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ON THE NONCOMMUTATIVE HYPERGEOMETRIC EQUATION

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ABSTRACT: Recently, J. A. Tirao [Proc. Nat. Acad. Sci. 100 (14) (2003), 8138–8141] considered a matrix-valued analogue of the ${}_2F_1$ Gauß hypergeometric function and showed that it is the unique solution of a matrix-valued hypergeometric equation analytic at $z = 0$ with value I, the identity matrix, at $z = 0$. We give an independent proof of Tirao's result, extended to the slightly more general setting of hypergeometric functions over an abstract unital Banach algebra. We provide a similar (but more complicated-looking) result for a second type of noncommutative ${}_2F_1$ Gauß hypergeometric function.

KEYWORDS: Noncommutative hypergeometric series, hypergeometric equation.
AMS SUBJECT CLASSIFICATION (2000): Primary 33C05; Secondary 33C99, 34G10, 34K30, 46H99.

1. Introduction

Hypergeometric series with noncommutative parameters and argument, in the special case involving square matrices, have been the subject of recent study, see e.g. [3, 5, 6, 9, 10, 11, 14, 15, 16]. (For the classical theory of hypergeometric series, cf. [2, 7, 8].) In particular, Tirao [16] considered a specific type of matrix-valued hypergeometric function ${}_2F_1$, and showed, among others results, that it satisfies a matrix-valued differential equation of the second order (a "matrix-valued hypergeometric equation"), and conversely that any solution of the latter is a matrix-valued hypergeometric function of the considered type. This result was reformulated by one of the present authors [15] in the more general setting of hypergeometric functions with parameters and argument over an unital Banach algebra R. Specifically, in [14, 15] two related types of noncommutative hypergeometric series were studied, "type I" and "type II", from the view-point of explicit summation theorems they

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satisfy. In the terminology of [14, 15], Tirao's extension of the Gauß hypergeometric function belongs to type I. As a matter of fact, the explicit form of the noncommutative hypergeometric equation satisfied by the type II Gauß hypergeometric function has so far not been determined. (A priori, it is not clear that the type II hypergeometric equation would be of second order or even have a reasonable compact form.) In this paper we give an independent derivation of Tirao's result for the type I Gauß hypergeometric function and succeed in providing an analogous (however, more complicated-looking) result for the type II case.

We refer to [4, 17] and [1, 12, 13] for comprehensive references on Banach algebras, and on differential equations in Banach spaces.

In the following section, we collect some definitions and notations, taken almost verbatim from [14, 15]. These are needed in Section 3 for the study of the type I and type II noncommutative hypergeometric equations.

2. Preliminaries

Let R be a unital Banach algebra, i.e., a ring with a multiplicative identity with some norm $\|\cdot\|$. Throughout this paper, the elements of R will be denoted by capital letters $A, B, C, \ldots$. In general these elements need not commute with each other; however, we may sometimes specify certain commutation relations explicitly. We denote the identity by I and the zero element by O. Whenever a multiplicative inverse element exists for any $A \in R$, we denote it by A^{-1}. (Since R is a unital ring, we have $AA^{-1} = A^{-1}A = I$.) On the other hand, as we shall implicitly assume that all the expressions which appear are well defined, whenever we write A^{-1} we assume its existence. For instance, in (1) and (2) we assume that $C_i + jI$ is invertible for all $1 \le i \le r$, $0 \le j < k$.

An important special case is when R is the ring of $n \times n$ square matrices (our notation is certainly suggestive with respect to this interpretation), or, more generally, one may view R as a space of some abstract operators.

For any nonnegative integers m and l with $m \ge l - 1$ we define the noncommutative product as follows:

$$\prod_{j=l}^{m} A_j = \begin{cases} I & m = l - 1 \\ A_l A_{l+1} \cdots A_m & m \ge l. \end{cases}$$

In [14, 15] a more general definition was given, which however we will not need here.

For nonnegative integers k and r we define the generalized *noncommutative shifted factorial of type I* by

$$\left\lceil \begin{matrix} A_1, A_2, \dots, A_r \\ C_1, C_2, \dots, C_r \end{matrix}; Z \right\rfloor_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (C_i + (k-j)I)^{-1}(A_i + (k-j)I) \right) Z \right], \tag{1}$$

and the *noncommutative shifted factorial of type II* by

$$\left\lfloor \begin{matrix} A_1, A_2, \dots, A_r \\ C_1, C_2, \dots, C_r \end{matrix}; Z \right\rceil_k := \prod_{j=1}^{k} \left[\left(\prod_{i=1}^{r} (C_i + (j-1)I)^{-1}(A_i + (j-1)I) \right) Z \right]. \tag{2}$$

Note the unusual usage of brackets ("floors" and "ceilings" are intermixed) on the left-hand sides of (1) and (2) which is intended to suggest that the products involve noncommuting factors in a prescribed order. In both cases, the product, read from left to right, starts with a denominator factor. The brackets in the form "$\lceil - \rfloor$" are intended to denote that the factors are *falling*, while in "$\lfloor - \rceil$" that they are *rising*.

We define the *noncommutative hypergeometric series of type I* by

$${}_{r+1}F_r \left\lceil \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r \end{matrix}; Z \right\rfloor := \sum_{k \geq 0} \left\lceil \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r, I \end{matrix}; Z \right\rfloor_k,$$

and the *noncommutative hypergeometric series of type II* by

$${}_{r+1}F_r \left\lfloor \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r \end{matrix}; Z \right\rceil := \sum_{k \geq 0} \left\lfloor \begin{matrix} A_1, A_2, \dots, A_{r+1} \\ C_1, C_2, \dots, C_r, I \end{matrix}; Z \right\rceil_k.$$

In each case, the series terminates if one of the upper parameters A_i is of the form $-nI$. If the series is nonterminating, then the series converges in R if $\|Z\| < 1$. If $\|Z\| = 1$ the series may converge in R for some particular choice of upper and lower parameters. Exact conditions depend on the Banach algebra R.

It is clear that given a valid identity of elements in R, one may obtain a new one by simply reversing all the products (of elements of the unit ring R) simultaneously on each side of the respective identities. This is clearly an involution. (For square matrices this is equivalent to transposition of matrices.) This operation on an expression $E \in R$ shall be denoted by ${}^{\sim}E$.

For instance (compare with (1)),

$$\tilde{}\left[\begin{matrix}A_1, A_2, \dots, A_r \\ C_1, C_2, \dots, C_r\end{matrix}; Z\right]_k = \prod_{j=1}^{k}\left(Z\prod_{i=1}^{r}(A_i + (j-1)I)(C_i + (j-1)I)^{-1}\right).$$

3. Type I and type II noncommutative hypergeometric equations

Tirao [16] proved the following result:

Proposition. *For a positive integer n, let $R = M_{n\times n}(\mathbb{C})$ be the ring of complex $n\times n$ square matrices. Let $A, B, C, F_0 \in R$ be such that the spectrum of C contains no negative integers, and let $z \in \mathbb{C}$. Then $F(z) = {}_2F_1\left[\begin{matrix}A, B \\ C\end{matrix}; zI\right] F_0$ is the unique solution analytic at $z = 0$ of the matrix-valued hypergeometric equation*

$$z(1-z)F''(z) + (C - z(1 + A + B))F'(z) - ABF(z) = 0,$$

where $F(0) = F_0$.

As was indicated without proof in [15, Remark 2.1] this extends easily to the following:

Theorem 1. *Let R be a unital Banach algebra with norm $\|\cdot\|$, let $A, B, C, F_0 \in R$ such that $C + jI$ is invertible for all nonnegative integers j. Further let Z be central (i.e., $Z \in \{X \in R : XY = YX,\ \forall Y \in R\}$) with $\|Z\| < 1$. Then*

$$F(Z) = {}_2F_1\left[\begin{matrix}A, B \\ C\end{matrix}; Z\right] F_0 \tag{3}$$

is the unique solution analytic at $Z = O$ of the noncommutative hypergeometric equation

$$Z(I - Z)F''(Z) + (C - Z(I + A + B))F'(Z) - ABF(Z) = O, \tag{4}$$

where $F(O) = F_0$.

We provide an operator proof of Theorem 1. On the contrary, Tirao's proof of the above Proposition given in [16] is essentially different. Starting with the matrix-valued hypergeometric equation it involves the computation of the coefficients F_k in the analytic series $F(z) = \sum_{k\geq 0} F_k z^k$ by a generic Ansatz.

Proof: First of all, the (right multiple of the) type I noncommutative hypergeometric series

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]F_0\\
&=\left[\sum_{k\geq 0}\left(\prod_{j=1}^{k}(C+(k-j)I)^{-1}(A+(k-j)I)(B+(k-j)I)\right)\frac{Z^k}{k!}\right]F_0
\end{aligned}
$$

is clearly analytic at $Z=O$ and ${}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};O\right]F_0=F_0$.

Next we show that ${}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]F_0$ is a solution of the differential equation (4). We define the linear operator

$$\mathcal{D}_T := T + Z\frac{\mathrm{d}}{\mathrm{d}\,Z},$$

where $T\in R$, acting (from the left) on functions of Z over R.

If $F(Z)$ is analytic at $Z=O$ we can write $F(Z)=\sum_{k\geq 0}F_kZ^k$, where $F_k\in R$ for any nonnegative integer k. It is immediate that

$$\mathcal{D}_T\ F(Z)=\sum_{k\geq 0}(T+kI)\,F_kZ^k.$$

Hence

$$\mathcal{D}_A\left(\mathcal{D}_B\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]\right)=\sum_{k\geq 0}(A+kI)(B+kI)\left[\begin{matrix}A,B\\C,I\end{matrix};Z\right]_k,$$

and

$$
\begin{aligned}
&\mathcal{D}_{C-I}\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right] = \sum_{k\geq 0}(C+(k-1)I)\left[\begin{matrix}A,B\\C,I\end{matrix};Z\right]_k\\
&=C-I+\sum_{k\geq 1}(A+(k-1)I)(B+(k-1)I)\\
&\times\left(\prod_{j=1}^{k-1}(C+(k-1-j)I)^{-1}(A+(k-1-j)I)(B+(k-1-j)I)\right)\frac{Z^k}{k!}\\
&=C-I+\sum_{k\geq 0}(A+kI)(B+kI)
\end{aligned}
$$

$$\times \left(\prod_{j=1}^{k} (C+(k-j)I)^{-1}(A+(k-j)I)(B+(k-j)I) \right) \frac{Z^{k+1}}{(k+1)!}.$$

Thus we have

$$\frac{\mathrm{d}}{\mathrm{d}\,Z}\left(\mathcal{D}_{C-I}\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]\right) = \mathcal{D}_A\left(\mathcal{D}_B\ {}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]\right). \tag{5}$$

Since the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\,Z}\left(\mathcal{D}_{C-I}\ F(Z)\right) = \mathcal{D}_A\left(\mathcal{D}_B\ F(Z)\right), \tag{6}$$

or, more explicitly,

$$\frac{\mathrm{d}}{\mathrm{d}\,Z}\left(C-I+Z\frac{\mathrm{d}}{\mathrm{d}\,Z}\right)F(Z) = \left(A+Z\frac{\mathrm{d}}{\mathrm{d}\,Z}\right)\left(B+Z\frac{\mathrm{d}}{\mathrm{d}\,Z}\right)F(Z),$$

is equivalent to (4), it follows from (5) (and multiplication of a constant from the right) that ${}_2F_1\left[\begin{matrix}A,B\\C\end{matrix};Z\right]F_0$ satisfies the differential equation (4).

The uniqueness of the solution (3) of (4) with $F(0) = F_0$ readily follows from the theorem of existence and uniqueness of solutions of differential equations in Banach spaces (hence in Banach algebras), cf. e.g. [13]. All we need to show is that if there were two solutions $F_1(Z)$ and $F_2(Z)$ then $F_1'(O) = F_2'(O)$. (As we are considering a second order differential equation, two initial conditions, fixing $F(O)$ and $F'(O)$, are required to make the solution unique.)

Asume that $F_1(Z)$ and $F_2(Z)$ are solutions of (4) with $F_1(O) = F_2(O) = F_0$. Then we have

$$\begin{aligned}&Z(I-Z)F_1''(Z) + (C - Z(A+B+I))F_1'(Z) - ABF_1(Z)\\ &= Z(I-Z)F_2''(Z) + (C - Z(A+B+I))F_2'(Z) - ABF_2(Z).\end{aligned}$$

Evaluating this equation in $Z = O$ we get $C\,F_1'(O) = C\,F_2'(O)$ and since C is invertible the claim follows. ■

Now we are ready to state and prove the following new result concerning type II noncommutative hypergeometric series. It appears to lie in the nature of the type II series that the result is not as simple and elegant as in the corresponding type I case. In particular, the following theorem as stated requires the condition $C(C-A-B)+AB$ being invertible, which has no counterpart in the type I case.

Theorem 2. *Let R be a unital Banach algebra with norm $\|\cdot\|$, let $A, B, C, F_0 \in R$ such that $C(C-A-B)+AB$ and $C+jI$ are invertible for all nonnegative integers j. Further let Z be central (i.e., $Z \in \{X \in R : XY = YX,\ \forall Y \in R\}$) with $\|Z\| < 1$. Then*

$$F(Z) = F_0\, {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \tag{7}$$

is the unique solution analytic at $Z = O$ of the noncommutative hypergeometric equation

$$\begin{aligned} Z(I-Z)F''(Z) + ZF'(Z)(C-I-A-B) + \big((I-Z)F'(Z) - F(Z)C^{-1}AB\big) \\ \times \big(C(C-A-B)+AB\big)^{-1} C\big(C(C-A-B)+AB\big) = O, \end{aligned} \tag{8}$$

where $F(O) = F_0$.

Proof: First of all, the (left multiple of the) type II noncommutative hypergeometric series

$$\begin{aligned} &F_0\, {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \\ &= F_0 \sum_{k\geq 0} \left(\prod_{j=1}^{k} (C+(j-1)I)^{-1}(A+(j-1)I)(B+(j-1)I) \right) \frac{Z^k}{k!} \end{aligned}$$

is clearly analytic at $Z = O$ and $F_0\, {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; O\right] = F_0$.

Next we show that $F_0\, {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; O\right]$ is a solution of the differential equation (8). We have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\,Z}\, {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \\ &= \sum_{k\geq 1} \left(\prod_{j=1}^{k} (C+(j-1)I)^{-1}(A+(j-1)I)(B+(j-1)I) \right) \frac{Z^{k-1}}{(k-1)!} \\ &= \sum_{k\geq 0} \left(\prod_{j=1}^{k+1} (C+(j-1)I)^{-1}(A+(j-1)I)(B+(j-1)I) \right) \frac{Z^k}{k!} \end{aligned}$$

$$= \sum_{k\geq 0} \left\lfloor \begin{matrix} A, B \\ C, I \end{matrix}; Z \right\rfloor_k (C+kI)^{-1}(A+kI)(B+kI). \tag{9}$$

We define the linear operator $\widetilde{\mathcal{D}}_T$ by

$$\widetilde{\mathcal{D}}_T := T + \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z} Z,$$

where $T \in R$, acting from the *right* on functions over R. Here $\frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z}$ is the differential operator applied from the right side. With other words

$$F(Z)\, \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z} = \frac{\mathrm{d}}{\mathrm{d}\, Z}\, F(Z),$$

and

$$F(Z)\, \widetilde{\mathcal{D}}_T = F(Z)\, T + Z \frac{\mathrm{d}}{\mathrm{d}\, Z}\, F(Z),$$

where $F(Z)$ is any function of Z (Z being central) over R.

In particular, we have

$$F(Z)\, \widetilde{\mathcal{D}}_T = {}^{\sim}\!\left(\mathcal{D}_T\left({}^{\sim}F(Z)\right)\right),$$

where the reversion operator $^{\sim}$ was defined at the end of Section 2.

If $F(Z)$ is analytic at $Z = O$ we can write $F(Z) = \sum_{k\geq 0} F_k Z^k$, where $F_k \in R$ for any nonnegative integer k. It is immediate that

$$F(Z)\, \widetilde{\mathcal{D}}_T = \sum_{k\geq 0} F_k Z^k\, (T + kI),$$

and

$$F(Z)\, \widetilde{\mathcal{D}}_U^{-1} = \sum_{k\geq 0} F_k Z^k\, (U + kI)^{-1},$$

provided $U + kI$ is invertible in R for all nonnegative integers k.

Hence

$$\left(\left({}_2F_1 \left\lfloor \begin{matrix} A, B \\ C \end{matrix}; Z \right\rceil \widetilde{\mathcal{D}}_C^{-1}\right) \widetilde{\mathcal{D}}_A\right) \widetilde{\mathcal{D}}_B$$
$$= \sum_{k\geq 0} \left\lfloor \begin{matrix} A, B \\ C, I \end{matrix}; Z \right\rfloor_k (C+kI)^{-1}(A+kI)(B+kI)$$

$$= {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z},$$

by (9).

It follows that

$$G(Z) = {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right] \widetilde{\mathcal{D}}_C^{-1}$$

is a solution of the differential equation

$$\Big(G(Z)\, \widetilde{\mathcal{D}}_A\Big)\, \widetilde{\mathcal{D}}_B = \Big(G(Z)\, \widetilde{\mathcal{D}}_C\Big) \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z}.$$

This is simply a "reversed" version of (6) with A and B interchanged and $C + I$ in place of C. It thus follows from Theorem 1 that $G(Z)$ satisfies the reversed type I noncommutative hypergeometric equation:

$$Z(I - Z)G''(Z) + G'(Z)(C + I - Z(I + A + B)) - G(Z)\, AB = O. \quad (10)$$

We now need to rewrite (10) in terms of $F(Z) = {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right]$. We have

$$F(Z) = G(Z)\, \widetilde{\mathcal{D}}_C = G(Z)C + ZG'(Z),$$

and

$$F'(Z) = G'(Z)(C + I) + ZG''(Z),$$

which, in conjunction with (10), gives

$$(I-Z)F'(Z)+F(Z)(C-A-B)-F(Z)\, \widetilde{\mathcal{D}}_C^{-1}\big(C(C-A-B)+AB\big) = O. \quad (11)$$

Next, we multiply both sides of (11) from the right with

$$\big(C(C - A - B) + AB\big)^{-1}\widetilde{\mathcal{D}}_C\big(C(C - A - B) + AB\big)$$

(which is $\big(C(C - A - B) + AB\big)^{-1}C\big(C(C - A - B) + AB\big) + \frac{\widetilde{\mathrm{d}}}{\mathrm{d}\, Z}Z$). After a series of computations, including the simplifiction

$$I-(C-A-B)\big(C(C-A-B)+AB\big)^{-1}C = C^{-1}AB\big(C(C-A-B)+AB\big)^{-1}C,$$

we eventually arrive at (8). Further, by multiplying a constant from the left, it follows that $F_0\, {}_2F_1\left[\begin{matrix} A, B \\ C \end{matrix}; Z\right]$ satisfies the differential equation (8).

Using the same argument as in the proof of Theorem 1, one readily establishes the uniqueness of the solution (7) of (8) with $F(O) = F_0$. ∎

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