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KEY POLYNOMIALS, INVARIANT FACTORS AND AN ACTION OF THE SYMMETRIC GROUP ON YOUNG TABLEAUX

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Abstract: We give a combinatorial description of the invariant factors associated with certain sequences of product of matrices, over a local principal ideal domain, under the action of the symmetric group by place permutation. Lascoux and Schützenberger have defined a permutation on a Young tableau to associate to each Knuth class a right and left key which they have used to give a combinatorial description of a key polynomial. The action of the symmetric group on the sequence of invariant factors generalizes this action of the symmetric group, by Lascoux and Schützenberger, to Young tableaux of the same shape and weight. As a dual translation, we obtain an action of the symmetric group on words congruent with key-tableaux based on nonstandard pairing of parentheses.

Keywords: Action of the symmetric group on Young tableaux, frank words, invariant factors, *jeu de taquin*, key polynomials and matrices over a local principal ideal domain.
AMS Subject Classification (2000): 05E05, 05E10, 15A45.

1. Introduction

The purpose of this paper is to give a combinatorial description of the hexagons defined by the invariant factors associated with a certain type of sequences of product of matrices, over a local principal ideal domain, under the action of the symmetric group by place permutation, and to show its relationship with the combinatorics developed by Lascoux and Schützenberger to give a combinatorial description of key polynomials. Key polynomials were combinatorially investigated by Lascoux and Schützenberger, in the case of the symmetric group, in [12, 13].

Given an n by n non-singular matrix A, with entries in a local principal ideal domain with prime p, by Gaußian elimination one can reduce A to a diagonal matrix Δ_α with diagonal entries $p^{\alpha_1}, \ldots, p^{\alpha_n}$, for unique nonnegative integers $\alpha_1 \geq \ldots \geq \alpha_n$, called the *Smith normal form* of A. The sequence

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$p^{\alpha_1}, \ldots, p^{\alpha_n}$ defines the invariant factors of A, and $\alpha = (\alpha_1, \ldots, \alpha_n)$ the *invariant partition* of A. It is known that α, β, γ are invariant partitions of nonsingular matrices A, B, and C such that $AB = C$ if and only if there exists a Littlewood-Richardson tableau T of type (α, β, γ), that is, a tableau of shape γ/α which rectifies to the key tableau of weight β (Yamanouchi tableau of weight β) [5, 6]. The relationship between invariant factors and the product of Schur functions was noticed earlier by several authors, with different approaches, as P. Hall, J. A. Green, T. Klein, R. C. Thompson *et al* [9, 10, 15, 1]. (For an overview and other interconnectedness, see the survey by W. Fulton [6] as well as [5, 7, 8].)

Let $m = (m_1, \cdots, m_t)$ be a weak composition. Let the symmetric group S_t act on weak compositions of length $\leq t$ *via* the left action $s_i m = (m_1, \ldots, m_{i+1}, m_i, \cdots, m_t)$ with s_i, $1 \leq i \leq t-1$, the simple transpositions of S_t. Let $\beta(m)$ be the unique partition in the orbit $S_t m$ and $\beta'(m)$ its conjugate. $K(m)$ denotes the key-tableau of weight m, that is, the tableau of weight m whose *column shape* is $\beta'(m)$, and $D_{[m_k]}$ the n by n diagonal matrix having the ith diagonal entry equals p whenever $i \in [m_k]$ and 1 otherwise. The invariant partition of $D_{[m_k]}$ is (1^{m_k}). We identify $K(m)$ with the sequence of diagonal matrices $(D_{[m_1]}, \cdots, D_{[m_t]})$ in the sense that the sequence of partitions $(1^{m_1}) \subseteq (1^{m_1}) + (1^{m_2}) \subseteq \cdots \subseteq (1^{m_1}) + \cdots + (1^{m_t}) = \beta'(m)$ defines the key $K(m)$ and, simultaniously, are the invariant partitions of the sequence of product of matrices $D_{[m_1]}$, $D_{[m_1]}D_{[m_2]}$, $\cdots$, $D_{[m_1]}D_{[m_2]} \cdots D_{[m_t]}$. (French notation is adopted.) For instance,

$$K(10325) = \begin{array}{llll} 5 & & & \\ 4 & 5 & & \\ 3 & 4 & 5 & \\ 1 & 3 & 3 & 5 \end{array} \qquad \text{is identified with } \; (D_{[1]}, D_{\emptyset}, D_{[3]}, D_{[2]}, D_{[4]}).$$

Let $T(m)$ be a tableau of skew-shape γ/α and weight m. Let J_k denote the column-word of length m_k defined by the set of column-indices of the letter k in $T(m)$, and put $J := J_t \cdots J_2 J_1$, called the indexing-set word of $T(m)$. The sequence of column lengths of J is $m^{\#}$ the reverse of m. Let w be the word of $T(m)$ defined by concatenation of the column-words of T left to right. Write $(\emptyset \leftarrow w) = (P(w), \mathcal{Q}(w))$ to mean that the row insertion of w produces the pair of tableaux $P = P(w)$ and $\mathcal{Q}(w)$ of the same shape, with $\mathcal{Q}(w)$ a standard tableau. We have $(\emptyset \leftarrow J) = (Q, \mathcal{Q}(J))$ such that $\mathcal{Q}(J) = (std(evac\, P))^t$ and $\mathcal{Q}(w) = (std\, P)^t$, where *evac* denotes evacuation,

to $w|\{4,5\}$ and $w|\{7,8\}$, and bracket all the successive factors 31 and 32 of w'. Thus, we have $w' = 3(31)2-12-$, with the letters 3 and 1 belonging to $\{2\}$ and $\{3\}$, respectively; and then, we have $w'_1 = (3--2)-12-$, with the letters 3 and 2 of this factor belonging to $\{1\}$ and $\{4,5\}$, respectively. Then, we get the decomposition

$$w = w|(\{1,4,5\}, \{7,8\}, \{2,3\}, \{6\}),$$

with $w|\{1,4,5\} = 321$, $w|\{7,8\} = 21$, $w|\{2,3\} = 31$ and $w|\{6\} = 1$. Finally, let $w'' = -31---21$ be the subword of w obtained by removing the subwords $w|(\{1,4,5\} = 321$ and $w|\{6\} = 1$. This word have only one factor 3121 and thus we get the decomposition

$$w = w|(\{2,3,7,8\}^*, \{1,4,5\}^*, \{6\}^*) = \underline{3}\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1},$$

where the underlined letters define 3121, the upperlined letters define 321 and the remaining letter define the shuffle component 1. It is easy to check that the parenthesis matching operations induced by this decomposition are the standard ones:

$$\begin{tikzcd}[column sep=small]
 & 3\,\overline{3\,2}\,\underline{2\,1}\,2\,\overline{2\,1} \arrow[r, dash, "\theta_2"] & 3\,\overline{3\,2}\,\underline{2\,1}\,3\,\overline{3\,1} \arrow[dr, dash, "\theta_1"] & \\
3\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & 3\,\overline{3\,2}\,\underline{2\,1}\,3\,\overline{3\,1} \\
 & 3\,\overline{3\,1}\,\underline{2\,1}\,1\,\overline{2\,1} \arrow[r, dash, "\theta_1"'] & 3\,\overline{3\,2}\,\underline{2\,1}\,2\,\overline{2\,1} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \quad . \tag{3.23}$$

Finally to each hexagon (3.4) corresponds an hexagon (3.1).

Theorem 3.12. [2] *Given an hexagon* (3.4)*, there exists an n by n unimodular matrix U such that, for some partition α, $\Delta_\alpha U K(\sigma\beta(m))$ with σ running in S_3 is a hexagon, whose indexing frank words are those of* (3.4)*.*

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Algorithm 3.10. Let $w \equiv K(\beta_1, \beta_2, \beta_3)$. Our algorithm is presented as a three step definition.

Step 1. Consider the subword $w_{|\{2,1\}}$ and bracket every factor 21 of $w_{|\{2,1\}}$. The letters which are not bracketed constitute a subword of $w_{|\{2,1\}}$. Then bracket every factor 21 of this subword. Again, the letters which are not bracketed constitute a subword. Continue this procedure until it stops, that is, until we get a word consisting of l_1 no bracketed letters $1's$ in w. This bracketing process enables us to decompose w as

$$w|(I_1, \ldots, I_{l_3+l_2}, J_1, \ldots, J_{l_3}, K_1, \ldots, K_{l_1}), \tag{3.22}$$

where $w|I_l = 21$, $l \in [l_3 + l_2]$, $w|J_l = 3$, $l \in [l_3]$, and $w|K_l = 1$, $l \in [l_1]$.

Step 2. Let w' be the subword of w obtained by removing all letters 1 belonging to the factors $w|I_l$, for all $l \in [l_3 + l_2]$. As in the previous step, we bracket all the successive factors 32 and 31 of w'. We get a refinement of the decomposition (3.22), by making the unions of k sets J_l with k sets K_l, for some integer $0 \leq q \leq \min\{l_3, l_1\}$, and making the unions of the remaining $l_3 - q$ sets J_l with $l_3 - q$ sets I_l:

$$w|(F_1, \ldots, F_q, G_1, \ldots, G_{l_3-q}, I_1, \ldots, I_{l_2+q}, K_1, \ldots, K_{l_1-q}),$$

where $w|F_l = 31$, $l \in [q]$, $w|G_l = 321$, $l \in [l_3 - q]$, $w|I_l = 21$, $l \in [l_2 + q]$, and $w|K_l = 1$, $l \in [l_1 - q]$.

Step 3. Finally, let w'' be the subword of w obtained by removing the subwords $w|G_l = 321$ and $w|K_l = 1$, for all $l \geq 1$. As before, we bracket all the successive factors 3121 of w''. This operation consists of the union of the q sets F_l with q sets I_l. The decomposition of w obtained in this way, is denoted by $w|(I_1^*, \ldots, I^*_{l_3+l_2+l_1-q})$, where $w|I_l^* = 3121$, $l \in [q]$, $w|I_l^* = 321$, $l \in [q+1, l_3]$, $w|I_l^* = 21$, $l \in [l_3 + 1, l_3 + l_2]$, and $w|I_l^* = 1$, $l \in [l_3 + l_2 + 1, l_3 + l_2 + l_1 - q]$.

In next example, we illustrate the application of the previous algorithm to a Yamanouchi word.

Example 3.11. Let $w = 33121121 \equiv K(4, 2, 2)$. Following the first step of algorithm 3.10, we bracket all the successive factors 21 of $w_{|\{1,2\}}$, that is, 331(21)1(21), obtaining in this way the decomposition

$$w = w|(\{4, 5\}, \{7, 8\}, \{1\}, \{2\}, \{3\}, \{6\}),$$

where $w|\{4, 5\} = w|\{7, 8\} = 21$, $w|\{1\} = w|\{2\} = 3$ and $w|\{3\} = w|\{6\} = 1$. Next, let $w' = 3312 - 12-$ (where $-$ indicates the place of the suppressed letters) be the subword of w obtained by removing the letters 1 belonging

^{t} transposition and std standardization. J is a frank word if and only if $P = K(m)$. Equivalently $\mathcal{Q}(J) = std(K(m^\#))^t$.

Let U be a n by n unimodular matrix, that is, a matrix whose determinant is not divided by p. Put $\Delta_\alpha UK(m)$ for the sequence Δ_α, $\Delta_\alpha UD_{[m_1]}$, $\Delta_\alpha UD_{[m_1]}D_{[m_2]}, \cdots, \Delta_\alpha UD_{[m_1]}D_{[m_2]}\cdots D_{[m_t]}$. The sequence of invariant partitions $\alpha^0 = \alpha \subseteq \alpha^1 \subseteq \ldots \subseteq \alpha^t$, associated with this sequence of matrices, satisfy for $k = 0, 1, \ldots, t-1$, $|\alpha^{k+1}| - |\alpha^k| = m_{k+1}$ and $\alpha_i^k \leq \alpha_i^{k+1} \leq \alpha_i^k + 1$, for any i. Thus $\Delta_\alpha UK(m)$ is identified with a tableau $T(m)$ of skew-shape γ/α and weight m, and it is shown in [3] that $P(w) = K(m)$ and J is a frank word. When we consider the action of the symmetric S_t on weak compositions of length $\leq t$ *via* the left action, we are at the same time defining an action of the symmetric group on the sequence of matrices $\Delta_\alpha UK(m)$, where U is a fixed unimodular matrix, and, therefore, on tableaux of skew-shape. We obtain two families of hexagons, which are dual translation of each other: one on frank words running over tableaux with the same shape and weight, rather than on the frank words within a Knuth class; and, the other one on key-tableaux based on nonstandard pairing of parentheses. However in each hexagon there is only one tableau and we may associate to it right and left keys. The construction which leads to the first hexagon is based on a particular row shuffle decomposition of a three-column frank word and on a variant of the *jeu de taquin* on a two-column tableau or contretableau. This means that the second hexagon is based on a column shuffle decomposition of a word congruent with a key over a three-letter alphabet and on a nonstandard pairing of parentheses. These hexagons, contain in particular, the ones defined, respectively, by the *jeu de taquin* operation, and by the operation based on the standard matching of parentheses.

2. Variants of the *jeu de taquin* on two-column frank words, pairing of parentheses and invariant factors

In this section, we describe the invariant factors, equivalently, the skew-tableaux on a two-letter alphabet, associated with the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1 m)$ with $m = (m_1, m_2)$. For this, we have to define *variants* of the *jeu de taquin* on two-column frank words and to show its relationship with pairings of parentheses on words congruent with keys over a two-letter alphabet.

We denote by Θ the *jeu de taquin* operation on a two-column tableau or contre-tableau (a two-column skew-tableau such that the pair of columns is

aligned at the top) $J_2\,J_1$, and by $\tilde{\Theta}$ a variant of Θ which runs as follows. If J_2J_1 is a contretableau (tableau), slide vertically the entries of the column J_2 (J_1) along the column J_1 (J_2) such that the row weak increasing order is preserved, and a common label to the two columns never has a vacant west (east) neighbor. Then exchange the vacant positions with the east (west) neighbors. In particular, when the first (second) column J_2 (J_1) is slided down (up) maximally such that the row weakly increasing order is preserved, we get the *jeu de taquin*. For instance,

$$\Theta : \begin{array}{cc} 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} \blacksquare & 5 \\ \blacksquare & 4 \\ 2 & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 4 & \blacksquare \\ 2 & 3 \\ 1 & 2 \end{array}, \tag{2.1}$$

$$\tilde{\Theta} : \begin{array}{cc} 2 & 5 \\ 1 & 4 \\ \blacksquare & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} \blacksquare & 5 \\ 2 & 4 \\ \blacksquare & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 2 & 4 \\ 3 & \blacksquare \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 5 & \blacksquare \\ 3 & \blacksquare \\ 2 & 4 \\ 1 & 2 \end{array}. \tag{2.2}$$

Clearly, $\tilde{\Theta}(J_2J_1)$ and $\Theta(J_2J_1)$ are not congruent unless $\tilde{\Theta} = \Theta$, but $\tilde{\Theta}(J_2J_1)$ is a frank word with the same shape and weight as $\Theta(J_2J_1)$.

Let $w = w_1w_2\dots w_k$ be a word on the two-letter alphabet $\{r, r+1\}$. A pairing of w is a set of indexed pairs (called r-pairs) (w_i, w_j) such that $1 \le i < j \le k$, $w_i = r+1$, and $w_j = r$, and if (w_l, w_s) is another pair, then i, l, j, s are pairwise distinct. View each r (resp. $r+1$) as a left (resp. right) parenthesis. The r-pairs of w are precisely the matched parentheses. Furthermore the subword of unpaired $r's$ and $(r+1)'s$ is a subword of w the form $r^k(r+1)^l$. In general, not every r-pairing gives the maximal number of r-pairs of w,and if $\tilde{\theta}_r$ is the operation which replaces the word $r^k(r+1)^l$ of unpaired $r's$ and $(r+1)'s$ in w (in the corresponding positions) by $r^l(r+1)^k$, unless certain conditions are imposed on the r-pairing, the maximal number of r-pairs of $\tilde{\theta}_r w$ and w may be different. However, when either $k = 0$ or $l = 0$, although w and $\tilde{\theta}_r w$ may have different r-pairings, they have always the same maximal number of r-pairs. We shall restrict ourselves to words w in these conditions, that is, w is a word on a two-letter alphabet congruent with a two-letter key. In this case, the operation $\tilde{\theta}_r$ can be reduced to a variant of *jeu de taquin* on two-column frank words. In particular, the operation

$k \ge 0$ words 3121, l_1 words 321, l_2 words 21 and $l_3 - k$ words 1, $w = sh((3121)^k, (321)^{l_1}, (21)^{l_2}, 1^{l_3-k})$, such that

(a) $\tilde{\theta}_i w = sh((\theta_i 3121)^k, (\theta_i 321)^{l_1}, (\theta_i 21)^{l_2}, (\theta_i 1)^{l_3-k})$, $i = 1, 2$;

(b) $\tilde{\theta}_i\tilde{\theta}_j w = sh((\theta_i\theta_j 3121)^k, (\theta_i\theta_j 321)^{l_1}, (\theta_i\theta_j 21)^{l_2}, (\theta_i\theta_j 1)^{l_3-k})$, $1 \le i \ne j \le 2$;

(c) $\tilde{\theta}_1\tilde{\theta}_2\tilde{\theta}_1 w = sh((\theta_1\theta_2\theta_1 3121)^k, (\theta_1\theta_2\theta_1 321)^{l_1}, (\theta_1\theta_2\theta_1 21)^{l_2}, (\theta_1\theta_2\theta_1 1)^{l_3-k})$.

That is, the hexagon (3.2) is a "shuffle" of the hexagons (3.17), (3.18), (3.19) and (3.20) with the appropriate multiplicities.

Example 3.8. The hexagon (3.13) gives gives rise to the hexagon, below, where the operations are based on nonstandard pairing of parentheses

$$\begin{tikzcd}[column sep=small]
& \underline{3}\,\underline{2}\,3\,2\,\bar{2}\,\underline{2}\,\underline{1} \arrow[r, dash, "\theta_2"] & \underline{3}\,\underline{1}\,3\,1\,\bar{1}\,\underline{2}\,\underline{1} \arrow[dr, dash, "\tilde{\theta}_1"] & \\
\underline{3}\,\underline{2}\,3\,2\,\bar{3}\,\underline{3}\,\underline{1} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\tilde{\theta}_2"'] & & & \underline{3}\,\underline{1}\,2\,1\,\bar{1}\,\underline{2}\,\underline{1} \\
& \underline{3}\,\underline{2}\,3\,1\,\bar{3}\,\underline{3}\,\underline{1} \arrow[r, dash, "\theta_1"'] & \underline{3}\,\underline{2}\,2\,1\,\bar{2}\,\underline{2}\,\underline{1} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.21}$$

(the bared letters indicate the subwords 3121 and 1 in the shuffle).

Remark 3.9. The following example is the translation of the previous remark to hexagon (3.2). The hexagon

$$\begin{tikzcd}[column sep=small]
& 3221 \arrow[r, dash, "\tilde{\theta}_2"] & 3321 \arrow[dr, dash, "\tilde{\theta}_1"] & \\
3211 \arrow[ur, dash, "\tilde{\theta}_1"] \arrow[dr, dash, "\tilde{\theta}_2"'] & & & 3321 \\
& 3211 \arrow[r, dash, "\tilde{\theta}_1"'] & 3221 \arrow[ur, dash, "\tilde{\theta}_2"'] &
\end{tikzcd}$$

is not a shuffle of the two hexagons (3.18) and (3.20).

We will show that this family of actions of S_3, induced by the different shuffle decompositions of a Yamanouchi word w over a three letter alphabet, includes the action defined by the operations θ_i, $i = 1, 2$. This is achieved in the following algorithm, where we exhibit a special shuffle decomposition for w. As a consequence, using (3.16), the hexagon (3.4) contains, in particular, the action defined by the *jeu de taquin*. We denote by $w_{|A}$ the subword of w obtained by suppressing the letters not in A. If $X \subseteq [l]$ with l the length of w, then $w|X$ is the subword of w defined by the letters of w in positions X.

where Σ is obtained by sorting the billeters of Π by weakly increasing rearrangement for the anti-lexicographic order with priority on the first row. Since $(J_3J_2J_1)\uparrow$ is a shuffle of X_1, X_2, X_3 and X_4, then w is a shuffle of 3121, 321, 21, 1 such that $w|X_1 = 3121$, $w|X_2 = 321$, $w|X_3 = 21$, $w|X_4 = 1$, and $w|X_i$ denotes the subword of w defined by the letters below the positions X_i of the top word in the biword Σ. Therefore the hexagon (3.2) is a "*shuffle*" of four hexagons,

$$\begin{tikzcd}[column sep=small]
& \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 3\ 1 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 1\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 3\ 1 \end{pmatrix} \\
& \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 1\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} c^2a^2b^4a^4 \\ 3\ 2\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.17}$$

$$\begin{tikzcd}[column sep=small]
& \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \\
& \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} c^5a^5b^5 \\ 3\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.18}$$

$$\begin{tikzcd}[column sep=small]
& \begin{pmatrix} b^3a^3 \\ 3\ 1 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} b^3a^3 \\ 3\ 2 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} b^3a^3 \\ 2\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} b^3a^3 \\ 3\ 2 \end{pmatrix} \\
& \begin{pmatrix} b^3a^3 \\ 2\ 1 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} b^3a^3 \\ 3\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd} \tag{3.19}$$

$$\begin{tikzcd}[column sep=small]
& \begin{pmatrix} a^1 \\ 1 \end{pmatrix} \arrow[r, dash, "\theta_2"] & \begin{pmatrix} a^1 \\ 2 \end{pmatrix} \arrow[dr, dash, "\theta_1"] & \\
\begin{pmatrix} a^1 \\ 1 \end{pmatrix} \arrow[ur, dash, "\theta_1"] \arrow[dr, dash, "\theta_2"'] & & & \begin{pmatrix} a^1 \\ 3 \end{pmatrix} \\
& \begin{pmatrix} a^1 \\ 2 \end{pmatrix} \arrow[r, dash, "\theta_1"'] & \begin{pmatrix} a^1 \\ 3 \end{pmatrix} \arrow[ur, dash, "\theta_2"'] &
\end{tikzcd}\ . \tag{3.20}$$

Indeed, by corollary 3.1, every Yamanouchi word w on a three-letter alphabet is a shuffle of $k \geq 0$ words 3121, l_1 words 321, l_2 words 21 and $l_3 - k$ words 1, that, by abuse of notation, we shall write $w = sh((3121)^k, (321)^{l_1}, (21)^{l_2}, 1^{l_3-k})$.

Theorem 3.7. *The vertices of the hexagon* (3.2) *are the words of the tableaux of skew-shape defined by the hexagon* (3.1) *only if there exist a shuffle of*

based on the standard r-pairing, denoted by θ_r, can be reduced to the *jeu de taquin*.

Suppose that w is congruent with the key of weight $(0^{r-1}, m_r, m_{r+1})$. Without loss of generality, assume $m_{r+1} \leq m_r$. Let $J_{r+1}J_r$ be a frank word of column shape $(m_{r+1}, m_r, 0^{r-1})$, such that by sorting the billeters of the biword $\Sigma' = \begin{pmatrix} J_{r+1}J_r \\ (r+1)^{m_{r+1}}r^{m_r} \end{pmatrix}$, by weakly increasing rearrangement of the billeters for the anti-lexicographic order with priority on the first row, we get $\Sigma = \begin{pmatrix} J_{r+1}J_r\uparrow \\ w \end{pmatrix}$, where $J_{r+1}J_r\uparrow$ indicates $J_{r+1}J_r$ by weakly increasing order. Consider an r-pairing in w defined by an increasing injection $i : J_{r+1} \longrightarrow J_r$, that is, $x \leq i(x)$, such that $J_r \cap J_{r+1} \subseteq i(J_{r+1})$. (We identify a column word with its underlying set.) To perform $\tilde{\theta}_r w$ based on this r-pairing means to apply an operation $\tilde{\Theta}$ on $J_{r+1}J_r$ (denoted by $\tilde{\Theta}_r$) which exchanges the vacant entries of the first column with the correspondent east neighbors consisting of $J_r \backslash i(J_{r+1})$ in the second column J_r. Conversely, an operation $\tilde{\Theta}_r$ on $J_{r+1}J_r$ means an operation $\tilde{\theta}_r$ on w, where the r-pairing on w is defined by any increasing injection $i : J_{r+1} \longrightarrow J_r$ such that $\tilde{\Theta}J_{r+1}J_r = [J_{r+1} \cup (J_r \backslash B)]B$, where $J_r \cap J_{r+1} \subseteq i(J_{r+1}) = B$. When $\tilde{\Theta}_r = \Theta_r$ we get the standard pairing of parentheses on w and thus θ_r. Thus the operations $\tilde{\Theta}_r$, Θ_r and $\tilde{\theta}_r$, θ_r are respectively translated into each other, according the following commutative diagram,

$$\begin{tikzcd}
\Sigma = \begin{pmatrix} J_{r+1}J_r\uparrow \\ w \end{pmatrix} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \Sigma' = \begin{pmatrix} J_{r+1}J_r \\ (r+1)^{m_{r+1}}r^{m_r} \end{pmatrix} \arrow[d, leftrightarrow] \\
\tilde{\Sigma} = \begin{pmatrix} \tilde{\Theta}(J_{r+1}J_r)\uparrow \\ \tilde{\theta}_r w \end{pmatrix} \arrow[r, leftrightarrow] & \tilde{\Sigma}' = \begin{pmatrix} \tilde{\Theta}(J_{r+1}J_r) \\ (r+1)^{m_r}r^{m_{r+1}} \end{pmatrix}
\end{tikzcd}\ . \tag{2.3}$$

If $(\emptyset \leftarrow w) = (P, Q)$ then $(\emptyset \leftarrow \tilde{\theta}_r w) = (\theta_r P, Q')$, where Q and Q' are distinct unless $\tilde{\theta}_r = \theta_r$. As $\tilde{\Theta}_r$ runs out of the congruence class, $\tilde{\theta}_r$ does not preserve the Q-symbol but we have $\theta_r w \equiv \tilde{\theta}_r w$. For instance, in (2.1), any increasing injection $\{1, 2\} \to \{2, 3\}$ defines a standard pairing of parentheses, giving rise to $\theta_1 : (2(21)1)1 \to (2(21)1)2$; and in (2.2), any increasing injection $\{1, 2\} \to \{2, 4\}$ defines a pairing of parentheses, giving rise to $\tilde{\theta}_1 : (2(21)11) \to (2(21)21)$.

We are now in conditions to describe the invariant factors, equivalently, the skew-tableaux on a two-letter alphabet associated with the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1m)$.

Lemma 2.1. [2] (*a*) *Let U be a n by n unimodular matrix. Then, there exists $\sigma \in \mathcal{S}_n$ such that $U = TP_\sigma QL$, where T is an n by n upper triangular matrix, with $1's$ along the main diagonal, Q is an n by n upper triangular matrix, with $1's$ along the main diagonal, and multiples of p above it, and L is an n by n lower triangular matrix, with units along the main diagonal.*

(*b*) *By elementary operations on the left and on the right, $\Delta_\alpha UK(m)$ may be considered equal to $\Delta_\alpha P_\sigma QK(m)$, with $\sigma \in \mathcal{S}_n$.*

(*c*) *The Smith normal form of $\Delta_\alpha P_\sigma QD_{[m_1]}$, with $\sigma \in \mathcal{S}_n$, is the diagonal matrix Δ_{α^1} where $\alpha \subseteq \alpha^1$ is the horizontal strip tableau of skew-shape α^1/α.*

Theorem 2.2. [2] *Let $m = (m_1, m_2)$. Let T and T' be respectively the tableaux defined by the sequences $\Delta_\alpha UK(m)$ and $\Delta_\alpha UK(s_1m)$, with indexing-set words J_2J_1, $J_2'J_1'$, and words w, w'. Then,*

(*a*) *J_2J_1, $J_2'J_1'$ are frank words such that $\tilde{\Theta}_1J_2J_1 = J_2'J_1'$.*

(*b*) *$w \equiv K(m)$ and $w' = \tilde{\theta}_1\omega \equiv K(s_1m)$.*

Conversely, if T and T' are respectively tableaux of skew-shape with indexing-set frank words J_2J_1 and $J_2'J_1'$ satisfying $J_2'J_1' = \tilde{\Theta}_1J_2J_1$, then there exist an unimodular matrix U such that $\Delta_\alpha UK(m)$ and $\Delta_\alpha U'K(s_1m)$ define the tableaux T and T' respectively.

Example 2.3. Let $U = P_{4321}T_{14}(p)$, where P_{4321} is the permutation matrix associated with $4321 \in S_4$ and $T_{14}(p)$ is the elementary matrix obtained from the identity by placing the prime p in position $(1,4)$. With $\alpha = (2,1)$ the sequences $\Delta_\alpha U(D_{[3]}, D_{[2]})$ and $\Delta_\alpha U(D_{[2]}, D_{[3]})$ define, respectively, $T = \begin{array}{llll} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$ and $T' = \begin{array}{llll} 2 & & & \\ \bullet & 2 & 2 & \\ \bullet & \bullet & 1 & 1 \end{array}$. The words $w = 21211$ of T and $w' = 22211$ of T' satisfy $\tilde{\theta}_1 w = w' \equiv \theta_1\omega$, where $\tilde{\theta}_1$ is the operation based on the parentheses matching $(21(21)1)$. However, if we choose $U' = P_{3241}T_{24}(p)$, the sequences $\Delta_\alpha U'$ $(D_{[3]}, D_{[2]})$ and $\Delta_\alpha U'$ $(D_{[2]}, D_{[3]})$ define, respectively, T and $T'' = \begin{array}{llll} 2 & & & \\ \bullet & 1 & 2 & \\ \bullet & \bullet & 1 & 2 \end{array}$. In this case, the word w'' of T'' satisfy $\theta_1 w = w''$. The corresponding operations on the indexing frank words are displayed as

$$\begin{tikzcd}
 & \begin{smallmatrix} & 3 & 3 \\ & 5 & 5 \\ 2 & 2 & \\ & 4 & \end{smallmatrix} \arrow[r, dash, "\Theta_2"] & \begin{smallmatrix} 3 & & 3 \\ 5 & & 5 \\ 2 & 2 & \\ 4 & & \end{smallmatrix} \arrow[dr, dash, "\Theta_1"] & \\
\begin{smallmatrix} & 3 & 3 \\ & 5 & 5 \\ 2 & & 2 \\ & & 4 \end{smallmatrix} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\Theta_2"'] & & & \begin{smallmatrix} 3 & & 3 \\ 5 & 5 & \\ 2 & 2 & \\ 4 & & \end{smallmatrix} \\
 & \begin{smallmatrix} & 3 & 3 \\ 5 & & 5 \\ 2 & & 2 \\ & & 4 \end{smallmatrix} \arrow[r, dash, "\Theta_1"'] & \begin{smallmatrix} & 3 & 3 \\ 5 & 5 & \\ 2 & 2 & \\ & 4 & \end{smallmatrix} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd} \quad . \tag{3.14}$$

The second one gives the frank words in the Knuth class of J.

We may now describe the hexagon (3.2). Without loss of generality, we may consider the hexagon (3.4) in the simplified form in the sense that the sets A_i^j are singular,

$$J = \begin{tikzcd}
 & \begin{smallmatrix} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ & b^3 & a^3 \\ c^2 & a^2 & \\ & a^1 & \end{smallmatrix} \arrow[r, dash, "\tilde{\Theta}_2"] & \begin{smallmatrix} c^5 & b^5 & a^5 \\ b^4 & & a^4 \\ b^3 & & a^3 \\ c^2 & a^2 & \\ a^1 & & \end{smallmatrix} \arrow[dr, dash, "\tilde{\Theta}_1"] & \\
\begin{smallmatrix} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ & b^3 & a^3 \\ c^2 & & a^2 \\ & & a^1 \end{smallmatrix} \arrow[ur, dash, "\tilde{\Theta}_1"] \arrow[dr, dash, "\tilde{\Theta}_2"'] & & & \begin{smallmatrix} c^5 & b^5 & a^5 \\ b^4 & & a^4 \\ b^3 & a^3 & \\ c^2 & a^2 & \\ a^1 & & \end{smallmatrix} \\
 & \begin{smallmatrix} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ b^3 & & a^3 \\ c^2 & & a^2 \\ & & a^1 \end{smallmatrix} \arrow[r, dash, "\tilde{\Theta}_1"'] & \begin{smallmatrix} c^5 & b^5 & a^5 \\ & b^4 & a^4 \\ b^3 & a^3 & \\ c^2 & a^2 & \\ & a^1 & \end{smallmatrix} \arrow[ur, dash, "\tilde{\Theta}_2"'] &
\end{tikzcd} \tag{3.15}$$

with $c^j \leq b^j \leq a^j$, $j = 1, \ldots, 5$, and $c^2 \leq a^2 < b^4 \leq a^4$. The contretableau J can be splitted into row words $X_1 = c^2a^2b^4a^4$, $X_2 = c^5b^5a^5$, $X_3 = b^3a^3$, and $X_4 = a^1$, with $c^j \leq b^j \leq a^j$, $j = 1, \ldots, 5$, and $c^2 \leq a^2 < b^4 \leq a^4$. We consider the biwords

$$\Sigma' = \begin{pmatrix} J_3 & J_2 & J_1 \\ 3^2 & 2^3 & 1^5 \end{pmatrix} \longleftrightarrow \Pi = \begin{pmatrix} c^2a^2b^4a^4 & c^5b^5a^5 & b^3a^3 & a^1 \\ 3\ 1\ 2\ 1 & 3\ 2\ 1 & 2\ 1 & 1 \end{pmatrix} \tag{3.16}$$

$$\longleftrightarrow \Sigma = \begin{pmatrix} (J_3J_2J_1)\uparrow \\ w \end{pmatrix},$$

Finally, since $F_3XH_1 = \tilde{\Theta}_2(L_3H_2H_1)$ we have

$$
\begin{aligned}
&X \subseteq H_2,\ |X| = |L_3|,\ L_3 \leq X,\ H_2 \cap L_{\subseteq} X \text{ and} \\
&F_3 = L_3 \cup (H_2 \setminus X).
\end{aligned} \tag{3.12}
$$

By (3.10) and by $A_2^5 \cup A_1^2 \cup A_1^3 = X_2 \subseteq H_2 = L_2 \cup A_1^1 \cup A_1^2 \cup A_1^3$, we conclude that $A_2^5 \subseteq L_2 \cup A_1^1$. But A_2^5 and A_1^1 are disjoint sets, it follows $A_2^5 \subseteq L_2$. Define $A_2^4 = L_2 \setminus A_2^5$ and $A_2^3 = J_2 \setminus L_2$. As $|L_2| = |H_1|$, we also have $|A_1^4| = |A_2^4|$, $|A_1^3| = |A_2^3|$, $(J_1 \setminus A_1^5) \cap A_2^4 \subseteq A_1^4$ and $(J_1 \setminus (A_1^5 \cup A_1^4)) \cap A_2^3 \subseteq A_1^3$. Moreover from the inequality $L_2 \leq H_1$, we get $A_2^4 \leq A_1^4$. By (3.9) and (3.5), we get $A_1^2 < A_2^4$ and by (3.11), we have $A_2^3 \leq A_1^3$. ■

¿From this hexagon we get, respectively, a right key $K_+ = \begin{array}{ccc} A_1^5 & A_1^5 & A_1^5 \\ A_1^4 & A_1^4 & A_1^4 \\ A_1^3 & A_1^3 & \\ A_1^2 & & \\ A_1^1 & & \end{array}$ and

a left key $K_- = \begin{array}{ccc} A_3^5 & A_3^5 & A_3^5 \\ A_2^4 & & \\ A_2^3 & A_2^3 & \\ A_3^2 & A_3^2 & A_3^2 \\ A_1^1 & & \end{array}$, with $K_+ \geq K_-$.

Example 3.6. For instance, considering the contretableau $J = \begin{array}{ccc} 3 & 5 & 5 \\ & 2 & 4 \\ & & 3 \\ & & 2 \end{array}$

we may consider the following decompositions of J which lead to different hexagons.

$$
\begin{tikzcd}[column sep=small]
& \begin{array}{ccc} & 3 & 3\\ & 5 & 5\\ 2 & 2 & \\ & 4 & \end{array} \arrow[r, dash, "\Theta_2"] & \begin{array}{ccc} 3 & & 3\\ 5 & & 5\\ 2 & 2 & \\ 4 & & \end{array} \arrow[dr, dash, "\tilde{\Theta}_1"] & \\
\begin{array}{ccc} & 3 & 3\\ & 5 & 5\\ 2 & & 2\\ & & 4\end{array} \arrow[ur, dash, "\Theta_1"] \arrow[dr, dash, "\tilde{\Theta}_2"'] & & & \begin{array}{ccc} 3 & 3 & \\ 5 & & 5\\ 2 & 2 & \\ 4 & & \end{array}, \\
& \begin{array}{ccc} 3 & & 3\\ & 5 & 5\\ 2 & & 2\\ & & 4\end{array} \arrow[r, dash, "\Theta_1"'] & \begin{array}{ccc} 3 & 3 & \\ & 5 & 5\\ 2 & 2 & \\ & 4 & \end{array} \arrow[ur, dash, "\Theta_2"'] &
\end{tikzcd} \tag{3.13}
$$

follows

$$
\Theta: \begin{array}{cc} \blacksquare & 4 \\ 3 & 3 \\ 1 & 2 \end{array} \longleftrightarrow \begin{array}{cc} 4 & \blacksquare \\ 3 & 3 \\ 1 & 2 \end{array} \qquad \tilde{\Theta}: \begin{array}{cc} 3 & 4 \\ 1 & 3 \\ \blacksquare & 2 \end{array} \longleftrightarrow \begin{array}{cc} 3 & 4 \\ 1 & 3 \\ 2 & \blacksquare \end{array} \longleftrightarrow \begin{array}{cc} 3 & \blacksquare \\ 2 & 4 \\ 1 & 3 \end{array}. \tag{2.4}
$$

The operations Θ_r (θ_r) can be extended to frank words with more than two columns (words on a t-letter alphabet, $t \geq 2$) [11, 14]. Under certain conditions, operations $\tilde{\Theta}_r$ ($\tilde{\theta}_r$) can be extended, as well, to frank words with more than two columns (words on a t-letter alphabet, $t \geq 2$). For this, we generalize a criterion, by Lascoux and Schützenberger in [13], to test whether the concatenation of a frank word with a column word is a frank word. Denote, respectively, by $L(J)$ and $R(J)$ the left and right columns of a frank word J.

Theorem 2.4. [13] *The concatenation JJ' of two frank words J, J' is frank if and only if $R(H)L(H')$ is frank for any pair of frank words H, H' such that $H \equiv J$ and $H' \equiv J'$.*

Notice that when J, J' are column-words, JJ' is frank if and only if JJ' is a tableau or a contretableau. Therefore, we deduce the following criterion for the concatenation of a column with a frank word.

Corollary 2.1. *Let $J = J_k \cdots J_1$ be a frank word and J_{k+1} a column. Then, $J_{k+1}J$ is frank if and only if $J_{k+1}J_k$ and $\overline{J}_k J_{k-1} \cdots J_1$ are frank words, where $\overline{J}_{k+1}\,\overline{J}_k = \Theta_k(J_{k+1}J_k)$.*

The criterion given by this corollary can be generalized to operations $\tilde{\Theta}$. Given two columns B, B', we write $B \leq B'$ [respectively, $B \triangleright B'$] if there is an increasing injection $B \to B'$ [respectively, decreasing injection $B \leftarrow B'$]. We put $|J|$ for the cardinal of J as a set.

Corollary 2.2. *Let $J = J_k \cdots J_1$ be a frank word and J_{k+1} a column. Then, $J_{k+1}J$ is frank if and only if $J_{k+1}J_k$ and $\widetilde{J}_k J_{k-1} \cdots J_1$ are frank words, where $\widetilde{J}_{k+1}\,\widetilde{J}_k = \tilde{\Theta}_k(J_{k+1}J_k)$ for some operation $\tilde{\Theta}_k$.*

Proof: The necessary condition is a consequence of the previous corollary. Reciprocally, assume the existence of an operation $\tilde{\Theta}_k$ in the required conditions, and let $\overline{J}_{k+1}\,\overline{J}_k = \Theta_k(J_{k+1}J_k)$. Clearly, we have $\overline{J}_k \leq \widetilde{J}_k$, and also $\overline{J}_{k+1} \triangleright \widetilde{J}_{k+1}$, since $|\overline{J}_k| = |\widetilde{J}_k|$. By the hypotheses, the product $\widetilde{J}_k L(H)$ is frank, for any frank word $H \equiv J_{k-1} \cdots J_1$. This means that either $\widetilde{J}_k \leq L(H)$, or

$\widetilde{J}_k \rhd L(H)$. By transitivity, we find that either $\overline{J}_k \leq L(H)$, or $\overline{J}_k \rhd L(H)$, *i.e.*, $\overline{J}_k L(H)$ is frank. Thus, by theorem 2.4, the word $\overline{J}_k J_{k-1} \cdots J_1$ is frank, and therefore, by the previous corollary, $J_{k+1}J$ is frank. ■

Theorem 2.5. *Let T be the tableau defined by $\Delta_\alpha U\ K(m)$, with word w and J the indexing set word. Then $P(w) = K(m)$ and J is a frank word of shape $m^\#$.*

Proof: Let $J = J_t \ldots J_1$. We will prove, by induction on $t \geq 1$, that $J_t \cdots J_1$ is a frank word. When $t = 1$ the result is trivial, and the case $t = 2$ is a consequence of theorem 2.2 (see [2]). So, let $t > 2$ and let T be the tableau defined by $\Delta_\alpha UK(m_1, \ldots, m_t)$. By the inductive step, the word $J_{t-1} \cdots J_1$ is frank, since the sequence $\Delta_\alpha UK(m_1, \ldots, m_{t-1})$ defines the tableau T' with indexing set word $J_{t-1} \ldots J_1$ and weight $(m_1, \ldots, m_{t-1})$.

By Smith normal form theorem, there is a partition $\bar{\alpha}$ and an unimodular matrix U' such that by elementary row operations, $\Delta_{\bar{\alpha}} UD_{[m_1]} \cdots D_{[m_{t-2}]}$ can be reduced to $\Delta_{\bar{\alpha}} U'$. The sequence $\Delta_{\bar{\alpha}} U'K(m_{t-1}, m_t)$ defines the tableau $\overline{T}$ with indexing sets J_{t-1}, J_t, and weight (m_{t-1}, m_t). By the case $t = 2$, the word $J_t J_{t-1}$ is frank. Moreover, by theorem 2.2, we find that if $\overline{T}'$ is the tableau defined by the sequence $\Delta_{\bar{\alpha}} U, K(m_t, m_{t-1})$, the indexing sets $\overline{J}_{t-1}, \overline{J}_t$ of $\overline{T}'$ satisfy $\overline{J}_t \overline{J}_{t-1} = \tilde{\Theta}_{t-1}(J_t J_{t-1})$ for some operation $\tilde{\Theta}_{t-1}$.

Finally, notice that $\Delta_\alpha UK(m_1, \ldots, m_{t-2}, m_t)$ defines the tableau $\widetilde{T}$ with indexing set word $\overline{J}_{t-1} J_{t-2} \ldots J_1$, and weight $(m_1, \ldots, m_{t-2},\ m_t)$. By the inductive step, $\overline{J}_{t-1} J_{t-2} \cdots J_1$ is a frank word. Thus, by corollary 2.2, the word $J_t \cdots J_1$ is frank, and therefore, $w \equiv K(m)$. ■

3. An action of the symmetric group on Young tableaux

Let U be an n by n unimodular matrix and $(\beta_1, \beta_2, \beta_3) = \beta(m_1, m_2, m_3)$. We consider the following hexagon

$$\begin{tikzcd}[column sep=small]
 & \Delta_\alpha UK(\beta_2,\beta_1,\beta_3) \arrow[r, dash, "s_2"] & \Delta_\alpha UK(\beta_2,\beta_3,\beta_1) \arrow[dr, dash, "s_1"] & \\
\Delta_\alpha UK(\beta_1,\beta_2,\beta_3) \arrow[ur, dash, "s_1"] \arrow[dr, dash, "s_2"'] & & & \Delta_\alpha UK(\beta_3,\beta_2,\beta_1). \\
 & \Delta_\alpha UK(\beta_1,\beta_3,\beta_2) \arrow[r, dash, "s_1"'] & \Delta_\alpha UK(\beta_3,\beta_1,\beta_2) \arrow[ur, dash, "s_2"'] &
\end{tikzcd} \tag{3.1}$$

¿From the discussion in the introduction, we may look at (3.1) as an hexagon whose vertices are tableaux of skew-shape such that the words

Again the frank word $F_3F_2G_1 = \tilde{\Theta}_2(J_3G_2G_1)$ satisfy (V) of corollary 3.1. Then

$$\begin{aligned} &F_2 \subseteq G_2,\ |F_2| = |J_3|,\ J_3 \leq F_2 \leq G_1,\ G_2 \cap J_3 \subseteq F_2 \text{ and} \\ &F_3 = J_3 \cup (G_2 \setminus F_2). \end{aligned} \tag{3.7}$$

By (3.5) and (3.7), we have $F_2 \subseteq G_2 = J_2 \cup (J_1 \setminus G_1)$. Thus, we may write $F_2 = A_2^5 \cup A_1^2$, with $A_2^5 \subseteq J_2$ and $A_1^2 \subseteq J_1 \setminus G_1$. Moreover,since $J_3 \leq F_2$, we may also write $J_3 = A_3^5 \cup A_3^2$, where $A_3^5 \leq A_2^5$ e $A_3^2 \leq A_1^2$ satisfy $|A_3^5| = |A_2^5|$, $|A_3^2| = |A_1^2|$, $G_2 \cap A_3^5 \subseteq A_2^5$ and $G_2 \cap A_3^2 \subseteq A_1^2$. We define $A_1^1 = J_1 \setminus (G_1 \cup A_1^2)$, therefore $J_1 \setminus G_1 = A_1^1 \cup A_1^2$.

The frank word $F_3XH_1 = \tilde{\Theta}_1 F_3F_2G_1$ satisfy (I) of corollary 3.1. Then

$$\begin{aligned} &H_1 \subseteq G_1,\ |H_1| = |F_2|,\ F_2 \leq H_1,\ F_2 \cap G_1 \subseteq H_1 \text{ and} \\ &F_3 \rhd X = F_2 \cup (G_1 \setminus H_1) \rhd H_1. \end{aligned} \tag{3.8}$$

Since $F_2 = A_2^5 \cup A_1^2 \leq H_1$, we can define

$$A_1^5 = \min\{Z \subseteq H_1 : |Z| = |A_2^5| \text{ and } A_2^5 \leq Z\},$$

where the minimum is taken with respect to $\leq$, and $A_1^4 = H_1 \setminus A_1^5$. As $H_1 \subseteq G_1$, put $A_1^3 = G_1 \setminus H_1$. We have $H_1 = A_1^5 \cup A_1^4$ and $X = A_2^5 \cup A_1^2 \cup A_1^3$. From $F_2 \leq H_1$ and the definition of A_1^5, we get

$$A_3^5 \leq A_2^5 \leq A_1^5 \text{ and } A_3^2 \leq A_1^2 < A_1^4,$$

where $A_1^2 < A_1^4$ means that $A_1^2 \leq A_1^4$ and $A_1^2 \cap A_1^4 = \emptyset$. Note that from (3.5) and (3.8), we obtain $J_1 \cap A_2^5 \subseteq A_1^5$. By lemma 3.2

$$F_2 \leq L_2. \tag{3.9}$$

Now we consider the bottom edges of our hexagon (3.3). Since the frank word $L_3H_2H_1 = \tilde{\Theta}_1(L_3L_2J_1)$ satisfy (II) of corollary 3.1 we have

$$\begin{aligned} &H_1 \subseteq J_1,\ |H_1| = |L_2|,\ L_2 \leq H_1,\ L_2 \cap J_1 \subseteq H_1 \text{ and} \\ &L_3 \leq H_2 = L_2 \cup (J_1 \setminus H_1) \rhd H_1. \end{aligned} \tag{3.10}$$

By lemma 3.2, (c), we have

$$G_2 \leq H_2. \tag{3.11}$$

(*b*) *The contretableau* J *has a decomposition, as below, giving rise to the hexagon*

$$\begin{tikzcd}[column sep=small]
 & \begin{array}{ccc}A_3^5&A_2^5&A_1^5\\ &A_2^4&A_1^4\\ &A_2^3&A_1^3\\ A_3^2&A_1^2&\\ &A_1^1&\end{array} \arrow[r, dash, "\tilde\Theta_2"] & \begin{array}{ccc}A_3^5&A_2^5&A_1^5\\ A_2^4&&A_1^4\\ A_2^3&&A_1^3\\ A_3^2&A_1^2&\\ A_1^1&&\end{array} \arrow[dr, dash, "\tilde\Theta_1"] & \\
J=\begin{array}{ccc}A_3^5&A_2^5&A_1^5\\ &A_2^4&A_1^4\\ &A_2^3&A_1^3\\ A_3^2&&A_1^2\\ &&A_1^1\end{array} \arrow[ur, dash, "\tilde\Theta_1"] \arrow[dr, dash, "\tilde\Theta_2"'] & & & \begin{array}{ccc}A_3^5&A_2^5&A_1^5\\ A_2^4&&A_1^4\\ A_2^3&A_1^3&\\ A_3^2&A_1^2&\\ A_1^1&&\end{array}, \\
 & \begin{array}{ccc}A_3^5&A_2^5&A_1^5\\ &A_2^4&A_1^4\\ A_2^3&&A_1^3\\ A_3^2&&A_1^2\\ &&A_1^1\end{array} \arrow[r, dash, "\tilde\Theta_1"'] & \begin{array}{ccc}A_3^5&A_2^5&A_1^5\\ &A_2^4&A_1^4\\ A_2^3&A_1^3&\\ A_3^2&A_1^2&\\ &A_1^1&\end{array} \arrow[ur, dash, "\tilde\Theta_2"'] &
\end{tikzcd} \tag{3.4}$$

where the sets A_i^j *are pairwise disjoint in each column* J_i, $A_{i+1}^j \leq A_i^j$, *with* $|A_{i+1}^j| = |A_i^j|$,

$$A_3^2 \leq A_1^2 < A_2^4 \leq A_1^4,$$

$|A_3^2| = |A_1^2| = |A_2^4| = |A_1^4|$, *and* $J_1 \cap A_2^5 \subseteq A_1^5$, $(J_1 \setminus A_1^5) \cap A_2^4 \subseteq A_1^4$, $[J_1 \setminus (A_1^5 \cup A_1^4)] \cap A_2^3 \subseteq A_1^3$, $[J_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^2 \subseteq A_1^2$, *and* $[J_2 \cup (A_1^2 \cup A_1^1)] \cap A_3^5 \subseteq A_2^5$, *where* $<$ *means* $\leq$ *without common elements.*

Proof: $(b) \Rightarrow (a)$ The vertices of the hexagon (3.4), by previous corollary, are frank words, and clearly satisfy (c) of lemma 3.2.

$(a) \Rightarrow (b)$ The frank words $J_3J_2J_1$ and $J_3G_2G_1$ are, respectively, in the conditions of (IV) and (II) of corollary 3.1 and satisfy $\tilde\Theta_1 J_3J_2J_1 = J_3G_2G_1$. Then

$$\begin{aligned}&G_1 \subseteq J_1,\ |G_1| = |J_2|,\ J_2 \leq G_1,\ J_1 \cap J_2 \subseteq G_1 \text{ and}\\ &G_2 = J_2 \cup (J_1 \setminus G_1),\ J_3 \leq G_2.\end{aligned} \tag{3.5}$$

Since the frank word $\tilde\Theta_2(J_3J_2J_1) = L_3L_2J_1$ is in the conditions of (III) of corollary 3.1 we have

$$\begin{aligned}&L_2 \subseteq J_2,\ |L_2| = |J_3|,\ J_3 \leq L_2 \leq J_1\ J_2 \cap J_3 \subseteq L_2 \text{ and}\\ &L_3 = J_3 \cup (J_2 \setminus L_2).\end{aligned} \tag{3.6}$$

are congruent with a key $K(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})$, and the indexing frank words have column shape $(\beta_{i_1}, \beta_{i_2}, \beta_{i_3})^{\#}$ with (i_1, i_2, i_3) running over the orbit $S_3\beta(m)$. Therefore, we have two hexagons, one defined by the words of the skew-tableaux and the other one defined by the indexing frank words. These hexagons are copies of each other since operations based on pairing of parentheses can be reduced to variations of the *jeu de taquin* on two-column frank words and *vice versa*. Taking into account theorems 2.2 and 2.5, the next statement follows from the hexagon above. Given $\sigma \in S_t$, put $\sigma^{\#} = rev\,\sigma$, where *rev* denotes the longest permutation of S_t.

Theorem 3.1. *Let* $\sigma \in< s_1, s_2 >$, $\theta \in < \theta_1, \theta_2 >$ *and* $\Theta \in < \Theta_1, \Theta_2 >$ *with the same reduced word. Let* $T(\sigma\beta(m))$ *be the tableau defined by* $\Delta_\alpha UK(\sigma\beta(m))$, *with word* σw *and indexing frank word* σJ *of shape* $\sigma^{\#}\beta(m)$. *Then* $\{T(\sigma\beta(m)) : \sigma \in< s_1, s_2 >\}$ *are the vertices of a hexagon such that*

(*a*) *there exist* $\tilde\theta_1$ *and* $\tilde\theta_2$ *satisfying the Moore-Coxeter relations of the symmetric group* S_3, *such that* $\tilde\theta \in < \tilde\theta_1, \tilde\theta_2 >$, *with the same reduced word as* θ, *verifies* $\sigma w = \tilde\theta w \equiv \theta K(\beta) = K(\sigma\beta(m))$.

(*b*) *there exist* $\tilde\Theta_1$ *and* $\tilde\Theta_2$ *satisfying the Moore-Coxeter relations of the symmetric group* S_3, *such that* $\tilde\Theta \in < \tilde\Theta_1, \tilde\Theta_2 >$ *with the same reduced word as* Θ, *verifies* $\sigma J = \tilde\Theta J$.

Our aim is therefore to describe explicitly the operations $\tilde\theta_i$ and $\tilde\Theta_i$ in the hexagons, defined in (a) and (b) of this theorem,

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & \tilde\theta_1 w \arrow[r, dash] & \tilde\theta_2\tilde\theta_1 w \arrow[dr, dash] & \\
w \arrow[ur, dash] \arrow[dr, dash] & & & \tilde\theta_1\tilde\theta_2\tilde\theta_1 w \\
 & \tilde\theta_2 w \arrow[r, dash] & \tilde\theta_1\tilde\theta_2 w \arrow[ur, dash] &
\end{tikzcd} \tag{3.2}$$

and

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & \tilde\Theta_1 J = J_3G_2G_1 \arrow[r, dash] & \tilde\Theta_2\tilde\Theta_1 J = F_3F_2G_1 \arrow[dr, dash] & \\
J = J_3J_2J_1 \arrow[ur, dash] \arrow[dr, dash] & & & \tilde\Theta_1\tilde\Theta_2\tilde\Theta_1 J = F_3XH_1. \\
 & \tilde\Theta_2 J = L_3L_2J_1 \arrow[r, dash] & \tilde\Theta_1\tilde\Theta_2 J = L_3H_2H_1 \arrow[ur, dash] &
\end{tikzcd} \tag{3.3}$$

In fact the hexagon (3.1) and, hence, hexagon (3.3), obey the following conditions. (The translation of these conditions to hexagon (3.2) will be done later.)

Lemma 3.2. [2] *Consider the hexagons (3.1) and (3.3). Then*

(a) *If L_3L_2 and F_3F_2 are, respectively, the indexing frank words of $\Delta_\alpha UK(\beta_1,\beta_3)$ and $\Delta_\alpha UK(\beta_2,\beta_3)$, then $F_2 \le L_2$.*

(b) *If L_3H_2 and J_3G_2 are, respectively, the indexing frank words of $\Delta_\alpha UK(\beta_3,\beta_1)$ and $\Delta_\alpha UK(\beta_2,\beta_1)$, then $G_2 \le H_2$.*

(c) *The operations $\tilde\Theta_1$ and $\tilde\Theta_2$ defining the hexagon (3.3) are such that $\tilde\Theta_2[\tilde\Theta_1 J] = F_3F_2G_1$ with $F_2 \le L_2$, and $\tilde\Theta_1[\tilde\Theta_2 J] = L_3H_2H_1$ with $G_2 \le H_2$.*

Remark 3.3. The conditions (c), in the previous lemma, imposed on the operations of the hexagon (3.3) do not come from the braid relations of the operations $\tilde\Theta_i$. As can be seen in the example below, there are operations $\tilde\Theta_1$ and $\tilde\Theta_2$ which close the hexagon and do not satisfy the conditions in (c). For instance,

$$\begin{tikzcd}
 & \begin{smallmatrix}1&3&\\ &2&4\end{smallmatrix} \arrow[r, "\tilde\Theta_2"] & \begin{smallmatrix}2&&\\1&3&4\end{smallmatrix} \arrow[dr, "\tilde\Theta_1"] & \\
\begin{smallmatrix}1&2&4\\&&3\end{smallmatrix} \arrow[ur, "\tilde\Theta_1"] \arrow[dr, "\tilde\Theta_2"'] & & & \begin{smallmatrix}2&&\\1&3&4\end{smallmatrix} \\
 & \begin{smallmatrix}1&2&4\\&&3\end{smallmatrix} \arrow[r, "\tilde\Theta_1"'] & \begin{smallmatrix}1&3&\\&2&4\end{smallmatrix} \arrow[ur, "\tilde\Theta_2"'] &
\end{tikzcd}.$$

We start to analyse the hexagon (3.3) under the conditions in (c), of the previous lemma. The Knuth class of a key over a three-letter alphabet as well as any frank word with three columns can be characterized in terms of the shuffling operation. This characterization gives a combinatorial explanation of our hexagons (3.1), (3.2) and (3.3). Indeed by Green's theorem the set of all shuffles of the columns of a key are contained in the Knuth class of a key. However under certain conditions we have equality.

Theorem 3.4. [3] *Let K be a key with first column A. Then, the Knuth class of K is equal to the set of all shuffles of its columns if and only if each of its column is either an interval of A or is obtained from an interval of A by removing a single letter.*

This criterion can be easily applied considering the planar representation of the weight of the key-tableau. For instance $K(2,0,1,2,4,2,3)$ is the shuffle of its columns, since each column in the planar representation of the weight

$(2,0,1,2,4,2,3)$,

$$\begin{array}{c|cccc}
7 & \bullet & \bullet & \bullet & \\
6 & \bullet & \bullet & & \\
5 & \bullet & \bullet & \bullet & \bullet \\
4 & \bullet & \bullet & & \\
3 & \bullet & & & \\
1 & \bullet & \bullet & & \\
\hline
 & 1 & 2 & 3 & 4
\end{array}$$

has at most, one gap of size 1. Each column is either an interval of $A = \{1,3,4,5,6,7\}$ or is obtained from an interval of A removing one letter.

Corollary 3.1. *If $K(m)$ is a key over a three-letter alphabet, then the Knuth class of $K(m)$ equals the set of all shuffles of its columns. Equivalently, if J is a three-column frank word of shape m, then J is a shuffle of rows whose lengths, by weakly decreasing order, is $\beta'(m)$, the conjugate shape of $K(m^\#)$. That is J has one of the following forms*

$$\textit{(I)}\ \begin{matrix}A_1^1&&\\A_1^2&A_2^2&\\A_1^3&A_2^3&A_3^3\end{matrix},\quad
\textit{(II)}\ \begin{matrix}&A_2^1&\\A_1^2&A_2^2&\\A_1^3&A_2^3&A_3^3\end{matrix},\quad
\textit{(III)}\ \begin{matrix}&&A_3^1\\A_1^2&&A_3^2\\A_1^3&A_2^3&A_3^3\end{matrix},$$

$$\textit{(IV)}\ \begin{matrix}&&A_3^1\\&A_2^2&A_3^2\\A_1^3&A_2^3&A_3^3\end{matrix},\quad
\textit{(V)}\ \begin{matrix}A_1^1&&\\A_1^2&&A_3^2\\A_1^3&A_2^3&A_3^3\end{matrix},\quad
\textit{(VI)}\ \begin{matrix}&A_2^1&\\&A_2^2&A_3^2\\A_1^3&A_2^3&A_3^3\end{matrix},$$

where $A_1^3 \le A_2^3 \le A_3^3$, with $|A_1^3| = |A_2^3| = |A_3^3|$; $A_i^r \cap A_i^s = \emptyset$, for $r \ne s$, $i = 1,2,3$, and $A_1^2 \le A_2^2$, $A_1^2 \le A_3^2$, $A_2^2 \le A_3^2$, with $|A_1^2| = |A_2^2| = |A_3^2|$.

Theorem 3.5. *Let $J = J_3J_2J_1$ be a contretableau. The following assertions are equivalent.*

(a) *There exist $\tilde\Theta_1$ and $\tilde\Theta_2$ defining the hexagon (3.3) such that $\tilde\Theta_2[\tilde\Theta_1 J] = F_3F_2G_1$ with $F_2 \le L_2$, and $\tilde\Theta_1[\tilde\Theta_2 J] = L_3H_2H_1$ with $G_2 \le H_2$.*