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DISCRETE NEGATIVE NORMS IN THE ANALYSIS OF SUPRACONVERGENT TWO DIMENSIONAL CELL-CENTERED SCHEMES

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Abstract: In this paper we study the convergence properties of cell-centered finite difference schemes for second order elliptic equations with variable coefficients. We prove that the finite difference schemes on nonuniform meshes although not even being consistent are nevertheless second order convergent. The convergence is studied with the aid of an appropriate negative norm. Numerical examples support the convergence result.

Keywords: cell-centered finite differences scheme, nonuniform mesh, stability, supraconvergence.

1. Introduction

In the last decades there has been a strong mathematical interest in numerical discretizations methods that have higher convergence order than expected by analyzing the truncation error in a standard way. In the context of finite difference schemes on nonequidistant grids this behavior is called supraconvergence. Different methods of proving supraconvergence of finite difference schemes for ordinary differential equations have been used by the various authors (see e.g. [3], [10], [12], [13], [16], [17], [21] and [24]). The phenomenon of supraconvergence in more than one space dimension has also been studied in the literature (see e.g. [6], [8], [9] and [19]). The topic in the context of finite element methods has been treated in the papers [3], [4], [8], [11], [14], [15], [18], [20], [22], [28].

We are interested in studying this phenomena in a variant of finite differences, the so called cell-centered schemes, which are used in many codes. In fact, these schemes are not even consistent but nevertheless second order convergent. This fact was noticed by Tikhonov and Samarskii ([26]). Russell and Wheeler ([23]) use the equivalence of a cell-centered finite difference method and a mixed finite element method with a special quadrature formula for proving first order convergence of the solution and the gradient.

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Manteuffel and White ([21]) show second order convergence in both vertex-centered finite difference schemes and cell-centered finite difference schemes for scalar problems, on nonuniform meshes. Supraconvergence results for two-dimensional cell-centered schemes were presented by Forsyth and Sammon ([9]) and also by Weiser and Wheeler ([27]), among others.

Our main purpose is to analyze how these two additional orders of convergence come out for more general problems than considered so far. The analysis of the present paper is based on using negative norms. The analysis of supraconvergence with one additional order of convergence in [3] and [6] is more or less explicitly based on the concept of negative norms which are related to the norm H^{-1}. The concept of negative norms in the analysis of supraconvergence was also used in [3], [4], [6], [7], [8], [12] and [15]. The idea in this paper is to work instead with a discrete version of the H^{-2}-norm. The convergence result relies in the stability inequality with respect to this norm. The analysis of supraconvergence with two additional order of convergence for the one-dimensional case is considered in [2], with the aid of so-called Spijker norms ([25]). These norms are applied twice corresponding to the two gained additional orders of convergence. The use of Spijker norms is restricted to one dimension. But they give the idea for a generalization to higher dimensions because they are related to the negative norms (for more details see [2]).

We consider the discretization of the differential equation with Dirichlet boundary condition

$$-(au_x)_x - (cu_y)_y + du_x + eu_y + fu = g \quad \text{on} \quad \Omega, \tag{1}$$

$$u = \psi \quad \text{on} \quad \partial\Omega. \tag{2}$$

The coefficients of A are assumed to satisfy $a(x,y) \geq \underline{a} > 0$, $c(x,y) \geq \underline{c} > 0$, $\forall (x,y) \in \Omega$, $a, c \in W^{3,\infty}(\Omega)$, $d, e, f \in W^{2,\infty}(\Omega)$. The domain Ω is an union of rectangles.

In order to prepare the definition of the cell-centered finite difference approximation of (1)–(2) let us first introduce the nonuniform grid G_H. In a rectangle $R = (x_{-1}, x_{N+1}) \times (y_{-1}, y_{M+1})$ which contains Ω we define the subset $G_H := R_1 \times R_2$, where

$$R_1 := \{x_{-1} < x_0 < \ldots < x_N < x_{N+1}\}$$

and

$$R_2 := \{y_{-1} < y_0 < \ldots < y_M < y_{M+1}\}.$$

Let

$$S_H \;:=\; \{(x_{j-1/2}, y_{\ell-1/2}) : j = 0, \ldots, N+1,\ \ell = 0, \ldots, M+1\},$$

where $x_{j-1/2} := (x_{j-1} + x_j)/2$, $y_{\ell-1/2} := (y_{\ell-1} + y_\ell)/2$. Our aim is to obtain numerical solutions in $\Omega_H := S_H \cap \Omega$. We define also $\partial\Omega_H := S_H \cap \partial\Omega$ and $\bar{\Omega}_H := \Omega_H \cup \partial\Omega_H$. The grid G_H is assumed to satisfy the following condition: the vertices of Ω are in the centers of the rectangles formed by G_H.

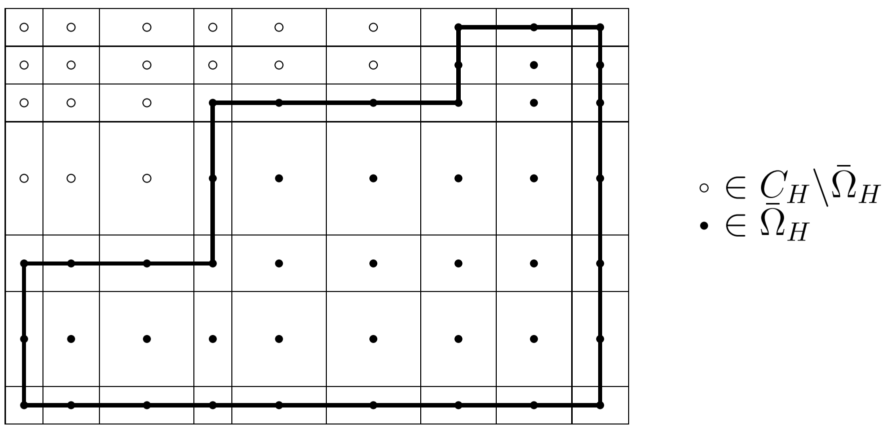

FIGURE 1. Domain and grid points.

Figure 1 illustrates the cell-centered grid in the domain.

If the case of a rectangular domain we allow $x_{-1} = x_0$, $x_{N+1} = x_N$, $y_{-1} = y_0$ and $y_{M+1} = y_M$.

For the formulation of the difference problem we use the centered difference quotients in x-direction

$$(\delta_x v_H)_{j,\ell+1/2} := \frac{v_{j+1/2,\ell+1/2} - v_{j-1/2,\ell+1/2}}{h_{j-1/2}},$$

$$(\delta_x w_H)_{j-1/2,\ell+1/2} := \frac{w_{j,\ell+1/2} - w_{j-1,\ell+1/2}}{h_{j-1}},$$

where $h_{j-1/2} := x_{j+1/2} - x_{j-1/2}$, $h_{j-1} := x_j - x_{j-1}$. Correspondingly, the finite centered difference quotients with respect to the y variable are defined, with the mesh-size vector k in place of h.

Let R_H and R_{G_H} be the operators that define restrictions to $\bar{\Omega}_H$ and $G_H \cap \Omega$, respectively.

The difference problem is to find $u_H \in \bar{\Omega}_H$ such that

$$A_H u_H = M_H R_{G_H} g \quad \text{on} \quad \Omega_H, \tag{3}$$

$$u_H = R_H \psi \quad \text{on} \quad \partial\Omega_H, \tag{4}$$

where the difference operator A_H is given by

$$A_H u_H := -\delta_x(a\delta_x u_H) - \delta_y(c\delta_y u_H) + M_x(d\delta_x u_H) + M_y(e\delta_y u_H) + f u_H, \tag{5}$$

and

$$\begin{aligned}
(M_x w_H)_{j-1/2,\ell-1/2} &:= \frac{w_{j-1,\ell-1/2} + w_{j,\ell-1/2}}{2},\\
(M_y w_H)_{j-1/2,\ell-1/2} &:= \frac{w_{j-1/2,\ell-1} + w_{j-1/2,\ell}}{2},\\
(M_H w_H)_{j-1/2,\ell-1/2} &:= \frac{w_{j-1,\ell-1} + w_{j-1,\ell} + w_{j,\ell-1} + w_{j,\ell}}{4},
\end{aligned}$$

for $(x_{j-1/2}, y_{\ell-1/2}) \in \Omega_H$. This last three quantities are zero for $(x_{j-1/2}, y_{\ell-1/2}) \in \partial\Omega_H$.

In the sequel we need norms for grid functions. To this end we introduce in the next section discrete versions of the Sobolev spaces $W_0^{m,2}(\Omega)$, $m = 0, 1, 2$.

2. Discrete $W_0^{m,2}(\Omega)$ spaces

Let $\overset{\circ}{W}{}_H^{m,2}(R)$, $m = 0, 1, 2$, be the space of grid functions defined in C_H, which are zero on the set

$$\{(x_{j-1/2}, y_{\ell-1/2}) : j = 0, N+1, \ell = 0, \dots, M+1 \vee j = 1, \dots, N,\ \ell = 0, M+1\},$$

equipped with the norm

$$\|v_H\|_{W_H^{m,2}(R)} := \Big(\sum_{r=0}^{m} |v_H|_{r,H}^2\Big)^{1/2},$$

where

$$|v_H|_{0,H}^2 := \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}|v_{j-1/2,\ell-1/2}|^2,$$

$$\begin{aligned}|v_H|_{1,H}^2 := & \sum_{j=0}^{N}\sum_{\ell=1}^{M} h_{j-1/2}k_{\ell-1}|\left(\delta_x v_H\right)_{j,\ell-1/2}|^2 \\ & +\sum_{j=1}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}|\left(\delta_y v_H\right)_{j-1/2,\ell}|^2,\end{aligned}$$

$$\begin{aligned}|v_H|_{2,H}^2 := & \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}\big(|\left(\delta_x^2 v_H\right)_{j-1/2,\ell-1/2}|^2 + |\left(\delta_y^2 v_H\right)_{j-1/2,\ell-1/2}|^2\big) \\ & +2\sum_{j=0}^{N}\sum_{\ell=0}^{M} h_{j-1/2}k_{\ell-1/2}|\left(\delta_{xy} v_H\right)_{j,\ell}|^2,\end{aligned}$$

with the difference quotient δ_{xy} given by

$$\left(\delta_{xy}v_H\right)_{j,\ell} := \frac{\left(\delta_x v_H\right)_{j,\ell+1/2} - \left(\delta_x v_H\right)_{j,\ell-1/2}}{k_{\ell-1/2}} = \frac{\left(\delta_y v_H\right)_{j+1/2,\ell} - \left(\delta_y v_H\right)_{j-1/2,\ell}}{h_{j-1/2}}.$$

Let P_{S_H} be the following operator that extends a grid function v_H in $\bar{\Omega}_H$ to S_H,

$$P_{S_H}v_H := v_H \quad \text{on} \quad \bar{\Omega}_H, \quad P_{S_H}v_H := 0 \quad \text{on} \quad S_H\backslash\bar{\Omega}_H.$$

We denote by $\overset{\circ}{W}{}_H^{m,2}(\Omega)$, $m = 0, 1, 2$, the space of functions defined in $\bar{\Omega}_H$, null in $\partial\Omega_H$, equipped with the norm

$$\|v_H\|_{m,H} := \Big(\sum_{r=0}^{m} |P_{S_H}v_H|_{r,H}^2\Big)^{1/2}, \quad m = 0, 1, 2.$$

The space $\overset{\circ}{W}{}_H^{0,2}(\Omega)$ (also denoted by $\overset{\circ}{L}{}_H^2(\Omega)$) is endowed by the inner product

$$(v_H, w_H)_H := \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}(P_{S_H}v_H)_{j-1/2,\ell-1/2}(P_{S_H}\bar{w}_H)_{j-1/2,\ell-1/2}.$$

When it is clear from the context that we use the extended function, we omit the notation P_{S_H}.

The discrete spaces introduced above form discrete approximations of their continuous counterparts in the sense that we explain in what follows.

Let Λ be a sequence of positive vectors of step-sizes , $H = (h, k)$, such that the maximum step-size, H_{max}, converges to zero. A sequence $(v_H)_\Lambda \in \Pi\ \overset{\circ}{L}{}^2_H(\Omega)$ converges discretely to $v \in L^2(\Omega)$ in $(L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega))$, $v_H \to v$ in $(L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega))$ $(H \in \Lambda)$, if for each $\epsilon > 0$ there exists $\varphi \in C_0^\infty(\Omega)$ such that

$$\|v - \varphi\|_{L^2(\Omega)} \leq \epsilon, \quad \lim_{H_{max}\to 0} \sup\{\|v_H - R_H\varphi\|_{0,H}\} \leq \epsilon.$$

A sequence $(v_H)_\Lambda \in \Pi\ \overset{\circ}{W}{}^{1,2}_H(\Omega)$ converges discretely to $v \in W_0^{1,2}(\Omega)$ in $(W_0^{1,2}(\Omega), \Pi\ \overset{\circ}{W}{}^{1,2}_H(\Omega))$, $v_H \to v$ in $(W_0^{1,2}(\Omega), \Pi\ \overset{\circ}{W}{}^{1,2}_H(\Omega))$ $(H \in \Lambda)$, if for each $\epsilon > 0$ there exists $\varphi \in C_0^\infty(\Omega)$ such that

$$\|v - \varphi\|_{W^{1,2}(\Omega)} \leq \epsilon, \quad \lim_{H_{max}\to 0} \sup\{\|v_H - R_H\varphi\|_{1,H}\} \leq \epsilon.$$

A sequence $(v_H)_\Lambda$ weakly converges to v in $(L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega))$, $v_H \rightharpoonup v$ in $(L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega))$ $(H \in \Lambda)$, if

$$(w_H, v_H)_H \to (w, v)_0 \quad (H \in \Lambda)$$

for all $w \in L^2(\Omega)$ and $(w_H)_\Lambda \in \Pi\ \overset{\circ}{L}{}^2_H(\Omega)$ such that $w_H \to w$ in $(L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega))$.

The following lemma ([1]) is an important technical tool in the stability analysis.

Lemma 1. *The sequence of imbeddings* $(J_H)_\Lambda$*,*

$$J_H : \overset{\circ}{W}{}^{1,2}_H(\Omega) \to \overset{\circ}{L}{}^2_H(\Omega) \quad (H \in \Lambda)$$

is discretely compact.

Lemma 2 ([?]) and Lemma 3 ([1]) give some more information about the imbedding of Lemma 1.

Lemma 2. *Let* $(v_H)_\Lambda \in \Pi\ \overset{\circ}{L}{}^2_H(\Omega)$ *be a bounded sequence. Then there exists a subsequence* $\Lambda' \subset \Lambda$ *and an element* $v \in L^2(\Omega)$ *such that*

$$v_H \rightharpoonup v \quad in \quad (L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda').$$

Lemma 3. *Let* $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}_H^{1,2}(\Omega)$ *be a bounded sequence and an element* $v \in L^2(\Omega)$ *such that*

$$v_H \rightharpoonup v \quad in \quad (L^2(\Omega), \Pi \overset{\circ}{L}{}_H^2(\Omega)) \quad (H \in \Lambda).$$

Then $v \in W_0^{1,2}(\Omega)$.

3. Stability

We introduce the discrete Laplace operator

$$\Delta_H v_H := \delta_x^2 v_H + \delta_y^2 v_H, \quad v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega),$$

and the norm

$$|||v_H|||_{2,H} := \|\Delta_H v_H\|_{0,H}, \quad v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega).$$

Some trivial algebraic manipulations lead to the next result ([2]).

Lemma 4. *The norms* $\|.\|_{2,H}$ *and* $|||.|||_{2,H}$ *are equivalent in* $\overset{\circ}{W}{}_H^{2,2}(\Omega)$.

Let A_H^* be the Hilbert adjoint operator from A_H. The following result gives a stability condition for A_H in the negative norm $\|.\|_{-L}$,

$$\|v_H\|_{-L} := \sup_{0 \neq \varphi_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega)} \frac{|(v_H, \varphi_H)_H|}{\|L\varphi_H\|_{0,H}}, \quad v_H \in \overset{\circ}{L}{}_H^2(\Omega),$$

where $L : \overset{\circ}{W}{}_H^{2,2}(\Omega) \to \overset{\circ}{L}{}_H^2(\Omega)$ is an injective operator.

We notice that for $v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega)$ holds

$$\begin{aligned}
\sup_{0 \neq \varphi_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega)} \frac{|(A_H v_H, \varphi_H)_H|}{\|L\varphi_H\|_{0,H}} &= \sup_{0 \neq w_H \in \overset{\circ}{L}{}_H^2(\Omega)} \frac{|\left(A_H v_H, (A_H^*)^{-1} w_H\right)_H|}{\|L(A_H^*)^{-1} w_H\|_{0,H}} \\
&= C \sup_{0 \neq w_H \in \overset{\circ}{L}{}_H^2(\Omega)} \frac{|(v_H, w_H)_H|}{\|L(A_H^*)^{-1} w_H\|_{0,H}},
\end{aligned}$$

and Lemma 5 follows.

Lemma 5. *Let L be an injective operator defined in $\overset{\circ}{W}{}_H^{2,2}(\Omega)$. If*

$$C\|L(A_H^*)^{-1}w_H\|_{0,H} \le \|w_H\|_{0,H} \quad \forall w_H \in \overset{\circ}{L}{}_H^2(\Omega), \tag{6}$$

for a constant $C > 0$, then

$$C\|v_H\|_{0,H} \le \sup_{0\neq\varphi_H\in\overset{\circ}{W}{}_H^{2,2}(\Omega)} \frac{|(A_H v_H, \varphi_H)_H|}{\|L\varphi_H\|_{0,H}} \quad \forall v_H \in\overset{\circ}{W}{}_H^{2,2}(\Omega). \tag{7}$$

Our aim in this section is to show that the inequality (7) hold for $L = \Delta_H$. For the proof we will use (6).

We first define explicitly the adjoint operator from A_H, $A_H^* :\overset{\circ}{W}{}_H^{2,2}(\Omega) \to\overset{\circ}{L}{}_H^2(\Omega)$,

$$A_H^* := A_H^{(2)*} + A_H^{(1)*},$$

with

$$A_H^{(2)*}v_H \;:=\; -\delta_x\left(a\delta_x v_H\right) - \delta_y\left(c\delta_y v_H\right) \quad \text{in} \quad \Omega_H,$$

$$A_H^{(1)*}v_H \;:=\; -\delta_x\left(\bar{d}M_x^* v_H\right) - \delta_y\left(\bar{e}M_y^* v_H\right) + \bar{f}v_H \quad \text{in} \quad \Omega_H,$$

where

$$(M_x^* v_H)_{j,\ell-1/2} := \frac{v_{j-1/2,\ell-1/2}h_{j-1} + v_{j+1/2,\ell-1/2}h_j}{2h_{j-1/2}}, \tag{8}$$

$$\left(M_y^* v_H\right)_{j-1/2,\ell} := \frac{v_{j-1/2,\ell-1/2}k_{\ell-1} + v_{j-1/2,\ell+1/2}k_\ell}{2k_{\ell-1/2}}, \tag{9}$$

$$A_H^{(1)*}v_H := A_H^{(2)*}v_H := 0 \quad \text{on} \quad \partial\Omega_H.$$

Let us first prove that A_H^* is $\overset{\circ}{W}{}_H^{1,2}(\Omega)$-regular, i.e., there exists $C > 0$ such that

$$\|v_H\|_{1,H} \le C\|A_H^* v_H\|_{0,H} \quad \forall v_H \in\overset{\circ}{W}{}_H^{2,2}(\Omega).$$

This will be made with the aid of lemmas 6–9.

Lemma 6. *If $(v_H)_\Lambda \in \Pi\, \overset{\circ}{W}{}_H^{1,2}(\Omega)$ is a bounded sequence and $\alpha \in C(\bar{\Omega})$ then $\big(M_x(\alpha\delta_x v_H)\big)_\Lambda$ and $\big(M_y(\alpha\delta_y v_H)\big)_\Lambda$ are bounded in $\Pi\, \overset{\circ}{L}{}_H^2(\Omega)$.*

Proof: Since

$$\begin{aligned} & h_{j-1}\big|\big(\alpha\delta_x v_H\big)_{j-1,\ell-1/2} + \big(\alpha\delta_x v_H\big)_{j,\ell-1/2}\big|^2 \\ & \quad \leq \ 4h_{j-3/2}\big|\big(\alpha\delta_x v_H\big)_{j-1,\ell-1/2}\big|^2 + 4h_{j-1/2}\big|\big(\alpha\delta_x v_H\big)_{j,\ell-1/2}\big|^2, \end{aligned}$$

then

$$\|M_x(\alpha\delta_x v_H)\|^2_{0,H} \leq 2\|\alpha\|^2_{L^\infty(\Omega)}\|v_H\|^2_{1,H}.$$

Analogously we have

$$\|M_y(\alpha\delta_y v_H)\|^2_{0,H} \leq 2\|\alpha\|^2_{L^\infty(\Omega)}\|v_H\|^2_{1,H}.$$

∎

Lemma 7. *Let $H \in \Lambda$. Then*

$$\big(-\delta_x(a\delta_x v_H) - \delta_y(c\delta_y v_H), v_H\big)_H \geq C_P\|v_H\|^2_{1,H} \quad \forall v_H \in \overset{\circ}{W}{}^{1,2}_H(\Omega), \tag{10}$$

and

$$\big(A_H v_H, v_H\big)_H \geq C_E\|v_H\|^2_{1,H} - C_K\|v_H\|^2_{0,H} \quad \forall v_H \in \overset{\circ}{W}{}^{1,2}_H(\Omega), \tag{11}$$

with $C_P > 0$, $C_E > 0$ and C_K not depending on H.

Proof: Since a has a lower bound $\underline{a}$ then

$$\begin{aligned} & \underline{a}\sum_{j=0}^{N}\sum_{\ell=1}^{M} h_{j-1/2}k_{\ell-1}\big|\big(\delta_x v_H\big)_{j,\ell-1/2}\big|^2 \\ & \leq \ -\sum_{j=1}^{N}\sum_{\ell=1}^{M} k_{\ell-1}\Big(\big(a\delta_x v_H\big)_{j,\ell-1/2} - \big(a\delta_x v_H\big)_{j-1,\ell-1/2}\Big)\bar{v}_{j-1/2,\ell-1/2} \\ & = \ \big(-\delta_x(a\delta_x v_H), v_H\big)_H. \end{aligned}$$

In the same way we can prove that

$$\underline{c}\sum_{j=1}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}\big|\big(\delta_y v_H\big)_{j-1/2,\ell-1}\big|^2 \leq \big(-\delta_y(c\delta_y v_H), v_H\big)_H.$$

Then (10) follows.

From Lemma 6, there exists $C > 0$ such that

$$\|M_x(d\delta_x v_H) + M_y(e\delta_y v_H)\|_{0,H} \leq C\|v_H\|_{1,H},$$

and then

$$\Big|\big(M_x(d\delta_x v_H) + M_y(e\delta_y v_H) + f v_H, v_H\big)_H\Big| \leq C_L\|v_H\|_{1,H}\|v_H\|_{0,H}.$$

Let $\epsilon > 0$. We can find C_ϵ such that

$$\|v_H\|_{1,H}\|v_H\|_{0,H} \leq \epsilon\|v_H\|^2_{1,H} + C_\epsilon\|v_H\|^2_{0,H}.$$

Taking $\epsilon = \dfrac{C_P}{2C_L}$ we conclude (11) with $C_E = \dfrac{C_P}{2}$ and $C_K = C_L C_\epsilon$.

■

Lemma 8. *Let* $(v_H)_\Lambda \in \Pi \overset{\circ}{W}{}^{1,2}_H(\Omega)$ *and* $v \in W^{1,2}_0(\Omega)$ *such that*

$$v_H \to v \text{ in } (W^{1,2}_0(\Omega), \Pi \overset{\circ}{W}{}^{1,2}_H(\Omega)) \quad (H \in \Lambda)$$

and let $\alpha \in C(\bar{\Omega})$. *Then*

$$M_x(\alpha\delta_x v_H) \to \alpha v_x \text{ and } M_y(\alpha\delta_y v_H) \to \alpha v_y \text{ in } (L^2(\Omega), \Pi \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda). \tag{12}$$

Proof: Let C satisfy $\|\alpha\|_{L^\infty(\Omega)} \leq C$. For any positive real number ϵ there exists $\varphi \in C^\infty(\bar{\Omega})$ such that

$$\|v - \varphi\|_{W^{1,2}(\Omega)} \leq \epsilon, \quad \lim_{H_{max}\to 0} \sup\{\|v_H - R_H\varphi\|_{1,H}\} \leq \frac{1}{4C}\epsilon.$$

Since

$$\|M_x(\alpha\delta_x v_H) - M_x(\alpha\delta_x R_H\varphi)\|_{0,H} \leq 2\|\alpha\|_{L^\infty(\Omega)}\|v_H - R_H\varphi\|_{1,H}$$

and for H_{max} small enough

$$\|M_x(\alpha\delta_x R_H\varphi) - R_H(\alpha\varphi_x)\|_{0,H} \leq \frac{\epsilon}{2},$$

then there exists a final section $H \in \Lambda$ such that

$$\|M_x(\alpha\delta_x v_H) - R_H(\alpha\varphi_x)\|_{0,H} \leq \epsilon.$$

Analogously we prove that

$$\|M_y(\alpha\delta_y v_H) - R_H(\alpha\varphi_y)\|_{0,H} \leq \epsilon.$$

Consequently (12) holds.

■

Lemma 9. *Let $(v_H)_\Lambda$ be a bounded sequence in $\Pi\ \overset{\circ}{W}{}^{1,2}_H(\Omega)$ and $\alpha \in C(\bar{\Omega})$. Then for a subsequence $\Lambda' \subseteq \Lambda$ there exists $v \in W^{1,2}_0(\Omega)$ such that*

$$v_H \to v \ \textit{in}\ (L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda')$$

and the following week convergence hold

$$M_x(\alpha\delta_x v_H) \rightharpoonup \alpha v_x \ \textit{and}\ \ M_y(\alpha\delta_y v_H) \rightharpoonup \alpha v_y \ \textit{in}\ (L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda').$$

Proof: It follows from Lemma 6 that $(M_x(\alpha\delta_x v_H))_\Lambda$ is bounded in $\overset{\circ}{L}{}^2_H(\Omega)$. Taking Lemma 2 into account we have

$$(M_x(\alpha\delta_x v_H))_\Lambda \rightharpoonup w \text{ in } (L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda''),$$

for a subsequence $\Lambda'' \subseteq \Lambda$ and $w \in L^2(\Omega)$. Then for any $\varphi \in C_0^\infty(\Omega)$

$$(R_H\varphi, M_x(\alpha\delta_x v_H))_H \to (\varphi, w)_0 \quad (H \in \Lambda''). \tag{13}$$

From Theorem 1 and Lemma 3, there exists $v \in W^{1,2}_0(\Omega)$ and $\Lambda' \subseteq \Lambda''$, such that

$$v_H \to v \text{ in } (L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda').$$

Let us prove that

$$\delta_x(\alpha M_x^* R_H\varphi) \rightharpoonup (\alpha\varphi)_x \text{ in } (L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda'), \tag{14}$$

with $(M_x^* R_H\varphi)_{j,\ell-1/2}$ given by (8). Let $\psi \in C_0^\infty(\Omega)$. From Lemma 8

$$\begin{aligned} \big(-\delta_x(\alpha M_x^* R_H\varphi), R_H\psi\big)_H &= \big(R_H\varphi, M_x(\alpha\delta_x R_H\psi)\big)_H \\ &\to (\varphi, \alpha\psi_x)_0, \end{aligned}$$

or equivalently,

$$\big(-\delta_x(\alpha M_x^* R_H\varphi), R_H\psi\big)_H \to \big(-(\alpha\varphi)_x, \psi\big)_0. \tag{15}$$

From Theorem 2, there exists $z \in L^2(\Omega)$ such that

$$\delta_x(\alpha M_x^* R_H\varphi) \rightharpoonup z \text{ in } (L^2(\Omega), \Pi\ \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda')$$

and consequently

$$\big(-\delta_x(\alpha M_x^* R_H\varphi), R_H\psi\big)_H \to (-z, \psi)_0. \tag{16}$$

From (15) and (16) we obtain (14).

Since

$$\big(R_H\varphi, M_x(\alpha\delta_x v_H)\big)_H = \big(-\delta_x(\alpha M_x^* R_H\varphi), v_H\big)_H \to \big(-(\alpha\varphi)_x, v\big)_0 = (\varphi, \alpha v_x)_0,$$

using (13) we conclude that

$$M_x(\alpha\delta_x v_H) \rightharpoonup \alpha v_x \text{ in } (L^2(\Omega), \Pi \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda').$$

We prove

$$M_y(\alpha\delta_y v_H) \rightharpoonup \alpha v_y \text{ in } (L^2(\Omega), \Pi \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda')$$

analogously.

■

Theorem 1. *There exists a final sequence $\Lambda' \subset \Lambda$ and C not depending on H such that*

$$\|v_H\|_{1,H} \le C\|A_H^* v_H\|_{0,H} \quad \forall v_H \in \overset{\circ}{W}{}^{1,2}_H(\Omega), \tag{17}$$

$H \in \Lambda'$.

Proof: Assuming (17) not to hold we can find a subsequence $\Lambda'' \subseteq \Lambda$ and elements v_H, $H \in \Lambda''$, such that

$$\|v_H\|_{1,H} = 1 \text{ and } \|A_H^* v_H\|_{0,H} \to 0 \quad (H \in \Lambda''). \tag{18}$$

Lemma 1 and Lemma 3 allow the sequence Λ'' and $v \in W_0^{1,2}(\Omega)$ to be chosen such that

$$v_H \to v \text{ in } (L^2(\Omega), \Pi \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda'').$$

Let $w \in W_0^{1,2}(\Omega)$ be the solution of

$$(aw_x, z_x)_0 + (cw_y, z_y)_0 = \big((dv)_x + (ev)_y + fv, z\big)_0 \quad \forall z \in W_0^{1,2}(\Omega) \tag{19}$$

and $(w_H)_\Lambda \in \Pi \overset{\circ}{W}{}^{1,2}_H(\Omega)$ such that

$$w_H \to w \text{ in } (W_0^{1,2}(\Omega), \Pi \overset{\circ}{W}{}^{1,2}_H(\Omega)) \quad (H \in \Lambda).$$

Let us prove the convergence

$$|z_H|_{1,H} \to 0, \tag{20}$$

for $z_H = v_H - w_H$. Lemma 7 gives the existence of $C > 0$ such that

$$|z_H|_{1,H}^2 \le C\big((A_H^* v_H, z_H)_H + a(w_H, z_H) + c(w_H, z_H) + (v_H, A_H^{(1)} z_H)_H\big), \tag{21}$$

where

$$a(w_H, z_H) := \sum_{j=0}^{N}\sum_{\ell=1}^{M} h_{j-1/2}k_{\ell-1}a(x_j, y_{\ell-1/2})\,(\delta_x w_H)_{j,\ell-1/2}\,(\delta_x \bar{z}_H)_{j,\ell-1/2}$$

and

$$c(w_H, z_H) := \sum_{j=1}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}c(x_{j-1/2}, y_\ell)\,(\delta_y w_H)_{j-1/2,\ell}\,(\delta_y \bar{z}_H)_{j-1/2,\ell}\,.$$

Since $\|A_H^* v_H\|_{0,H} \to 0$ then $(A_H^* v_H, z_H)_H \to 0$. Let $z = v - w$. Our aim is to prove that

$$a(w_H, z_H) \to (a w_x, z_x)_0 \quad (H \in \Lambda''), \tag{22}$$

and

$$c(w_H, z_H) \to (c w_y, z_y)_0 \quad (H \in \Lambda''). \tag{23}$$

Lemma 8 yields

$$M_x(\delta_x w_H) \to w_x \text{ in } (L^2(\Omega), \Pi\, \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda).$$

Since $(z_H)_\Lambda$, $H \in \Lambda''$, is bounded in $\Pi\, \overset{\circ}{W}{}^{1,2}_H(\Omega)$, Lemma 9 allows a subsequence $\Lambda' \subset \Lambda''$ to be chosen such that

$$(M_x(a\delta_x z_H))_\Lambda \rightharpoonup a z_x \text{ in } (L^2(\Omega), \Pi\, \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda'),$$

and consequently (22) holds. We prove (23) analogously.

For the last term of (21) we have

$$(v_H, A_H^{(1)} z_H)_H \to (v, A^{(1)} z)_0 \quad (H \in \Lambda'').$$

Since w is the solution of (19) then

$$a(w_H, z_H) + c(w_H, z_H) + (v_H, A_H^{(1)} z_H)_H \to 0 \quad (H \in \Lambda'')$$

and (20) follows. Then

$$v_H = z_H + w_H \to w \text{ in } (L^2(\Omega), \Pi\, \overset{\circ}{L}{}^2_H(\Omega)) \quad (H \in \Lambda'),$$

and

$$(Aw, z)_0 = 0 \quad \forall z \in W_0^{1,2}(\Omega).$$

For A being injective $\|v_H\|_{1,H} = 1$ is not possible.

■

Let us now prove a stability result for $A_H^{(2)*}$.

Lemma 10. *There exists C not depending on H such that*

$$\|v_H\|_{2,H} \leq C\big(\|A_H^{(2)*}v_H\|_{0,H} + \|v_H\|_{1,H}\big) \quad \forall v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega), \tag{24}$$

$H \in \Lambda$.

Proof: Let $v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega)$. We define

$$\begin{aligned} B_x^{(1)}v_H \;:=\; & \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}(\delta_x^2\bar{v}_H)_{j-1/2,\ell-1/2} \\ & \times\Big[\frac{a(x_{j-1/2},y_{\ell-1/2}) - a(x_{j-1},y_{\ell-1/2})}{h_{j-1}}(\delta_x v_H)_{j-1,\ell-1/2} \\ & +\frac{a(x_j,y_{\ell-1/2}) - a(x_{j-1/2},y_{\ell-1/2})}{h_{j-1}}(\delta_x v_H)_{j,\ell-1/2}\Big] \\ & +\sum_{j=0}^{N}\sum_{\ell=0}^{M} h_{j-1}k_{\ell-1/2}\frac{c(x_j,y_\ell) - c(x_{j-1/2},y_\ell)}{h_{j-1}}(\delta_y v_H)_{j-1/2,\ell}(\delta_{xy}\bar{v}_H)_{j,\ell} \\ & +\sum_{j=0}^{N}\sum_{\ell=0}^{M} h_j k_{\ell-1/2}\frac{c(x_{j+1/2},y_\ell) - c(x_j,y_\ell)}{h_j}(\delta_y v_H)_{j+1/2,\ell}(\delta_{xy}\bar{v}_H)_{j,\ell} \end{aligned}$$

$$\begin{aligned} B_x^{(2)}v_H \;:=\; & \sum_{j=1}^{N}\sum_{\ell=1}^{M} h_{j-1}k_{\ell-1}a(x_{j-1/2},y_{\ell-1/2})|(\delta_x^2 v_H)_{j-1/2,\ell-1/2}|^2 \\ & +\sum_{j=0}^{N}\sum_{\ell=0}^{M} h_{j-1/2}k_{\ell-1/2}c(x_j,y_\ell)|(\delta_{xy}v_H)_{j,\ell}|^2, \end{aligned}$$

$B_y^{(1)}$ and $B_y^{(2)}$ similar to $B_x^{(1)}$ and $B_x^{(2)}$, respectively, replacing a with c, x with y and the indexes in a obvious way. We have

$$\big(A_H^{(2)*}v_H, \delta_x^2 v_H + \delta_y^2 v_H\big)_H = -B_H^{(1)}v_H - B_H^{(2)}v_H, \tag{25}$$

where $B_H^{(1)} := B_x^{(1)} + B_y^{(1)}$ and $B_H^{(2)} := B_x^{(2)} + B_y^{(2)}$.

The conditions assumed for the coefficients a and c give the existence of $C_E > 0$ and $C_L > 0$ such that

$$C_E|v_H|_{2,H}^2 \leq B_H^{(2)}v_H$$

and

$$B_H^{(1)} v_H \le C_L |v_H|_{1,H} |v_H|_{2,H},$$

which together with (25) yield

$$\begin{aligned} C_E |v_H|_{2,H}^2 &\le |\big(A_H^{(2)*} v_H, \delta_x^2 v_H + \delta_y^2 v_H\big)_H| + |B_H^{(1)} v_H| \\ &\le \|A_H^{(2)*} v_H\|_{0,H} |v_H|_{2,H} + C_L |v_H|_{1,H} |v_H|_{2,H}. \end{aligned}$$

Then (24) follows with $C = \max\{1/C_E, C_L/C_E\}$.

■

The main result of this section is the following stability theorem.

Theorem 2. *There exists $C > 0$ and a final section $\Lambda' \subset \Lambda$ such that*

$$\|v_H\|_{0,H} \le C \sup_{0 \neq w_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega)} \frac{|(A_H v_H, w_H)_H|}{\|w_H\|_{2,H}} \quad \forall v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega), \tag{26}$$

$H \in \Lambda'$.

Proof: Let $v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega)$. Since $A_H^{(1)*}$: $\overset{\circ}{W}{}_H^{1,2}(\Omega) \to \overset{\circ}{L}{}_H^2(\Omega)$ is bounded then there exists $C_L > 0$ such that

$$\|A_H^{(2)*} v_H\|_{0,H} \le \|A_H^* v_H\|_{0,H} + \|A_H^{(1)*} v_H\|_{0,H} \le \|A_H^* v_H\|_{0,H} + C_L \|v_H\|_{1,H}.$$

Lemma 10 gives the existence of $C' > 0$ such that

$$\begin{aligned} \|v_H\|_{2,H} &\le C'(\|A_H^{(2)*} v_H\|_{0,H} + \|v_H\|_{1,H}) \\ &\le C' \|A_H^* v_H\|_{0,H} + (C' + C' C_L) \|v_H\|_{1,H}. \end{aligned}$$

From Theorem 1 follows the existence of $C > 0$ such that

$$\|v_H\|_{2,H} \le C \|A_H^* v_H\|_{0,H} \quad \forall v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega). \tag{27}$$

Finally, we observe that (27) is equivalent to

$$\|(A_H^*)^{-1} w_H\|_{2,H} \le C \|w_H\|_{0,H} \quad \forall w_H \in \overset{\circ}{L}{}_H^2(\Omega).$$

■

The estimate (26) can be given the alternative form which uses a negative norm

$$\|v_H\|_{0,H} \le C \|A_H v_H\|_{-\Delta_H} \quad \forall v_H \in \overset{\circ}{W}{}_H^{2,2}(\Omega).$$

4. Convergence

The main result of this paper in Theorem 3 relies in the stability result of Theorem 2. An estimate for $\|R_H u - u_H\|_{0,H}$ will be obtained with the aid of (26) replacing v_H by $R_H u - u_H$ and bounding

$$\big(A_H(R_H u) - M_H(R_{G_H} g), v_H\big)_H.$$

The bounds in lemmas 11–13 are for that purpose.

In what follows we use the notation $\sum\limits_{\Omega_H}$ for the sum over the set of indexes (j,ℓ) such that $(x_{j+1/2}, y_{\ell+1/2}) \in \Omega_H$.

Lemma 11. *Let $u \in H^4(\Omega)$. Then there holds*

$$\begin{aligned} &|\big(-\delta_x(a\delta_x u), v_H\big)_H - \big(M_H R_{G_H}(au_x)_x, v_H\big)_H| \\ &\leq\; C\|a\|_{W^{3,\infty}(\Omega)}\Big(\sum_{\Omega_H}(h_j^2+k_\ell^2)^2\|u\|^2_{H^4((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\|v_H\|_{2,H} \end{aligned} \tag{28}$$

and

$$\begin{aligned} &|\big(-\delta_y(c\delta_y u), v_H\big)_H - \big(M_H R_{G_H}(cu_y)_y, v_H\big)_H| \\ &\leq\; C\|c\|_{W^{3,\infty}(\Omega)}\Big(\sum_{\Omega_H}(h_j^2+k_\ell^2)^2\|u\|^2_{H^4((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\|v_H\|_{2,H}, \end{aligned} \tag{29}$$

for all $v_H \in \overset{\circ}{W}{}^{2,2}_H(\Omega)$.

Proof: Let $v_H \in \overset{\circ}{W}{}^{2,2}_H(\Omega)$. We consider, in first place, only the terms in $\big(\delta_x a\delta_x u, v_H\big)_H$ and $\big(M_H R_{G_H}(au_x)_x, v_H\big)_H$ which have the factor $\bar{v}_{j+1/2,\ell+1/2}$, for some j, with ℓ given. Let us suppose, without loss of generality, that the set of the points in the form $(., y_{\ell+1/2})$ belonging to Ω_H is

$$\{(x_{p_\ell+1/2}, y_{\ell+1/2}), (x_{p_\ell+3/2}, y_{\ell+1/2}), \ldots, (x_{p_\ell+N_\ell-1/2}, y_{\ell+1/2})\}.$$

Let

$$S_1 := \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j k_\ell \big(\delta_x a\delta_x u\big)_{j+1/2,\ell+1/2}\bar{v}_{j+1/2,\ell+1/2}$$

and

$$S_1^{(1)} := -\sum_{j=p_\ell}^{p_\ell+N_\ell}\Big(\int_{y_\ell}^{y_{\ell+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} a(x_j,y)u_x(x,y)\,dx\,dy\Big)\big(\delta_x\bar v_H\big)_{j,\ell+1/2}.$$

We have

$$\begin{aligned} S_1 &= \sum_{j=p_\ell}^{p_\ell+N_\ell-1} k_\ell\Big(\big(a\delta_x u\big)_{j+1,\ell+1/2}-\big(a\delta_x u\big)_{j,\ell+1/2}\Big)\bar v_{j+1/2,\ell+1/2} \\ &= -\sum_{j=p_\ell}^{p_\ell+N_\ell} h_{j-1/2}k_\ell\big(a\delta_x u\big)_{j,\ell+1/2}\big(\delta_x\bar v_H\big)_{j,\ell+1/2} \\ &= -\sum_{j=p_\ell}^{p_\ell+N_\ell} k_\ell\int_{x_{j-1/2}}^{x_{j+1/2}} a(x_j,y_{\ell+1/2})u_x(x,y_{\ell+1/2})\,dx\big(\delta_x\bar v_H\big)_{j,\ell+1/2}. \end{aligned}$$

The functional

$$\lambda(g) := g\Big(\frac12\Big)-\int_0^1 g(\xi)\,d\xi$$

is bounded in $W^{2,1}(0,1)$ and vanishes for $g=1$ and ξ. Thus the Bramble-Hilbert Lemma (see e.g. [5]) gives the existence of a positive constant C such that

$$|\lambda(g)|\le C\|g''\|_{L^1(0,1)}.$$

From the last estimate applied to $g=w$, where

$$w(\xi) := a(x_j,y_\ell+\xi k_\ell)\int_{x_{j-1/2}}^{x_{j+1/2}} u_x(x,y_\ell+\xi k_\ell)\,dx \quad \xi\in[0,1],$$

follows

$$S_1 = S_1^{(1)}-\sum_{j=p_\ell}^{p_\ell+N_\ell}\int_{y_\ell}^{y_{\ell+1}}\int_{x_{j-1/2}}^{x_{j+1/2}} E_{j,\ell}\big(\delta_x\bar v_H\big)_{j,\ell+1/2},$$

with

$$|E_{j,\ell}|\le Ck_\ell^2\Big|a(x_j,.)\int_{x_{j-1/2}}^{x_{j+1/2}} u_x(x,.)\,dx\Big|_{W^{2,1}((y_\ell,y_{\ell+1}))}.$$

Let

$$S_2 := \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_jk_\ell\big(M_HR_{G_H}(au_x)_x\big)_{j+1/2,\ell+1/2}\bar v_{j+1/2,\ell+1/2},$$

which can be written in the form

$$S_2 \;=\; S_2^{(1)} + \sum_{j=p_\ell}^{p_\ell+N_\ell-1} F_{j,\ell}\,\bar v_{j+1/2,\ell+1/2},$$

where

$$S_2^{(1)} := \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \int_{y_\ell}^{y_{\ell+1}} \int_{x_j}^{x_{j+1}} (au_x)_x(x,y)\,dx\,dy\,\bar v_{j+1/2,\ell+1/2}$$

and

$$F_{j,\ell} := \big(M_H R_{G_H}(au_x)_x\big)_{j+1/2,\ell+1/2} - \int_{y_\ell}^{y_{\ell+1}} \int_{x_j}^{x_{j+1}} (au_x)_x(x,y)\,dx\,dy.$$

$F_{j,\ell}$ can be bounded with the aid of the Bramble-Hilbert Lemma. Let the function w be defined by

$$w(\xi,\eta) := (au_x)_x(x_j+\xi h_j, y_\ell+\eta k_\ell), \quad (\xi,\eta)\in(0,1)\times(0,1).$$

Then

$$F_{j,\ell} \;=\; h_j k_\ell\Big(\frac{w(0,0)+w(1,0)+w(0,1)+w(1,1)}{4} - \int_0^1\int_0^1 w(\xi,\eta)\,d\xi\,d\eta\Big).$$

The functional

$$\lambda(g) := \frac{g(0,0)+g(1,0)+g(0,1)+g(1,1)}{4} - \int_0^1\int_0^1 g(\xi,\eta)\,d\xi\,d\eta,$$

$g\in W^{2,1}((0,1)\times(0,1))$, is bounded and vanishes for $g=1$, ξ and η. Again, by Bramble-Hilbert Lemma the estimate

$$|\lambda(g)| \le C|g|_{W^{2,1}((0,1)\times(0,1))}$$

holds and we obtain the bound

$$\begin{aligned}|F_{j,\ell}| \le C\Big(& h_j^2\|(au_x)_{xxx}\|_{L^1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\\ &+k_\ell h_j\|(au_x)_{xxy}\|_{L^1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} + k_\ell^2\|(au_x)_{xyy}\|_{L^1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big).\end{aligned}$$

Let us finally consider the difference $S_1^{(1)} - S_2^{(1)}$. For $S_2^{(1)}$ we have

$$
\begin{aligned}
S_2^{(1)} &= \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \int_{y_\ell}^{y_{\ell+1}} \Big((au_x)(x_{j+1},y) - (au_x)(x_j,y) \Big)\, dy \bar{v}_{j+1/2,\ell+1/2} \\
&= -\sum_{j=p_\ell}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} h_{j-1/2}(au_x)(x_j,y)\, dy \big(\delta_x \bar{v}_H\big)_{j,\ell+1/2} \\
&= -\sum_{j=p_\ell}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} \int_{x_{j-1/2}}^{x_{j+1/2}} (au_x)(x_j,y)\, dx\, dy \big(\delta_x \bar{v}_H\big)_{j,\ell+1/2}
\end{aligned}
$$

and then

$$
S_1^{(1)} - S_2^{(1)} = (T_1 + T_2)/2 + T_3 + T_4,
$$

with

$$
\begin{aligned}
T_1 \;:=\; & -\sum_{j=p_\ell+1}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{j-1}}{2} \Big(u_x(x_{j-1},y) + u_x(x_j,y) \Big) - \int_{x_{j-1}}^{x_j} u_x(x,y)dx \right] \\
& \times \Big(a(x_{j-1},y)(\delta_x \bar{v}_H)_{j-1,\ell+1/2} + a(x_j,y)(\delta_x \bar{v}_H)_{j,\ell+1/2} \Big)\, dy,
\end{aligned}
$$

$$
\begin{aligned}
T_2 \;:=\; & -\sum_{j=p_\ell+1}^{p_\ell+N_\ell} \int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{j-1}}{2} \Big(u_x(x_j,y) - u_x(x_{j-1},y) \Big) \right. \\
& \left. + \int_{x_{j-1}}^{x_{j-1/2}} u_x(x,y)dx - \int_{x_{j-1/2}}^{x_j} u_x(x,y)dx \right] \\
& \times \Big(a(x_j,y)(\delta_x \bar{v}_H)_{j,\ell+1/2} - a(x_{j-1},y)(\delta_x \bar{v}_H)_{j-1,\ell+1/2} \Big)\, dy,
\end{aligned}
$$

$$
T_3 := -\int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{p_\ell-1}}{2} u_x(x_{p_\ell},y) - \int_{x_{p_\ell-1/2}}^{x_{p_\ell}} u_x(x,y)dx \right] a(x_{p_\ell},y)\, dy (\delta_x \bar{v}_H)_{p_\ell,\ell+1/2},
$$

and

$$
\begin{aligned}
T_4 \;:=\; & -\int_{y_\ell}^{y_{\ell+1}} \left[\frac{h_{p_\ell+N_\ell}}{2} u_x(x_{p_\ell+N_\ell},y) - \int_{x_{p_\ell+N_\ell}}^{x_{p_\ell+N_\ell+1/2}} u_x(x,y)dx \right] a(x_{p_\ell+N_\ell},y)\, dy \\
& \times (\delta_x \bar{v}_H)_{p_\ell+N_\ell,\ell+1/2}.
\end{aligned}
$$

The sum in T_1 contains the errors of the trapezoidal rule that can be bounded with the aid of the Bramble-Hilbert Lemma by

$$\begin{aligned}|T_1| \ \leq \ & C \sum_{j=p_\ell+1}^{p_\ell+N_\ell} h_{j-1}^2 \|u_{xxx}\|_{L^1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \|a\|_{L^\infty((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \\ & \times\Big(|(\delta_x\bar{v}_H)_{j-1,\ell+1/2}| + |(\delta_x\bar{v}_H)_{j,\ell+1/2}|\Big).\end{aligned}$$

For T_2 we have only the first order bound but the factor

$$a(x_j,y)(\delta_x\bar{v}_H)_{j,\ell+1/2} - a(x_{j-1},y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2}$$

allows to estimate T_2 with the same order as T_1. We have

$$\begin{aligned}& a(x_j,y)(\delta_x\bar{v}_H)_{j,\ell+1/2} - a(x_{j-1},y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2} \\ = \ & a(x_{j-1/2},y)\Big((\delta_x\bar{v}_H)_{j,\ell+1/2} - (\delta_x\bar{v}_H)_{j-1,\ell+1/2}\Big) \\ & +\Big(a(x_{j-1/2},y) - a(x_{j-1},y)\Big)(\delta_x\bar{v}_H)_{j-1,\ell+1/2} \\ & +\Big(a(x_j,y) - a(x_{j-1/2},y)\Big)(\delta_x\bar{v}_H)_{j,\ell+1/2} \\ = \ & h_{j-1}a(x_{j-1/2},y)(\delta_x^2\bar{v}_H)_{j-1/2,\ell+1/2} \\ & +\frac{h_{j-1}}{2}\Big(a_x(\eta_1,y)(\delta_x\bar{v}_H)_{j-1,\ell+1/2} + a_x(\eta_2,y)(\delta_x\bar{v}_H)_{j,\ell+1/2}\Big),\end{aligned}$$

for some $\eta_1 \in [x_{j-1}, x_{j-1/2}]$, $\eta_2 \in [x_{j-1/2}, x_j]$, and then

$$\begin{aligned}|T_2| \ \leq \ & C \sum_{j=p_\ell+1}^{p_\ell+N_\ell} h_{j-1}^2 \|u_{xx}\|_{L^1((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \|a\|_{W^{1,\infty}((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))} \\ & \times\Big(|(\delta_x^2\bar{v}_H)_{j-1/2,\ell+1/2}| + |(\delta_x\bar{v}_H)_{j-1,\ell+1/2}| + |(\delta_x\bar{v}_H)_{j,\ell+1/2}|\Big).\end{aligned}$$

For T_3 and T_4 we have

$$|T_3| \leq \int_{y_\ell}^{y_{\ell+1}} \frac{h_{p_\ell-1}}{8} \|u_{xx}(.,y)\|_{L^1((x_{p_\ell-1/2},x_{p_\ell}))} |a(x_{p_\ell},y)|\, dy |(\delta_x\bar{v}_H)_{p_\ell,\ell+1/2}|$$

and

$$\begin{aligned}|T_4| \ \leq \ & \int_{y_\ell}^{y_{\ell+1}} \frac{h_{p_\ell+N_\ell}}{8} \|u_{xx}(.,y)\|_{L^1((x_{p_\ell+N_\ell},x_{p_\ell+N_\ell+1/2}))} |a(x_{p_\ell+N_\ell},y)|\, dy \\ & \times|(\delta_x\bar{v}_H)_{p_\ell+N_\ell,\ell+1/2}|.\end{aligned}$$

Considering the equality

$$(\delta_x \bar{v}_H)_{p_\ell,\ell+1/2} = -\sum_{i=p_\ell}^{j} h_i (\delta_x^2 \bar{v}_H)_{i+1/2,\ell+1/2} + (\delta_x \bar{v}_H)_{j+1,\ell+1/2},$$

$j = p_\ell, \ldots, p_\ell + N_\ell - 1$, follows

$$\begin{aligned} \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j (\delta_x \bar{v}_H)_{p_\ell,\ell+1/2} &= \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j \Big(\sum_{i=p_\ell}^{j} h_i (\delta_x^2 \bar{v}_H)_{i+1/2,\ell+1/2} \Big) \\ &\quad + \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j (\delta_x \bar{v}_H)_{j+1,\ell+1/2}, \end{aligned}$$

and then

$$\begin{aligned} |(\delta_x \bar{v}_H)_{p_\ell,\ell+1/2}| &\leq \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j |(\delta_x^2 \bar{v}_H)_{j+1/2,\ell+1/2}| \\ &\quad + \frac{1}{x_{p_\ell+N_\ell} - x_{p_\ell}} \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j |(\delta_x \bar{v}_H)_{j+1,\ell+1/2}|. \end{aligned}$$

For T_3 we have

$$\begin{aligned} |T_3| &\leq \frac{h_{p_\ell-1}}{8} \|u_{xx}\|_{L^1((x_{p_\ell-1/2},x_{p_\ell})\times(y_\ell,y_{\ell+1}))} \|a(x_{p_\ell},.)\|_{L^\infty((y_\ell,y_{\ell+1}))} \\ &\times \Big(\sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j |(\delta_x^2 \bar{v}_H)_{j+1/2,\ell+1/2}| + \frac{1}{x_{p_\ell+N_\ell} - x_{p_\ell}} \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j |(\delta_x \bar{v}_H)_{j+1,\ell+1/2}| \Big), \end{aligned}$$

and in the same way for T_4 we obtain

$$\begin{aligned} |T_4| &\leq \frac{h_{p_\ell+N_\ell}}{8} \|u_{xx}\|_{L^1((x_{p_\ell+N_\ell},x_{p_\ell+N_\ell+1/2})\times(y_\ell,y_{\ell+1}))} \|a(x_{p_\ell+N_\ell},.)\|_{L^\infty((y_\ell,y_{\ell+1}))} \\ &\times \Big(\sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j |(\delta_x^2 \bar{v}_H)_{j+1/2,\ell+1/2}| + \frac{1}{x_{p_\ell+N_\ell} - x_{p_\ell}} \sum_{j=p_\ell}^{p_\ell+N_\ell-1} h_j |(\delta_x \bar{v}_H)_{j+1,\ell+1/2}| \Big). \end{aligned}$$

Using the Schwarz inequality we obtain (28).
The proof of (29) is analogous.

■

Lemma 12. *Let $u \in H^3(\Omega)$. Then the following estimates hold*

$$|\big(M_x(d\delta_x u), v_H\big)_H - \big(M_H R_{G_H}(du_x), v_H\big)_H| \\ \leq C\|d\|_{W^{2,\infty}(\Omega)}\Big(\sum_{\Omega_H}(h_j^2+k_\ell^2)^2\|u_{xxx}\|^2_{L^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\|v_H\|_{1,H} \tag{30}$$

and

$$|\big(M_y(e\delta_y u), v_H\big)_H - \big(M_H R_{G_H}(eu_y), v_H\big)_H| \\ \leq C\|e\|_{W^{2,\infty}(\Omega)}\Big(\sum_{\Omega_H}(h_j^2+k_\ell^2)^2\|u_{yyy}\|^2_{L^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\|v_H\|_{1,H} \tag{31}$$

for all $v_H \in \overset{\circ}{W}{}^{1,2}_H(\Omega)$.

Proof: Let us consider the terms in $\big(M_x(d\delta_x u), v_H\big)_H$ and $\big(M_H R_{G_H}(du_x), v_H\big)_H$ which have the factor $\bar{v}_{j+1/2,\ell+1/2}$, for some j, with ℓ given. We obtain for $\big(M_x(d\delta_x u), v_H\big)_H$ and $\big(M_H R_{G_H}(du_x), v_H\big)_H$, respectively,

$$\begin{aligned}
&\sum_{j=p_\ell}^{p_\ell+N_\ell-1} k_\ell h_j\big(M_x(d\delta_x u)\big)_{j+1/2,\ell+1/2}\bar{v}_{j+1/2,\ell+1/2}\\
&= \sum_{j=p_\ell}^{p_\ell+N_\ell-1} k_\ell\Big[\sum_{i=p_\ell}^{j} h_i\big(M_x(d\delta_x u)\big)_{i+1/2,\ell+1/2}\\
&\quad -\sum_{i=p_\ell}^{j-1} h_i\big(M_x(d\delta_x u)\big)_{i+1/2,\ell+1/2}\Big]\bar{v}_{j+1/2,\ell+1/2}\\
&= -\sum_{j=p_\ell}^{p_\ell+N_\ell} k_\ell\sum_{i=p_\ell}^{j-1} h_i\big(M_x(d\delta_x u)\big)_{i+1/2,\ell+1/2}(\bar{v}_{j+1/2,\ell+1/2}-\bar{v}_{j-1/2,\ell+1/2})\\
&= -\sum_{j=p_\ell}^{p_\ell+N_\ell} k_\ell h_{j-1/2}\sum_{i=p_\ell}^{j-1} h_i\big(M_x(d\delta_x u)\big)_{i+1/2,\ell+1/2}(\delta_x\bar{v}_H)_{j,\ell+1/2}
\end{aligned}$$

and

$$
\begin{aligned}
&\sum_{j=p_\ell}^{p_\ell+N_\ell-1} k_\ell h_j \big(M_H R_{G_H}(du_x)\big)_{j+1/2,\ell+1/2} \bar{v}_{j+1/2,\ell+1/2} \\
&= \; -\sum_{j=p_\ell}^{p_\ell+N_\ell} k_\ell h_{j-1/2} \sum_{i=p_\ell}^{j-1} h_i \big(M_x(du_x)\big)_{i+1/2,\ell+1/2} (\delta_x \bar{v}_H)_{j,\ell+1/2} \\
&\quad + \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \frac{h_j}{2} k_\ell \Big((E_y)_{j,\ell+1/2} + (E_y)_{j+1,\ell+1/2}\Big) \bar{v}_{j+1/2,\ell+1/2},
\end{aligned}
$$

where

$$
(E_y)_{j,\ell+1/2} := \frac{(du_x)_{j,\ell} + (du_x)_{j,\ell+1}}{2} - (du_x)_{j,\ell+1/2}.
$$

Let $w(\xi) := (du_x)(x_j, y_\ell + \xi k_\ell)$, $\xi \in [0,1]$. Then

$$
(E_y)_{j,\ell+1/2} = \frac{w(0) + w(1)}{2} - w\Big(\frac{1}{2}\Big).
$$

The functional

$$
\lambda(g) := \frac{g(0) + g(1)}{2} - g\Big(\frac{1}{2}\Big)
$$

is bounded in $W^{2,1}(0,1)$ and vanishes for $g = 1$ and ξ. Again by the Bramble-Hilbert Lemma the estimate

$$
|\lambda(g)| \le C \|g''\|_{L^1(0,1)}, \quad g \in W^{2,1}(0,1),
$$

holds and we obtain the bound

$$
\begin{aligned}
&\sum_{j=p_\ell}^{p_\ell+N_\ell-1} \frac{h_j}{2} k_\ell |(E_y)_{j,\ell+1/2} + (E_y)_{j+1,\ell+1/2}| |v_{j+1/2,\ell+1/2}| \\
&\le \; \sum_{j=p_\ell}^{p_\ell+N_\ell-1} \frac{h_j}{2} k_\ell^2 \left(\|\big((du_x)_{xx}\big)(x_j, .)\|_{L^1(I_\ell)} + \|\big((du_x)_{xx}\big)(x_{j+1}, .)\|_{L^1(I_\ell)} \right) \\
&\quad \times |v_{j+1/2,\ell+1/2}|. \qquad (32)
\end{aligned}
$$

We have

$$\begin{aligned}
&\sum_{i=p_\ell}^{j-1} h_i \left[\big(M_x(d\delta_x u)\big)_{i+1/2,\ell+1/2} - \big(M_x(du_x)\big)_{i+1/2,\ell+1/2} \right] \\
&\quad = \sum_{i=p_\ell+1}^{j-1} h_{i-1/2} d_{i,\ell+1/2}\big((\delta_x u)_{i,\ell+1/2} - u_x(x_i, y_{\ell+1/2})\big) \\
&\qquad + \frac{h_{j-1}}{2} d_{j,\ell+1/2}\big((\delta_x u)_{j,\ell+1/2} - u_x(x_j, y_{\ell+1/2})\big) \\
&\qquad + \frac{h_{p_\ell}}{2} d_{p_\ell,\ell+1/2}\big((\delta_x u)_{p_\ell,\ell+1/2} - u_x(x_{p_\ell}, y_{\ell+1/2})\big).
\end{aligned}$$

Using (32) we obtain the bound (30). The proof of (31) is analogous.

■

Lemma 13. *Let* $w \in H^2(\Omega)$. *Then*

$$\begin{aligned}
&|\big(R_H w, v_H\big)_H - \big(M_H R_{G_H} w, v_H\big)_H| \\
&\leq \; C\Big(\sum_{\Omega_H} (h_j^2 + k_\ell^2)^2 \|w\|^2_{H^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2} \|v_H\|_{0,H} \qquad (33)
\end{aligned}$$

for all $v_H \in \overset{\circ}{L}{}^2_H(\Omega)$.

Proof: We can write

$$\begin{aligned}
\big(M_H R_{G_H} w\big)_{j+1/2,\ell+1/2} \; &= \; w_{j+1/2,\ell+1/2} + (E_x)_{j+1/2,\ell} + (E_x)_{j+1/2,\ell+1} \\
&\quad + (E_y)_{j+1/2,\ell+1/2},
\end{aligned}$$

where

$$(E_x)_{j+1/2,\ell} := \frac{w_{j,\ell} + w_{j+1,\ell}}{4} - \frac{w_{j+1/2,\ell}}{2}$$

and

$$(E_y)_{j+1/2,\ell+1/2} := \frac{w_{j+1/2,\ell} + w_{j+1/2,\ell+1}}{2} - w_{j+1/2,\ell+1/2}.$$

Using the Bramble-Hilbert Lemma as before we obtain (33).

■

Let us consider in (33) $w = fu$. We obtain

$$\begin{aligned}
&|\big(fu, v_H\big)_H - \big(M_H R_{G_H}(fu), v_H\big)_H| \\
&\leq \; C\|f\|_{W^{2,\infty}(\Omega)} H^2_{max} \Big(\sum_{\Omega_H} \|u\|^2_{H^2((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2} \|v_H\|_{0,H} \qquad (34)
\end{aligned}$$

for all $v_H \in \overset{\circ}{L}{}^2_H(\Omega)$.

The next convergence theorem follows from Theorem 2 and from the bounds (28), (29), (30), (31) and (34).

Theorem 3. *Let Ω be a union of rectangles. Assume that the solution u of (1)–(2) lies in $H^4(\Omega)$. Then for $H \in \Lambda$, with H_{max} small enough, the discrete problem (3)–(4) has a unique solution u_H which satisfies*

$$\begin{aligned}\|R_H u - u_H\|_{0,H} &\leq C\Big(\sum_{\Omega_H}(h_j^2+k_\ell^2)^2\|u\|^2_{H^4((x_j,x_{j+1})\times(y_\ell,y_{\ell+1}))}\Big)^{1/2}\\ &\leq CH^2_{max}\|u\|_{H^4(\Omega)}.\end{aligned}$$

5. Numerical results

We present numerical results for the problem

$$\begin{aligned}-\Delta u = f \quad &\text{on} \quad \Omega = (0,1)\times(0,1),\\ u = 0 \quad &\text{on} \quad \partial\Omega,\end{aligned}$$

which has the solution $u(x,y) = [x(x-1)y(y-1)]^2$.

Figure 2 shows the numerical solution on 500 random meshes ($N-1\times M-1$ points placed in Ω at random), where N and M ranges from 10 to 110. The logarithm of the norm of the error, $\log(\|R_H u - u_H\|_{0,H})$, is plotted versus the logarithm of the maximum step-size. The straight line is the least-squares fit to the points and has the slope 2.1721, which confirms the estimates given in Theorem 3.

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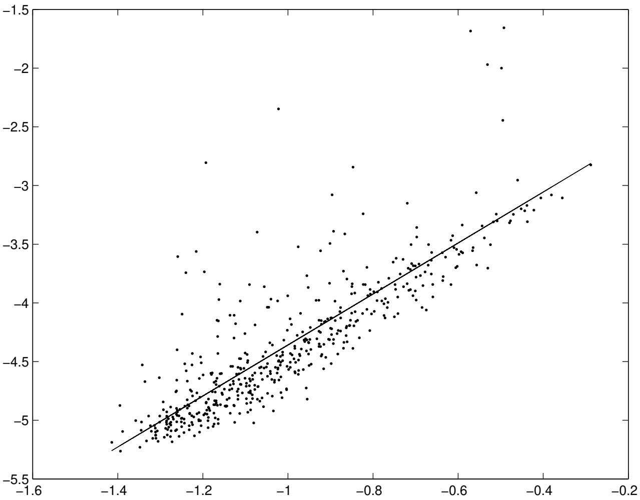

FIGURE 2. $\log(\|R_H u - u_H\|_{0,H})$ *versus* $\log(h_{max})$.

References

[1] S. Barbeiro, 2006, Discretely compact imbeddings in spaces of cell-centerd grid functions, Pré-Publicações do Departamento de Matemática, Universidade de Coimbra, 06-46.

[2] S. Barbeiro, 2005, Normas Duais Discretas em Problemas Elípticos, PhD Thesis, Universidade de Coimbra.

[3] S. Barbeiro, J.A. Ferreira, R.D. Grigorieff, 2005, Supraconvergence of a finite difference scheme for solutions in $H^s(0, L)$, IMA J. Numer. Anal., 25, 797-811.

[4] D. Bojović, B.S. Jovanović, 2001, Fractional order convergence rate estimates of finite difference method on nonuniform meshes, Comput. Methods Appl. Math. 1, 213-221.

[5] J.H. Bramble, S.R. Hilbert, 1970/1971, Bounds for a class of linear functionals with applications to Hermite interpolation, Numer. Math. 16, 362-369.

[6] J.A. Ferreira, R.D. Grigorieff, 2006, Supraconvergence and supercloseness of a scheme for elliptic equations on non-uniform grids, Numerical Functional Analysis and Optimization, 27 (5-6). 539-564.

[7] J.A. Ferreira, 1997, On the convergence on nonrectangular grids, J. Comp. Appl. Math. 85, 333-344.

[8] J.A. Ferreira, R.D. Grigorieff, 1998, On the supraconvergence of elliptic finite difference schemes, Appl. Num. Math. 28, 275-292.

[9] P.A. Forsyth, P.H. Sammon, 1988, Quadratic convergence for cell-centered grids, Appl. Num. Math. 4, 377-394.

[10] B. Garcia-Archila, 1992, A supraconvergent scheme for the Korteweg-de Vries equation, Numer. Math. 61, 292-310.

[11] G. Goodsell, J.R. Whiteman, 1989, A unified treatment of superconvergent recovered gradient functions for piecewise linear finite element approximations, Internat. J. Numer. Methods Engrg., 27, 469-481.

[12] R.D. Grigorieff, 1988, Some stability inequalities for compact finite difference operators, Math. Nachr. 135, 93-101.

[13] F. de Hoog, D. Jackett, 1985, On the rate of convergence of finite difference schemes on nonuniform grids, J. Austral. Math. Soc. Ser. B 26, 247-256.

[14] B.S. Jovanović, L.D. Ivanović, E.E. Süli, 1987, Convergence of finite difference schemes for elliptic equations with variable coefficients, IMA J. Numer. Anal. 7, 301-305.
[15] B.S. Jovanović, B.Z. Popović, 2001, Convergence of a finite difference scheme for the third boundary-value problem for an elliptic equation with variable coefficients, Comput. Methods Appl. Math. 1, 356-366.
[16] H.O. Kreiss, T.A. Manteuffel, B. Swartz, B. Wendroff, A.B. White Jr., 1986, Supraconvergent schemes on irregular grids, Math. Comp. 47, 537-554.
[17] R. Lazarov, V. Makarov, W. Weinelt, 1984, On the convergence of difference schemes for the approximation of solutions $u \in W_2^m$ $(m > 0.5)$ of elliptic equations with mixed derivatives, Numer. Math. 44, 223-232.
[18] P. Lesaint, M. Zlámal, 1979, Superconvergence of the gradient of finite element solutions, R.A.I.R.O. Analyse Numérique 13, 139-166.
[19] C.D. Levermore, T.A. Manteuffel, A.B. White Jr., 1987, Numerical solutions of partial differential equations on irregular grids, Computational techniques and applications: CTAC-87, 417-426, North-Holland, Amsterdam.
[20] N. Levine, 1985, Superconvergent recovery of the gradient from piecewise linear finite element approximations, IMA J. Numer. Anal., 5, 407-427.
[21] T.A. Manteuffel, A.B. White Jr., 1986, The numerical solution of second order boundary value problems on nonuniform meshes, Math. Comp. 47, 511-535.
[22] L.A. Oganesjan, L.A. Ruhovec, 1969, An investigation of the rate of convergence of variation-difference schemes for second order elliptic equations in a two-dimensional region with smooth boundary, Ž. Vyčisl. Mat. i Mat. Fiz., 9, 1102-1120.
[23] T.F. Russell, M.F. Wheeler, 1983, Finite element and finite difference methods for continuous flows in porous media, The Mathematics of Reservoir Simulation, R. E. Ewing, ed., SIAM, Philadelphia, 35-106.
[24] A.A. Samarskii, 1963, Local one dimensional schemes on nonuniform nets, U.S.S.R. Comput. Math. and Math. Phys. 3, 572-619.
[25] M.N. Spijker, 1968, Stability and Convergence of Finite-Difference Methods, Thesis, Leiden.
[26] A.N. Tikhonov, A.A. Samarskii, 1962, Homogeneous difference schemes on nonuniform nets, U.S.S.R. Comput. Math. and Math. Phys. 2, 927-953.
[27] A. Weiser, M.F. Wheeler, 1988, On convergence of block-centered finite differences for elliptic problems, SIAM J. Numer. Anal. 25, 351-375.
[28] M. Zlámal, 1978, Superconvergence and reduced integration in finite element method, Math. Comp. 32, 663-685.

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