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EXPONENTIABLE FUNCTORS BETWEEN QUANTALOID-ENRICHED CATEGORIES

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Abstract: Exponentiable functors between quantaloid-enriched categories are characterized in elementary terms. The proof goes as follows: the elementary conditions on a given functor translate into existence statements for certain adjoints that obey some lax commutativity; this, in turn, is precisely what is needed to prove the existence of partial products with that functor; so that the functor's exponentiability follows from the works of Niefield [1980] and Dyckhoff and Tholen [1987].

1. Introduction

The study of exponentiable morphisms in a category $\mathcal{C}$, in particular of exponentiable functors between (small) categories (i.e. Conduché fibrations), has a long history; see [Niefield, 2001] for a short account. Recently M. M. Clementino and D. Hofmann [2006] found simple necessary-and-sufficient conditions for the exponentiability of a functor between $\mathcal{V}$-enriched categories, where $\mathcal{V}$ is a symmetric quantale which has its top element as unit for its multiplication and whose underlying sup-lattice is a locale. Our aim here is to prove the following characterization of the exponentiable functors between $\mathcal{Q}$-enriched categories, where now $\mathcal{Q}$ is *any (small) quantaloid*, thus considerably generalizing the aforementioned result of [Clementino and Hofmann, 2006].

Theorem 1.1. *A functor $F\colon \mathbb{A}\longrightarrow\mathbb{B}$ between $\mathcal{Q}$-enriched categories is exponentiable, i.e. the functor "product with F"*

$$-\times F\colon \mathsf{Cat}(\mathcal{Q})/_{\mathbb{B}}\longrightarrow\mathsf{Cat}(\mathcal{Q})/_{\mathbb{B}}$$

admits a right adjoint, if and only if the following two conditions hold:

(1) *for every $a,a'\in\mathbb{A}$ and $\bigvee_i f_i\leq\mathbb{B}(Fa',Fa)$,*

$$\Big(\bigvee_i f_i\Big)\wedge\mathbb{A}(a',a)=\bigvee_i\Big(f_i\wedge\mathbb{A}(a',a)\Big),$$

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(2) *for every* $a, a'' \in \mathbb{A}$, $b' \in \mathbb{B}$, $f \leq \mathbb{B}(b', Fa)$ *and* $g \leq \mathbb{B}(Fa'', b')$,

$$(g \circ f) \wedge \mathbb{A}(a'', a) = \bigvee_{a' \in F^{-1}b'} \Big((g \wedge \mathbb{A}(a'', a')) \circ (f \wedge \mathbb{A}(a', a)) \Big).$$

These conditions are "elementary" in the sense that they are simply equalities (of infima, suprema and compositions) of morphisms in the base quantaloid $\mathcal{Q}$. The second condition is precisely what [Clementino and Hofmann, 2006] had too, albeit in their more restrictive setting; but they did not discover the first condition *an sich*: because it is obviously always true if the base category is a locale.

The proof of our theorem goes as follows. In section 3 we first translate conditions 1.1–1 and 1.1–2 into existence statements for certain adjoints obeying some lax commutativity. Next, in section 4, we show that these latter adjoints are precisely what is needed to prove the existence of partial products in $\mathsf{Cat}(\mathcal{Q})$ over $F\colon \mathbb{A} \longrightarrow \mathbb{B}$. The result then follows from R. Dyckhoff and W. Tholen's [1987] observation, complementary to S. Niefield's [1982] work, that a morphism $f\colon A \longrightarrow B$ in a category $\mathcal{C}$ with finite limits is exponentiable if and only if $\mathcal{C}$ admits partial products over f.

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2. Preliminaries

For the basics on $\mathcal{Q}$-enriched categories we refer to [Stubbe, 2005]; all our notations are as in that paper. Here we shall just observe that $\mathsf{Cat}(\mathcal{Q})$ has pullbacks and a terminal – and therefore all finite limits [Borceux, 1994, Proposition 2.8.2] – and fix some notations.

The terminal object in $\mathsf{Cat}(\mathcal{Q})$, write it as $\mathbb{T}$, has:

- objects: $\mathbb{T}_0 = \mathcal{Q}_0$, with types $tX = X$,
- hom-arrows: $\mathbb{T}(Y, X) = \top_{X,Y}$ = the top element of $\mathcal{Q}(X, Y)$.

For two functors $F\colon \mathbb{A} \longrightarrow \mathbb{C}$ and $G\colon \mathbb{B} \longrightarrow \mathbb{C}$ with common codomain, their pullback $\mathbb{A} \times_{\mathbb{C}} \mathbb{B}$ has:

- objects: $(\mathbb{A} \times_{\mathbb{C}} \mathbb{B})_0 = \{(a, b) \in \mathbb{A}_0 \times \mathbb{B}_0) \mid Fa = Gb\}$ with $t(a, b) = ta = tb$,
- hom-arrows: $(\mathbb{A} \times_{\mathbb{C}} \mathbb{B})((a', b'), (a, b)) = \mathbb{A}(a', a) \wedge \mathbb{B}(b', b)$,

where one computes Φ^F with the aid of *any chosen representing co-span* for Φ, is (well-defined and) the right adjoint in (4). We end by showing that it satisfies the lax commutativity of the diagram in figure 5; thereto it is important that, in the map prescription of (6), any chosen representing co-span for a given distributor will do.

For $b, b', b'' \in \mathbb{B}$, $\Phi\colon \mathbb{A}_b \nrightarrow \mathbb{A}_{b'}$ and $\Psi\colon \mathbb{A}_{b'} \nrightarrow \mathbb{A}_{b''}$, consider the $\mathcal{Q}$-category $\mathbb{C}$ like so:

- objects: $\mathbb{C}_0 = (\mathbb{A}_b)_0 \uplus (\mathbb{A}_{b'})_0 \uplus (\mathbb{A}_{b''})_0$ with "inherited types",
- hom-arrows: for all $a, a_1 \in \mathbb{A}_b$, $a', a'_1 \in \mathbb{A}_{b'}$ and $a'', a''_1 \in \mathbb{A}_{b''}$, put $\mathbb{C}(a_1, a) = \mathbb{A}_b(a_1, a)$, $\mathbb{C}(a'_1, a') = \mathbb{A}_{b'}(a'_1, a')$, $\mathbb{C}(a''_1, a'') = \mathbb{A}_{b''}(a''_1, a'')$, $\mathbb{C}(a', a) = \Phi(a', a)$, $\mathbb{C}(a'', a') = \Psi(a'', a')$ and $\mathbb{C}(a'', a) = (\Psi \otimes \Phi)(a'', a)$, all other hom-arrows are zero.

The co-spans of full embeddings

$$\mathbb{A}_b \xrightarrow{S} \mathbb{C} \xleftarrow{T} \mathbb{A}_{b'} \ , \ \mathbb{A}_{b'} \xrightarrow{T} \mathbb{C} \xleftarrow{U} \mathbb{A}_{b''} \ , \ \mathbb{A}_b \xrightarrow{S} \mathbb{C} \xleftarrow{U} \mathbb{A}_{b''}$$

represent respectively Φ, Ψ and $\Psi \otimes \Phi$. Writing $(\mathbb{P}, P, E)$ for the partial product of F and $\mathbb{C}$, the compostion-inequality

$$\mathbb{P}((b'', U), (b', T)) \circ \mathbb{P}((b', T), (b, S)) \leq \mathbb{P}((b'', U), (b, S))$$

says precisely that $\Psi^F \circ \Phi^F \leq (\Psi \otimes \Phi)^F$, as wanted. □

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Proof: For $b, b' \in \mathbb{B}$ and $\Phi\colon \mathbb{A}_b \nrightarrow \mathbb{A}_{b'}$, choose a representing co-span

$$\mathbb{A}_b \xrightarrow{S} \mathbb{C} \xleftarrow{T} \mathbb{A}_{b'}\ .$$

Considering the partial product of F with $\mathbb{C}$, say $(\mathbb{P}, P, E)$, it is a fact that the hom-arrow $\mathbb{P}((b', T), (b, S))$ is a $\mathcal{Q}$-arrow smaller than $\mathbb{B}(b', b)$. Now, any $\mathcal{Q}$-arrow $f\colon X \longrightarrow Y$ determines a $\mathcal{Q}$-category* $\mathbb{P}_f$ like so:

- objects: $(\mathbb{P}_f)_0 = \{X\} \uplus \{Y\}$ with $tX = X \in \mathcal{Q}$ and $tY = Y \in \mathcal{Q}$,
- hom-arrows: $\mathbb{P}_f(Y, X) = f$, $\mathbb{P}_f(X, X) = 1_X$, $\mathbb{P}_f(Y, Y) = 1_Y$ and $\mathbb{P}_f(X, Y) = 0_{Y,X}$.

The inequality $f \leq \mathbb{B}(b', b)$ holds if and only if

$$P_f\colon \mathbb{P}_f \longrightarrow \mathbb{B}\colon X \mapsto b, Y \mapsto b'$$

is a functor; and similarly the collection of inequalities $f \wedge \mathbb{A}(a', a) \leq \Phi(a', a)$ (one for each $a \in \mathbb{A}_b, a' \in \mathbb{A}_{b'}$) is equivalent to

$$E_f\colon \mathbb{P}_f \times_{\mathbb{B}} \mathbb{A} \longrightarrow \mathbb{C}\colon (X, a) \mapsto a, (Y, a') \mapsto a'$$

being a functor. Using the universal property of the partial product $(\mathbb{P}, P, E)$ one easily checks that P_f and E_f determine and are determined by the single functor

$$K\colon \mathbb{P}_f \longrightarrow \mathbb{P}\colon X \mapsto (b, S), Y \mapsto (b', T),$$

whose functoriality in turn is equivalent to the inequality $f \leq \mathbb{P}((b', T), (b, S))$.

The above argument is actually independent of the chosen representing co-span for Φ: if another co-span

$$\mathbb{A}_b \xrightarrow{S'} \mathbb{C}' \xleftarrow{T'} \mathbb{A}_{b'}$$

also represents Φ, and $(\mathbb{P}', P', E')$ denotes the partial product of F with $\mathbb{C}'$, then the "same" argument shows that, for any $\mathcal{Q}$-arrow $f \leq \mathbb{B}(b', b)$, the collection of inequalities $f \wedge \mathbb{A}(a', a) \leq \Phi(a', a)$ (one for each $a \in \mathbb{A}_b, a' \in \mathbb{A}_{b'}$) is equivalent to the single inequality $f \leq \mathbb{P}'((b', T'), (b, S'))$. Thus it follows that

$$\mathbb{P}((b', T), (b, S)) = \mathbb{P}'((b', T'), (b, S')).$$

As a result the map

$$\mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b'}) \longrightarrow \downarrow \mathbb{B}(b', b)\colon \Phi \mapsto \Phi^F := \mathbb{P}((b', T), (b, S)), \tag{6}$$

*This is actually an instance of the universal representing co-span, when viewing the $\mathcal{Q}$-arrow $f\colon X \longrightarrow Y$ as a one-element distributor $(f)\colon *_X \nrightarrow *_Y$ between one-object $\mathcal{Q}$-categories.

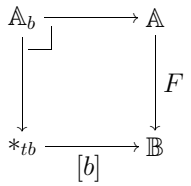

FIGURE 1. a specific pullback

and comes with the obvious projections. All verifications are entirely straightforward.

For an $X \in \mathcal{Q}$, the one-object $\mathcal{Q}$-category with hom-arrow 1_X is written as $*_X$. There is an obvious bijection between the objects of type X in some $\mathcal{Q}$-category $\mathbb{B}$ and the functors from $*_X$ to $\mathbb{B}$. Thus, let $[b]\colon *_{tb} \longrightarrow \mathbb{B}$ stand for the functor "pointing at" $b \in \mathbb{B}$. Given a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and an object $b \in \mathbb{B}$ in its codomain, we shall write $\mathbb{A}_b$ for the pullback in figure 1. That is to say, $\mathbb{A}_b$ has

- objects: $(\mathbb{A}_b)_0 = F^{-1}b = \{a \in \mathbb{A} \mid b = Fa\}$, all of type tb,
- hom-arrows: $\mathbb{A}_b(a', a) = 1_{tb} \wedge \mathbb{A}(a', a)$.

Note that $\mathbb{A}_b = \emptyset$ if and only if $b \notin F(\mathbb{A})$.

3. Adjoints obeying a lax commutativity

In this section we shall translate the elementary conditions in 1.1 into existence statements of certain adjoints obeying some lax commutative diagrams.

Lemma 3.1. *For a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories, the following are equivalent conditions:*

(1) *condition 1.1–1 holds,*
(2) *for every $a, a' \in \mathbb{A}$, the order-preserving map*

$$\downarrow \mathbb{B}(Fa', Fa) \longrightarrow \mathcal{Q}(ta, ta')\colon f \mapsto f \wedge \mathbb{A}(a', a) \tag{1}$$

has a right adjoint,
(3) *for every $b, b' \in F(\mathbb{A})$, the order-preserving map*

$$\downarrow \mathbb{B}(b', b) \longrightarrow \mathsf{Matr}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b'})\colon f \mapsto \Big(f \wedge \mathbb{A}(a', a)\Big)_{(a,a') \in \mathbb{A}_b \times \mathbb{A}_{b'}} \tag{2}$$

has a right adjoint.

(4) *for every* $b, b' \in F(\mathbb{A})$, *the order-preserving map*

$$\downarrow \mathbb{B}(b', b) \longrightarrow \mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b'}) \colon f \mapsto \Big(f \wedge \mathbb{A}(a', a)\Big)_{(a,a') \in \mathbb{A}_b \times \mathbb{A}_{b'}} \tag{3}$$

has a right adjoint.

(5) *for every* $b, b' \in \mathbb{B}$, *the order-preserving map in (3) has a right adjoint.*

Proof : The equivalence of the first two statements is trivial: an order-preserving map between complete lattices has a right adjoint if and only if it preserves arbitrary suprema.

Next, if we use $g \mapsto g^F$ as generic notation for the right adjoints to the maps in (1), then

$$M \mapsto M^F := \bigwedge \{M(a', a)^F \mid (a, a') \in \mathbb{A}_b \times \mathbb{A}_{b'}\}$$

is the right adjoint to the map in (2). Conversely, if $M \mapsto M^F$ is the right adjoint to the map in (2), then for any $a, a' \in \mathbb{A}$

$$g \mapsto g^F := \Big(T^{(a,a')}(g)\Big)^F$$

is the right adjoint to the map in (1), with $T^{(a,a')}(g)$ standing for the $\mathcal{Q}$-matrix from $\mathbb{A}_{Fa}$ to $\mathbb{A}_{Fa'}$ all of whose elements are set to the top element in $\mathcal{Q}(ta, ta')$ except for the element indexed by (a, a') which is set to g.

The equivalence of 3 and 4 follows straightforwardly from two facts: First, the *matrix*

$$\widehat{f} := \Big(f \wedge \mathbb{A}(a', a)\Big)_{a \in \mathbb{A}_b, a' \in \mathbb{A}_{b'}}$$

is always a *distributor* from $\mathbb{A}_b$ to $\mathbb{A}_{b'}$: because for any $a, a_1 \in \mathbb{A}_b$ and $a', a'_1 \in \mathbb{A}_{b'}$ it is automatic that

$$\begin{aligned} \widehat{f}(a', a_1) \circ \mathbb{A}_b(a_1, a) &= \Big(f \wedge \mathbb{A}(a', a_1)\Big) \circ \Big(1_{ta} \wedge \mathbb{A}(a_1, a)\Big) \\ &\leq \Big(f \circ 1_{ta}\Big) \wedge \Big(\mathbb{A}(a', a_1) \circ \mathbb{A}(a_1, a)\Big) \\ &\leq f \wedge \mathbb{A}(a', a) \\ &= \widehat{f}(a', a) \end{aligned}$$

and similarly $\mathbb{A}_{b'}(a'_1, a') \circ \widehat{f}(a', a) \leq \widehat{f}(a', a)$. And second, the inclusion

$$\mathsf{Dist}(\mathcal{Q})(\mathbb{A}_{b'}, \mathbb{A}_b) \longrightarrow \mathsf{Matr}(\mathcal{Q})(\mathbb{A}_{b'}, \mathbb{A}_b) \colon \Phi \mapsto \Phi$$

has both a left and a right adjoint; namely, its left adjoint is $M \mapsto \mathbb{A}_{b'} \circ M \circ \mathbb{A}_b$ and its right adjoint is $M \mapsto [\mathbb{A}_{b'}, \{\mathbb{A}_b, M\}]$. (In both expressions, $\mathbb{A}_{b'}$

Whereas the identity inequality

$$1_{t(b,H)} \leq \mathbb{P}((b, H), (b, H))$$

reduces to the fact that $H \colon \mathbb{A}_b \longrightarrow \mathbb{C}$ is a functor, it is the assumed 3.2–5 together with the composition inequality in the $\mathcal{Q}$-category $\mathbb{C}$ that assures the composition inequality:

$$\mathbb{P}((b'', H''), (b', H')) \circ \mathbb{P}((b', H'), (b, H)) \leq \mathbb{P}((b'', H''), (b, H)).$$

This construction clearly makes P and E functorial. As for the universal property of $(\mathbb{P}, P, E)$, given a $\mathcal{Q}$-category $\mathbb{P}'$ and functors $P' \colon \mathbb{P}' \longrightarrow \mathbb{B}$ and $E' \colon \mathbb{P}' \times_{\mathbb{B}} \mathbb{A} \longrightarrow \mathbb{C}$, it is straightforward to verify that

$$K \colon \mathbb{P}' \longrightarrow \mathbb{P} \colon x \mapsto K(x) := \Big(P'x,\ E'(x, -) \colon \mathbb{A}_{P'x} \longrightarrow \mathbb{C} \colon a \mapsto E'(x, a)\Big)$$

is the required unique factorization. □

Finally we shall show that conditions 3.1–5 and 3.2–5 are not only sufficient but also necessary for $\mathsf{Cat}(\mathcal{Q})$ to admit partial products over $F \colon \mathbb{A} \longrightarrow \mathbb{B}$. Thereto we shall use an auxiliary construction concerning distributors between $\mathcal{Q}$-categories that we better recall beforehand: given a distributor $\Phi \colon \mathbb{X} \nrightarrow \mathbb{Y}$, we shall say that a co-span of functors like

$$\mathbb{X} \xrightarrow{S} \mathbb{C} \xleftarrow{T} \mathbb{Y}$$

represents Φ when $\Phi = \mathbb{C}(T-, S-)$. Any Φ admits at least one such representing co-span: let $\mathbb{C}_0 = \mathbb{X}_0 \uplus \mathbb{Y}_0$ and for all $a, a' \in \mathbb{X}_0$ and $b, b' \in \mathbb{Y}_0$ put $\mathbb{C}(a', a) = \mathbb{X}(a', a)$, $\mathbb{C}(b', b) = \mathbb{Y}(b', b)$, $\mathbb{C}(b, a) = \Phi(b, a)$, $\mathbb{C}(a, b) = 0_{tb,ta}$, so that the co-span of full embeddings

$$\mathbb{X} \xrightarrow{S_{\mathbb{X}}} \mathbb{C} \xleftarrow{S_{\mathbb{Y}}} \mathbb{Y}$$

surely represents Φ. (This latter co-span is universal amongst all representing co-spans for Φ; M. Grandis and R. Paré [1999] speak, in the context of double colimits in double categories, of the cotabulator (or gluing, or collage) of Φ. This is however not important for us here; on the contrary, further on it is crucial to consider non-universal representing co-spans.)

Lemma 4.2. *If* $\mathsf{Cat}(\mathcal{Q})$ *admits partial products over* $F \colon \mathbb{A} \longrightarrow \mathbb{B}$, *then* $F \colon \mathbb{A} \longrightarrow \mathbb{B}$ *satisfies 3.1–5 and 3.1–5.*

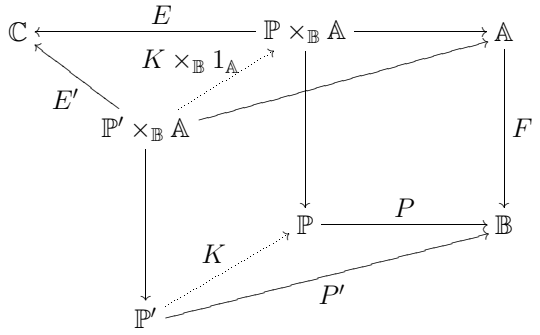

FIGURE 6. the definition of a partial product

having a right adjoint, i.e. to the *exponentiability* of F.

Suppose now that $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ and $\mathbb{C}$ are given, and that we want to construct their partial product $(\mathbb{P}, P, E)$. Putting $\mathbb{P}' = *_X$ in the diagram in figure 6 and letting X range over all objects of $\mathcal{Q}$, the universal property of the partial product dictates at once what the object-set $\mathbb{P}_0$ and the object-maps $P\colon \mathbb{P}_0 \longrightarrow \mathbb{C}_0$ and $E\colon (\mathbb{P} \times_{\mathbb{B}} \mathbb{A})_0 \longrightarrow \mathbb{C}_0$ must be:

- $\mathbb{P}_0 = \{(b, H) \mid b \in \mathbb{B}$ and $H\colon \mathbb{A}_b \longrightarrow \mathbb{C}$ is a functor$\}$, with types $t(b, H) = tb$,
- for $(b, H) \in \mathbb{P}_0$, $P(b, H) = b$,
- for $((b, H), a) \in (\mathbb{P} \times_{\mathbb{B}} \mathbb{A})_0$, $E((b, H), a) = Ha$.

Thus we are left to find a $\mathcal{Q}$-enrichment of the object-set $\mathbb{P}_0$, making it a $\mathcal{Q}$-category $\mathbb{P}$ and making P and E functors with the required universal property; the next lemma tells us how to do this.

Lemma 4.1. *If $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ satisfies 3.1–5 and 3.2–5, then $\mathsf{Cat}(\mathcal{Q})$ admits partial products over $F\colon \mathbb{A} \longrightarrow \mathbb{B}$.*

Proof: Assuming 3.1–5 it makes sense to define

$$\begin{aligned} \mathbb{P}((b', H'), (b, H)) &:= \mathbb{C}(H'-, H-)^F \\ &= \text{the outcome of applying the right adjoint to the map} \\ &\quad \text{in (3) on the distributor } \mathbb{C}(H'-, H-)\colon \mathbb{A}_b \nrightarrow \mathbb{A}_{b'}. \end{aligned}$$

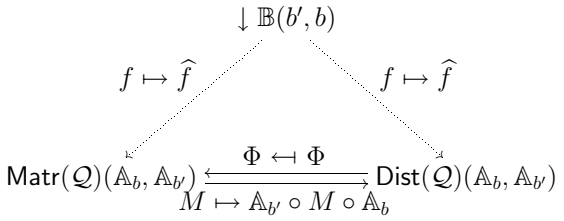

FIGURE 2. a diagram for the proof of 3.1

and $\mathbb{A}_b$ are viewed as monads in the quantaloid $\mathsf{Matr}(\mathcal{Q})$, and we compute composition, resp. lifting and extension, of matrices.) Hence both triangles in figure 2 commute and both solid arrows are left adjoints, so it follows that one dashed arrow is a left adjoint if and only if the other one is.

Finally, the only difference between the fourth and the fifth statement is that in the latter it may be that $\mathbb{A}_b$ or $\mathbb{A}_{b'}$ is empty; but then $\mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b'})$ is a singleton (containing the empty distributor) in which case the right adjoint to (3) always exists. □

In the statement of the next lemma we shall write

$$\downarrow \mathbb{B}(b', b) \underset{\Phi^F \longmapsto \Phi}{\overset{f \mapsto \widehat{f}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b'}) \tag{4}$$

for the adjunctions (one for each pair (b, b') of objects of $\mathbb{B}$) that 3.1–5 alludes to.

Lemma 3.2. *For a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ between $\mathcal{Q}$-categories for which the equivalent conditions in 3.1 hold, the following are equivalent conditions:*

(1) *condition 1.1–2 holds,*
(2) *for every $a, a'' \in \mathbb{A}$ and $b' \in \mathbb{B}$, the diagram in figure 3, in which the horizontal arrows are given by composition (in $\mathsf{Dist}(\mathcal{Q})$, resp. $\mathcal{Q}$), the left vertical arrow is*

$$(f, g) \mapsto \Big((f \wedge \mathbb{A}(a', a))_{a' \in \mathbb{A}_{b'}}, (g \wedge \mathbb{A}(a'', a'))_{a' \in \mathbb{A}_{b'}} \Big) \tag{5}$$

and the right vertical arrow is as in (1), is lax commutative as indicated,

$$\begin{array}{ccc}
\mathsf{Dist}(\mathcal{Q})(*_{ta}, \mathbb{A}_{b'}) \times \mathsf{Dist}(\mathcal{Q})(\mathbb{A}_{b'}, *_{ta''}) & \xrightarrow{- \otimes -} & \mathsf{Dist}(\mathcal{Q})(*_{ta}, *_{ta''}) \\
\uparrow & & \| \\
 & \geq & \mathcal{Q}(ta, ta'') \\
\downarrow \mathbb{B}(b', Fa) \times \downarrow \mathbb{B}(Fa'', b') & \xrightarrow[- \circ -]{} & \downarrow \mathbb{B}(Fa'', Fa) \uparrow
\end{array}$$

FIGURE 3. the diagram for 3.2–2

$$\begin{array}{ccc}
\mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b'}) \times \mathsf{Dist}(\mathcal{Q})(\mathbb{A}_{b'}, \mathbb{A}_{b''}) & \xrightarrow{- \otimes -} & \mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b''}) \\
(\widehat{-}) \times (\widehat{-}) \uparrow & & \uparrow (\widehat{-}) \\
 & \geq & \\
\downarrow \mathbb{B}(b', b) \times \downarrow \mathbb{B}(b'', b') & \xrightarrow[- \circ -]{} & \downarrow \mathbb{B}(b'', b)
\end{array}$$

FIGURE 4. the diagram for 3.2–3 and 3.2–4

(3) *for every* $b, b'' \in F(\mathbb{A})$ *and* $b' \in \mathbb{B}$*, the diagram in figure 4 is lax commutative as indicated,*
(4) *for every* $b, b', b'' \in \mathbb{B}$*, the diagram in figure 4 is lax commutative as indicated,*
(5) *for every* $b, b', b'' \in \mathbb{B}$*, the diagram in figure 5 is lax commutative as indicated.*

Proof : First it is easily verified, in an analogous manner as in the previous proof, that the map in (5) is well-defined, i.e. that we indeed defined *distributors*

$$(f \wedge \mathbb{A}(a', a))_{a' \in \mathbb{A}_{b'}}, \text{ resp. } (g \wedge \mathbb{A}(a'', a'))_{a' \in \mathbb{A}_{b'}}$$

from $*_{ta}$ to $\mathbb{A}_{b'}$, resp. from $\mathbb{A}_{b'}$ to $*_{ta''}$. Now the equivalence of the first two statements is immediate; the "oplax commutativity" of the diagram in figure 3 is always true, thus explaining why in 1.1–2 there is an equality instead of

$$\begin{array}{ccc}
\mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b'}) \times \mathsf{Dist}(\mathcal{Q})(\mathbb{A}_{b'}, \mathbb{A}_{b''}) & \xrightarrow{- \otimes -} & \mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b''}) \\
(-)^F \times (-)^F \downarrow & \leq & \downarrow (-)^F \\
\downarrow \mathbb{B}(b', b) \times \downarrow \mathbb{B}(b'', b') & \xrightarrow[- \circ -]{} & \downarrow \mathbb{B}(b'', b)
\end{array}$$

FIGURE 5. the diagram for 3.2–5

an inequality. That the second and the third statement are equivalent, is because all order-theoretic operations on a distributor are done "elementwise"; and the third and fourth are equivalent because in case $\mathbb{A}_b$ or $\mathbb{A}_{b''}$ is empty, $\mathsf{Dist}(\mathcal{Q})(\mathbb{A}_b, \mathbb{A}_{b''})$ is a singleton, hence all is trivial. Finally, the equivalence of the two last statements follows from the respective vertical arrows being adjoint. □

4. Partial products

In this section we link the conditions in 3.1 and 3.2 on a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ to the existence of so-called partial products in $\mathsf{Cat}(\mathcal{Q})$ with F: this completes the proof of 1.1.

First recall R. Dyckhoff and W. Tholen's [1987] definition (which they gave for any morphism $f\colon A \longrightarrow B$ and any object C in any category $\mathcal{C}$ with finite limits, but here it is for $\mathcal{Q}$-categories): the *partial product* of a functor $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ with a $\mathcal{Q}$-category $\mathbb{C}$ is a $\mathcal{Q}$-category $\mathbb{P}$ together with functors $P\colon \mathbb{P} \longrightarrow \mathbb{B}$, $E\colon \mathbb{P} \times_{\mathbb{B}} \mathbb{A} \longrightarrow \mathbb{C}$ such that, for any other $\mathcal{Q}$-category $\mathbb{P}'$ and functors $P'\colon \mathbb{P}' \longrightarrow \mathbb{B}$, $E'\colon \mathbb{P}' \times_{\mathbb{B}} \mathbb{A} \longrightarrow \mathbb{C}$ there exists a unique functor $K\colon \mathbb{P}' \longrightarrow \mathbb{P}$ satisfying $P \circ K = P'$ and $E \circ (K \times_{\mathbb{B}} 1_{\mathbb{A}}) = E'$ (see figure 6). This is really just the explicit description of the coreflection of $\mathbb{C}$ along the functor "pullback with F"

$$- \times_{\mathbb{B}} \mathbb{A}\colon \mathsf{Cat}(\mathcal{Q})_{/\mathbb{B}} \longrightarrow \mathsf{Cat}(\mathcal{Q}).$$

Hence $\mathsf{Cat}(\mathcal{Q})$ admits all partial products with $F\colon \mathbb{A} \longrightarrow \mathbb{B}$ if and only if this functor has a right adjoint. S. Niefield [1982] proved that this in turn is equivalent to the functor "product with F"

$$- \times F\colon \mathsf{Cat}(\mathcal{Q})_{/\mathbb{B}} \longrightarrow \mathsf{Cat}(\mathcal{Q})_{/\mathbb{B}}$$