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HARDY-TYPE THEOREM FOR ORTHOGONAL FUNCTIONS WITH RESPECT TO THEIR ZEROS. THE JACOBI WEIGHT CASE

L. D. ABREU ⋆, F. MARCELLAN† AND S.B. YAKUBOVICH‡

ABSTRACT: Motivated by G. H. Hardy's 1939 results [4] on functions orthogonal with respect to their real zeros $\lambda_n,\ n = 1, 2, \ldots$, we will consider, within the same general conditions imposed by Hardy, functions satisfying an orthogonality with respect to their zeros with Jacobi weights on the interval $(0, 1)$, that is, the functions $f(z) = z^\nu F(z),\ \nu \in \mathbb{R}$, where F is entire and

$$\int_0^1 f(\lambda_n t)f(\lambda_m t)t^\alpha(1-t)^\beta dt = 0, \quad \alpha > -1-2\nu,\ \beta > -1,$$

when $n \neq m$. Considering all possible functions on this class we are lead to the discovery of a new family of generalized Bessel functions including Bessel and Hyperbessel functions as special cases.

KEYWORDS: Zeros of special functions, Orthogonality, Jacobi weights, Mellin transform on distributions, Entire functions, Bessel functions, Hyperbessel functions.
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1. Introduction

In his 1939 paper [4], G. H. Hardy proved that, under certain conditions, the only functions satisfying

$$\int_0^1 f(\lambda_n t)f(\lambda_m t)dt = 0$$

if $m \neq n$, are the Bessel functions.

With a view to extend Hardy's results to the q-case, it was observed in [1] that a substantial part of Hardy's argument could be carried out without virtually any change, when the measure dt is replaced by an arbitrary positive

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measure $du(t)$ on the interval $(0,1)$ and functions satisfying

$$\int_0^1 f(\lambda_n t) f(\lambda_m t) du(t) = 0 \tag{1}$$

if $m \neq n$, are considered.

In particular, the completeness of the set $\{f(\lambda_n t)\}$ holds for such a general orthogonal system. Furthermore, there exists an associated Lagrange-type sampling theorem for functions defined by integral transforms whose kernel is defined by means of f [1].

It is however not possible to identify, under such a degree of generalization, the whole class of functions satisfying (1). Therefore, we raise the question: Given a specific positive real measure on the interval $(0,1)$, what are the resulting orthogonal functions? In the case where this measure is the $d_q x$ arising from Jackson's q-integral, it was shown in [1] that the corresponding functions are the Jackson q-Bessel functions of the third type. In this note we will answer this question in the case where $du(t) = t^\alpha (1-t)^\beta dt$ is the measure associated with the orthogonal Jacobi polynomials on the interval $(0,1)$. That is, we will characterize in some sense the functions $f(z) = z^\nu F(z)$, where $\nu \in \mathbb{R}$, F is entire, which satisfy the equality

$$\int_0^1 f(\lambda_n t) f(\lambda_m t) t^\alpha (1-t)^\beta dt = 0, \ \alpha > -1-2\nu, \ \beta > -1, \tag{2}$$

with membership of the classes $\mathcal{A}$ or $\mathcal{B}$ being specified in the following definition (the same general restrictions are imposed in [4] and [1]).

Definition 1. *Let $\nu, \alpha \in \mathbb{R}$ be such that $2\nu+\alpha > -1$ and $f(z) = z^\nu F(z)$. The class $\mathcal{A}$ is constituted by entire functions F of order less than two or of order two and minimal type having real positive zeros $\lambda_n > 0, \ n = 1, 2, \ \ldots$ with $\sum \lambda_n^{-2} < \infty$ situated symmetrically about the origin. The class $\mathcal{B}$, in turn, contains entire functions $F(z)$ which have real but not necessarily positive zeros $\lambda_n, \ n = 1, 2, \ \ldots$ with $\sum \lambda_n^{-1} < \infty$ and of order less than one or of order one and minimal type with $F(0) = 1$.*

In this paper we will deal mainly with functions of the class $\mathcal{B}$. Namely, we will show that these functions satisfy the Abel-type integral equation of the second kind [7] and will study their important properties and series representations. However, in the last section we will derive similar integral equation for the class $\mathcal{A}$ and will also give its explicit solutions in terms of the series.

We will appeal in the sequel to the theory of the Mellin transform [6], [7], [8]. As it is known the Mellin direct and inverse transforms are defined by the formulas

$$f^*(s) = \int_0^\infty f(x)x^{s-1}dx, \tag{1.3}$$

$$f(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} f^*(s)x^{-s}ds,\ s=\gamma+i\tau,\ x>0, \tag{1.4}$$

where integrals (1.3)- (1.4) exist as Lebesgue ones if we assume the conditions $f \in L_1(\mathbb{R}_+; x^{\gamma-1}dx)$, $f^* \in L_1((\gamma - i\infty, \gamma + i\infty); d\tau)$, respectively. However functions from the classes $\mathcal{A}$ or $\mathcal{B}$ can have a non- integrable singularity at infinity and the classical Mellin transform (1.3) generally does not exist. Therefore owing to [8], Ch. 4 we introduce the Mellin transform for distributions from the dual space M'_{γ_1,γ_2} to Zemanian's testing -function space M_{γ_1,γ_2} into the space of analytic functions in the open vertical strip $\Omega_f := \{s \in \mathbb{C} : \gamma_1 < \mathrm{Re} s < \gamma_2\}$ by the formula

$$f^*(s) := \langle f(x), x^{s-1}\rangle,\ s \in \Omega_f. \tag{1.5}$$

In this case the inversion integral (1.4) is convergent in $\mathcal{D}'(\mathbb{R}_+)$, i.e. for any smooth function $\theta \in \mathcal{D}(\mathbb{R}_+) \subset M_{\gamma_1,\gamma_2}$ with compact support on $\mathbb{R}_+$ it holds

$$\langle f(x), \theta(x)\rangle = \lim_{r\to\infty} \langle \frac{1}{2\pi i}\int_{\gamma-ir}^{\gamma+ir} f^*(s)x^{-s}ds,\ \theta(x)\rangle. \tag{1.6}$$

2. The Hardy type integral equation

Suppose that $f \in \mathcal{B}$ and satisfies (2). Let

$$A_n = \int_0^1 [f(\lambda_n t)]^2 t^\alpha (1-t)^\beta dt,\ \alpha > -1-2\nu,\ \beta > -1.$$

If $a_n(z)$ stands for the Fourier coefficients of the expansion of $f(zt)$ in terms of the orthonormal basis $\{A_n^{-\frac{1}{2}} f(\lambda_n t)\}$, then it is possible to show, following the arguments in [4], that

$$a_n(z) = \frac{1}{A_n^{1/2}}\int_0^1 f(zt)f(\lambda_n t)t^\alpha(1-t)^\beta dt = \frac{A_n f(z)}{f'(\lambda_n)(z-\lambda_n)}. \tag{2.1}$$

In the sequel we will essentially continue to go along the lines of [4], but since considerable number of adaptations have to be made in order to deal with the presence of the Jacobi weight, we find it better to provide the proof.

Using the Parseval identity for inner products we get

$$\int_0^1 f(zt)f(\zeta t)t^{\alpha}(1-t)^{\beta}dt = \sum_{n=0}^{\infty} a_n(z)a_n(\zeta). \tag{2.2}$$

Theorem 1. *Let $f \in \mathcal{B}$ and satisfy* (2) *with $\alpha > -1-2\nu,\ \nu \in \mathbb{R},\ \beta > -1$. Then it satisfies the integral equation*

$$a\int_0^z u^{\nu+\alpha+1}(z-u)^{\beta}f(u)du = (az+1)\int_0^z u^{\nu+\alpha}(z-u)^{\beta}f(u)du$$

$$+z^{\nu+\alpha+\beta+1}f(z)A, \tag{2.3}$$

where $a = F'(0)$ and $A = -B(2\nu+\alpha+1, \beta+1)$, B denotes the Beta- function [3].

Proof: Substituting (2.1) into (2.2), we obtain, after some simplifications,

$$\int_0^1 f(zt)f(\zeta t)t^{\alpha}(1-t)^{\beta}dt = -f(z)f(\zeta)\frac{q(z)-q(\zeta)}{z-\zeta}, \tag{2.4}$$

where

$$q(z) = \sum_{n=1}^{\infty} \frac{A_n\lambda_n}{\{f'(\lambda_n)\}^2}\left[\frac{1}{z-\lambda_n} + \frac{1}{\lambda_n}\right].$$

Letting $\zeta \to 0$ in (2.4), the result is

$$\int_0^1 t^{\nu+\alpha}f(zt)(1-t)^{\beta}dt = -\frac{f(z)q(z)}{z}.$$

The change of variables $u = zt$ gives

$$\int_0^z u^{\nu+\alpha}(z-u)^{\beta}f(u)du = -z^{\nu+\alpha+\beta}f(z)q(z). \tag{2.5}$$

Now, when z is small enough, $f(z)z^{\nu}$ and $q(z) \sim zq'(0)$. Therefore, when $z \to 0$ we have

$$\int_0^z u^{2\nu+\alpha}(z-u)^{\beta}du \sim -z^{2\nu+\alpha+\beta+1}q'(0)$$

and, as a consequence,

$$q'(0) = -B(2\nu+\alpha+1, \beta+1).$$

Rewriting (2.4) in the form

$$\int_0^1 t^{2\nu+\alpha} F(zt)F(\zeta t)(1-t)^\beta dt = -F(z)F(\zeta)\frac{q(z)-q(\zeta)}{z-\zeta}$$

and differentiating in ζ we obtain

$$\int_0^1 t^{2\nu+\alpha+1} F'(\zeta t)F(zt)(1-t)^\beta dt$$
$$= -F(z)F'(\zeta)\frac{q(z)-q(\zeta)}{z-\zeta} - F(z)F(\zeta)\frac{-q'(\zeta)(z-\zeta)+q(z)-q(\zeta)}{(z-\zeta)^2}.$$

Setting $\zeta = 0$, it gives

$$F'(0)\int_0^1 t^{\nu+\alpha+1}(1-t)^\beta f(zt)dt = -\frac{f(z)}{z^2}\left[q(z)(F'(0)z+1) - Az\right].$$

So, if $a = F'(0)$, the change of variable $zt = u$ yields

$$a\int_0^z u^{\nu+\alpha+1}(z-u)^\beta f(u)du = -z^{\nu+\alpha+\beta} f(z)\left[(za+1)q(z) - Az\right].$$

Combining with (2.5) we get the integral equation (2.3). ■

3. The Hardy-type theorem

In this section we will prove the Hardy type theorem [4], which will characterize all possible orthogonal functions with respect their zeros from the class $\mathcal{B}$ under the Jacobi weight. Indeed, we have

Theorem 2. *Let f satisfy conditions of Theorem 1. Then*

$$f(z) = const.\ z^\nu \sum_{n=0}^\infty a_n z^n, \tag{3.1}$$

where

$$a_n = \frac{(a(\beta+1))^n}{\Gamma(\mu+n)} \prod_{j=1}^n \frac{(\mu)_j}{(\mu+\beta+1)_j - (\mu)_j},\ n = 0, 1, \dots . \tag{3.2}$$

Here $\mu = 2\nu+\alpha+1$, $\Gamma(z)$ is Euler's Gamma-function, $(b)_j$ is the Pochhammer symbol [3] and the empty product is equal to 1. Moreover, series (3.1) represents an entire function of the order $\rho = \frac{1}{\beta+2} < 1$, when $\beta > -1$. In particular, the case $\beta = 0$ drives at the Hardy solutions in terms of the Bessel functions (cf. [4], p.43).

Proof: Making again an elementary substitution $u = zt$ and cutting the multiplier $z^{\nu+\alpha+\beta+1}$ we write (2.3) in the form

$$az\int_0^1 t^{\nu+\alpha+1}(1-t)^{\beta}f(zt)dt = (az+1)\int_0^1 t^{\nu+\alpha}(1-t)^{\beta}f(zt)dt$$

$$-B(2\nu+\alpha+1,\beta+1)f(z). \tag{3.3}$$

Hence we observe that (3.3) is a second kind integral equation containing two Erdlyi-Kober fractional integration operators with linear coefficients (see [7], Ch. 3). Seeking a possible solution in terms of the series $z^{\nu}\sum_{n=0}^{\infty}a_n z^n$ where $a_n \neq 0,\ a_0 = 1$ and no two consecutive a_n vanish (see [4], p. 43) we substitute it in (3.3). After changing the order of integration and summation via the uniform convergence, calculation the inner Beta-integrals, and elementary substitutions we come out with the equality

$$a\sum_{n=1}^{\infty}a_{n-1}z^{n+\nu}\left[B(\mu+n,\beta+1)-B(\mu+n-1,\beta+1)\right]$$

$$=\sum_{n=1}^{\infty}a_n z^{n+\nu}\left[B(\mu+n,\beta+1)-B(\mu,\beta+1)\right],\ \mu=2\nu+\alpha+1. \tag{3.4}$$

Hence equating coefficients of the series and taking into account that $B(\mu+n,\beta+1)-B(\mu,\beta+1)\neq 0, n\in\mathbb{N}, \mu>0,\ \beta>-1$ we obtain the following recurrence relations

$$a_n = a\ a_{n-1}\frac{B(\mu+n,\beta+1)-B(\mu+n-1,\beta+1)}{B(\mu+n,\beta+1)-B(\mu,\beta+1)}$$

$$= a^n\prod_{j=1}^{n}\frac{B(\mu+j,\beta+1)-B(\mu+j-1,\beta+1)}{B(\mu+j,\beta+1)-B(\mu,\beta+1)}, \tag{3.5}$$

where $n\in\mathbb{N}_0$ and the empty product is equal to 1. Moreover, invoking the definition of the Pochhammer symbol $(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)}$, the reduction formula for Gamma-function $\Gamma(z+1) = z\Gamma(z)$ and representing Beta-function via $B(b,c) = \frac{\Gamma(b)\Gamma(c)}{\Gamma(b+c)}$ we write (3.5) in the form (3.2).

Further, it is easily seen from (3.5) and asymptotic formula for the ratio of Gamma-functions [3] that

$$\left|\frac{a_n}{a_{n-1}}\right| = \left|a\frac{B(\mu+n,\beta+1)-B(\mu+n-1,\beta+1)}{B(\mu+n,\beta+1)-B(\mu,\beta+1)}\right|$$
$$= \frac{\Gamma(\mu+n-1)}{\Gamma(\mu+n+\beta+1)}\frac{|a|(\beta+1)}{B(\mu,\beta+1)-B(\mu+n,\beta+1)}$$
$$= O\left(n^{-(\beta+2)}\right) \to 0,\ n\to\infty,\ \beta>-1.$$

Thus $F(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function. Let us compute its order ρ. To do this we appeal to the familiar in calculus Stolz's theorem. Hence we find

$$\rho = \mathrm{limsup}_{n\to\infty}\frac{n\log n}{\log|a_n|^{-1}} = \lim_{n\to\infty}\frac{(n+1)\log(n+1)-n\log n}{\log|a_{n+1}|^{-1}-\log|a_n|^{-1}}$$
$$= \lim_{n\to\infty}\frac{\log n}{\log n^{\beta+2}} = \frac{1}{\beta+2} < 1,\ \beta>-1.$$

Thus same arguments as in [4] guarantee that all our solutions belong to the class $\mathcal{B}$. In particular, for $\beta = 0$ we easily get from (3.2) that

$$a_n = \frac{a^n}{\Gamma(\mu+n)}\prod_{j=1}^{n}\frac{(\mu)_j}{(\mu+1)_j-(\mu)_j} = \frac{(a\mu)^n}{\Gamma(\mu+n)n!},\ n=0,1,\ldots.$$

Then the series representation for the Bessel function [3] yields

$$f(z) = \text{const. } z^{-\alpha/2}J_{\mu-1}\left(cz^{1/2}\right),\ c^2 = -4a\mu,$$

which slightly generalizes Hardy's solutions [4], p.43. Theorem 2 is proved. ∎

4. Properties of solutions and their particular cases

First in this section we will apply the Mellin transform (1.5) to the integral equation (2.3), reducing it to a certain functional equation. Then we will solve this equation by methods of the calculus of finite differences [5] to obtain the value of the Mellin transform for solutions (3.1).

In fact, returning to (3.3) we consider $z = x \in \mathbb{R}_+$ and we apply through the Mellin transform (1.5) taking into account the operational formula

$$\langle f(xt), x^{s-1}\rangle = t^{-s}f^*(s).$$

Then using values of the elementary Beta-integrals we come out with the following homogeneous functional equation

$$aB(\nu+\alpha-s+1,\beta+1)f^*(s+1) = aB(\nu+\alpha-s,\beta+1)f^*(s+1)$$

$$+B(\nu+\alpha-s+1,\beta+1)f^*(s) - B(2\nu+\alpha+1,\beta+1)f^*(s), \tag{4.1}$$

where s is a parameter of the Mellin transform (1.5) such that $s+1 \in \Omega_f$. Hence denoting by $h(s) = B(\nu+\alpha-s+1,\beta+1)$ and invoking the condition $a \neq 0$ (see [4], p.43) we rewrite (3.2) as

$$f^*(s+1) = H(s)f^*(s), \tag{4.2}$$

where the kernel $H(s)$ is given by

$$H(s) = \frac{1}{a}\frac{h(s)-h(-\nu)}{h(s)-h(s+1)}. \tag{4.3}$$

We can simplify $H(s)$ appealing to the basic properties for Beta-functions [3]. After straightforward calculations we get finally

$$H(s) = \frac{1}{a(\beta+1)}\frac{(s-\nu-\alpha)(h(s)-h(-\nu))}{h(s)}. \tag{4.4}$$

Meanwhile,

$$h(s)-h(-\nu) = \int_0^1 (1-t)^\beta (t^{\nu+\alpha-s} - t^{2\nu+\alpha})dt = \sum_{n=0}^\infty \frac{(-\beta)_n}{n!}$$

$$\times \left[\frac{1}{\nu+\alpha-s+n+1} - \frac{1}{2\nu+\alpha+n+1}\right] = (\nu+s)\chi(s),$$

where

$$\chi(s) = \sum_{n=0}^\infty \frac{(-\beta)_n}{n!(2\nu+\alpha+n+1)(\nu+\alpha+n+1-s)}.$$

It is easily seen that $\chi(s)$ and $h(s)$ have the same simple poles at the points $s = \nu+\alpha+n+1,\ n = 0,1,\ldots$. As a consequence, taking into account the expression of $h(s)$ in terms of Gamma-functions, we find that $H(s)$ is meromorphic with simple poles $s = \nu+\alpha+\beta+n+2,\ n \in \mathbb{N}_0$. Furthermore, via the asymptotic behavior of the Beta-function [3], we get $H(s) = O(s^{\beta+2}),\ s \to \infty$.

The functional equation (4.2) can be formally solved by methods proposed in [5], Ch. 11. So it has a general solution

$$f^*(s) = \omega(s) \exp\left(\sum_c^s \log H(z) \Delta z \right), \tag{4.5}$$

where $\omega(s)$ is an arbitrary periodic function of s of period 1, $\sum_c^s$ means the operation of general summation [5] with a given constant c and the function $\log H(s)$ is summable in this sense (cf. [5] Ch. 8). Clearly that it may be necessary to make suitable cuts in the s plane in view of the possible multi-valued nature of the right-hand side of (4.5).

Let us consider interesting particular cases of the solutions (3.1) when $\beta = k \in \mathbb{N}_0$. Indeed, putting $\beta = k$ in (3.2) and using the definition of Pochhammer's symbol we obtain

$$\frac{(\mu+k+1)_j}{(\mu)_j} - 1 = \frac{\Gamma(\mu+k+j+1)}{\Gamma(\mu+k+1)(\mu)_j} - 1 = \frac{(\mu)_{k+j+1}}{(\mu)_{k+1}(\mu)_j} - 1$$

$$= \frac{(\mu+j)(\mu+j+1)\dots(\mu+k+j)}{(\mu)_{k+1}} - 1 = \frac{jP_k(j)}{(\mu)_{k+1}},$$

where we denote by $P_k(j) = (j-\alpha_1)\dots(j-\alpha_k)$, $\alpha_i \in \mathbb{C}$, $i = 1,\dots,k$, a polynomial of degree k with respect to j and evidently $P_k(j) > 0$, $j \in \mathbb{N}$. Substituting the above expression into (3.2) it becomes

$$a_n = \frac{(a(k+1)(\mu)_{k+1})^n}{\Gamma(\mu+n)n!} \prod_{j=1}^{n} \prod_{i=1}^{k} \frac{1}{j-\alpha_i}, \ n,k \in \mathbb{N}_0 \tag{4.6}$$

and the empty products are equal to 1. Consequently, solutions (3.1) represent the so-called hyperbessel functions [7], Ch. 19 of the order $k+1$, namely

$$f(z) = \text{const. } z^\nu$$

(4.7)
$$\times {}_0F_{k+1}\left(2\nu+\alpha+1,\ 1-\alpha_1,\dots,\ 1-\alpha_k;\ a(k+1)(2\nu+\alpha+1)_{k+1}z\right),\ k \in \mathbb{N}_0.$$

As it is known [7] functions

$$F(z) = {}_0F_{k+1}\left(2\nu+\alpha+1,\ 1-\alpha_1,\dots,\ 1-\alpha_k;\ a(k+1)(2\nu+\alpha+1)_{k+1}z\right)$$

in (4.7) satisfy the following $k+2$-th order linear differential equation

$$\frac{d}{dz}\left(z\frac{d}{dz}+2\nu+\alpha\right)\prod_{i=1}^{k}\left(z\frac{d}{dz}-\alpha_i\right)F-a(k+1)(2\nu+\alpha+1)_{k+1}F=0. \tag{4.8}$$

Remark 1. We point out that in [2] the authors considered other generalizations of the Bessel functions which also satisfy higher order differential equations.

Remark 2. In particular, for $k=0$ we easily get Hardy's solutions in terms of the ${}_0F_1$ - functions and strictly prove their orthogonality (1.2) by using the corresponding second order differential equation (4.8). However, even for $k=1$ the direct proof of the orthogonality property (1.2) for solutions (4.7) is a difficult task since we deal in this case with the third order differential equation. As we aware, the orthogonality property for the hyperbessel functions with respect to their zeros is yet unknown. So, following conclusions of Theorem 2 we conjecture here this fact as well as the orthogonality for all solutions (3.1) of the class $\mathcal{B}$.

5. Orthogonal functions of the class $\mathcal{A}$

In the case $f\in\mathcal{A}$ we recall again the arguments in [4] to write accordingly, the formula for Fourier coefficients $a_n(z)$ of $f(zt)$ (see (2.1)) as

$$a_n(z)=\frac{2A_n\lambda_n}{f'(\lambda_n)}\frac{f(z)}{z^2-\lambda_n^2}. \tag{5.1}$$

We have

Theorem 3. *If $f\in\mathcal{A}$ and satisfy (2) with $\alpha>-1-2\nu,\ \nu\in\mathbb{R},\ \beta>-1$, then the integral equation holds*

$$a\int_0^z u^{\nu+\alpha+2}(z-u)^{\beta}f(u)du$$

$$=(az^2+2)\int_0^z u^{\nu+\alpha}(z-u)^{\beta}f(u)du+z^{\nu+\alpha+\beta+1}f(z)A, \tag{5.2}$$

where $a=F''(0)$ and $A=-2B(2\nu+\alpha+1,\beta+1)$.

Proof: As in the proof of Theorem 1 we substitute (5.1) in (2.2) and we get the equality

$$\int_0^1 f(zt)f(\zeta t)t^{\alpha}(1-t)^{\beta}dt=-f(z)f(\zeta)\frac{q(z)-q(\zeta)}{z^2-\zeta^2}, \tag{5.3}$$

where now

$$q(z) = 4\sum_{n=1}^{\infty} \frac{A_n \lambda_n^2}{\{f'(\lambda_n)\}^2} \left[\frac{1}{z^2-\lambda_n^2} + \frac{1}{\lambda_n^2}\right].$$

Letting $\zeta \to 0$ in (5.3), we obtain

$$\int_0^1 t^\nu f(zt) t^\alpha (1-t)^\beta dt = -\frac{f(z)q(z)}{z^2},$$

which yields after the change $u = zt$

$$\int_0^z u^{\nu+\alpha}(z-u)^\beta f(u)du = -z^{\nu+\alpha+\beta-1} f(z)q(z). \tag{5.4}$$

When z is small, $f(z)z^\nu$ and $q(z) \sim \frac{z^2}{2}q''(0)$. Therefore, as $z \to 0$ we get

$$2\int_0^z u^{2\nu+\alpha}(z-u)^\beta du \sim -z^{2\nu+\alpha+\beta+1} q''(0).$$

Thus $q''(0) = -2B(2\nu+\alpha+1, \beta+1) = A$. If we rewrite (5.3) in the form

$$\int_0^1 t^{2\nu+\alpha} F(zt)F(\zeta t)(1-t)^\beta dt = -F(z)F(\zeta)\frac{q(z)-q(\zeta)}{z^2-\zeta^2}.$$

then after differentiation with respect to ζ we get

$$\begin{aligned} &\int_0^1 t^{2\nu+\alpha+1} F'(\zeta t)F(zt)(1-t)^\beta dt \\ = \;& -F(z)F'(\zeta)\frac{q(z)-q(\zeta)}{z^2-\zeta^2} - F(z)F(\zeta)\frac{-q'(\zeta)(z^2-\zeta^2)+2\zeta(q(z)-q(\zeta))}{(z^2-\zeta^2)^2}. \end{aligned}$$

Differentiating again in ζ and setting $\zeta = 0$, we deduce after some simplifications

$$F''(0)\int_0^1 t^{\nu+\alpha+2}(1-t)^\beta f(zt)dt = -f(z)\left[\frac{F''(0)q(z)}{z^2} - \frac{A}{z^2} + \frac{2q(z)}{z^4}\right].$$

Letting $a = F''(0)$ and making the change $zt = u$ we obtain the equation

$$a\int_0^z u^{\nu+\alpha+2}(z-u)^\beta f(u)du = -z^{\nu+\alpha+\beta-1} f(z)\left[(z^2 a + 2)q(z) - Az^2\right].$$

Taking into account (5.4) we finally obtain the integral equation (5.2). ∎

An analog of the Hardy theorem for the class $\mathcal{A}$ is

Theorem 4. *Let f satisfy conditions of Theorem 3. Then*

$$f(z) = const.z^{\nu}\sum_{n=0}^{\infty} a_{2n} z^{2n}, \tag{5.5}$$

where

$$a_{2n} = \left(\frac{a(\beta+1)}{2}\right)^{n} \prod_{j=1}^{n} \frac{(\beta + 2(\mu+2j-1))(\mu)_{2(j-1)}}{(\mu+\beta+1)_{2j} - (\mu)_{2j}}, \ n = 0, 1, \dots,$$

$\mu = 2\nu + \alpha + 1$ *and the empty product is equal to* 1.

Proof: The substitution $u = zt$ and the value of A reduce equation (5.2) to

$$az^2 \int_0^1 t^{\nu+\alpha}(t^2-1)(1-t)^{\beta} f(zt)dt$$

$$= 2\left[\int_0^1 t^{\nu+\alpha}(1-t)^{\beta} f(zt)dt - f(z)B(2\nu+\alpha+1, \beta+1)\right]. \tag{5.6}$$

Since the case $f \in \mathcal{A}$ presumes the following series representation

$$f(z) = \text{const.}z^{\nu}\sum_{n=0}^{\infty} a_{2n} z^{2n},$$

with $a_{2n} \neq 0$ for any $n \in \mathbb{N}_0$, we substitute it into (5.6), change the order of integration and summation, and calculate the inner Beta-integrals. Therefore denoting by $\mu = 2\nu + \alpha + 1$ we obtain

$$a\sum_{n=1}^{\infty} a_{2(n-1)} z^{2n} \left[B(\mu+2n, \beta+1) - B(\mu+2(n-1), \beta+1)\right]$$

$$= 2\sum_{n=1}^{\infty} a_{2n} z^{2n} \left[B(\mu+2n, \beta+1) \ - \ B(\mu, \beta+1)\right].$$

Hence equating coefficients of the series the following recurrence relations appear

$$a_{2n} = \frac{a}{2} a_{2(n-1)} \frac{B(\mu+2n, \beta+1) - B(\mu+2(n-1), \beta+1)}{B(\mu+2n, \beta+1) \ - \ B(\mu, \beta+1)}$$

$$= \left(\frac{a}{2}\right)^{n} \prod_{j=1}^{n} \frac{B(\mu+2j, \beta+1) - B(\mu+2(j-1), \beta+1)}{B(\mu+2j, \beta+1) - B(\mu, \beta+1)}, \tag{5.7}$$

where $n = 0, 1, 2, \ldots$, and the empty product is equal to 1. In the same way as (3.2) it can be written in the form

$$a_{2n} = \left(\frac{a(\beta+1)}{2}\right)^n \prod_{j=1}^{n} \frac{(\beta + 2(\mu + 2j - 1))(\mu)_{2(j-1)}}{(\mu+\beta+1)_{2j} - (\mu)2j}, \ n = 0, 1, \ldots .$$

■

Putting $\beta = 0$ in (5.7) by straightforward calculations functions (5.5) become

$$f(z) = \text{const.} z^{\frac{1-\alpha}{2}} J_{\frac{\mu}{2}-1}\left((-a\mu)^{1/2} z\right),$$

which slightly generalizes Hardy's solutions [4], p.42.

It is easily seen that the series in (5.5) is an entire function. Finally we compute the order of these solutions. By similar arguments as in Theorem 2 we have

$$\rho = \text{limsup}_{n\to\infty} \frac{2n \log 2n}{\log |a_{2n}|^{-1}} = \lim_{n\to\infty} \frac{2(n+1)\log 2(n+1) - 2n \log 2n}{\log |a_{2(n+1)}|^{-1} - \log |a_{2n}|^{-1}}$$

$$= \lim_{n\to\infty} \frac{2 \log n}{\log n^{\beta+2}} = \frac{2}{\beta+2} < 2, \ \beta > -1.$$

Therefore all solutions belong to the class $\mathcal{A}$.

Remark 3. Our final conjecture is that solutions $f \in \mathcal{A}$ of the Hardy-type integral equation (5.2) are orthogonal with respect to their own zeros for the Jacobi weight.

References

[1] L. D. Abreu, *Functions q-orthogonal with respect to their own zeros,* Proc. Amer. Math. Soc. 133 (2005), 1197-1203.

[2] W. N. Everitt, C. Markett, *On a generalization of Bessel functions satisfying higher-order differential equations.* J. Comput. Appl. Math. 54 (1994), 325–349.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger, and F.G. Tricomi, *Higher Transcendental Functions*, Vol. I, McGraw-Hill, New York, London, Toronto (1953).

[4] G. H. Hardy, *Notes on special systems of orthogonal functions II: On functions orthogonal with respect to their own zeros.* J. London Math. Soc., 14 (1939), 37-44.

[5] L. M. Milne-Thomson, *The Calculus of Finite Differences*, Macmillan and Co. Limited, London (1933).

[6] E.C. Titchmarsh, *Introduction to the Theory of Fourier Integrals*, Clarendon Press, Oxford (1937).

[7] S.B. Yakubovich and Yu.F. Luchko, *The Hypergeometric Approach to Integral Transforms and Convolutions*, (Kluwer Ser. Math. and Appl.: Vol. 287), Dordrecht, Boston, London (1994).

[8] A.H. Zemanian, *Generalized Integral Transformations*, Dover Publ., New York (1987).

L. D. Abreu ⋆
Departamento de Matemática da Universidade de Coimbra, 3001-454 Coimbra, Portugal

F. Marcellan[†]
Department of Mathematics, University Carlos III de Madrid, Avenida de la Universidad 30, 28911 Leganés, Spain

S.B. Yakubovich[‡]
Department of Pure Mathematics, Faculty of Sciences, University of Porto, Campo Alegre st., 687, 4169-007 Porto, Portugal

E-mail address: ⋆daniel@mat.uc.pt
E-mail address: [†]pacomarc@ing.uc3m.es
E-mail address: [‡]syakubov@fc.up.pt