$$W_{\psi}f(x,s) = \langle f, T_x D_s\psi(t)\rangle_{L^2(\mathbf{R})} = \int_{-\infty}^{\infty} f(t)\overline{\psi^{x,s}(t)}dt.$$

A function $\psi \in L^2(R)$ is said to be admissible if

$$\int_0^{\infty} |\mathcal{F}\psi(s)|^2 \frac{ds}{s} = K$$

where K is a constant. If ψ is admissible, then for all $f \in L^2(R)$ we have

$$\int_0^\infty \int_{-\infty}^\infty s^{-2} \left| W_\psi f(x,s) \right|^2 dxds = K \left\| f \right\|^2 \tag{7}$$

We will restrict ourselves to parameters $s > 0$ and functions $f \in H^2(\mathbf{U})$, where $H^2(\mathbf{U})$ is the Hardy space in the upper half plane $\mathbf{U} = \{z = x + is : s > 0\}$

$$H^2(\mathbf{U}) = \{f : f \text{ is analytic in } \mathbf{U} \text{ and } \sup_{0<s<\infty} \int_{-\infty}^\infty \left| f(x+is) \right|^2 dx < \infty\}.$$

Let $\mathcal{F}$ denote the Fourier transform

$$(\mathcal{F}f)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty e^{-itx} f(x)dx$$

By the Paley-Wiener theorem, $H^2(\mathbf{U})$ is constituted by the functions whose Fourier transform is supported in $(0, +\infty)$ and belongs to $L^2(0, \infty)$.

Now take the special window $\psi_\alpha(t)$ defined in (1). Since

$$\mathcal{F}\psi_\alpha^{-x,s}(t) = \mathbf{1}_{[0,\infty]} s^{\alpha+\frac{1}{2}} t^\alpha e^{-(-s+ix)t} \tag{8}$$

then

$$W_{\psi_\alpha} f(-x,s) = s^{\alpha+\frac{1}{2}} \int_0^\infty t^\alpha e^{i\overline{z}t} (\mathcal{F}f)(t)dt \tag{9}$$

where the function defined by the integral is analytic in $z = x + si$. The identity (7) gives

$$\int\int_{\mathbf{U}} \left| W_{\psi_\alpha} f(-x,s) \right|^2 s^{-2} dxds = \|f\|_{H^2(\mathbf{R})} \tag{10}$$

This motivates the definition of the *Bergman transform*, or the *analytic wavelet transform* by

$$Ber^\alpha \ f(z) = s^{-\alpha-\frac{1}{2}} W_{\psi_\alpha} f(-x,s) \tag{11}$$

where $z = x + is$ (see for instance [12], where the authors use this Bergman transform with the same normalization in the case $\alpha = 1$). Introducing the scale of weighted Bergman spaces

$$A_\alpha(\mathbf{U}) = \{f \text{ analytic in } \mathbf{U} \text{ such that } \int\int_{\mathbf{U}} \left| f(z) \right|^2 s^{\alpha-2} dxds < \infty\},$$

it is clear from (10) and (11) that $Ber^\alpha\ f(z) \in A_{2\alpha+1}(\mathbf{U})$. We have therefore an isometric transformation

$$Ber^\alpha : H^2(\mathbf{R}) \to A_{2\alpha+1}(\mathbf{U})$$

3.2. Fourier transforms of Laguerre functions. The Laguerre polynomials will play a central role in our discussion.One way to define them is by means of the Rodrigues formula

$$L_n^\alpha(x) = \frac{e^x x^{-\alpha}}{n!} \frac{d^n}{dx^n} \left[e^{-x} x^{\alpha+n}\right] \tag{12}$$

and this gives, in power series form

$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k}{(\alpha+1)_k} \frac{x^k}{k!}.$$

For information on specific systems of orthogonal polynomials, we suggest [13]. The Laguerre functions are defined as

$$l_n^\alpha(x) = \mathbf{1}_{[0,\infty]}(x) e^{-x/2} x^{\alpha/2} L_n^\alpha(x)$$

and they are known to constitute an orthogonal basis for the space $L^2(0,\infty)$. By the Paley-Wiener theorem, the Fourier transform is an isomorphism between $H^2(\mathbf{U})$ and $L^2(0,\infty)$. Therefore, the Fourier transform of the functions l_n^α form an orthogonal basis for the space $H^2(\mathbf{U})$. From now on we will set

$$S_n^\alpha(t) = \mathcal{F} l_n^\alpha(t) \tag{13}$$

The functions $S_n^\alpha(t)$ can be evaluated explictly using (12). This was already done by Shen in [5], where the author was interested in describing the orthogonal polynomials which arise from an application of the Fourier transform on the Laguerre polynomials. A description of this method of generating new families of orthogonal polynomials with an interesting historical account is in the paper [17] where a link is provided between Jacobi and Meixner-Polaczek polynomials. Following [5], we have

$$S_n^\alpha(t) = \frac{\Gamma\left(\frac{\alpha}{2}+1\right)(1+\alpha)_n}{n!} \sum_{k=0}^{n} \frac{(-n)_k (\frac{\alpha}{2}+1)_k}{k!(\alpha+1)_k} \left(\frac{1}{\frac{1}{2}-it}\right)^{k+\frac{\alpha}{2}+1} \tag{14}$$

4. Bergman transform of $S_n^\alpha(t)$

In this section the correspondences (4) and (5) will be established via a direct calculation. The complex linear fractional transformations will play an important role, in a style that is reminiscent of the way they are used in the discrete series representation of $SL(2,\mathbf{R})$ over the Bergman space [6, chapter IX].

We first define a set of functions that, as we shall see later, constitute a basis of $A_\alpha(\mathbf{U})$. For every $n \geq 0$ and $\alpha > -1$ let

$$\Psi_n^\alpha(z) = \left(\frac{i\overline{z}-\frac{1}{2}}{i\overline{z}+\frac{1}{2}}\right)^n \left(i\overline{z}+\frac{1}{2}\right)^{-\alpha-1}.$$

Now define a map T_α such that for every function $f \in A_\alpha(U)$ the action of T_α is

$$T_\alpha f(w) = f\left[\frac{i}{2}\frac{w+1}{w-1}\right]\left(\frac{1}{1-w}\right)^{\alpha+1} \tag{15}$$

The range space of T_α is a weighted Bergman space in the unit disc. The weighted Bergman spaces in the unit disc are denoted by $A_\alpha(\mathbf{D})$ and defined as

$$A_\alpha(\mathbf{D}) = \left\{ f \text{ analytic in } D \text{ such that } \int\int_D |f(z)|^2 (1-|z|)^{\alpha-2} dxdy < \infty \right\}.$$

Lemma 4.1 *The map*

$$T_\alpha : A_\alpha(\mathbf{U}) \to A_\alpha(\mathbf{D})$$

is an unitary isometry between Hilbert spaces.

Proof: Argue as in the proof of Lema 1 in [6, pag.185]. ■

Proposition 4.1 *For $n = 0, 1, \ldots$, the following relations hold:*

$$Ber^\alpha(S_n^{2\alpha}) = c_n^\alpha \Psi_n^\alpha \text{ , with } c_n^\alpha = (-1)^{\alpha+1}(2\alpha+n)!/n! \tag{16}$$

and, for $|z| < 1$,

$$T_\alpha(\Psi_n^\alpha) = z^n \tag{17}$$

In other words, the function $S_n^{2\alpha}$ is the pre-image, under $T_\alpha \circ \left(\frac{1}{c_n^\alpha} Ber^\alpha\right)$, of $z^n \in A_\alpha(D)$.

Proof: Using the Plancherel theorem for the Fourier transform, formula (8) and (13) we have

$$\begin{aligned} Ber^{\alpha}\ S_n^{2\alpha}(z) &= s^{-\alpha-\frac{1}{2}} \int_{-\infty}^{\infty} S_n^{2\alpha}(t)\overline{\psi_{\alpha}^{-x,s}(t)}dt \\ &= -\int_0^{\infty} e^{-(i\overline{z}+\frac{1}{2})t} t^{2\alpha} L_n^{2\alpha}(t)dt \end{aligned}$$

Applying Rodrigues formula (12) for the Laguerre polynomials, the result is

$$\begin{aligned} Ber^{\alpha}\ S_n^{2\alpha}(z) &= \frac{(-1)^{\alpha+1}}{n!}\mathcal{L}\left[\frac{d^n}{dx^n}\left[e^{-x}x^{2\alpha+n}\right]\right](i\overline{z}-\frac{1}{2}) \\ &= c_n^{\alpha}\Psi_n^{2\alpha}(z), \end{aligned}$$

where $\mathcal{L}$ stands for the Laplace transform, whose well known properties establish the last identity. Now, the linear fractional transformation

$$w = \frac{2\overline{z}+i}{2\overline{z}-i} = \frac{i\overline{z}-\frac{1}{2}}{i\overline{z}+\frac{1}{2}}$$

is an analytic isomorphism between the upper half plane and the unit circle. Since the inverse of this transformation is given by

$$\overline{z} = \frac{i}{2}\frac{w+1}{w-1},$$

a short calculation with the definition of T_{α} gives (17). ■

Corollary 1. $\{\Psi_n^{\alpha}(z)\}$ *is a basis of* $A_{\alpha}(U)$ *and the map* T_{α} *is an unitary isomorphism between* $A_{\alpha}(U)$ *and* $A_{\alpha}(\mathbf{D})$.

Proof: Since $\{z^n\}$ is an orthogonal basis of the space $A_{\alpha}(\mathbf{D})$ [6, pag. 186] and it is contained in the range of T_{α}, T_{α} is onto and therefore an unitary isomorphism.The functions $\{\Psi_n^{\alpha}(z)\}$ form a basis for the space $A_{\alpha}(\mathbf{U})$ since they are the pre-images of the basis $\{z^n\}$. ■

As a consequence we obtain a new proof of the (known) isomorphic property of the Bergman transform.

Corollary 2. *The transform* $Ber^{\alpha} : H^2(\mathbf{R}) \to A_{2\alpha+1}(\mathbf{U})$ *is an isometric isomorphism.*

Proof: The isometry is a consequence of the isometric property of the wavelet transform, so we need only to prove that the $Ber^{\frac{\alpha}{2}}$ is onto. But in view of the preceding result, the range of $Ber^{\frac{\alpha}{2}}$ contains a basis of $A_\alpha(\mathbf{U})$. Therefore, $Ber^{\frac{\alpha}{2}}$ is onto. ■

Remark 1. *Similar calculations as we have seen here also play a role in* [4]*, in the context of Laplace transformations and group representations and in* [3]*, to obtain an explicit formula for the eigenvalues of the time-scale localization operator.*

5. Wavelet frames with Fourier transforms of Laguerre functions

We wish to construct wavelet frames with analysing wavelets S_n^α, by using the common discretization of the continuous wavelet transform via the hyperbolic lattice. A sequence of functions $\{e_j\}$ is said to be a frame in a Hilbert space H if there exist constants A and B such that

$$A\left\|f\right\|^2 \leq \sum_j \left|\left\langle f, e_j\right\rangle\right|^2 \leq B\left\|f\right\|^2. \tag{18}$$

Discretizing the scale parameter s in the Wavelet transform by a sequence a^j and the parameter x by $a^j bk$ gives

$$W_\psi f(a^j bk, a^j) = \left\langle f, T_{a^j bk} D_{a^j} \psi\right\rangle. \tag{19}$$

We want to know conditions in a and b under which $\{T_{a^j bk} D_{a^j}\psi\}$ is a frame, for a given wavelet ψ.

In spaces of analytic functions a related concept to frames is the one of a set of sampling. A set $\Gamma = \{z_j\}$ is said to be a set of sampling for the Bergman space $A_\alpha(\mathbf{U})$ if there exist positive constants A and B such that

$$A\int\int_{\mathbf{U}} \left|f(z)\right|^2 y^{\alpha-2} dxdy \leq \sum_j \left|f(z_j)\right|^2 y_j^\alpha \leq B\int\int_{\mathbf{U}} \left|f(z)\right|^2 y^{\alpha-2} dxdy \tag{20}$$

The following theorem is due to Seip [22].

Theorem A *Let $\Gamma(a,b) = \{z_{jk}\}_{j,k\in\mathbf{Z}}$, where $z_{jk} = a^j(bk+i)$. $\Gamma(a,b)$ is a set of sampling for $A_\alpha(\mathbf{U})$ if and only if $b\log a < \frac{4\pi}{\alpha-1}$.*

Remark 2. *Observe that $\{T_{a^j bk} D_{a^j}\psi_\alpha\}_{j,k\in\mathbf{Z}}$ is a frame of $H^2(\mathbf{U})$ iff $\Gamma(a,b) = \{z = a^j bk + a^j i\}_{j,k\in\mathbf{Z}}$ is a set of sampling for $A_{2\alpha+1}(\mathbf{U})$. Indeed, since functions in $A_{2\alpha+1}(\mathbf{U})$ can be identified with Ber^α transforms of $H^2(\mathbf{U})$ functions, it follows from (20) and (11) that $\Gamma(a,b)$ is a set of sampling for $A_{2\alpha+1}(\mathbf{U})$*

if and only if

$$A\int\int_{\mathbf{U}}|W_{\psi_\alpha}f(x,s)|^2\frac{dxds}{s^2}\leq\sum_{j,k}\left|W_{\psi_\alpha}f(a^jbk,a^j)\right|^2\leq B\int\int_{\mathbf{U}}|W_{\psi_\alpha}f(x,s)|^2\frac{dxds}{s^2}$$

using (10) this is equivalent to

$$A\left\|f\right\|^2_{H^2(\mathbf{U})}\leq\sum_{j,k}|\langle f,T_{a^jbk}D_{a^j}\psi_\alpha\rangle|^2\leq B\left\|f\right\|^2_{H^2(\mathbf{U})}$$

which says that $\{T_{a^jbk}D_{a^j}\psi_\alpha\}_{j,k}$ *is a frame in* $H^2(\mathbf{U})$.

The next Lemma is crucial and it is just a simple modification of the representation (14).

Lemma 5.2 *The functions* S_n^α *can written as the linear combination (6) of analysing wavelets* $\psi_{k+\frac{\alpha}{2}}(t)$ *defined by (1) with the coefficients* $C^{\alpha,n}$ *and* $a_k^{\alpha,n}$ *given as*

$$\begin{aligned} C^{\alpha,n} &= \frac{\Gamma\left(\frac{\alpha}{2}+1\right)(1+\alpha)_n}{n!} \\ a_k^{\alpha,n} &= (2i)^{k+\frac{\alpha}{2}+1}\frac{(-n)_k(\frac{\alpha}{2}+1)_k}{k!(\alpha+1)_k}. \end{aligned}$$

Our sufficient density result follows from Theorem A and Lemma 2.

Theorem 1. *If* $b\log a<\frac{4\pi}{2n+\alpha}$, *then* $\{T_{a^jbk}D_{a^j}S_n^\alpha(\frac{t}{2})\}_{j,k}$ *is a frame of* $H^2(\mathbf{U})$.

Proof: The definition of the Wavelet transform gives, using (6),

$$\begin{aligned} W_{S_n^\alpha(\frac{t}{2})}f(-x,s) &= \left\langle f,T_{-x}D_sS_n^\alpha(\frac{t}{2})\right\rangle \\ &= C^{\alpha,n}\sum_{k=0}^n a_k^{\alpha,n}\left\langle f,T_{-x}D_s\psi_{k+\frac{\alpha}{2}}\right\rangle \\ &= C^{\alpha,n}\sum_{k=0}^n a_k^{\alpha,n}W_{\psi_{k+\frac{\alpha}{2}}}f(-x,s). \end{aligned}$$

Therefore, defining the function $F(z)$ as

$$F(z)=s^{-n-\frac{\alpha}{2}-\frac{1}{2}}W_{S_n^\alpha(\frac{t}{2})}f(-x,s)$$

we can write

$$F(z)=C^{\alpha,n}\sum_{k=0}^n a_k^{\alpha,n}s^{k-n}Ber^{k+\frac{\alpha}{2}}f(z)$$

Since $Ber^{k+\frac{\alpha}{2}}f \in A_{2k+\alpha+1}(\mathbf{U})$, it is clear that $s^{k-n}Ber^{k+\frac{\alpha}{2}}f(z) \in A_{2n+\alpha+1}(\mathbf{U})$ for all $k < n$ and this is sufficient to assure that $F \in A_{2n+\alpha+1}(\mathbf{U})$. Since, by hypothesis, $b \log a < \frac{4\pi}{2n+\alpha}$, then $\Gamma(a,b)$ is a set of sampling for $A_{2n+\alpha+1}(\mathbf{U})$. From this we infer that F verifies the inequality

$$A \int\int_{\mathbf{U}} |F(z)|^2 s^{2n+\alpha-1} dxds \leq \sum_j |F(z_j)|^2 s_j^{2n+\alpha+1} \leq B \int\int_{\mathbf{U}} |F(z)|^2 s^{2n+\alpha-1} dxds \tag{21}$$

or

$$\begin{aligned} & A \int_0^\infty \int_{-\infty}^\infty \left| W_{S_n^\alpha(\frac{t}{2})} f(-x,s) \right|^2 s^{-2} dxds \qquad (22) \\ \leq & \sum_{j,k} \left| W_{S_n^\alpha(\frac{t}{2})} f(a^j bk, a^j) \right|^2 \leq B \int_0^\infty \int_{-\infty}^\infty \left| W_{S_n^\alpha(\frac{t}{2})} f(-x,s) \right|^2 s^{-2} dxds \end{aligned}$$

taking into account that $S_n^\alpha(\frac{t}{2})$ is admissible, we can apply (7) and (19) to write the above inequalities as

$$A \|f\|^2 \leq \sum_{j,k} \left| \left\langle f, T_{a^j bk} D_{a^j} S_n^\alpha(\frac{t}{2}) \right\rangle \right|^2 \leq B \|f\|^2$$

and conclude that $\{T_{a^j bk} D_{a^j} S_n^\alpha(\frac{t}{2})\}_{j,k}$ is a frame. ∎

Remark 3. *Theorem 2 parallels Theorem 3.1 in* [10], *which states that, in the case of the uniform lattice for the time-frequency plane, if the size of the lattice* Λ *is* $< (n+1)^{-1}$ *then the Gabor system* $\{e^{2\pi i \lambda_2 t} H_n(t - \lambda_1) : \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$, *where* H_n *stands for the Hermite function of order* n, *is a frame for* $L^2(\mathbf{R})$. *In particular, if one is dealing with the Von Neumann lattice with parameters* a *and* b, *the condition is* $ab < (n+1)^{-1}$.

It is possible to state an analogue of Theorem 2 in terms of the deeper Beurling type density results from [23]. In order to do this we need some definitions. Define the pseudohyperbolic metric on the unit disk by $\varrho(z,\zeta) = \left|\frac{z-\zeta}{1-\overline{\zeta}z}\right|$. A sequence $\Gamma = \{z_j\} \subset \mathbf{D}$ is separated if $\inf_{n \neq j} \left|\frac{z_j - z_n}{z_j - \overline{z}_n}\right| > 0$ and its lower density D^- is given by

$$D^-(\Gamma) = \lim_{r \to 1} \inf \inf_z \frac{\sum_{\varrho(z_j,z)<r} (1 - \varrho(z_j, z))}{\log \frac{1}{1-r}}$$

The next Theorem is from [23]:

Theorem B *A separated sequence* $\Gamma \subset \mathbf{D}$ *is a sampling sequence for* $A^\alpha(\mathbf{D})$ iff $D^-(\Gamma) > \frac{\alpha}{2}$.

From this we obtain the following, with the same proof as in Theorem 1.

Theorem 2. *Let* $\Gamma \subset \mathbf{D}$ *denote a uniformed separated sequence obtained from mapping the sequence* $\{z_{k,j} = a_k + ib_j\} \subset \mathbf{U}$ *into the unit disk via a Cayley transform. If* $D^-(\Gamma) > n + \frac{\alpha+1}{2}$ *then* $\{T_{a_k} D_{b_j} S_n^\alpha(\frac{t}{2})\}_{j,k}$ *is a frame of* $H^2(\mathbf{U})$

Remark 4. *Observe that combining the Paley-Wiener with the Plancherel theorem, we have* $\|f\|_{H^2(\mathbf{U})} = \|f\|_{L^2(0,\infty)}$ *and the results of Theorem 1 and 2 say also that we have a frame for* $L^2(0,\infty)$.

6. Further properties of the functions $S_n^\alpha(t)$

The notation

$$F(a,b;c;x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{k!(c)_k} x^k.$$

for the hypergeometric function is used in this section. Rewriting S_n^α in this notation gives

$$S_n^\alpha(t) = C^{\alpha,n} \left(\frac{1}{2} - it\right)^{-\frac{\alpha}{2}-1} F(-n, \frac{\alpha}{2}+1; \alpha+1; \frac{1}{\frac{1}{2}-it})$$

(observe that the infinite sum becomes a polynomial of order n, since $(-n)_k = 0$ if $k > 0$). Composing the functions $S_n^\alpha(t)$ with the fractional linear transformation

$$z = \frac{2t-i}{2t+i} \tag{23}$$

the result is

$$S_n^\alpha(t) = \Gamma\left(\frac{\alpha}{2}+1\right)(1-z)^{\frac{\alpha}{2}+1} g_n^\alpha(z) \tag{24}$$

where

$$g_n^\alpha(z) = \frac{(\alpha/2)_n}{n!} F(-n, \frac{\alpha}{2}+1; -n-\alpha/2+1; z)$$

is a polynomial in z of degree n. This was pointed out in [5]. It was also shown that these polynomials satisfy the orthogonality

$$\int_{|z|=1} g_n^\alpha(z)\overline{g_n^\alpha(z)}\, |1-z|^\alpha \frac{dz}{z} = 0 \text{ if } m \neq n \tag{25}$$

and therefore are orthogonal on the unit circle with respect to the weight

$$w(z) = \sin^{\alpha}\frac{\theta}{2}d\theta. \tag{26}$$

This fact implies many properties, since there exists a very rich theory for orthogonal polynomials on the unit circle (see [24] and references therein and also chapter 8 of [13]). For example, the general theory assures that all the zeros of $g_n^{\alpha}(z)$ lay within the unit disc.

Remark 5. *Setting $a = \frac{\alpha}{w}$ in Example 8.2.5 at* [13], *and using the identity $1 - e^{i\theta} = 4\sin^2\frac{\theta}{2}$ to write the measure (8.2.21) as (26) we recognize that the polynomials $g_n^{\alpha}(z)$ are, up to a normalization, a family of orthogonal polynomials on the unit circle known as the* circular Jacobi orthogonal polynomials.

Remark 6. *From (25) the functions $z^{-\frac{1}{2}}(1-z)^{\frac{\alpha}{2}}g_n^{\alpha}(z)$ are orthogonal on the circle and form a basis of the Hardy space on the unit disc*

$$H^2(\mathbf{D}) = \{f : f \textit{ is analytic in } \mathbf{D} \textit{ and } \sup_{r<1}\int_0^{2\pi}\left|f(re^{it})\right|^2 dt < \infty\}$$

it is also clear that $S_n^{\alpha}(t)$ are orthogonal on the real line (the boundary of the upper half place). Since $g_n^{\alpha}(z)$ are the circular Jacobi orthogonal polynomials, the basis functions $z^{-\frac{1}{2}}(1-z)^{\frac{\alpha}{2}}g_n^{\alpha}(z)$ are the circular Jacobi orthogonal functions and it is therefore natural to call $S_n^{\alpha}(t)$ the rational Jacobi orthogonal functions.

Remark 7. *From the general theory* [13, (8.2.10)] *follows that, if κ_n is the leading coefficient of the polynomial, then the sequence of polynomials $\{g_n^{\alpha}(z)\}$ satisfies a three term recurrence relation*

$$\begin{aligned}
& \kappa_n g_n^{\alpha}(0)g_{n+1}^{\alpha}(z) + \kappa_{n-1}g_{n+1}^{\alpha}(0)zg_{n-1}^{\alpha}(z) \\
= \; & [\kappa_n g_{n+1}^{\alpha}(0) + \kappa_{n+1}g_n^{\alpha}(0)z]g_n^{\alpha}(z)
\end{aligned}$$

where κ_n is the leading coefficient of the polynomial. From the explicit representation of the polynomials $g_n^{\alpha}(z)$ it is easily seen that

$$\begin{aligned}
\kappa_n &= \frac{(\frac{\alpha}{2}+1)_n}{n!} \\
\phi_n(0) &= \frac{\alpha(\frac{\alpha}{2}+1)_{n-1}}{2n!}
\end{aligned}$$

This three term recurrence relation provides a effective method for computational purposes: To evaluate the functions $S_n^{\alpha}(t)$ it is sufficient to combine this recurrence relation with formulas (23) and (24).

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