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ON QUASI-JACOBI AND JACOBI-QUASI BIALGEBROIDS

J. M. NUNES DA COSTA AND F. PETALIDOU

ABSTRACT: We study quasi-Jacobi and Jacobi-quasi bialgebroids and their relationships with twisted Jacobi and quasi Jacobi manifolds. We show that we can construct quasi-Lie bialgebroids from quasi-Jacobi bialgebroids, and conversely, and also that the structures induced on their base manifolds are related via a "quasi Poissonization".

KEYWORDS: Quasi-Jacobi bialgebroid, Jacobi-quasi bialgebroid, quasi-Jacobi bialgebra, twisted Jacobi manifold, quasi Jacobi manifold.
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1. Introduction

The notion of *quasi-Lie bialgebroid* was introduced in [19]. It is a structure on a pair (A, A^*) of vector bundles, in duality, over a differentiable manifold M that is defined by a Lie algebroid structure on A^*, a skew-symmetric bracket on the space of smooth sections of A and a bundle map $a : A \to TM$, satisfying some compatibility conditions. These conditions are expressed in terms of a section φ of $\bigwedge^3 A^*$, which turns to be an obstruction to the Lie bialgebroid structure on (A, A^*). A quasi-Lie bialgebroid will be denoted by (A, A^*, φ). In the case where A is a Lie algebroid and its dual vector bundle A^* is equipped with a skew-symmetric bracket on its space of smooth sections and a bundle map $a_* : A^* \to TM$ and the compatibility conditions are expressed in terms of a section Q of $\bigwedge^3 A$, the triple (A, A^*, Q) is called a *Lie-quasi bialgebroid* [9]. When $\varphi = 0$ and $Q = 0$, quasi-Lie and Lie-quasi bialgebroids are just Lie bialgebroids. We note that, while the dual of a Lie bialgebroid is itself a Lie bialgebroid, the dual of a quasi-Lie bialgebroid is a Lie-quasi bialgebroid, and conversely [9]. The quasi-Lie and Lie-quasi bialgebroids are particular cases of proto-bialgebroids [9]. As in the case of a Lie bialgebroid, the doubles $A \oplus A^*$ of a quasi-Lie and of a Lie-quasi bialgebroid are endowed with a Courant algebroid structure [19], [9].

It was shown in [20] that the theory of quasi-Lie bialgebroids is the natural framework in which we can treat *twisted Poisson manifolds*. These structures were introduced in [21], under the name of Poisson manifolds with a

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closed 3-form background, motivated by problems of string theory [18] and of topological field theory [8], and since then deserved a lot of interest (see e.g [9], [11]).

The notion of *Jacobi bialgebroid* and the equivalent one of *generalized Lie bialgebroid* were introduced, respectively, in [3] and [5], in such a way that a Jacobi manifold has a Jacobi bialgebroid canonically associated and conversely. A Jacobi bialgebroid over M is a pair $((A, \phi), (A^*, W))$ of Lie algebroids over M, in duality, endowed with 1-cocycles $\phi \in \Gamma(A^*)$ and $W \in \Gamma(A)$ in their Lie algebroid cohomology complex with trivial coefficients, respectively, that satisfies a compatibility condition. Also, its double $(A \oplus A^*, \phi + W)$ is endowed with a Courant-Jacobi algebroid structure [4], [15].

In order to adapt to the framework of Jacobi manifolds the concepts of twisted Poisson manifold and quasi-Lie bialgebroid, we have recently introduced in [16] the notions of *twisted Jacobi manifold* and *quasi-Jacobi bialgebroid*. The purpose of the present paper is to develop the theory of quasi-Jacobi bialgebroids, as well as of its dual concept of *Jacobi-quasi bialgebroids*, and to establish a very close relationship between quasi-Jacobi and quasi-Lie bialgebroids.

The paper contains four sections, besides the Introduction, and one Appendix (section 5). In section 2 we recall the definition of quasi-Jacobi bialgebroid, we present some basic results established in [16], we develop the examples of quasi-Jacobi and Jacobi-quasi bialgebroids associated to twisted Jacobi manifolds and to quasi Jacobi manifolds, and, finally, we study the triangular quasi-Jacobi bialgebroids. Section 3 is devoted to the study of the structures induced on the base manifolds of quasi-Jacobi and Jacobi-quasi bialgebroids. Several examples are presented. In section 4 we establish an one to one correspondence between quasi-Jacobi bialgebroids structures $((A, \phi), (A^*, W), \varphi)$ over a manifold M and quasi-Lie bialgebroids structures $(\tilde{A}, \tilde{A}^*, \tilde{\varphi})$ over $\tilde{M} = M \times \mathbb{R}$. Also, we prove that the structure induced on $\tilde{M} = M \times \mathbb{R}$ by $(\tilde{A}, \tilde{A}^*, \tilde{\varphi})$ is the "quasi Poissonization" of the structure induced on M by $((A, \phi), (A^*, W), \varphi)$. The dual version of these results is also presented. Finally, in the Appendix, we define the concept of action of a Lie algebroid with 1-cocycle on a differentiable manifold that is used in this paper.

Notation: If (A, ϕ) is a Lie algebroid with 1-cocycle ϕ, we denote by d^ϕ the differential operator d of A modified by ϕ, i.e., $d^\phi \alpha = d\alpha + \phi \wedge \alpha$, for any $\alpha \in \Gamma(\bigwedge^k A^*)$. Moreover, we denote by δ the usual de Rham differential operator on a manifold M and by d the differential operator of the Lie algebroid $TM \times \mathbb{R}$, $\mathrm{d}(\alpha, \beta) = (\delta\alpha, -\delta\beta)$, for $(\alpha, \beta) \in \Gamma(\bigwedge^k(T^*M \times \mathbb{R})) \equiv \Gamma(\bigwedge^k T^*M) \times \Gamma(\bigwedge^{k-1} T^*M)$. We also consider the identification $\Gamma(\bigwedge^k(TM \times \mathbb{R})) \equiv \Gamma(\bigwedge^k TM) \times \Gamma(\bigwedge^{k-1} TM)$. For the Schouten bracket and the interior product of a form with a multi-vector field, we use the convention of sign indicated by Koszul [12].

2. Quasi-Jacobi bialgebroids and Jacobi-quasi bialgebroids

Let $((A, \phi), (A^*, W))$ be a pair of dual vector bundles over a differentiable manifold M, each one endowed with a 1-form ϕ and W, respectively, and φ a 3-form of A.

Definition 2.1. *A* quasi-Jacobi bialgebroid *structure on* $((A, \phi), (A^*, W), \varphi)$ *consists of a Lie algebroid structure with* 1*-cocycle* $([\cdot,\cdot]_*, a_*, W)$ *on* A^**, a bundle map* $a : A \to TM$ *and a skew-symmetric operation* $[\cdot,\cdot]$ *on* $\Gamma(A)$ *satisfying, for all* $X, Y, Z \in \Gamma(A)$ *and* $f \in C^\infty(M, \mathbb{R})$*, the following conditions:*

1) $[X, fY] = f[X, Y] + (a(X)f)Y$;
2) $a([X, Y]) = [a(X), a(Y)] - a_*(\varphi(X, Y, \cdot))$;
3) $[[X, Y], Z] + c.p. = -d_*^W(\varphi(X, Y, Z)) - ((i_{\varphi(X,Y,\cdot)} d_*^W Z) + c.p.)$;
4) $d\phi - \varphi(W, \cdot, \cdot) = 0$, *where* d *is the quasi-differential operator on* $\Gamma(\bigwedge A^*)$ *determined by the structure* $([\cdot,\cdot], a)$ *on* A;
5) $d^\phi \varphi = 0$, *where* d^ϕ *is given, for any* $\beta \in \Gamma(\bigwedge^k A^*)$, *by* $d^\phi(\beta) = d\beta + \phi \wedge \beta$;
6) $d_*^W[P, Q]^\phi = [d_*^W P, Q]^\phi + (-1)^{p+1}[P, d_*^W Q]^\phi$, *with* $P \in \Gamma(\bigwedge^p A)$ *and* $Q \in \Gamma(\bigwedge A)$.

As in the case of quasi-Lie and Lie-quasi bialgebroids, by interchanging the roles of (A, ϕ) and (A^*, W) in the above definition, we obtain the notion of *Jacobi-quasi bialgebroid* over a differentiable manifold M.

Definition 2.2. *A* Jacobi-quasi bialgebroid *structure on* $((A, \phi), (A^*, W), Q)$*,* A *and* A^* *being dual vector bundles over a differentiable manifold* M *and* Q *a section of* $\bigwedge^3 A$*, consists of a Lie algebroid structure with* 1*-cocycle* $([\cdot,\cdot], a, \phi)$ *on* A*, a bundle map* $a_* : A^* \to TM$*, a skew-symmetric operation* $[\cdot,\cdot]_*$ *on* $\Gamma(A^*)$ *and a section* $W \in \Gamma(A)$*, satisfying the conditions 1)-6) of Definition 2.1 in their dual versions.*

Hence, we can easily conclude:

Proposition 2.3. *If* $((A,\phi),(A^*,W),\varphi)$ *is a quasi-Jacobi bialgebroid over a differentiable manifold* M*, then* $((A^*,W),(A,\phi),\varphi)$ *is a Jacobi-quasi bialgebroid over* M*, and conversely.*

In the case where both 1-cocycles ϕ and W are zero, we recover, from Definitions 2.1 and 2.2, the notions of quasi-Lie and Lie-quasi bialgebroid, respectively. On the other hand, if $\varphi = 0$ in Defintion 2.1 (resp. $Q = 0$ in Definition 2.2), then $((A,\phi),(A^*,W),0) \equiv ((A,\phi),(A^*,W))$ is a Jacobi bialgebroid over M.

Remark 2.4. In [16], we proved that the double of a quasi-Jacobi bialgebroid is a Courant-Jacobi algebroid ([4], [15]). By a similar computation, we may conclude that the double of a Jacobi-quasi bialgebroid is also a Courant-Jacobi algebroid.

The rest of this section is devoted to some important examples of quasi-Jacobi and Jacobi-quasi bialgebroids.

2.1. Quasi-Jacobi and Jacobi-quasi bialgebras. A *quasi-Jacobi bialgebra* is a quasi-Jacobi bialgebroid over a point, that is a triple $((\mathcal{G},\phi),(\mathcal{G}^*,W),\varphi)$, where $(\mathcal{G}^*,[\cdot,\cdot]_*,W)$ is a real Lie algebra of finite dimension with 1-cocycle $W \in \mathcal{G}$ in its Chevalley-Eilenberg cohomology, $(\mathcal{G},\phi)$ is the dual space of $\mathcal{G}^*$ endowed with a bilinear skew-symmetric bracket $[\cdot,\cdot]$ and an element $\phi \in \mathcal{G}^*$ and $\varphi \in \bigwedge^3 \mathcal{G}^*$, such that conditions 3)-6) of Definition 2.1 are satisfied.

By dualizing the above notion, we get a *Jacobi-quasi bialgebra*, that is a Jacobi-quasi bialgebroid over a point.

In the particular case where $\varphi = 0$, we recover the concept of Jacobi bialgebra [5]. When $\phi = 0$ and $W = 0$, we recover the notion of quasi-Lie bialgebra due to Drinfeld [2].

We postpone the study of quasi-Jacobi bialgebras to a future paper, in preparation. We believe that they can be considered as the infinitesimal invariants of Lie groups endowed with a certain type of twisted Jacobi structures that can be constructed from the solutions of a twisted Yang-Baxter equation.

2.2. The quasi-Jacobi and the Jacobi-quasi bialgebroids of a twisted Jacobi manifold. We recall that a *twisted Jacobi manifold* [16] is a differentiable manifold M equipped with a section (Λ, E) of $\bigwedge^2(TM \times \mathbb{R})$ and a

2-form ω such that

$$\frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)} = (\Lambda, E)^{\#}(\delta\omega, \omega),^{*} \tag{1}$$

where $[\cdot,\cdot]^{(0,1)}$ denotes the Schouten bracket of the Lie algebroid $(TM \times \mathbb{R}, [\cdot,\cdot], \pi)$ modified by the 1-cocycle $(0,1)$, $\pi : TM \times \mathbb{R} \to TM$ is the projection on the first factor and $(\Lambda, E)^{\#}$ is the natural extension of the homomorphism of $C^\infty(M,\mathbb{R})$-modules $(\Lambda, E)^{\#} : \Gamma(T^*M \times \mathbb{R}) \to \Gamma(TM \times \mathbb{R})$, $(\Lambda, E)^{\#}(\alpha, f) = (\Lambda^{\#}(\alpha) + fE, -\langle \alpha, E\rangle)$, $(\alpha, f) \in \Gamma(T^*M \times \mathbb{R})$, to a homomorphism from $\bigwedge^k(T^*M \times \mathbb{R})$ to $\bigwedge^k(TM \times \mathbb{R})$, $k \in \mathbb{N}$, given, for all $f \in C^\infty(M,\mathbb{R})$, by $(\Lambda, E)^{\#}(f) = f$, and, for any $(\eta, \xi) \in \Gamma(\bigwedge^k(T^*M \times \mathbb{R}))$ and $(\alpha_1, f_1), \dots, (\alpha_k, f_k) \in \Gamma(T^*M \times \mathbb{R})$, by

$$\begin{aligned}&(\Lambda, E)^{\#}(\eta, \xi)((\alpha_1, f_1), \dots, (\alpha_k, f_k))\\ &\quad = (-1)^k(\eta, \xi)((\Lambda, E)^{\#}(\alpha_1, f_1), \cdots, (\Lambda, E)^{\#}(\alpha_k, f_k)).\end{aligned}$$

In [16], we presented several examples of twisted Jacobi manifolds such as *twisted exact Poisson manifolds* and *twisted locally conformal symplectic manifolds*. In a very recent Note [17], where we discuss the characteristic foliation of a twisted Jacobi manifold, we introduced the notion of *twisted contact Jacobi manifold* which produces another example of twisted Jacobi manifold. Next, we recall this last example and present a new one.

Examples 2.5.

1) Twisted contact Jacobi manifolds: A twisted contact Jacobi manifold is a $(2n+1)$-dimensional differentiable manifold M equipped with a 1-form ϑ and a 2-form ω such that $\vartheta \wedge (\delta\vartheta + \omega)^n \neq 0$, everywhere in M. We consider on M the vector field E, given by

$$i_E\vartheta = 1 \qquad \text{and} \qquad i_E(\delta\vartheta + \omega) = 0,$$

and the bivector field Λ whose associated morphism $\Lambda^{\#} : \Gamma(T^*M) \to \Gamma(TM)$ is defined, for any $\alpha \in \Gamma(T^*M)$, by

$$\Lambda^{\#}(\vartheta) = 0 \qquad \text{and} \qquad i_{\Lambda^{\#}(\alpha)}(\delta\vartheta + \omega) = -(\alpha - (i_E\alpha)\vartheta).$$

Then, the triple $(M, (\Lambda, E), \omega)$ is a twisted Jacobi manifold.

*Since, for any $(\varphi, \omega) \in \Gamma(\bigwedge^3(T^*M \times \mathbb{R}))$, $\mathrm{d}^{(0,1)}(\varphi, \omega) = (\delta\varphi, \varphi - \delta\omega)$ and $\mathrm{d}^{(0,1)}(\varphi, \omega) = (0,0) \Leftrightarrow \varphi = \delta\omega$, equation (1) means that $\frac{1}{2}[(\Lambda, E), (\Lambda, E)]^{(0,1)}$ is the image by $(\Lambda, E)^{\#}$ of a $\mathrm{d}^{(0,1)}$-closed 3-form of $TM \times \mathbb{R}$.

2) Twisted conformal Jacobi structures: Let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold and f a function on M that never vanishes. We can define a new twisted Jacobi structure $((\Lambda^f,E^f),\omega^f)$ on M, which is said to be *f-conformal* to $((\Lambda,E),\omega)$, by setting

$$\Lambda^f = f\Lambda\,; \quad E^f = \Lambda^{\#}(\delta f) + fE\,; \quad \omega^f = \frac{1}{f}\,\omega.$$

In the sequel, let $(M,(\Lambda,E),\omega)$ be a twisted Jacobi manifold and $(T^*M\times \mathbb{R}, [\cdot,\cdot]^{\omega}_{(\Lambda,E)}, \pi\circ(\Lambda,E)^{\#}, (-E,0))$ its canonically associated Lie algebroid with 1-cocycle, [16]. The Lie bracket $[\cdot,\cdot]^{\omega}_{(\Lambda,E)}$ on $\Gamma(T^*M\times\mathbb{R})$ is given, for all $(\alpha,f),(\beta,g)\in\Gamma(T^*M\times\mathbb{R})$, by

$$\begin{aligned}[(\alpha,f),(\beta,g)]^{\omega}_{(\Lambda,E)} &= [(\alpha,f),(\beta,g)]_{(\Lambda,E)}\\ &\quad + (\delta\omega,\omega)((\Lambda,E)^{\#}(\alpha,f),(\Lambda,E)^{\#}(\beta,g),\cdot),\end{aligned}$$

where $[\cdot,\cdot]_{(\Lambda,E)}$ is the usual bracket on $\Gamma(T^*M\times\mathbb{R})$ associated to a section (Λ,E) of $\bigwedge^2(TM\times\mathbb{R})$ ([7], [5]):

$$\begin{aligned}[(\alpha,f),(\beta,g)]_{(\Lambda,E)} &= \mathcal{L}^{(0,1)}_{(\Lambda,E)^{\#}(\alpha,f)}(\beta,g) - \mathcal{L}^{(0,1)}_{(\Lambda,E)^{\#}(\beta,g)}(\alpha,f)\\ &\quad - \mathrm{d}^{(0,1)}((\Lambda,E)((\alpha,f),(\beta,g))).\end{aligned}\tag{2}$$

We consider, on the vector bundle $TM\times\mathbb{R}\to\mathbb{R}$, the Lie algebroid structure over M with 1-cocycle $([\cdot,\cdot],\pi,(0,1))$ and also a new bracket $[\cdot,\cdot]'$ on the space of its smooth sections given, for all $(X,f),(Y,g)\in\Gamma(TM\times\mathbb{R})$, by

$$[(X,f),(Y,g)]' = [(X,f),(Y,g)] - (\Lambda,E)^{\#}((\delta\omega,\omega)((X,f),(Y,g),\cdot)).$$

We have shown in [16]:

Theorem 2.6. *The triple* $((TM\times\mathbb{R},[\cdot,\cdot]',\pi,(0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]^{\omega}_{(\Lambda,E)},\pi\circ(\Lambda,E)^{\#},(-E,0)),(\delta\omega,\omega))$ *is a quasi-Jacobi bialgebroid over* M.

Furthermore, we have:

Theorem 2.7. *The triple* $((TM\times\mathbb{R},[\cdot,\cdot],\pi,(0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]_{(\Lambda,E)},\pi\circ(\Lambda,E)^{\#},(-E,0)),(\Lambda,E)^{\#}(\delta\omega,\omega))$ *is a Jacobi-quasi bialgebroid over* M.

Proof: It suffices to check that all conditions of Definition 2.2 are satisfied. Condition 1) can be checked directly, using the definition (2) of the bracket

$[\cdot,\cdot]_{(\Lambda,E)}$. For 2), we take into account that $((\Lambda,E),\omega)$ is a twisted Jacobi structure, hence (1) holds, and we apply the general formula

$$\begin{aligned}(\Lambda,E)^{\#}([(\alpha,f),(\beta,g)]_{(\Lambda,E)}) &= [(\Lambda,E)^{\#}(\alpha,f),(\Lambda,E)^{\#}(\beta,g)] \\ &- \frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}((\alpha,f),(\beta,g),\cdot).\end{aligned}$$

By projection, we obtain 2). Condition 3) can be checked directly, after a long computation. In order to prove 4), we remark that the quasi-differential operator d_* determined by $([\cdot,\cdot]_{(\Lambda,E)},\pi\circ(\Lambda,E)^{\#})$ is given [5], for all $(R,S)\in\Gamma(\bigwedge^k(TM\times\mathbb{R}))$, by

$$d_*(R,S)=([\Lambda,R]+kE\wedge R+\Lambda\wedge S,-[\Lambda,R]+(1-k)E\wedge S+[E,R]).$$

So, $d_*(-E,0)=([E,\Lambda],0)$, and since $(M,(\Lambda,E),\omega)$ is a twisted Jacobi manifold, we may write (see Proposition 3.1 of [16])

$$\begin{aligned}d_*(-E,0) &= ((\Lambda^{\#}\otimes 1)(\delta\omega)(E)-(((\Lambda^{\#}\otimes 1)(\omega)(E))\wedge E),0) \\ &= ((\Lambda,E)^{\#}(\delta\omega,\omega))((0,1),\cdot,\cdot),\end{aligned}$$

where $(\Lambda^{\#}\otimes 1)(\omega)(E)$ is given, for any 1-form α on M, by $(\Lambda^{\#}\otimes 1)(\omega)(E)(\alpha)=\omega(\Lambda^{\#}(\alpha),E)$. On the other hand, since $d_*^{(-E,0)}(R,S)=[(\Lambda,E),(R,S)]^{(0,1)}$, we have

$$\begin{aligned}d_*^{(-E,0)}((\Lambda,E)^{\#}(\delta\omega,\omega)) &= [(\Lambda,E),(\Lambda,E)^{\#}(\delta\omega,\omega)]^{(0,1)} \\ &= \frac{1}{2}[(\Lambda,E),[(\Lambda,E),(\Lambda,E)]^{(0,1)}]^{(0,1)}=0,\end{aligned}$$

whence we get condition 5). Finally, 6) can be established, as in the proof of Theorem 8.2 in [16], by a straightforward but long computation. ■

Remark 2.8. In the case of twisted Poisson manifolds the above results were treated in [20] and [9].

2.3. The Jacobi-quasi bialgebroid of a quasi Jacobi manifold. Let $(\mathcal{G},[\cdot,\cdot])$ be a Lie algebra, ϕ a 1-cocycle in its Chevalley-Eilenberg cohomology and $(\cdot,\cdot)$ a nondegenerate symmetric bilinear form on $\mathcal{G}$. We denote by ψ the canonical 3-form on $\mathcal{G}$ defined by $\psi(X,Y,Z)=\frac{1}{2}(X,[Y,Z])$, for all $X,Y,Z\in\mathcal{G}$, and by $Q_\psi\in\bigwedge^3\mathcal{G}$ its dual trivector that is given, for all $\mu,\nu,\xi\in\mathcal{G}^*$, by

$$Q_\psi(\mu,\nu,\xi)=\psi(X_\mu,X_\nu,X_\xi),$$

where X_μ,X_ν,X_ξ are, respectively, dual to μ,ν,ξ via $(\cdot,\cdot)$.

A $(\mathcal{G},\phi)$-manifold M is a differentiable manifold on which $(\mathcal{G},\phi)$ acts infinitesimally by $a^\phi : \mathcal{G} \to TM \times \mathbb{R}$, $a^\phi(X) = a(X) + \langle \phi, X\rangle$, for all $X \in \mathcal{G}$ (see Appendix). We keep the same notation a^ϕ for the induced maps on exterior algebras.

A natural generalization of the notion of quasi Poisson manifold, given in [1], is the concept of $(\mathcal{G},\phi)$-*quasi Jacobi manifold*, that we introduce as follows.

Definition 2.9. *A* $(\mathcal{G},\phi)$-quasi Jacobi manifold *is a* $(\mathcal{G},\phi)$*-manifold* M *equipped with a section* $(\Lambda, E) \in \Gamma(\bigwedge^2(TM \times \mathbb{R}))$ *such that*

$$\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)} = a^\phi(Q_\psi).$$

Theorem 2.10. *Let* (M,Λ,E) *be a* $(\mathcal{G},\phi)$*-quasi Jacobi manifold. Then,* $((TM \times \mathbb{R}, [\cdot,\cdot], \pi, (0,1)), (T^*M \times \mathbb{R}, [\cdot,\cdot]_{(\Lambda,E)}, \pi \circ (\Lambda,E)^\#, (-E,0)), a^\phi(Q_\psi))$ *is a Jacobi-quasi bialgebroid over* M*.*

Proof: We can check, without any difficulty, that all conditions of Definition 2.2 are satisfied (see also the proof of Theorem 2.7). Only, in order to establish 4), we note that

$$\begin{aligned} d_*(-E,0) &= ([E,\Lambda],0) = \frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}((0,1),\cdot,\cdot) \\ &= a^\phi(Q_\psi)((0,1),\cdot,\cdot). \end{aligned}$$

■

Remark 2.11. In the case where M is a $\mathcal{G}$-manifold equipped with a quasi Poisson structure, i.e a bivector field Λ on M such that $[\Lambda,\Lambda] = 2a(Q_\psi)$, a similar result holds: The triple $((TM,[\cdot,\cdot],id),(T^*M,[\cdot,\cdot]_\Lambda,\Lambda^\#),a(Q_\psi))$ is a Lie-quasi bialgebroid over M, where $[\cdot,\cdot]_\Lambda$ is the Koszul bracket associated to Λ.

2.4. Triangular quasi-Jacobi and Jacobi-quasi bialgebroids. Let $(A,[\cdot,\cdot],a,\phi)$ be a Lie algebroid with 1-cocycle over a differentiable manifold M, Π a section of $\bigwedge^2 A$ and Q a trivector on A such that

$$\frac{1}{2}[\Pi,\Pi]^\phi = Q.$$

We shall discuss what happens on the dual vector bundle A^* of A when we consider the vector bundle map $a_* : A^* \to TM$, $a_* = a \circ \Pi^\#$, $\Pi^\# : A^* \to A$

being the bundle map associated to Π, and the Koszul bracket $[\cdot,\cdot]_\Pi$ on the space $\Gamma(A^*)$ of its smooth sections given, for all $\alpha,\beta\in\Gamma(A^*)$, by

$$[\alpha,\beta]_\Pi = \mathcal{L}^{\phi}_{\Pi^{\#}(\alpha)}\beta - \mathcal{L}^{\phi}_{\Pi^{\#}(\beta)}\alpha - d^{\phi}(\Pi(\alpha,\beta)). \tag{3}$$

Let us set $W=-\Pi^{\#}(\phi)$. Taking into account that, for all $\alpha,\beta,\gamma\in\Gamma(A^*)$,

$$[[\alpha,\beta]_\Pi,\gamma]_\Pi + c.p. = -d^{\phi}(Q(\alpha,\beta,\gamma)) - ((i_{Q(\alpha,\beta,\cdot)}d^{\phi}\gamma)+c.p),$$

we can directly prove that

Proposition 2.12. *The triple* $((A,[\cdot,\cdot],a,\phi),(A^*,[\cdot,\cdot]_\Pi,a_*,W),Q)$ *is a Jacobi-quasi bialgebroid over* M*.*

Respecting the tradition, we shall call to the Jacobi-quasi bialgebroid constructed above, a *triangular Jacobi-quasi bialgebroid.* Clearly, the Lie-quasi bialgebroid associated to a twisted Poisson manifold [20] and the Jacobi-quasi bialgebroid associated to a twisted Jacobi manifold (see Theorem 2.7) are special cases of triangular Jacobi-quasi bialgebroids. Another important type of triangular quasi-Jacobi bialgebroid is the triangular quasi-Jacobi bialgebra, where Π is a solution of a Yang-Baxter's type equation.

Now, we consider the particular case where Q is the image by $\Pi^{\#}$ of a d^{ϕ}-closed 3-form φ of A, i.e.

$$\frac{1}{2}[\Pi,\Pi]^{\phi} = \Pi^{\#}(\varphi), \tag{4}$$

and the spaces of smooth sections of A^* and A are equipped, respectively, with the brackets

$$[\alpha,\beta]^{\varphi}_{\Pi} = [\alpha,\beta]_\Pi + \varphi(\Pi^{\#}(\alpha),\Pi^{\#}(\beta),\cdot), \quad \text{for all } \alpha,\beta\in\Gamma(A^*),$$

$[\cdot,\cdot]_\Pi$ being the Koszul bracket (3), and

$$[X,Y]' = [X,Y] - \Pi^{\#}(\varphi(X,Y,\cdot)), \quad \text{for all } X,Y\in\Gamma(A).$$

Under the above assumptions, by a straightforward calculation, we get:

Proposition 2.13. *The vector bundle* $A^*\to M$ *endowed with the structure* $([\cdot,\cdot]^{\varphi}_{\Pi},a_*)$ *is a Lie algebroid over* M *with* 1*-cocycle* $W=-\Pi^{\#}(\phi)$*.*

Also, we have:

Theorem 2.14. *The triple* $((A,[\cdot,\cdot]',a,\phi),(A^*,[\cdot,\cdot]^{\varphi}_{\Pi},a_*,W),\varphi)$ *is a triangular quasi-Jacobi bialgebroid over* M*.*

Proof: The proof is analogous to that of Theorem 8.2 in [16] and so it is omitted. ■

Remark 2.15. Obviously, if A is $TM \times \mathbb{R}$ equipped with the usual Lie algebroid structure with 1-cocycle, $([\cdot,\cdot],\pi,(0,1))$, and $\Pi = (\Lambda, E) \in \Gamma(\bigwedge^2(TM \times \mathbb{R}))$ satisfies (4), then the manifold M is endowed with a twisted Jacobi structure. The Lie algebroid structure on $A^* = T^*M \times \mathbb{R}$ given by Proposition 2.13, is the Lie algebroid structure canonically associated with the twisted Jacobi structure on M.

3. The structure induced on the base manifold of a quasi-Jacobi bialgebroid

In this section we will investigate the structure induced on the base manifold of a quasi-Jacobi bialgebroid. Similar results hold for a Jacobi-quasi bialgebroid.

Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. In [16], we have already considered the bracket $\{\cdot,\cdot\}$ on $C^\infty(M,\mathbb{R})$ defined, for all $f, g \in C^\infty(M,\mathbb{R})$, by

$$\{f,g\} = \langle d^\phi f, d_*^W g\rangle. \tag{5}$$

We have proved that it is $\mathbb{R}$-bilinear, skew-symmetric and a first order differential operator on each argument [16]. For the quasi differential operator d on $\Gamma(\bigwedge A^*)$ determined by $(a,[\cdot,\cdot])$, we have that it is a derivation with respect to the usual product of functions. Therefore, the map $(f,g) \mapsto \langle df, d_*g\rangle$ is a derivation on each argument and so, there exists a bivector field Λ on M such that, for all $f,g \in C^\infty(M,\mathbb{R})$,

$$\Lambda(\delta f,\delta g) = \langle df, d_*g\rangle = -\langle dg, d_*f\rangle.$$

If E is the vector field $a_*(\phi) = -a(W)$ on M then, from (5) and because $\langle \phi, W\rangle = 0$ holds [16], we get

$$\{f,g\} = \langle \mathrm{d}^{(0,1)}g, (\Lambda,E)^\#(\mathrm{d}^{(0,1)}f)\rangle.^\dagger \tag{6}$$

Since, for all $f \in C^\infty(M,\mathbb{R})$, $d^\phi f = (a^\phi)^*(\mathrm{d}^{(0,1)}f)$ and $d_*^W f = (a_*^W)^*(\mathrm{d}^{(0,1)}f)$ [16], where $(a^\phi)^*$ and $(a_*^W)^*$ denote, respectively, the transpose of a^ϕ and a_*^W, we obtain

$$(\Lambda,E)^\# = -a^\phi \circ (a_*^W)^* = a_*^W \circ (a^\phi)^*. \tag{7}$$

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†We note that the contraction between sections of $TM \times \mathbb{R}$ and $T^*M \times \mathbb{R}$ is given, for any $(\alpha,f) \in \Gamma(T^*M \times \mathbb{R})$ and $(X,g) \in \Gamma(TM \times \mathbb{R})$, by $\langle(\alpha,f),(X,g)\rangle = \langle \alpha, X\rangle + fg$.

It is well known ([13], [5]) that any bracket of type (6) satisfies the following relation:

$$\{f,\{g,h\}\}+c.p.=\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h). \tag{8}$$

Therefore, for the bracket defined by (5), in general, the Jacobi identity does not hold.

Proposition 3.1. *Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. Then, the bracket (5) satisfies, for all $f,g,h\in C^\infty(M,\mathbb{R})$, the following identity:*

$$\{f,\{g,h\}\}+c.p.=a_*^W(\varphi)(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h). \tag{9}$$

In (9), a_*^W denotes the natural extension of $a_*^W:\Gamma(A^*)\to\Gamma(TM\times\mathbb{R})$ to a bundle map from $\Gamma(\bigwedge^3 A^*)$ to $\Gamma(\bigwedge^3(TM\times\mathbb{R}))$.

Proof: Let f,g and h be any three functions on $C^\infty(M,\mathbb{R})$. Since $d_*^W\{f,g\}=[d_*^W g,d_*^W f]$ (see [16]), we compute

$$\begin{aligned}
\{h,\{f,g\}\} &= \langle d^\phi h,[d_*^W g,d_*^W f]\rangle=\langle \mathrm{d}^{(0,1)}h,a^\phi[(a_*^W)^*(\mathrm{d}^{(0,1)}g),(a_*^W)^*(\mathrm{d}^{(0,1)}f)]\rangle\\
&= \langle \mathrm{d}^{(0,1)}h,[a^\phi\circ(a_*^W)^*(\mathrm{d}^{(0,1)}g),a^\phi\circ(a_*^W)^*(\mathrm{d}^{(0,1)}f)]-a_*^W(\varphi(d_*^W g,d_*^W f,\cdot))\rangle\\
&= \langle \mathrm{d}^{(0,1)}h,[-(\Lambda,E)^\#(\mathrm{d}^{(0,1)}g),-(\Lambda,E)^\#(\mathrm{d}^{(0,1)}f)]\rangle-\varphi(d_*^W g,d_*^W f,d_*^W h)\\
&= \langle \mathrm{d}^{(0,1)}h,(\Lambda,E)^\#(\mathrm{d}^{(0,1)}\{g,f\})+\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}(\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}f,\cdot)\rangle\\
&\quad +\varphi(d_*^W f,d_*^W g,d_*^W h)\\
&= \{\{g,f\},h\}-\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h)\\
&\quad +a_*^W(\varphi)(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h).
\end{aligned}$$

Consequently,

$$\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)}(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h)=a_*^W(\varphi)(\mathrm{d}^{(0,1)}f,\mathrm{d}^{(0,1)}g,\mathrm{d}^{(0,1)}h). \tag{10}$$

Hence, from (8) and (10), we obtain (9). ∎

Looking at equation (10), we remark that the obstruction for (M,Λ,E) to be a Jacobi manifold, i.e. to have $[(\Lambda,E),(\Lambda,E)]^{(0,1)}=(0,0)$, is the image by a_*^W of the element φ in $\Gamma(\bigwedge^3 A^*)$. This obstruction can also be viewed as the image of φ under the infinitesimal action of the Lie algebroid with 1-cocycle (A^*,W) on M (see Appendix). Thus, inspired by the analogous terms of *quasi Poisson $\mathcal{G}$-manifold* ([1], [9]) and of $(\mathcal{G},\phi)$-*quasi Jacobi manifold* (see Section 2.3), we say that the pair (Λ, E) defines on M a (A^*, W)-*quasi Jacobi structure.*

Thus, we have proved:

Theorem 3.2. *Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. Then, the bracket $\{\cdot,\cdot\} : C^\infty(M,\mathbb{R}) \times C^\infty(M,\mathbb{R}) \to C^\infty(M,\mathbb{R})$ given by*

$$\{f,g\} = \langle d^\phi f, d_*^W g\rangle, \quad for\ f,g \in C^\infty(M,\mathbb{R}),$$

defines a (A^*,W)-quasi Jacobi structure on M.

Remark 3.3. In the case where $((A,\phi),(A^*,W),Q)$ is a Jacobi-quasi bialgebroid over M, then, we can easily prove that the Jacobi identity of the bracket defined by (5) is violated by the image of Q under a^ϕ. For this reason, we shall call to the structure (Λ, E) induced on M, an (A,ϕ)-*quasi Jacobi structure.* We note that, for the proof of this result, we use the relation $[d^\phi f, d^\phi g]_* = d^\phi\{f,g\}$, $f,g \in C^\infty(M,\mathbb{R})$, which leads to

$$(\Lambda, E)^\# = a^\phi \circ (a_*^W)^* = -a_*^W \circ (a^\phi)^*. \tag{11}$$

Examples 3.4.

1) A^*-Quasi Poisson structures: If $((A,\phi),(A^*,W),\varphi)$ is a quasi-Lie bialgebroid over M, i.e. both 1-cocycles ϕ and W are zero, Theorem 3.2 establishes the existence of a structure on M, defined by the bracket

$$\{f,g\} = \langle df, d_* g\rangle, \qquad f,g \in C^\infty(M,\mathbb{R}),$$

on $C^\infty(M,\mathbb{R})$, which is associated to a bivector filed Λ on M satisfying $[\Lambda,\Lambda] = 2a_*(\varphi)$. In our terminology, Λ endows M with a A^*-*quasi Poisson structure.* We remark that this result was obtained in [6] by different techniques.

2) Jacobi structures: When $\varphi = 0$, i.e. $((A,\phi),(A^*,W))$ is a Jacobi bialgebroid over M, the structure (Λ, E) on M determined by Theorem 3.2 is a Jacobi structure, and we recover the well known result of [5].

3) Twisted Jacobi structures: When φ is the image of an element $(\varphi_M, \omega_M) \in \Gamma(\bigwedge^3(T^*M \times \mathbb{R}))$ by the transpose map $(a^\phi)^* : \Gamma(\bigwedge^3(T^*M\times\mathbb{R})) \to \Gamma(\bigwedge^3 A^*)$ of a^ϕ, i.e. $\varphi = (a^\phi)^*(\varphi_M, \omega_M)$, then,

$$\frac{1}{2}[(\Lambda,E),(\Lambda,E)]^{(0,1)} = a_*^W(\varphi) = a_*^W((a^\phi)^*(\varphi_M,\omega_M)) \overset{(7)}{=} (\Lambda,E)^\#(\varphi_M,\omega_M).$$

Also, we have

$$d^{\phi}(\underbrace{(a^{\phi})^*(\varphi_M,\omega_M)}_{=\varphi}) = 0 \Leftrightarrow (a^{\phi})^*(\mathrm{d}^{(0,1)}(\varphi_M,\omega_M)) = 0,$$

which means that (φ_M,ω_M) is $\mathrm{d}^{(0,1)}$-closed on the distribution $Im(a^{\phi})$. This distribution is not, in general, involutive due to condition 2) of Definition 2.1. However, when $Im(a^{\phi})$ is involutive, as in the case where a^{ϕ} is surjective, $((\Lambda,E),\omega_M)$ defines a twisted Jacobi structure on the leaves of $Im(a^{\phi})$.

4) The case of the quasi-Jacobi bialgebroid associated to a twisted Jacobi manifold: Let $(M,(\Lambda_1,E_1),\omega)$ be a twisted Jacobi manifold and let $((TM\times \mathbb{R},[\cdot,\cdot]',\pi,(0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]^{\omega}_{(\Lambda_1,E_1)},\pi\circ(\Lambda_1,E_1)^{\#},(-E_1,0)),(\delta\omega,\omega))$ be its associated quasi-Jacobi bialgebroid. Then, the $(T^*M\times\mathbb{R},(-E_1,0))$-quasi Jacobi structure induced on M coincides with the initial structure (Λ_1,E_1). In fact, for any $f,g\in C^{\infty}(M,\mathbb{R})$ and taking into account that $\mathrm{d}'^{(0,1)}f=\mathrm{d}^{(0,1)}f$, where d' is the quasi-differential of $TM\times\mathbb{R}$ determined by the structure $([\cdot,\cdot]',\pi)$, and that $(d_*^{\omega})^{(-E_1,0)}g = -(\Lambda_1,E_1)^{\#}(\mathrm{d}^{(0,1)}g)$, we have

$$\{f,g\} = \langle \mathrm{d}'^{(0,1)}f,(d_*^{\omega})^{(-E_1,0)}g\rangle = \langle \mathrm{d}^{(0,1)}f,-(\Lambda_1,E_1)^{\#}(\mathrm{d}^{(0,1)}g)\rangle = \{f,g\}_1,$$

where $\{\cdot,\cdot\}_1$ denotes the bracket associated to (Λ_1,E_1).

Moreover, if we consider the Jacobi-quasi bialgebroid $((TM\times\mathbb{R},[\cdot,\cdot],\pi,(0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]_{(\Lambda_1,E_1)},\pi\circ(\Lambda_1,E_1)^{\#},(-E_1,0)),(\Lambda_1,E_1)^{\#}(\delta\omega,\omega))$ associated to the twisted Jacobi manifold $(M,(\Lambda_1,E_1),\omega)$, we get that the $(TM\times\mathbb{R},(0,1))$-quasi Jacobi structure (Λ,E) induced on M is the opposite of (Λ_1,E_1). It suffices to remark that

$$\begin{aligned}(\Lambda,E)^{\#} &\overset{(11)}{=} \pi^{(0,1)}\circ((\pi\circ(\Lambda_1,E_1)^{\#})^{(-E_1,0)})^* \\ &= \pi^{(0,1)}\circ((\Lambda_1,E_1)^{\#})^*\circ(\pi^{(0,1)})^* = -(\Lambda_1,E_1)^{\#}. \qquad (12)\end{aligned}$$

5) The induced structure on a quasi Jacobi manifold: We consider the Jacobi-quasi bialgebroid

$$((TM\times\mathbb{R},[\cdot,\cdot],\pi,(0,1)),(T^*M\times\mathbb{R},[\cdot,\cdot]_{(\Lambda_1,E_1)},\pi\circ(\Lambda_1,E_1)^{\#},(-E_1,0)),a^{\phi}(Q_{\psi}))$$

associated to a $(\mathcal{G},\phi)$-quasi Jacobi manifold (M,Λ_1,E_1). Then, repeating the computation (12), we conclude that, as in the previous case, the $(TM\times\mathbb{R},(0,1))$-quasi Jacobi structure (Λ,E) induced on M is the opposite of (Λ_1,E_1).

6) The case of a triangular quasi-Jacobi bialgebroid: If we consider a triangular quasi-Jacobi bialgebroid over M of type $((A,[\cdot,\cdot]',a,\phi),(A^*,[\cdot,\cdot]^{\varphi}_{\Pi}, a_*,W),\varphi)$, presented in Theorem 2.14, then, for all $f\in C^\infty(M,\mathbb{R})$,

$$d_*^W f = -\Pi^\#(d^\phi f) = -(\Pi^\# \circ (a^\phi)^*)(\mathrm{d}^{(0,1)}f).$$

So, the bracket (5) in $C^\infty(M,\mathbb{R})$ is given by

$$\{f,g\} = \langle d^\phi f, d_*^W g\rangle = \langle \mathrm{d}^{(0,1)}g, (a^\phi\circ\Pi^\#\circ(a^\phi)^*)\mathrm{d}^{(0,1)}f\rangle.$$

On the other hand, considering the (A^*,W)-quasi Jacobi structure (Λ,E) on M, we also have

$$\{f,g\} = \langle \mathrm{d}^{(0,1)}g, (\Lambda,E)^\#(\mathrm{d}^{(0,1)}f)\rangle.$$

Hence,

$$(\Lambda,E)^\# = a^\phi\circ\Pi^\#\circ(a^\phi)^*,$$

which means that (Λ,E) is the image by a^ϕ of Π and that a^ϕ is a type of "twisted Jacobi morphism" between (A,ϕ,Π) and $(TM\times\mathbb{R},(0,1),(\Lambda,E))$.

4. Quasi-Lie bialgebroids associated to quasi-Jacobi bialgebroids

Given a Lie algebroid $(A,[\cdot,\cdot],a)$ over M, we can endow the vector bundle $\tilde{A}=A\times\mathbb{R}\to M\times\mathbb{R}$ with a Lie algebroid structure over $M\times\mathbb{R}$ as follows. The sections of $\tilde{A}$ can be identified with the t-dependent sections of A, t being the canonical coordinate on $\mathbb{R}$, i.e., for any $\tilde{X}\in\Gamma(\tilde{A})$ and $(x,t)\in M\times\mathbb{R}$, $\tilde{X}(x,t)=\tilde{X}_t(x)$, where $\tilde{X}_t\in\Gamma(A)$. This identification induces, in a natural way, a Lie bracket on $\Gamma(\tilde{A})$, also denoted by $[\cdot,\cdot]$:

$$[\tilde{X},\tilde{Y}](x,t)=[\tilde{X}_t,\tilde{Y}_t](x),\quad \tilde{X},\tilde{Y}\in\Gamma(\tilde{A}),\ (x,t)\in M\times\mathbb{R},$$

and a bundle map, also denoted by a, $a:\tilde{A}\to T(M\times\mathbb{R})\equiv TM\oplus T\mathbb{R}$ with $a(\tilde{X})=a(\tilde{X}_t)$, in such a way that $(\tilde{A},[\cdot,\cdot],a)$ becomes a Lie algebroid over $M\times\mathbb{R}$. If ϕ is a 1-cocycle of the Lie algebroid A, we know from [5] that $\tilde{A}$ can be equipped with two other Lie algebroid structures over $M\times\mathbb{R}$, $([\cdot,\cdot]\tilde{}^{\phi},\widetilde{a}^{\phi})$ and $([\cdot,\cdot]\hat{}^{\phi},\widehat{a}^{\phi})$ given, for all $\tilde{X},\tilde{Y}\in\Gamma(\tilde{A})$, by

$$[\tilde{X},\tilde{Y}]\tilde{}^{\phi}=[\tilde{X}_t,\tilde{Y}_t]+\langle\phi,\tilde{X}_t\rangle\frac{\partial\tilde{Y}}{\partial t}-\langle\phi,\tilde{Y}_t\rangle\frac{\partial\tilde{X}}{\partial t};\tag{13}$$

$$\widetilde{a}^{\phi}(\tilde{X})=a(\tilde{X}_t)+\langle\phi,\tilde{X}\rangle\frac{\partial}{\partial t};\tag{14}$$

and

$$[\tilde{X},\tilde{Y}]\hat{}^{\phi} = e^{-t}\left([\tilde{X}_t,\tilde{Y}_t] + \langle\phi,\tilde{X}_t\rangle(\frac{\partial\tilde{Y}}{\partial t} - \tilde{Y}) - \langle\phi,\tilde{Y}_t\rangle(\frac{\partial\tilde{X}}{\partial t} - \tilde{X})\right); \qquad (15)$$

$$\widehat{a}^{\phi}(\tilde{X}) = e^{-t}(\widetilde{a}^{\phi}(\tilde{X})). \qquad (16)$$

Let $((A,\phi),(A^*,W),\varphi)$ be a quasi-Jacobi bialgebroid over M. Then, $(A^*,[\cdot,\cdot]_*,a_*,W)$ is a Lie algebroid with 1-cocycle and we can consider on $\tilde{A}^*$ the Lie algebroid structure $([\cdot,\cdot]\hat{}_*^{W},\widehat{a}_*^{W})$ defined by (15) and (16). Although A is not endowed with a Lie algebroid structure, we still can consider on $\Gamma(\tilde{A})$ a bracket $[\cdot,\cdot]\tilde{}^{\phi}$ and a bundle map $\widetilde{a}^{\phi}$ given by (13) and (14), respectively. We set $\tilde{\varphi} = e^t\varphi$.

Theorem 4.1. *Under the above assumptions, we have:*

1) *The triple $((A,\phi),(A^*,W),\varphi)$ is a quasi-Jacobi bialgebroid over M if and only if $(\tilde{A},\tilde{A}^*,\tilde{\varphi})$ is a quasi-Lie bialgebroid over $M\times \mathbb{R}$.*
2) *If $\tilde{\Lambda}$ is the induced $\tilde{A}^*$-quasi Poisson structure on $M\times\mathbb{R}$, then it is the "quasi Poissonization" of the induced (A^*,W)-quasi Jacobi structure (Λ,E) on M.*

Proof: 1) Let us suppose that $((A,\phi),(A^*,W),\varphi)$ is a quasi-Jacobi bialgebroid over M and let $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ be three arbitrary sections in $\Gamma(\tilde{A})$ and $\tilde{f}\in C^\infty(M\times\mathbb{R},\mathbb{R})$. A straightforward computation gives

$$[\tilde{X},\tilde{f}\tilde{Y}]\tilde{}^{\phi} = \tilde{f}[\tilde{X},\tilde{Y}]\tilde{}^{\phi} + (\widetilde{a}^{\phi}(\tilde{X})\tilde{f})\tilde{Y}. \qquad (17)$$

Moreover,

$$\begin{aligned}
\widetilde{a}^{\phi}([\tilde{X},\tilde{Y}]\tilde{}^{\phi}) &= \widetilde{a}^{\phi}([\tilde{X}_t,\tilde{Y}_t] + \langle\phi,\tilde{X}\rangle\frac{\partial\tilde{Y}}{\partial t} - \langle\phi,\tilde{Y}\rangle\frac{\partial\tilde{X}}{\partial t})\\
&= [a(\tilde{X}_t),a(\tilde{Y}_t)] - a_*(\varphi(\tilde{X}_t,\tilde{Y}_t,\cdot)) + \langle\phi,[\tilde{X}_t,\tilde{Y}_t]\rangle\frac{\partial}{\partial t}\\
&\quad +\langle\phi,\tilde{X}\rangle\left(a(\frac{\partial\tilde{Y}}{\partial t}) + \langle\phi,\frac{\partial\tilde{Y}}{\partial t}\rangle\frac{\partial}{\partial t}\right) - \langle\phi,\tilde{Y}\rangle\left(a(\frac{\partial\tilde{X}}{\partial t}) + \langle\phi,\frac{\partial\tilde{X}}{\partial t}\rangle\frac{\partial}{\partial t}\right).
\end{aligned}$$

But, since $d\phi - \varphi(W,\cdot,\cdot) = 0$,

$$\langle\phi,[\tilde{X}_t,\tilde{Y}_t]\rangle = a(\tilde{X}_t)\langle\phi,\tilde{Y}_t\rangle - a(\tilde{Y}_t)\langle\phi,\tilde{X}_t\rangle - \varphi(\tilde{X}_t,\tilde{Y}_t,W).$$

Also,

$$\widehat{a}_*^W(\tilde\varphi(\tilde X,\tilde Y,\cdot)) = a_*(\varphi(\tilde X_t,\tilde Y_t,\cdot)) + \varphi(\tilde X_t,\tilde Y_t,W)\frac{\partial}{\partial t}.$$

Hence, we get

$$\widetilde{a}^\phi([\tilde X,\tilde Y]\widetilde{\ }^\phi) = [\widetilde{a}^\phi(\tilde X),\widetilde{a}^\phi(\tilde Y)] - \widehat{a}_*^W(\tilde\varphi(\tilde X,\tilde Y,\cdot)). \tag{18}$$

On the other hand,

$$\begin{aligned}
[[\tilde X,\tilde Y]\widetilde{\ }^\phi,\tilde Z]\widetilde{\ }^\phi + c.p. &= \left([[\tilde X_t,\tilde Y_t],\tilde Z_t] - d\phi(\tilde X_t,\tilde Y_t)\frac{\partial \tilde Z_t}{\partial t}\right) + c.p. \\
&= -d_*^W(\varphi(\tilde X_t,\tilde Y_t,\tilde Z_t)) - \left(\left(i_{\varphi(\tilde X_t,\tilde Y_t,\cdot)}d_*^W\tilde Z_t + \varphi(W,\tilde X_t,\tilde Y_t)\frac{\partial \tilde Z_t}{\partial t}\right) + c.p.\right) \\
&= -\widehat{d}_*^W(\tilde\varphi(\tilde X,\tilde Y,\tilde Z)) - \left(i_{\tilde\varphi(\tilde X,\tilde Y,\cdot)}\widehat{d}_*^W\tilde Z + c.p.\right),
\end{aligned} \tag{19}$$

where $\widehat{d}_*^W$ denotes the differential operator of the Lie algebroid $(\tilde A^*,[\cdot,\cdot]_*^{\widehat{\ }W},\widehat{a}_*^W)$. From $d^\phi\varphi=0$, we deduce

$$\widetilde{d}^\phi\tilde\varphi = 0, \tag{20}$$

where $\widetilde{d}^\phi$ is the quasi-differential operator determined by the structure $([\cdot,\cdot]\widetilde{\ }^\phi,\widetilde{a}^\phi)$ on $\tilde A$. Finally, after a very long computation we obtain

$$\widehat{d}_*^W[\tilde P,\tilde Q]\widetilde{\ }^\phi = [\widehat{d}_*^W\tilde P,\tilde Q]\widetilde{\ }^\phi + (-1)^{p+1}[\tilde P,\widehat{d}_*^W\tilde Q]\widetilde{\ }^\phi, \tag{21}$$

for $\tilde P\in\Gamma(\bigwedge^p\tilde A)$ and $\tilde Q\in\Gamma(\bigwedge\tilde A)$. From relations (17) to (21), we conclude that $(\tilde A,\tilde A^*,\tilde\varphi)$ is a quasi-Lie bialgebroid over $M\times\mathbb{R}$.

Now, let us suppose that $(\tilde A,\tilde A^*,\tilde\varphi)$ is a quasi-Lie bialgebroid over $M\times\mathbb{R}$ and take three sections X, Y and Z of A and $f\in C^\infty(M,\mathbb{R})$. These sections can be viewed as sections of $\tilde A$ that don't depend on t, as well as the function f can also be viewed as a function on $C^\infty(M\times\mathbb{R},\mathbb{R})$. Condition 1) of Definition 2.1 is immediate from $[X,fY]\widetilde{\ }^\phi = f[X,Y]\widetilde{\ }^\phi + (\widetilde{a}^\phi(X)f)Y$. The condition $\widetilde{a}^\phi([X,Y]\widetilde{\ }^\phi) = [\widetilde{a}^\phi(X),\widetilde{a}^\phi(Y)] - \widehat{a}_*^W(\tilde\varphi(X,Y,\cdot))$ is equivalent to

$$\begin{aligned}
a([X,Y]) + \langle\phi,[X,Y]\rangle\frac{\partial}{\partial t} &= [a(X),a(Y)] - a_*(\varphi(X,Y,\cdot)) \\
&\quad + (a(X)\langle\phi,Y\rangle - a(Y)\langle\phi,X\rangle + \varphi(X,Y,W))\frac{\partial}{\partial t},
\end{aligned}$$

which gives conditions 2) and 4) of Definition 2.1. From $\widetilde{d}^\phi\tilde\varphi=0$ we deduce $d^\phi\varphi=0$. Finally, by similar computations, we obtain the two remaining conditions that lead to the conclusion that $((A,\phi),(A^*,W),\varphi)$ is a quasi-Jacobi bialgebroid over M.

2) Let $\tilde{\Lambda}$ be the $\tilde{A}^*$-quasi Poisson structure induced by $(\tilde{A},\tilde{A}^*,\tilde{\varphi})$ on $M\times I\!R$. For all $\tilde{f},\tilde{g}\in C^\infty(M\times I\!R, I\!R)$, we have

$$\{\tilde{f},\tilde{g}\}=\tilde{\Lambda}(\delta\tilde{f},\delta\tilde{g})$$

and, on the other hand,

$$\{\tilde{f},\tilde{g}\}=\langle\tilde{d}^\phi\tilde{f},\hat{d}_*^W\tilde{g}\rangle=e^{-t}(\langle d\tilde{f},d_*\tilde{g}\rangle+\frac{\partial\tilde{f}}{\partial t}a_*(\phi)(\tilde{g})+\frac{\partial\tilde{g}}{\partial t}a(W)(\tilde{f})).$$

If (Λ,E) is the (A^*,W)-quasi Jacobi structure induced by $((A,\phi),(A^*,W),\varphi)$ on M, since $E=a_*(\phi)=-a(W)$ and $\Lambda(\delta\tilde{f},\delta\tilde{g})=\langle d\tilde{f},d_*\tilde{g}\rangle$, we get that $\tilde{\Lambda}=e^{-t}(\Lambda+\frac{\partial}{\partial t}\wedge E)$. ∎

For the case of Jacobi-quasi bialgebroids we can prove a similar result. Let $((A,\phi),(A^*,W),Q)$ be a Jacobi-quasi bialgebroid over M. We consider on $\tilde{A}$ the Lie algebroid structure $(\widehat{[\cdot,\cdot]}^\phi,\widehat{a}^\phi)$ defined by (15) and (16), on $\tilde{A}^*$ the structure $(\widetilde{[\cdot,\cdot]}_*^W,\widetilde{a}_*^W)$ defined by (13) and (14), and we set $\tilde{Q}=e^tQ$.

Theorem 4.2. *Under the above assumptions, we have:*

1) *The triple $((A,\phi),(A^*,W),Q)$ is a Jacobi-quasi bialgebroid over M if and only if $(\tilde{A},\tilde{A}^*,\tilde{Q})$ is a Lie-quasi bialgebroid over $M\times I\!R$.*
2) *If $\tilde{\Lambda}$ is the $\tilde{A}$-quasi Poisson structure induced on $M\times I\!R$, then it is the "quasi Poissonization" of the (A,ϕ)-quasi Jacobi structure (Λ,E) induced on M.*

5. Appendix: Actions of Lie algebroids with 1-cocycles

In this Appendix, we extend the definition of Lie algebroid action [14], [10], to that of Lie algebroid with 1-cocycle action. We recall

Definition 5.1. *([10]) Let $(A,[\cdot,\cdot],a)$ be a Lie algebroid on M and $\varpi:F\to M$ a fibered manifold with base M, i.e. $\varpi:F\to M$ is a surjective submersion onto M. An* infinitesimal action of A on F *is a $I\!R$-linear map* $\mathbf{ac}:\Gamma(A)\to\Gamma(TF)$ *such that:*
(1) for each $X\in\Gamma(A)$, $\mathbf{ac}(X)$ is projectable to $a(X)$,
(2) the map $\mathbf{ac}$ *preserves brackets,*
(3) the map $\mathbf{ac}$ *is $C^\infty(M,I\!R)$-linear in the following sense: for each $f\in$*

$C^\infty(M, \mathbb{R})$ *and each* $X \in \Gamma(A)$,

$$\mathbf{ac}(fX) = (f \circ \varpi)\mathbf{ac}(X).$$

We extend the above concept as follows:

Definition 5.2. *Let* (A, ϕ) *be a Lie algebroid with* 1*-cocycle over a manifold* M*,* $\varpi : F \to M$ *a fibered manifold with base* M*, i.e.* $\varpi : F \to M$ *is a surjective submersion onto* M*, and* $\mathbf{ac} : \Gamma(A) \to \Gamma(TF)$ *an infinitesimal action of* A *on* F*. An* infinitesimal action of (A, ϕ) on F *is a* $\mathbb{R}$*-linear map* $\mathbf{ac}^\phi : \Gamma(A) \to \Gamma(TF \times \mathbb{R})$ *given, for each* $X \in \Gamma(A)$*, by*

$$\mathbf{ac}^\phi(X) = \mathbf{ac}(X) + \langle \phi, X \rangle.$$

In the particular case where M is a point and therefore A is a Lie algebra, we obtain, from Definition 5.2, the notion of infinitesimal action of a Lie algebra with 1-cocycle on a manifold F, used on the definition of quasi Jacobi structures, in section 2.3.

If, in the Definition 5.2, $F = M$ and $\varpi : M \to M$ is the identity, we get the concept of infinitesimal action of (A, ϕ) on the base manifold M that we have used to characterize the structure induced on the base manifold of a quasi-Jacobi bialgebroid.

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J. M. Nunes da Costa
Departamento de Matemática, Universidade de Coimbra, Apartado 3008, ,3001-454 Coimbra, PORTUGAL

E-mail address: jmcosta@mat.uc.pt

F. Petalidou
Faculty of Sciences and Technology, University of Peloponnese, 22100 Tripoli, GREECE

E-mail address: petalido@uop.gr