Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 06–47

ENERGY ESTIMATES FOR DELAY DIFFUSION-REACTION EQUATIONS

J.A. FERREIRA AND P.M. DA SILVA

Abstract: In this paper we consider nonlinear delay diffusion-reaction equations with initial and Dirichlet boundary conditions. The behavior and the stability of the solution of such initial boundary value problems (IBVPs) are studied using energy method. Simple numerical methods are considered on the computation of numerical approximations to the solution of the nonlinear IBVPs. Using the discrete energy method we study the behavior of the numerical approximations, its stability and its convergence. Numerical experiments illustrating the theoretical results established are also included.

Key words: Delay diffusion reaction equation, energy method, stability, convergence.

Mathematics Subject Classification (2000): 65M06, 65M20, 65M15

1. Introduction

Initial boundary value problems defined by the nonlinear delay diffusion-reaction equation

$$\frac{\partial u}{\partial t}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}(x,t) + f(u(x,t), u(x,t-\tau)), \qquad (x,t) \in (a,b) \times (0,T] \tag{1}$$

where τ is a delay parameter, and by the conditions

$$u(a,t) = u_a(t),\ u(b,t) = u_b(t),\ t \in (0,T], \tag{2}$$

$$u(x,t) = u_0(x,t),\ x \in (a,b),\ t \in [-\tau, 0], \tag{3}$$

or systems of delay diffusion-reaction equations of type (1), are largely used on the description of biological phenomena. The most simple model is the one obtained replacing the diffusion Verhulst equation by the logistic delay equation (1) with the reaction term

$$f(u(x,t), u(t-\tau)) = ru(x,t)\big(1 - \frac{u(x,t-\tau)}{\beta}\big),$$

Received October 24, 2006.
This work was supported by Center for Mathematics of University of Coimbra and by Instituto Superior de Engenharia de Coimbra .

where r and β are positive constants. Others version of equation (1) were also used in the studies of grow population phenomena. For instance, the x-independent version of equation (1) with

$$f(u(x,t), u(x,t-\tau)) = be^{-au(x,t-\tau)-d_1\tau}u(x,y-\tau) - du(x,t),$$

where a, b, d and d_1 are positive parameters, was proposed in [3] when a grow birth population was studied. Equation (1) with

$$f(u(x,t), u(x,t-\tau)) = bu(x,t-\tau)(1-u(x,t)) - cu(x,t)$$

where b and c are positive parameters, is considered in [9] independent of x for epidemic propagations phenomena.

Systems of delay partial differential equations of type (1) have been also used describing mathematically biological phenomena. In [8] the x-independent version of the system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \alpha_1 \dfrac{\partial^2 u_1}{\partial x^2} - R_0 u_1(x,t)u_2(x,t-\tau) + u_2(x,t), \\ \dfrac{\partial u_2}{\partial t} = \alpha_2 \dfrac{\partial^2 u_2}{\partial x^2} + R_0 u_1(x,t)u_2(x,t-\tau) - u_2(x,t), \end{cases} \tag{4}$$

where u_1 and u_2 represent the ratio of susceptible and infected individuals and $\alpha_i, i = 1, 2$, R_0 are positive constants, was used to describe a epidemic propagation.

On biological phenomena context the qualitative properties of the solution of the nonlinear problem (1)-(3) have an important role on the description of the dynamic of the species that we are studying. Such qualitative properties depend on the behavior of reaction terms.

Attending that we are not able to compute explicit expression for the solution of (1)-(3), numerical methods are the only way to get quantitative information to the nonlinear problem (1)-(3).

The study, from analytical and numerical point of view, of delay Cauchy problems or delay IBVPs was very fruitful in the last twenty years as it can be seen, for instance, in the books [1], [2] and in the references contained there. Nevertheless in ours days the study of mathematical models containing delay equations continues to be a fruitful topic. We mention, without be exhaustive, the papers [4],[6], [7] containing the analysis of some biological systems, [5] presenting a qualitative study of the solution of a hyperbolic delay equation, and [10] where spectral collocation methods for a parabolic reaction-diffusion equation of type (1) are studied.

The characterization of the behavior of the solution u of the initial boundary value problem (IBVP) (1)-(3) and the solution u_h^n of their discretizations using the behavior of the reaction term f is the aim of this paper. This characterization has an important role on the description of the behavior of all system.

Using energy method we establish estimates for u and u_h^n that depend on the derivatives of the reaction term f. As a consequence of these estimates we will conclude the stability of the solutions when the initial condition u_0 is perturbed.

The paper is organized as follows. In Section 2 we consider IBVPs (1)-(3) with the reaction term depending only on $u(x, t-\tau)$. In Section 2.1 the behavior os the solution of and its stability are studied. In Section 2.2 a numerical method of the Euler implicit type with centered finite difference discretization of the spatial derivative is considered. We study the behavior of the finite difference solution and a discrete version of the result established in the continuous context is obtained. The stability and the convergence of the numerical method are also proved. The procedures used for the continuous and discrete models with a reaction term dependent on $u(x, t-\tau)$ are easily extended in Section 3 to delay partial differential equations with a reaction term dependent on $u(x,t)$ and $u(x, t-\tau)$. Numerical simulations illustrating the theoretical results obtained in this paper are included in Section 4.

2. Reaction term depending on $u(x, t-\tau)$

2.1. Continuous models. The stability of the IBVP (1)-(3) is studied with respect to the L^2-norm. In what follows we assume the following assumptions:

$$u(x,t) \in [c,d], x \in [a,b] \times [-\tau, T], \tag{5}$$

for some c, d, and

$$f \text{ is continuously differentiable in } [c,d]\,, f(0)=0, \text{ and } f'_{max} = \max_{y\in[c,d]} f'(y). \tag{6}$$

We assume that $T = k\tau$, for some $k \in \mathbb{N}$.

Let v be a function defined in $[a,b] \times [-\tau, T]$, then for each t, v is a function of x which is denote by $v(t)$.

Theorem 1. *Let u be a solution of the IBVP (1)-(3) with homogeneous boundary conditions. Let us suppose that u satisfies (5) and for the reaction f term holds (6).*

If $\dfrac{\partial u}{\partial t}, \dfrac{\partial^j u}{\partial x^\ell} \in L^2, \ell = 1, 2,$ *then, for* $t \in [(m-1)\tau, m\tau] \subseteq [0, T]$*, holds*

$$\|u(t)\|^2_{L^2} \leq e^{\gamma m\tau} \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}, \tag{7}$$

with

$$\gamma = \frac{f'^2_{max}(b-a)^2}{2\alpha}.$$

Proof: Multiplying equation (1) by $u(t)$ it can be show that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} = -\alpha\|\frac{\partial u}{\partial x}\|^2_{L^2} + (f(u(t-\tau)), u(t)). \tag{8}$$

Attending that

$$(f(u(t-\tau)), u(t)) \leq \eta^2 f'^2_{max}\|u(t)\|^2_{L^2} + \frac{1}{4\eta^2}\|u(t-\tau)\|^2_{L^2}, \tag{9}$$

where $\eta \neq 0$ is an arbitrary constant, and considering the Poincaré-Friedrichs inequality, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2} \leq \big(-\frac{\alpha}{(b-a)^2} + \eta^2 f'^2_{max}\big)\|u(t)\|^2_{L^2} + \frac{1}{4\eta^2}\|u(t-\tau)\|^2_{L^2}. \tag{10}$$

Let η be such that

$$-\frac{\alpha}{(b-a)^2} + \eta^2 f'^2_{max} = 0.$$

Then, from (10) we get

$$\frac{d}{dt}\|u(t)\|^2_{L^2} \leq \frac{1}{2\eta^2}\|u(t-\tau)\|^2_{L^2} \tag{11}$$

which is equivalent to

$$\frac{d}{dt}\Big(\|u(t)\|^2_{L^2} - \frac{1}{2\eta^2}\int_0^t \|u(s-\tau)\|^2_{L^2} ds\Big) \leq 0. \tag{12}$$

From (12) we conclude that, for $t \in [0, \tau]$, holds

$$\|u(t)\|^2_{L^2} \leq \big(1 + \tau\frac{f'^2_{max}(b-a)^2}{2\alpha}\big) \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}.$$

The las inequality can be easily extended for $t \in [(m-1)\tau, m\tau]$

$$\|u(t)\|^2_{L^2} \leq \big(1 + \tau\frac{f'^2_{max}(b-a)^2}{2\alpha}\big)^m \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2},$$

which allows us to conclude (7). ■

Let u_1 and u_2 be solutions of IBVP (1)-(3) with initial conditions $u_{0,1}$ and $u_{0,2}$ respectively. Then $w = u_1 - u_2$ satisfies the nonlinear delay equation

$$\frac{\partial w}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + f(u_1(t-\tau)) - f(u_2(t-\tau)).$$

Following the proof of Theorem 1 and attending that $f(u_1(t-\tau)) - f(u_2(t-\tau)) = f'(\xi)w(t-\tau)$, with ξ in the segment with end points $u_1(t-\tau)$, $u_2(t-\tau)$, and w satisfies homogeneous boundary conditions, the next stability result can be proved.

Theorem 2. *Let u_1, u_2 be solutions of the IBVP (1)-(3), with initial conditions $u_{0,1}, u_{0,2}$ respectively. If u_1 and u_2 satisfy (5), then, under the assumptions of Theorem 1, for $t \in [(m-1)\tau, m\tau] \subseteq [0, T]$, we have*

$$\|u_1(t) - u_2(t)\|^2_{L^2} \le e^{\gamma m \tau} \max_{s\in[-\tau,0]} \|u_{0,1}(s) - u_{0,2}(s)\|^2_{L^2}, \tag{13}$$

with

$$\gamma = \frac{f'^2_{max}(b-a)^2}{2\alpha}.$$

■

The behavior of $u_1 - u_2$ is tremendously determined by the the magnitude of $\|u_{0,1} - u_{0,2}\|_{L^2}$ and by the behavior of the reaction term. Nevertheless, independently of f'^2_{max}, if $\|u_{0,1}(s) - u_{0,2}(s)\|_{L^2}$, for $s \in [0, \tau]$, is small enough then $\|u_1(t) - u_2(t)\|_{L^2}$ is also small enough in $[0, T]$. As a consequence of Theorem 2 we conclude that if u_1 and u_2 are solutions of the IBVP (1)- (3) then $u_1 = u_2$.

Finally we remark that Theorem 1 and Theorem 2 can be extended to solutions $u = (u_1, u_2)$ of the system of delay diffusion-reaction equations

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = \alpha_1 \dfrac{\partial^2 u_1}{\partial x^2} + f_1(u_1(t-\tau), u_2(t-\tau)) \\ \dfrac{\partial u_2}{\partial t} = \alpha_2 \dfrac{\partial^2 u_2}{\partial x^2} + f_2(u_1(t-\tau), u_2(t-\tau)) \quad \text{in } (a,b) \times (0,T], \end{cases} \tag{14}$$

with

$$\begin{aligned} &u_1(a,t) = u_{1,a}(t),\ u_1(b,t) = u_{1,b}(t),\ t \in (0,T], \\ &u_2(a,t) = u_{2,a}(t),\ u_2(b,t) = u_{2,b}(t),\ t \in (0,T], \end{aligned} \tag{15}$$

and

$$\begin{aligned} &u_1(x,t) = u_{1,0}(x,t),\ x \in (a,b),\ t \in [-\tau, 0], \\ &u_2(x,t) = u_{2,0}(x,t),\ x \in (a,b),\ t \in [-\tau, 0]. \end{aligned} \tag{16}$$

In fact, assuming that for the components of the solution, $u = (u_1, u_2)$, of the last problem, holds the assumption (5) and for $f_i, i = 1, 2$ holds

$$\begin{aligned} & f_i \text{ is continuously differentiable in } [c,d]\times[c,d],\ f(0,0)=0,\\ & \Big(\frac{\partial f_i}{\partial x}\Big)_{max} = \max_{(x,y)\in[c,d]\times[c,d]} \frac{\partial f_i}{\partial x}, \quad \Big(\frac{\partial f_i}{\partial y}\Big)_{max} = \max_{(x,y)\in[c,d]\times[c,d]} \frac{\partial f_i}{\partial y}, \end{aligned} \tag{17}$$

for $i = 1, 2$, we can prove the next result:

Theorem 3. *Let $u = (u_1, u_2)$ be a solution of the IBVP (14)-(16) with homogeneous boundary condition, such that $u_i, i = 1, 2$, satisfy (5) and $\frac{\partial u_i}{\partial t}, \frac{\partial^\ell u_i}{\partial x^\ell} \in L^2$, $\ell = 1, 2$. If the reaction terms f_i, $i = 1, 2$, satisfy (17) then, for $t \in [(m-1)\tau, m\tau] \subseteq [0, T]$, we have*

$$\|u(t)\|^2_{L^2} \le e^{\gamma m\tau} \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}, \tag{18}$$

where

$$\gamma = \frac{(b-a)^2 \max_{i=1,2}\Big(\big(\frac{\partial f_i}{\partial x}\big)^2_{max} + \big(\frac{\partial f_i}{\partial y}\big)^2_{max}\Big)}{\min_{i=1,2}\alpha_i} \tag{19}$$

and $u_0 = (u_{1,0}, u_{2,0})$. ■

In Theorem 3 we used the notation: if $v = (v_1, v_2)$ is such that $v_i \in L^2$ then $\|v\|^2_{L^2} = \|v_1\|^2_{L^2} + \|v_2\|^2_{L^2}$. The stability of u when the initial conditions $u_{1,0}, u_{2,0}$ are perturbed can be also established. As a consequence of such stability result we conclude that if u and v are solutions of the IBVP (14)-(16) then $u = v$.

2.2. Discrete models. In this section our aim is to study the behavior of numerical approximations to the solutions of IBVPs, considered in Section 2.1, defined by the standard centered implicit Euler method.

In $[a, b]$ we introduce the grid $I_h = \{x_i, i = 0, \dots, N\}$ with $x_0 = a, x_N = b$ and $x_{i+1} = x_i + h, i = 0, \dots, N-1$. Let Δt be the temporal stepsize and let $j \in \mathbb{N}$ be such that $j = \frac{\tau}{\Delta t}$. In $[-\tau, T]$ we consider the grid $\{t_\ell, \ell = -j, \dots, M\}$ defined by

$$t_{-j} = -\tau, t_{\ell+1} = t_\ell + \Delta t,\ \ell = -j, \dots, M-1.$$

Let $u_h^{n+1}(x_i)$ be the fully discrete approximation to $u(x_i, t_{n+1})$ defined by

$$u_h^{n+1}(x_i) = u_h^n(x_i) + \Delta t\alpha D_2 u_h^{n+1} + \Delta t f(u_h^{n+1-j}(x_i)), \tag{20}$$

for $i = 1, \dots, N-1, n = 1, \dots, M-1$, and such that

$$u_h^n(x_0) = u_a(t_n),\ u_h^n(x_N) = u_b(t_n), \quad n = 1, \dots, M, \tag{21}$$

$$u_h^n(x_i) = u_0(x_i, t_n), \quad i = 0, \dots, N, \quad n = -j+1, \dots, 0. \tag{22}$$

In (20) the difference operator D_2 is the usual second order centered finite difference operator

$$D_2 v_h(x_i) = \frac{v_h(x_{i+1}) - 2v_h(x_i) + v_h(x_{i-1})}{h^2} 1,\ i = 1, \dots, N-1.$$

The stability and convergence analysis are established with respect to a L^2 discrete norm which is defined in what follows. By $L^2(I_h)$ we denote the space of grid functions v_h such that $v_h(x_0) = v_h(x_N) = 0$. In $L^2(I_h)$ we introduce the inner product

$$(v_h, w_h)_h = h \sum_{i=1}^{N-1} v_h(x_i) w_h(x_i),\ v_h, w_h \in L^2(I_h). \tag{23}$$

By $\|.\|_{L^2(I_h)}$ we denote the norm induced by the inner product (23).

Let D_{-x} be the usual backward finite difference operator. The following relations

$$(D_2 v_h, w_h)_h = -h \sum_{i=1}^{N} D_{-x} v_h(x_i) D_{-x} w_h(x_i),\ v_h, w_h \in L^2(I_h), \tag{24}$$

$$\|v_h\|^2_{L^2(I_h)} \leq (b-a)^2 \sum_{i=1}^{N} h (D_{-x} v_h(x_i))^2,\ v_h \in L^2(I_h), \tag{25}$$

have a central role on the proof of the main result of this section - Theorem 4. Identity (24) can be proved using summation by parts. The second relation is known as a discrete Poincaré-Friedrichs inequality and it is a discrete version of the well known Poincaré-Friedrichs inequality.

The next result is a discrete version of Theorem 1 and establishes a characterization of the solution of (20), (22) when homogeneous boundary conditions are considered.

Theorem 4. *Let u_h^{n+1} be defined by (20)-(22) with homogeneous boundary conditions and such that $u_h^\ell(x_i) \in [c, d], i = 1, \dots, N-1, \ell = -j+1, \dots, M$. If the reaction term f satisfies (6) then*

$$\|u_h^{n+1}\|^2_{L^2(I_h)} \leq \mathbf{C} \max_{\mu=-j+1,\dots,0} \|u_0(t_\mu)\|^2_{L^2(I_h)},\ n = 0, \dots, M-1, \tag{26}$$

with

$$\mathbf{C} = 1 + T\gamma + \sum_{m=2}^{k} \big((T-(m-1)\tau)\gamma\big)^m, \tag{27}$$

and

$$\gamma = \frac{(b-a)^2 f'^2_{max}}{2\alpha}.$$

Proof: Multiplying (20) by u_h^{n+1} with respect to the inner product $(.,.)_h$, we have

$$\|u_h^{n+1}\|^2_{L^2(I_h)} = (u_h^n, u_h^{n+1})_h + \Delta t\alpha(D_2 u_h^{n+1}, u_h^{n+1})_h + \Delta t(f(u_h^{n+1}), u_h^{n+1})_h, \tag{28}$$

where $f(u_h^{n+1-j})(x_i) = f(u_h^{n+1-j}(x_i)), i = 1, \ldots, N-1$.
Considering in (28) identity (24) with $v_h = w_h = u_h^{n+1}$, the Poincaré-Friedrichs inequality and the Cauchy-Schwarz inequality we obtain

$$\big(\frac{1}{2} + \Delta t\frac{\alpha}{(b-a)^2}\big)\|u_h^{n+1}\|^2_{L^2(I_h)} \leq \frac{1}{2}\|u_h^n\|^2_{L^2(I_h)} + \Delta t(f(u_h^{n+1-j}), u_h^{n+1})_h. \tag{29}$$

Attending that

$$(f(u_h^{n+1-j}), u_h^{n+1})_h \leq \eta^2 f'^2_{max}\|u_h^{n+1}\|^2_{L^2(I_h)} + \frac{1}{4\eta^2}\|u_h^{n+1-j}\|^2_{L^2(I_h)}$$

where $\eta \neq 0$ is an arbitrary constant, from (29), we deduce

$$\big(\frac{1}{2} + \Delta t\big(\frac{\alpha}{(b-a)^2} - \eta^2 f'^2_{max}\big)\big)\|u_h^{n+1}\|^2_{L^2(I_h)} \leq \frac{1}{2}\|u_h^n\|^2_{L^2(I_h)} + \frac{\Delta t}{4\eta^2}\|u_h^{n+1-j}\|^2_{L^2(I_h)}. \tag{30}$$

Fixing η by

$$\eta^2 = \frac{\alpha}{(b-a)^2 f'^2_{max}},$$

we obtain

$$\|u_h^{n+1}\|^2_{L^2(I_h)} \leq \|u_h^n\|^2_{L^2(I_h)} + \Delta t\gamma\|u_h^{n+1-j}\|^2_{L^2(I_h)}, \tag{31}$$

with

$$\gamma = \frac{(b-a)^2 f'^2_{max}}{2\alpha}.$$

We remark that the grids considered in $[-\tau, T]$ are such that $j\Delta t = \tau$, $M\Delta t = T$ and $kj\Delta t = T$ (k is such that $k\tau = T$). In what follows we establish an estimate for $\|u_h^{\ell j}\|_{L^2(I_h)}$ with $\ell \in \{1, \ldots, k-1\}$, being the estimate for $\|u_h^{\ell j+q}\|_{L^2(I_h)}$, with $q \in \{1, \ldots, j\}$, established following the same steps.

From (31) it can be shown that

$$
\begin{aligned}
\|u_h^{\ell j}\|^2_{L^2(I_h)} \leq & \Big(1 + \ell j \Delta t\gamma + \big(\Delta t\gamma\big)^2 \sum_{\mu_1=1}^{(\ell-1)j} \sum_{\mu_2=1}^{\mu_1} 1 \\
& + \big(\Delta t\gamma\big)^3 \sum_{\mu_1=1}^{(\ell-2)j} \sum_{\mu_2=1}^{\mu_1} \sum_{\mu_3=1}^{\mu_2} 1 + \dots \\
& + \big(\Delta t\gamma\big)^\ell \sum_{\mu_1=1}^{j} \sum_{\mu_2=1}^{\mu_1} \cdots \sum_{\mu_\ell=1}^{\mu_{\ell-1}} 1\Big) \max_{n=-j+1,\dots,0} \|u_0(t_n)\|^2_{L^2(I_h)}.
\end{aligned}
\tag{32}
$$

From (32), the following estimate is easily established

$$
\begin{aligned}
\|u_h^{\ell j}\|^2_{L^2(I_h)} \leq & \Big(1 + \ell j \Delta t\gamma \\
& + \sum_{\mu=2}^{\ell} \big((\ell - \mu + 1) j \Delta t\gamma\big)^\mu\Big) \max_{n=-j+1,\dots,0} \|u_0(t_n)\|^2_{L^2(I_h)}.
\end{aligned}
\tag{33}
$$

Inequality (26) with $n+1 = \ell j$ follows from (33) attending that $j\Delta = \tau$ and $\ell j \Delta t \leq T$.

■

We establish in the next result the stability of method (20)-(22).

Theorem 5. *Let $u_h^{n+1}, \tilde{u}_h^{n+1}$ be defined by (20)-(22) with initial conditions u_0 and $\tilde{u}_0$ respectively, and such that $u_h^\ell(x_i), \tilde{u}_h^\ell(x_i) \in [c,d], i = 1,\dots,N-1, \ell = -j+1,\dots,M$. If the reaction term f_i satisfies (6) then*

$$
\|u_h^{n+1} - \tilde{u}_h^{n+1}\|^2_{L^2(I_h)} \leq \mathbf{C} \max_{\mu=-j+1,\dots,0} \|u_0(t_\mu) - \tilde{u}_0(t_\mu)\|^2_{L^2(I_h)},\ n = 0,\dots,M-1,
\tag{34}
$$

with $\mathbf{C}$ defined by (27).

Proof: Let v_h^{n+1} be defined by $v_h^{n+1} = u_h^{n+1} - \tilde{u}_h^{n+1}$. We have

$$
v_h^{n+1}(x_i) = v_h^n(x_i) + \Delta t\alpha D_2 v_h^{n+1} + \Delta t\big(f(u_h^{n+1-j}(x_i)) - f(\tilde{u}_h^{n+1-j}(x_i))\big). \tag{35}
$$

and

$$
v_h^\ell(x_0) = v_h^\ell(x_N) = 0, \ell = 0,\dots,M,
$$

$$
v_h^\ell(x_i) = u_0(x_i,t_\ell) - \tilde{u}_0(x_i,t_\ell), \quad i = 0,\dots,N, \quad \ell = -j+1,\dots,0.
$$

Attending that $f(u_h^{n+1-j}(x_i)) - f(\tilde{u}_h^{n+1-j}(x_i)) = f'(\xi_i) v_h^{n+1-j}$, where ξ_i belongs to the segment with the end points $u_h^{n+1-j}(x_i)$ and $\tilde{u}_h^{n+1-j}(x_i)$, the proof of (34) follows the proof of (26). ■

Theorem 5 establish that method (20) is unconditionally stable with stability coefficient $\mathbf{C}$ defined by (27).

The convergence of method (20) can be shown from the consistence and following the proof of Theorem 4. Let $e_h^{n+1}(x_i) = u(x_i, t_{n+1}) - u_h^{n+1}(x_i), i = 1, \ldots, N-1$, be the global error. This error satisfies the finite difference equation

$$\begin{aligned} e_h^{n+1}(x_i) = e_h^n(x_i) + \Delta t \alpha D_2 e_h^{n+1} + \Delta t\big(f(u(x_i, t_{n+1-j}) - f(u_h^{n+1-j}(x_i))\big) \\ +\Delta t T_h^{n+1}(x_i),\ i = 1, \ldots, N-1, \end{aligned} \tag{36}$$

and

$$e_h^{\mu}(x_i) = 0,\ i = 0, \ldots, N, \quad \mu = -j+1, \ldots, 0, e_h^n(x_0) = e_h^n(x_N) = 0,\ n = 1, \ldots, M. \tag{37}$$

In (36), $T_h^{n+1}(x_i)$ denotes the truncation error at point (x_i, t_{n+1}) which is an $O(\Delta t, h^2)$.

Following the procedure used in the proof of Theorem 4 and attending that

$$(T_h^{n+1}, e_h^{n+1})_h \le \eta^2 \|e_h^{n+1}\|^2_{L^2(I_h)} + \frac{1}{4\eta^2}\|T_h^{n+1}\|^2_{L^2(I_h)},$$

where $\eta \neq 0$ is an arbitrary constant, it can be shown that

$$\begin{aligned} \Big(1 + \Delta t\big(\frac{\alpha}{(b-a)^2} - \eta^2(f'^2_{max} + 1)\big)\|e_h^{n+1}\|^2_{L^2(I_h)} \\ \le \|e_h^n\|^2_{L^2(I_h)} + \Delta t \frac{1}{2\eta^2}\big(\|e_h^{n+1-j}\|^2_{L^2(I_h)} + \|T_h^{n+1}\|^2_{L^2(I_h)}\big) \end{aligned} \tag{38}$$

Fixing in (38) η by

$$\eta^2 = \frac{\alpha}{(b-a)^2(f'^2_{max} + 1)},$$

we obtain

$$\|e_h^{n+1}\|^2_{L^2(I_h)} \le \|e_h^n\|^2_{L^2(I_h)} + \Delta t \frac{(b-a)^2(f'^2_{max} + 1)}{2\alpha}\big(\|e_h^{n+1-j}\|^2_{L^2(I_h)} + \|T_h^{n+1}\|^2_{L^2(I_h)}\big). \tag{39}$$

We establish in what follows an estimate for $\|e_h^{\ell j}\|_{L^2(I_h)}$, with $\ell \in \{1, \ldots, k-1\}$. An analogous estimate for $\|e_h^{\ell j+q}\|_{L^2(I_h)}$, with $q \in \{1, \ldots, j\}$, can be also established following the same procedure.

As, from (39), we have

$$\begin{aligned} \|e_h^{\ell j}\|^2_{L^2(I_h)} &\leq \Delta t\gamma \sum_{\mu_1=1}^{\ell j} \|T_h^{\mu_1}\|^2_{L^2(I_h)} + (\Delta t\gamma)^2 \sum_{\mu_1=1}^{(\ell-1)j} \sum_{\mu_2=1}^{\mu_1} \|T_h^{\mu_2}\|^2_{L^2(I_h)} + \dots \\ &+(\Delta t\gamma)^\ell \sum_{\mu_1=1}^{j} \sum_{\mu_2=1}^{\mu_1} \cdots \sum_{\mu_\ell=1}^{\mu_{\ell-1}} \|T_h^{\mu_\ell}\|^2_{L^2(I_h)} \end{aligned} \tag{40}$$

with

$$\gamma = \frac{(b-a)^2(f'^2_{max}+1)}{2\alpha},$$

for the error $\|e_h^{\ell j}\|_{L^2(I_h)}$ we get the following estimate

$$\|e_h^{\ell j}\|_{L^2(I_h)} \leq \max_{m=1,\dots,M} \|T_h^m\|^2_{L^2(I_h)} \sum_{m=1}^{\ell} \big((\ell-m+1)j\Delta t\gamma\big)^m. \tag{41}$$

Attending that $j\Delta t = \tau$ and $\ell\tau \leq T$, estimate (41) implies

$$\|e_h^{\ell j}\|^2_{L^2(I_h)} \leq \max_{m=1,\dots,M} \|T_h^m\|^2_{L^2(I_h)} \sum_{m=1}^{k} \big((T-(m+1)\tau)\gamma\big)^m. \tag{42}$$

Finally, as $\|e_h^{\ell j+q}\|_{L^2(I_h)}$, with $q \in \{1,\dots,j\}$, is bounded by the upper bound of (42), we conclude the following result:

Theorem 6. *Under the assumptions of Theorem 1 and Theorem 5 for the error* $e_h^{n+1}, n = 1,\dots,M-1$, *we have*

$$\|e_h^{n+1}\|^2_{L^2(I_h)} \leq \mathbf{C}_e \max_{\mu=1,\dots,M} \|T_h^\mu\|^2_{L^2(I_h)},\ n = 0,\dots,M-1, \tag{43}$$

with

$$\mathbf{C}_e = \sum_{m=1}^{k} \big((T-(m-1)\tau)\gamma\big)^m, \tag{44}$$

and

$$\gamma = \frac{(b-a)^2(f'^2_{max}+1)}{2\alpha}.$$

■

If the truncation error is an $O(\Delta t, h^2)$, by Theorem 6, we conclude that $\max_{n=0,\dots,M} \|e_h^{n+1}\|_{L^2(I_h)} = O(\Delta t, h^2)$.

Let us consider now the numerical approximation $u_n^{n+1}(x_i) = (u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i))$ for the solution $u(x_i, t_{n+1}) = (u_1(x_i, t_{n+1}), u_2(x_i, t_{n+1}))$ of the IBVP (14)-(16) defined by

$$\begin{cases} \dfrac{u_{1,h}^{n+1}(x_i) - u_{1,h}^{n}(x_i)}{\Delta t} = \alpha_1 D_2 u_{1,h}^{n+1}(x_i) + f_1(u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)) \\ \dfrac{u_{2,h}^{n+1}(x_i) - u_{2,h}^{n}(x_i)}{\Delta t} = \alpha_2 D_2 u_{2,h}^{n+1}(x_i) + f_2(u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)) \end{cases} \tag{45}$$

for $i = 1, \ldots, N-1, n = 0, \ldots, M-1$, with

$$\begin{array}{l} u_{1,h}^{\ell}(x_0) = u_{1,a}(t_\ell),\ u_{1,h}^{\ell}(x_N) = u_{1,b}(t_\ell), \\ u_{2,h}^{\ell}(x_0) = u_{2,a}(t_\ell),\ u_{2,h}^{\ell}(x_N) = u_{2,b}(t_\ell), \end{array} \tag{46}$$

for $\ell = 1, \ldots, M$, and

$$\begin{array}{l} u_{1,h}^{\ell}(x_i) = u_{1,0}(x_i, t_\ell),\ u_{2,h}^{\ell}(x_i) = u_{2,0}(x_i, t_\ell), \\ i = 1, \ldots, N-1, \ell = -j, \ldots, 0. \end{array} \tag{47}$$

If we assume that the reaction terms $f_i, i = 1, 2$, satisfy (17) and $u_{1,h}^{\ell}(x_i)$, $u_{2,h}^{\ell}(x_i) \in [c, d], i = 0, \ldots, N, \ell = -j, \ldots, M$, then it can be shown

$$\|u_h^{n+1}\|_{L^2(I_h)}^2 \leq \|u_h^n\|_{L^2(I_h)}^2 + \Delta t \gamma \|u_h^{n+1-j}\|_{L^2(I_h)}^2 \tag{48}$$

where

$$\gamma = (b-a)^2 \frac{\min_{i=1,2} \left(\frac{\partial f_i}{\partial x}\right)_{max}^2 + \left(\frac{\partial f_i}{\partial y}\right)_{max}^2}{\min_{i=1,2} \alpha_i}, \tag{49}$$

provided homogeneous boundary conditions are considered. In (48) we use the discrete $L^2 \times L^2$ norm $\|v_h\|_{L^2(I_h)}^2 = \|(v_{1,h}, v_{2,h})\|_{L^2(I_h)}^2 = \|v_{1,h}\|_{L^2(I_h)}^2 + \|v_{2,h}\|_{L^2(I_h)}^2$. Inequality (48) allows us to conclude that for the solution of (45)-(47) with homogeneous Dirichlet boundary conditions holds an inequality analogous to inequality (26) with γ defined by (49).

A stability result and a convergence result analogous to Theorem 5 and Theorem 6 respectively holds for the solution of (45)-(47).

3. Reaction term depending on $u(x,t)$ and $u(x,t-\tau)$

3.1. Continuous models. Let us consider the IBVP (1)-(3) with the reaction term $f(u(t), u(t-\tau))$. We suppose that the solution of the mentioned problem satisfies assumption (5). Assumption (6) is replaced by the following one:

$$\begin{aligned}&f \text{ is continuously differentiable in } [c,d]\times[c,d]\,, f(0,0)=0,\\ &\big(\frac{\partial f}{\partial x}\big)_{max} = \max_{(x,y)\in[c,d]\times[c,d]}\frac{\partial f}{\partial x},\ \big(\frac{\partial f}{\partial y}\big)_{max} = \max_{(x,y)\in[c,d]\times[c,d]}\frac{\partial f}{\partial y}\,.\end{aligned} \tag{50}$$

We prove, in what follows, that for the solution of the IBVP (1)-(3) with the reaction term $f(u(t),u(t-\tau))$ holds an extension of Theorem 1. We start by mention that holds the following inequality

$$(f(u(t),u(t-\tau)),u(t)) \leq \Big(\big(\frac{\partial f}{\partial x}\big)_{max}+\eta^2\big(\frac{\partial f}{\partial y}\big)^2_{max}\Big)\|u(t)\|^2_{L^2}+\frac{1}{4\eta^2}\|u(t-\tau)\|^2_{L^2},$$

where $\eta\neq 0$ is a arbitrary constant.

If the reaction term and the diffusion coefficient satisfy

$$-\big(\frac{\partial f}{\partial x}\big)_{max}+\frac{\alpha}{(b-a)^2}>0\,, \tag{51}$$

then fixing η by

$$\eta^2=\frac{-\big(\frac{\partial f}{\partial x}\big)_{max}+\frac{\alpha}{(b-a)^2}}{\big(\frac{\partial f}{\partial y}\big)^2_{max}},$$

and following the proof of Theorem 1 we conclude (11). Else, it can be shown that

$$\frac{d}{dt}\|u(t)\|^2_{L^2}\leq 2\gamma\|u(t)\|^2_{L^2}+\|u(t-\tau)\|^2_{L^2}, \tag{52}$$

with

$$\gamma=\frac{1}{2}\big(\frac{\partial f}{\partial y}\big)^2_{max}-\frac{\alpha}{(b-a)^2}+\big(\frac{\partial f}{\partial x}\big)_{max}.$$

For (52) we obtain, for $t\in[(m-1)\tau,m\tau]\subseteq[0,T]$, the estimate

$$\|u(t)\|^2_{L^2}\leq(1+\tau)^m e^{2\gamma m\tau}\max_{s\in[-\tau,0]}\|u_0(s)\|^2_{L^2}. \tag{53}$$

We proved the following extension of Theorem 1 which establishes an estimate for the total energy of the solution of the IBVP (1)-(3) when the reaction term f is $u(t)$ and $u(t-\tau)$ dependent:

Theorem 7. *Let u be a solution of the IBVP (1)-(3) with homogeneous boundary conditions and such that (5) holds and* $\frac{\partial u}{\partial t},\frac{\partial^\ell u}{\partial x^\ell}\in L^2,\ell=1,2$. *If* *the reaction term f satisfies (50) then, for $t \in [(m-1)\tau, m\tau] \subseteq [0,T]$, we have*

$$\|u(t)\|^2_{L^2} \leq e^{\gamma m\tau} \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}, \tag{54}$$

where

$$\gamma = \frac{1}{2}\frac{(b-a)^2\big(\frac{\partial f}{\partial y}\big)^2_{max}}{\alpha - (b-a)^2\big(\frac{\partial f}{\partial x}\big)_{max}}$$

provided that the diffusion coefficient and the f satisfy (51), and

$$\|u(t)\|^2_{L^2} \leq e^{\Big(1+2\gamma\Big)m\tau} \max_{s\in[-\tau,0]} \|u_0(s)\|^2_{L^2}. \tag{55}$$

with

$$\gamma = \frac{1}{2}\Big(\frac{\partial f}{\partial y}\Big)^2_{max} - \frac{\alpha}{(b-a)^2} + \Big(\frac{\partial f}{\partial x}\Big)_{max},$$

provided that f does not satisfies (51). ■

We observe that, for IBVP (1)-(3) with a reaction term f depending on $u(t)$ and $u(t-\tau)$, holds a stability result analogous to Theorem 2.

Let us consider now the system (14) where f_i depends on $u_1(t), u_2(t), u_1(t-\tau)$ and $u_2(t-\tau)$, with conditions (15), (16). We suppose that the components of the solution, $u = (u_1, u_2)$, of this problem, satisfy (5) and the reaction terms $f_i, i = 1, 2$, verify the following:

$$\begin{aligned} &f_i \text{ is continuously differentiable in } [c,d]^4\ , f(0,0,0,0) = 0, \\ &\Big(\frac{\partial f_i}{\partial x_\ell}\Big)_{max} = \max_{x\in[c,d]^4} \frac{\partial f_i}{\partial x_\ell},\ i = 1,2, \ell = 1,2,3,4. \end{aligned} \tag{56}$$

In (56) we use the notation $[c,d]^4 = \{x = (x_1, x_2, x_3, x_4) : x_i \in [c,d], i = 1,2,3,4\}$.

Theorem 3 has in the present context the following formulation:

Theorem 8. *Let $u = (u_1, u_2)$ be a solution of the IBVP (14)-(16) with homogeneous boundary conditions and such that $u_i, i = 1,2$, satisfy (5). If the reaction terms $f_i, i = 1,2$, depending on $u_\ell(x,t), u_\ell(x,t-\tau), \ell = 1,2$, satisfy (56) then, for $t \in [(m-1)\tau, m\tau] \subseteq [0,T]$, holds (18) with*

$$\gamma = \frac{\max_{i=1,2}\big(\frac{\partial f_i}{\partial x_3}\big)^2_{max} + \big(\frac{\partial f_i}{\partial x_4}\big)^2_{max}}{\frac{\min_{i=1,2}\alpha_i}{(b-a)^2} - \max_{i=1,2}\big(\frac{\partial f_i}{\partial x_i}\big)_{max} - \frac{1}{2}\Big(\big|\big(\frac{\partial f_1}{\partial x_2}\big)_{max}\big| + \big|\big(\frac{\partial f_2}{\partial x_1}\big)_{max}\big|\Big)}, \tag{57}$$

provided that

$$\frac{\min_{i=1,2}\alpha_i}{(b-a)^2}-\max_{i=1,2}\Big(\frac{\partial f_i}{\partial x_i}\Big)_{max}-\frac{1}{2}\Big(\Big|\Big(\frac{\partial f_1}{\partial x_2}\Big)_{max}\Big|+\Big|\Big(\frac{\partial f_2}{\partial x_1}\Big)_{max}\Big|\Big)>0 \tag{58}$$

and, for $t\in[(m-1)\tau,m\tau]\subseteq[0,T]$, *we have*

$$\|u(t)\|_{L^2}^2\le e^{\Big(1+2\gamma\Big)m\tau}\max_{s\in[-\tau,0]}\|u_0(s)\|_{L^2}^2, \tag{59}$$

with

$$\gamma=\max_{i=1,2}\Big(\frac{\partial f_i}{\partial x_3}\Big)^2_{max}+\Big(\frac{\partial f_i}{\partial x_4}\Big)^2_{max}-\frac{\min_{i=1,2}\alpha_i}{(b-a)^2}+\frac{1}{2}\Big(\Big|\Big(\frac{\partial f_1}{\partial x_2}\Big)_{max}\Big|+\Big|\Big(\frac{\partial f_2}{\partial x_1}\Big)_{max}\Big|\Big)$$
$$+\max_{i=1,2}\Big(\frac{\partial f_i}{\partial x_i}\Big)_{max}, \tag{60}$$

provided the diffusion coefficient and the reaction terms $f_i, i=1,2,$ *do not satisfy (58).* ■

A stability result for the solution of (14)-(16) with reaction terms f_i depending on $u_\ell(x,t), u_\ell(x,t-\tau), \ell=1,2$, when the initial conditions are perturbed can be also established. Such stability result enable us to conclude that if u_1 and u_2 are solutions of the IBVP under consideration then $u_1=u_2$.

3.2. Discrete model. Let us consider now the numerical method (20) with $f(u_h^{n+1-j}(x_i))$ replaced by $f(u_h^{n+1}(x_i),u_h^{n+1-j}(x_i))$. A discrete version of Theorem 7 which can be seen as an extension of Theorem 4 in this case, can be proved. In fact, for homogeneous boundary conditions and considering that the diffusion coefficient and the reaction term satisfy (51), following the proof of Theorem 4, inequality (31) is proved with

$$\gamma=\frac{\Big(\frac{\partial f}{\partial y}\Big)^2_{max}}{2\Big(\frac{\alpha}{(b-a)^2}-\Big(\frac{\partial f}{\partial x}\Big)_{max}\Big)}. \tag{61}$$

Such inequality allows us to conclude (26) with γ defined by (61).
If the diffusion coefficient and the reaction term do not satisfy (51), then inequality (31) is replaced by

$$\|u_h^{n+1}\|^2_{L^2(I_h)}\le\frac{1}{1-2\Delta t\gamma}\Big(\|u_h^n\|^2_{L^2(I_h)}+\Delta t\|u_h^{n+1-j}\|^2_{L^2(I_h)}\Big), \tag{62}$$

provided that Δt is such that

$$1 - 2\gamma\Delta t > 0. \tag{63}$$

In (62) and (63), γ is defined by

$$\gamma = \frac{1}{2}\Big(\frac{\partial f}{\partial y}\Big)^2_{max} + \Big(\frac{\partial f}{\partial x}\Big)_{max} - \frac{\alpha}{(b-a)^2}. \tag{64}$$

From inequality (62) it can be shown that

$$\|u_h^n\|^2_{L^2(I_h)} \leq e^{\frac{2\gamma T}{1-2\gamma\Delta t}}\Big(1+jk\Delta t+\sum_{\ell=2}^{k-1}(j\Delta t)^\ell\sum_{i=0}^{\ell}(k-\ell)^i+(j\Delta t)^k\Big)\max_{\mu=-j,\dots,0}\|u_0(t_\mu)\|^2_{L^2(I_h)}, \tag{65}$$

for $n = 1, \dots, M$. Attending that $j\Delta t = \tau, k\tau = T$, from (65) we conclude the next result:

Theorem 9. *Let u_h^{n+1} be defined by (20)-(22) with $f(u_h^{n+1-j}(x_i))$ replaced by $f(u_h^{n+1}(x_i), u_h^{n+1-j}(x_i))$ and with homogeneous boundary conditions and such that $u_h^\ell(x_i) \in [c,d], i = 1,\dots,N-1, \ell = -j+1,\dots,M$. If the diffusion coefficient and the reaction term satisfy (51) then holds (26) with γ defined by (61). Else holds (51) with*

$$\mathbf{C} = e^{\frac{2\gamma T}{1-2\gamma\Delta t}}\Big(1+T+\sum_{\ell=2}^{k-1}\tau^\ell\sum_{i=0}^{\ell}(k-\ell)^i+\tau^k\Big),$$

γ defined by (64) and provided that Δt satisfies (63). ■

The stability of the method (20) with $f(u_h^{n+1-j}(x_i))$ replaced by $f(u_h^{n+1}(x_i), u_h^{n+1-j}(x_i))$ can be also established.

In what concerns the convergence, if (51) holds then the global error $e_h^n, n = 1,\dots,M$, satisfies inequality (43) with γ defined by

$$\gamma = \frac{\alpha - (b-a)^2\big(\frac{\partial f}{\partial x}\big)_{max}}{(b-a)^2\big(\big(\frac{\partial f}{\partial y}\big)_{max}+1\big)}. \tag{66}$$

If the diffusion coefficient and the reaction term do not satisfy (51), then the global error satisfies the following inequality

$$\|e_h^{n+1}\|^2_{L^2(I_h)} \leq \frac{1}{1-2\gamma\Delta t}\Big(\|e_h^n\|^2_{L^2(I_h)} + \Delta t\|e_h^{n+1-j}\|^2_{L^2(I_h)} + \Delta t\|T_h^{n+1}\|^2_{L^2(I_h)}\Big), \tag{67}$$

for $n = 0, \ldots, M-1$, provided that Δt satisfies (63) with γ defined by

$$\gamma = (\frac{\partial f}{\partial x})_{max} + \frac{1}{2}((\frac{\partial f}{\partial y})_{max} + 1) - \frac{\alpha}{(b-a)^2}. \tag{68}$$

In (67) $T_h^{n+1}(x_i)$ denotes the truncation error at $(x_i, t_{n+1}))$.
Inequality (67) implies for the global error the estimate

$$\|e_h^{n+1}\|^2_{L^2(I_h)} \leq \max_{\mu=1,\ldots,n+1} \|T_h^\mu\|^2_{L^2(I_h)} e^{\frac{2\gamma T}{1-2\gamma\Delta t}} \sum_{m=1}^{k} (T-(m-1)\tau)^m. \tag{69}$$

We proved the following convergence result:

Theorem 10. *Let u be a solution of the IBVP (1)-(3) satisfying (5). Let u_h^{n+1} the the fully discrete approximation defined by (20) with $f(u_h^{n+1-j}(x_i))$ replaced by $f(u_h^{n+1}(x_i), u_h^{n+1-j}(x_i))$ where the reaction term f satisfies (50).*

If (51) holds then the global error $e_h^\ell, \ell = 1, \ldots, M$, satisfies inequality (43) with γ defined by (66). Else the global error satisfies (69) provided that the time step size satisfies (63) with γ defined by (68). ■

Let us consider now the numerical approximation $u_h^{n+1}(x_i) = (u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i))$ defined by (45)-(47) with $f_i(u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i))$ replaced by $f_i(u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i), u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i))$ for $i = 1, 2$. The behavior of u_h^{n+1} can be studied using the arguments considered along this paper, being established results analogous to Theorem 9 and to the convergence result- Theorem 10. We only remark that, in this case, if the condition (58) is satisfied by the reaction terms and the diffusion coefficients then we obtain stability without any condition on the stepsizes. Otherwise, if the condition (58) does not holds then we obtain stability provided that the time stepsize satisfies (63) with γ defined by (60).

4. Numerical results

In this section we consider some numerical results which illustrate the theoretical studies presented in this paper.

Example 1. *Let us consider equation (1) with the reaction term*

$$f(u(x,t), u(t-\tau)) = ru(x,t)\big(1 - u(x,t-\tau)\big),$$

$[a, b] = [0, 100]$, *complemented with the initial condition*

$$u_0(x) = \begin{cases} 1, & x \leq 50 \\ 0, & x > 50, \end{cases}$$

and with the boundary conditions defined by

$$u(0, t) = 1,\ u(100, t) = 0, t \geq 0.$$

The solution of this problem is a traveling wave connecting the stationary states $u = 0$ *with* $u = 1$*. The results were obtained with* $h = 0.1$ *and* $\Delta t = 0.05$*. In Figure 1 we plot the numerical results obtained with method (20) with* $r = 1$ *and* $\tau = 0.2$*. We remark that the reaction term* f *and the diffusion coefficient do not satisfy condition (51) and the time stepsize* Δt *satisfies (63) with* γ *defined by (64).*

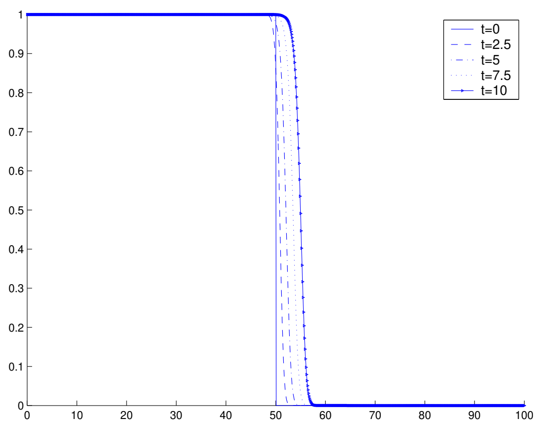

FIGURE 1. Numerical solution obtained with $r = 1$, $\tau = 0.2$.

The behavior of the solution when the delay parameter increases is illustrated in Figure 2. We observe that an increasing of τ *implies a decreasing on the propagation speed of the front.*

An increasing of the reaction parameter r *implies an increasing of the propagation speed of the front. Figure 3 illustrates the previous behavior.*

If $r = 4.8$ *then condition (51) does not holds and for* $\Delta t = 0.05$ *condition (63) does not also holds, then method (20) fails. In Figure 4 we plot the numerical results in the last case.*

Example 2. *The system (4) is considered in what follows with* $[a, b] = [0, 2]$, $R_0 = 5$, $\alpha_1 = \alpha_2 = \alpha$, *the boundary conditions defined by*

$$u_1(0, t) = u_1(2, t) = 0.98, \quad t > 0, u_2(0, t) = u_2(2, t) = 0.02, \quad t > 0,$$

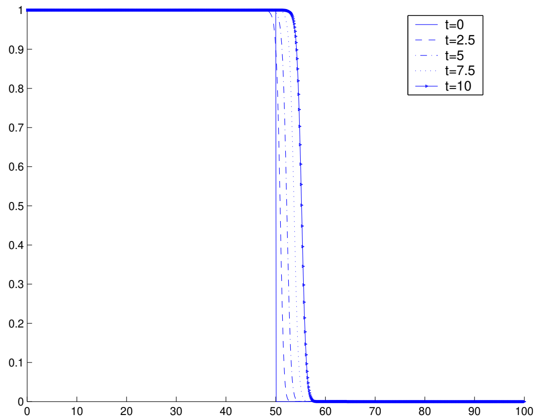

FIGURE 2. Numerical solution obtained with $r = 1$, $\tau = 0.5$.

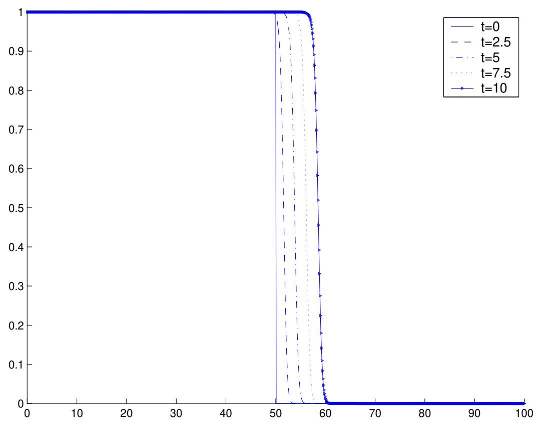

FIGURE 3. Numerical solution obtained with $r = 2$, $\tau = 0.2$.

and with the initial conditions

$$u_1(x,0) = 0.98, \quad x \in [0,2], u_2(x,0) = 0.02, \quad x \in [0,2].$$

The dependent variable u_1 *represents the infected individuals being the susceptible individuals represented by* u_2.

We consider method (45)-(47) with $f_\ell(u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)), \ell = 1,2,$ *replace by*

$$f_1(u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i), u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)) = -R_0 u_{1,h}^{n+1}(x_i) u_{2,h}^{n+1-j}(x_i) + u_{2,h}^{n+1}(x_i),$$

$$f_2(u_{1,h}^{n+1}(x_i), u_{2,h}^{n+1}(x_i), u_{1,h}^{n+1-j}(x_i), u_{2,h}^{n+1-j}(x_i)) = R_0 u_{1,h}^{n+1}(x_i) u_{2,h}^{n+1-j}(x_i) - u_{2,h}^{n+1}(x_i),$$

respectively.

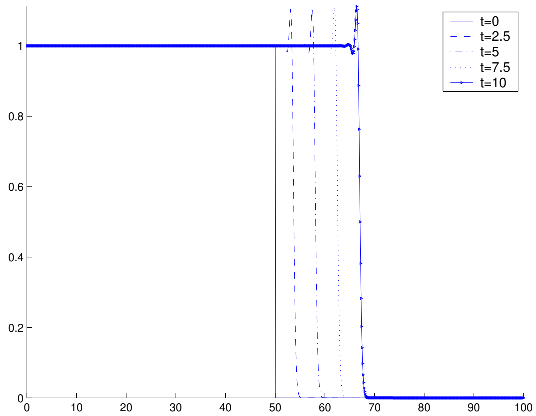

FIGURE 4. Numerical solution obtained with $r = 4.8$, $\tau = 0.2$.

In the numerical experiments we consider $h = 0.1$ *and* $\Delta = 0.05$. *We remark that, in this case, the reaction terms and the diffusion coefficients do not satisfy condition (58) but for the time stepsize holds condition (63) with* γ *defined by (60).*

Figure 5 and Figure 6 illustrate the behavior of the infected and susceptible individuals when the diffusion of all individuals is equal to one.

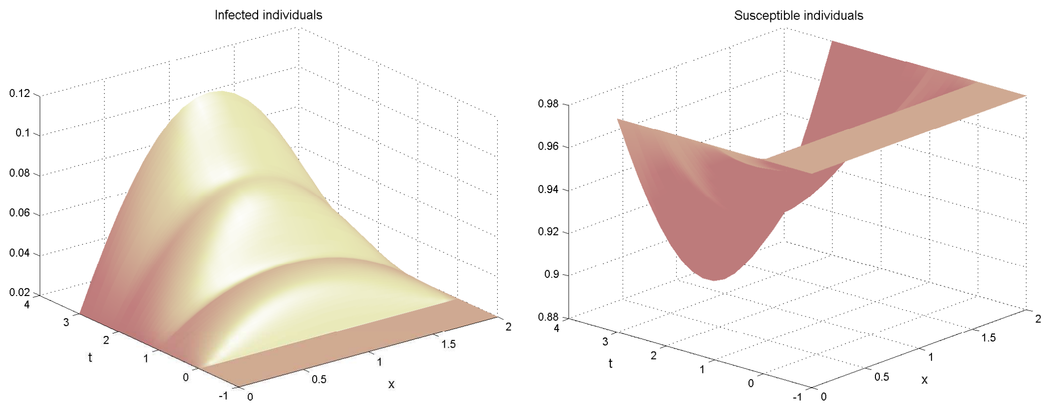

FIGURE 5. Numerical approximation for the infected individuals and for the susceptible individuals with $\alpha = 1$.

The influence of the diffusion coefficient on the dynamics of infected individuals is illustrated in Figure 7.

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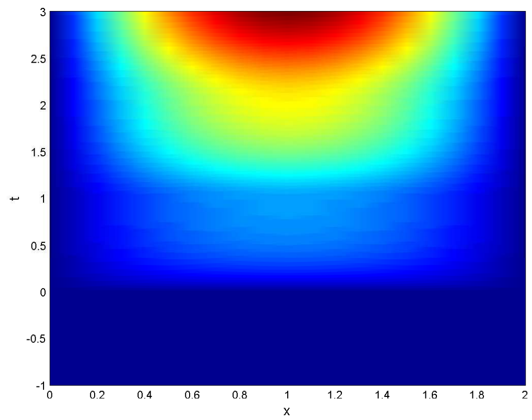

FIGURE 6. Numerical approximation for the infected individuals for $\alpha = 1$.

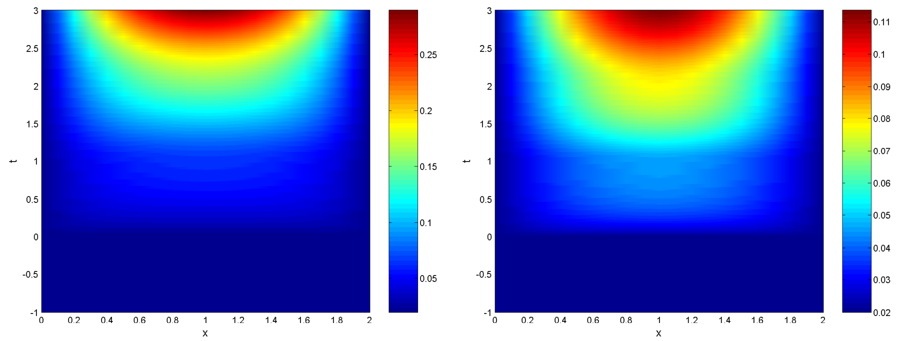

FIGURE 7. Numerical approximation for the infected individuals for $\alpha = 0.2, 1$.

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J.A. Ferreira
Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Portugal

E-mail address: ferreira@mat.uc.pt
URL: http://www.mat.uc.pt/~ferreira

P.M. da Silva
Departamento de Física e Matemática, Instituto Superior de Engenharia de Coimbra, Rua Pedro Nunes-Quinta da Nora, 3030-199 Coimbra, Portugal. email:pascals@isec.pt

E-mail address: pascals@isec.pt