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MODULAR CLASSES OF POISSON-NIJENHUIS LIE ALGEBROIDS

RAQUEL CASEIRO

ABSTRACT: The modular vector field of a Poisson-Nijenhuis Lie algebroid A is defined and we prove that, in case of non-degeneracy, this vector field defines a hierarchy of bi-Hamiltonian A-vector fields. This hierarchy covers an integrable hierarchy on the base manifold, which may not have a Poisson-Nijenhuis structure.

KEYWORDS: Poisson-Nijenhuis structures, Lie algebroids, modular vector fields, integrable hierarchies.

1. Introduction

The relative modular class of a Lie algebroid morphism was first discussed by Grabowski, Marmo and Michor in [11]. Kosmann-Schwarzbach and Weinstein in [16] showed that this relative class could be seen as a generalization of the notion of modular class introduced by Weinstein in [17]. In [8], Damianou and Fernandes introduced the modular vector field of a Poisson-Nijehuis manifold and showed that it is intimately related with integrable hierarchies (see, also, the alternative approach offered by Kosmann-Schwarzbach and Magri in [14]). In this paper, we generalize this construction and consider the modular vector field of a Poisson-Nijenhuis Lie algebroid.

Recall (see, e.g, [15]) that a Nijenhuis operator $N : A \to A$ on a Lie algebroid $(A, [\ ,\ ], \rho)$ allows us to define a deformed Lie algebroid structure $A_N = (A, [\ ,\ ]_N, \rho \circ N)$ such that $N : A_N \to A$ is a Lie algebroid morphism. Our first result states that the modular class of this morphism has a canonical representative:

Proposition 1. *The relative modular class $N : A_N \to A$ is represented by $\mathrm{d}_A \operatorname{Tr} N$.*

Let us assume now that A is equipped with a Poisson structure π compatible with N. Then we can define two Lie algebroid structures on A^*, namely $(A^*, [\ ,\ ]_\pi, \rho \circ \pi^\sharp)$ obtained by dualization from π, and $A^*_{N^*} = (A^*, [\ ,\ ]_{N\pi}, \rho \circ N\pi^\sharp)$ obtained from the first one by deformation

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along N^*. Again, $N^* : A^*_{N^*} \to A^*$ is a Lie algebroid morphism and, by the proposition above, its relative modular class has the canonical representative $X_{(N,\pi)} := \mathrm{d}_\pi(\operatorname{Tr} N)$, which we will call the *modular vector field* of the Poisson-Nijenhuis Lie algebroid (A, π, N). When $A = TM$ we recover the construction of Damianou and Fernandes [8] up to a factor of 1/2 (for the same reason that the modular class associated with a Poisson manifold differs from the modular class of its cotangent Lie algebroid by a factor of 1/2).

The modular vector field $X_{(N,\pi)}$ of the Poisson-Nijenhuis Lie algebroid (A, π, N) is always a $\mathrm{d}_{N\pi}$-cocycle. If N is non-degenerated it is also a $\mathrm{d}_{N\pi}$-coboundary. In this case we have the following generalization of a result of Damianou and Fernandes for a Poisson-Nijenhuis manifold:

Theorem 2. *Let (A, π, N) be a Poisson-Nijenhuis Lie algebroid with N a non-degenerated Nijenhuis operator. Then there exists a hierarchy of A-vector fields*

$$X^{i+j}_{(N,\pi)} = N^{i+j} X_{(N,\pi)} = \mathrm{d}_{N^i\pi} h_j = \mathrm{d}_{N^j\pi} h_i, \quad (i, j \in \mathbb{Z})$$

where

$$h_0 = \ln(\det N) \text{ and } h_i = \frac{1}{i} \operatorname{Tr} N^i, \quad (i \neq 0).$$

The hierarchy of flows on A, given by this theorem, covers a hierarchy of (ordinary) multi-Hamiltonian flows on the base manifold M. Although the hierarchy on A is generated by a Nijenhuis operator, it may happen that the base hierarchy is not generated by one. We will see that this is precisely the case for the A_n-Toda lattice. This gives a new explanation for the existence of a hierarchy of Poisson structures and flows associated with a bi-Hamiltonian system which may not have a Nijenhuis operator.

This paper is organized as follows. In Section 2, we present the necessary background on Poisson-Nijenhuis Lie algebroids. In Section 3, we introduce the modular vector field of a Poisson-Nijenhuis Lie algebroid, state its basic properties, and prove Theorem 2. Section 4 is concerned with integrable hierarchies and discusses the example of the A_n-Toda lattice. In the last section, we discuss our results in the context of three basic classes of Lie algebroids: The first class is the extreme case where the base manifold is a point, i.e., a Lie algebra. The second class, is the case of the tangent bundle of a manifold where we recover the results of [8], and the last class is the Lie algebroid associated with the dynamical Yang-Baxter equation of Etingof and Varchenko ([4]) and discovered by Xu in [18].

2. Poisson-Nijenhuis Lie Algebroids

In this section we will recall some basic facts about Nijenhuis operators and Poisson structures on Lie algebroids which we will need later. A general reference for Lie algebroids is the book by Cannas da Silva and Weinstein [3]. Nijenhuis operators are discussed in detail in the article by Kosmann-Schwarzbach and Magri [15], while PN-structures on Lie algebroids are discussed by Kosmann-Schwarzbach in [13] and by Grabowski and Urbanski in [12].

2.1. Cartan calculus on Lie algebroids. Let us recall that a Lie algebroid is a kind of generalized tangent bundle, which carries a generalized Cartan calculus.

First, for any Lie algebroid $(A, [\,,\,], \rho)$ we have a complex of *A-differential forms* $\Omega^k(A) := \Gamma(\wedge^k A^*)$ with the differential given by:

$$\begin{aligned}\mathrm{d}_A\omega(X_0, \dots, X_k) :=& \sum_{i=0}^{n}(-1)^i \rho(X_i)\cdot\omega(X_0, \dots, \hat{X}_i, \dots, X_k)\\ &+ \sum_{0\le i\le j\le k}(-1)^{i+j}\omega\left([X_i, X_j]_A, X_0, \dots, \hat{X}_i, \dots, X_k\right),\end{aligned}$$

where $X_0, \dots, X_k \in \Gamma(A)$. The corresponding *Lie algebroid cohomology* is denoted by $H^\bullet(A)$.

Dually, the space of *A-multivector fields* $\mathfrak{X}^\bullet(A) = \bigoplus_{k\in\mathbb{Z}} \mathfrak{X}^k(A) := \bigoplus_{k\in\mathbb{Z}} \Gamma(\wedge^k A)$ carries a super-Lie bracket $[\ ,\ ]_A$, extending the Lie bracket on $\Gamma(A)$, and satisfying the following super-commutation, super-derivation and super-Jacobi identities:

$$[P, Q] = -(-1)^{(p-1)(q-1)}[P, Q]$$
$$[P, Q\wedge R] = [P, Q]\wedge R + (-1)^{(p-1)q} Q\wedge [P, R]$$
$$(-1)^{(p-1)(r-1)}[P, [Q, R]] + (-1)^{(q-1)(p-1)}[Q, [R, P]] + (-1)^{(r-1)(q-1)}[R, [P, Q]] = 0$$

where $P \in \mathfrak{X}^p(A)$, $Q \in \mathfrak{X}^q(A)$ and $R \in \mathfrak{X}^r(A)$. The triple $(\mathfrak{X}^\bullet(A), [\ ,\ ]_A, \wedge)$ is a *Gerstenhaber algebra*.

If $X \in \Gamma(A)$ and $\omega \in \Omega^k(A)$ the *Lie derivative* of ω along X is the A-differential form $\mathcal{L}_X\omega \in \Omega^k(A)$ defined by

$$\mathcal{L}_X\omega := \mathrm{d}_A i_X\omega + i_X \mathrm{d}_A\omega,$$

where $i_X : \Omega^k(A) \to \Omega^{k-1}(A)$ is defined by

$$i_X\eta(X_1, \dots X_{k-1}) := \eta(X, X_1, \dots, X_k), \quad X_1, \dots, X_{k-1} \in \Gamma(A),$$

for $k > 1$. If $k = 1$, then $i_X\eta := \eta(X)$ and, for $k \leq 0$ we say that $i_X\eta = 0$.

A morphism $\phi : A \to B$ (over the identity) of Lie algebroids over M induces by transposition a chain map of the complexes of differential forms:

$$\phi^* : (\Omega^k(B), d_B) \to (\Omega^k(A), \mathrm{d}_A).$$

Hence, we also have a well defined map at the level of cohomology, which we will denote by the same letter $\phi^* : H^\bullet(B) \to H^\bullet(A)$.

2.2. Nijenhuis operators. Let $(A, [\, , \,], \rho)$ be a Lie algebroid over a manifold M. Recall that a *Nijenhuis operator* is a bundle map $N : A \to A$ (over the identity) such that the induced map on the sections (denoted by the same symbol N) has vanishing torsion:

$$T_N(X, Y) := N[X, Y]_N - [NX, NY] = 0, \quad X, Y \in \Gamma(A), \tag{1}$$

where $[\, , \,]_N$ is defined by

$$[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y], \quad X, Y \in \Gamma(A).$$

Let us set $\rho_N := \rho \circ N$. For a Nijenhuis operator N, one checks easily that the triple $A_N = (A, [\, , \,]_N, \rho_N)$ is a new Lie algebroid, and then $N : A_N \to A$ is a Lie algebroid morphism.

Since N is a Lie algebroid morphism, its transpose gives a chain map of the complexes of differential forms $N^* : (\Omega^k(A), d_A) \to (\Omega^k(A_N), \mathrm{d}_{A_N})$. Hence we also have a map at the level of algebroid cohomology $N^* : H^\bullet(A) \to H^\bullet(A_N)$.

2.3. Poisson structures on Lie algebroids. Let $\pi \in \mathfrak{X}^2(A)$ be a bivector on the Lie algebroid $(A, [\, , \,], \rho)$ and denote by $\pi^\sharp$ the usual bundle map

$$\begin{array}{rccc} \pi^\sharp : & A^* & \longrightarrow & A \\ & \alpha & \longmapsto & \pi^\sharp(\alpha) = i_\alpha\pi. \end{array}$$

We say that π defines a *Poisson structure on* A if $[\pi, \pi]_A = 0$. In this case, the bracket on the sections of A^* defined by

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha - \mathrm{d}_A\left(\pi(\alpha, \beta)\right), \quad \alpha, \beta \in \Gamma(A^*),$$

is a Lie bracket and $(A^*, [\, , \,]_{A^*}, \rho \circ \pi^\sharp)$ is a Lie algebroid. The differential of this Lie algebroid is given by $\mathrm{d}_\pi X = [\pi, X]_A$, $X \in \Omega(A^*)$, and the pair (A, A^*) is a special kind of Lie bialgebroid, called a *triangular Lie bialgebroid*.

2.4. Poisson-Nijenhuis Lie algebroids. The basic notion to be used in this paper is the following:

Definition 3. *A* ***Poisson-Nijenhuis Lie algebroid*** *(in short, a PN-algebroid) is a Lie algebroid* $(A, [\,,\,]_A\,, \rho)$ *equipped with a Poisson structure* π *and a Nijenhuis operator* N *which are* compatible.

The compatibility condition between N and π means that:

$$[\,,\,]_{N\pi} = [\,,\,]_{N^*}\,,$$

where $[\,,\,]_{N\pi}$ is the Lie bracket defined by the bivector field $N\pi \in \mathfrak{X}^2(A)$, and $[\,,\,]_{N^*}$ is the Lie bracket obtained from the Lie bracket $[\,,\,]_\pi$ by deformation along the Nijenhuis tensor N^*.

As a consequence, $N\pi$ defines a new Poisson structure on A, compatible with π:

$$[\pi, N\pi]_A = [N\pi, N\pi]_A = 0,$$

and one has a commutative diagram of morphisms of Lie algebroids:

$$\begin{tikzcd}
(A^*, [\cdot,\cdot]_{N\pi}) \arrow[r, "N^*"] \arrow[d, "\pi^\sharp"'] \arrow[dr, "N\pi^\sharp"] & (A^*, [\cdot,\cdot]_\pi) \arrow[d, "\pi^\sharp"] \\
(A, [\cdot,\cdot]_N) \arrow[r, "N"] & (A, [\cdot,\cdot]_A)
\end{tikzcd}$$

In fact, we have a whole hierarchy $N^k\pi$ $(k \in \mathbb{N})$ of pairwise compatible Poisson structures on A.

3. Modular class of a Poisson-Nijenhuis Lie algebroid

In this section we will state and prove our main results.

3.1. Modular class of a Lie algebroid. Let $(A, [\,,\,]\,, \rho)$ be a Lie algebroid over the manifold M. For simplicity we will assume that both M and A are orientable, so that there exist non-vanishing sections $\eta \in \mathfrak{X}^{\mathrm{top}}(A)$ and $\mu \in \Omega^{\mathrm{top}}(M)$.

The *modular form* of the Lie algebroid A with respect to $\eta\otimes\mu$ is the A-form $\xi_A \in \Omega^1(A)$, defined by

$$\langle \xi_A, X\rangle \eta \otimes \mu = \mathcal{L}_X\eta \otimes \mu + \eta \otimes \mathcal{L}_{\rho(X)}\mu, \quad X \in \Gamma(A). \tag{2}$$

This is a 1-cocycle of the Lie algebroid cohomology of A. If one makes a different choice of sections η' and μ', then $\eta'\otimes\mu' = f\eta\otimes\mu$, for some non-vanishing smooth function $f\in C^\infty(M)$. One checks easily that the modular form ξ'_A associated with this new choice is given by:

$$\xi'_A = \xi_A - \mathrm{d}_A \log|f|, \tag{3}$$

so that the cohomology class $[\xi_A]\in H^1(A)$ is independent of the choice of η and μ. This cohomology class is called the *modular class* of A and we will denoted it by $\operatorname{mod} A := [\xi_A]$.

Proposition 4. *Let N be a Nijenhuis operator on a Lie algebroid A and fix non-vanishing sections η and μ as above. The modular form ξ_{A_N} of the Lie algebroid A_N and the modular form ξ_A of A are related by:*

$$\xi_{A_N} = \mathrm{d}_A(\operatorname{Tr} N) + N^*\xi_A. \tag{4}$$

Proof: Around any point, we can always choose a local base $\{e_1,\dots,e_r\}$ of sections of A and local coordinates $(x_1,\dots,x_n)$ of M such that $\eta = e_1\wedge\dots\wedge e_r$ and $\mu = dx_1\wedge\dots\wedge dx_n$. In these coordinates, we have the following expressions for the anchor ρ and the Nijenhuis operator N:

$$\rho(e_i) = \sum_{u=1}^{n} p_i^u \frac{\partial}{\partial x_u} \text{ and } N(e_i) = \sum_{j=1}^{r} N_i^j e_j, \quad (i=1,\dots,r).$$

Now, for $i=1,\dots,r$, we compute:

$$\mathcal{L}_{e_i}\eta = \mathcal{L}_{e_i}(e_1\wedge\dots\wedge e_r) = \sum_{k,j=1}^{r}\left[N_i^k C_{kj}^j - \sum_{u=1}^{n}\left(\rho_j^u\frac{\partial N_i^j}{\partial x_u} + \rho_i^u\frac{\partial N_j^j}{\partial x_u}\right)\right]\eta$$

and

$$\mathcal{L}_{\rho_N(e_i)}\mu = \mathcal{L}_{\rho\circ N(e_i)}(dx_1\wedge\dots\wedge dx_n) = \sum_{k=1}^{r}\sum_{u=1}^{n}\left(\frac{\partial N_i^k}{\partial x_u}\rho_k^u + \frac{\partial \rho_k^u}{\partial x_u}N_i^k\right)\mu.$$

So

$$\begin{aligned}
\xi_A^N(e_i) &= \sum_j\left(\sum_k N_i^j C_{jk}^k + \sum_u\left(\frac{\partial\rho_j^u}{\partial x_u}N_i^j + \frac{\partial N_i^j}{\partial x_u}\rho_i^u\right)\right)\\
&= \sum_j\left(\sum_k N_i^j C_{jk}^k + \sum_u \frac{\partial\rho_j^u}{\partial x_u}N_i^j\right) + \mathrm{d}_A \operatorname{Tr} N(e_i)\\
&= \left(N^*\xi_A + \mathrm{d}_A\operatorname{Tr} N\right)(e_i).
\end{aligned}$$

By linearity this holds for any section of A, so the result follows. ■

The theorem shows that the A_N-form $\xi_{A_N} - N^*\xi_A = \mathrm{d}_A \operatorname{Tr} N$ is independent of the choice of section of $\eta \otimes \mu \in \mathfrak{X}^{\mathrm{top}}(A) \otimes \Omega^{\mathrm{top}}(M)$.

Remark 5. *This can also be checked directly using relation (3): If ξ_A and ξ_{A_N} are the modular forms associated to the choice $\eta \otimes \mu$, ξ'_A and ξ'_{A_N} are the modular forms associated with another choice $f\eta \otimes \mu$, then:*

$$\begin{aligned} \xi'_A &= \xi_A - \mathrm{d}_A \ln |f|, \\ \xi'_{A_N} &= \xi_{A_N} - \mathrm{d}_{A_N} \ln |f|. \end{aligned}$$

For any function $g \in C^\infty(M)$, we have $\mathrm{d}_{A_N} g = N^* d_A g$, so it follows from these relations that:

$$\xi'_{A_N} - N^*\xi'_A = \xi_{A_N} - N^*\xi_A.$$

Recall (see [11, 16]) that for any Lie algebroid morphism over the identity $\phi : (A, [\,,\,], \rho_A) \to (B, [\,,\,]_B, \rho_B)$ one defines its *relative modular class* to be the cohomology class $\operatorname{mod}^\phi(A, B) \in H^1(A)$ given by:

$$\operatorname{mod}^\phi(A, B) := \operatorname{mod} A - \phi^* \operatorname{mod} B. \tag{5}$$

Therefore we have the following immediate corollary of Proposition 4:

Corollary 6. *The relative modular class of the algebroid morphism $N : A_N \to A$ is a A_N-cohomology class with canonical representative the A_N-form $\mathrm{d}_A \operatorname{Tr} N$.*

Note that the class $[\mathrm{d}_A \operatorname{Tr} N] \in H^1(A_N)$ maybe non-trivial: in general, the differentials d_A and d_{A_N} will be distinct.

3.2. Modular class of a Poisson-Nijenhuis Lie algebroid.

Now we consider a Poisson-Nijenhuis Lie algebroid (A, N, π). Then N^* is a Nijenhuis operator of the dual Lie algebroid $(A^*, [\ ,\ ]_\pi, \rho \circ \pi^\sharp)$ and, by Corollary 6, its relative modular class has the canonical representative $\mathrm{d}_\pi(\operatorname{Tr} N^*)$, so that:

$$\operatorname{mod}^{N^*}(A^*_{N^*}, A^*) = [\mathrm{d}_\pi(\operatorname{Tr} N^*)] = [\mathrm{d}_\pi(\operatorname{Tr} N)].$$

Definition 7. *The **modular vector field** of the Poisson-Nijenhuis Lie algebroid (A, π, N) is defined by*

$$X_{(N,\pi)} = \xi_{A^*_{N^*}} - N\xi_{A^*} = \mathrm{d}_\pi(\operatorname{Tr} N) \in \mathfrak{X}(A).$$

Notice that the modular vector field of a PN-algebroid is a $\mathrm{d}_{N\pi}$-cocycle.

Proposition 8. *Let (A, π, N) be a PN-algebroid. Then*

$$N^k X_{(N,\pi)} = \frac{1}{k-i+1} X_{(N^{k-i+1},\, N^i\pi)}, \quad i < k \in \mathbb{N}.$$

Proof: The operator N is Nijenhuis so it satisfies the identity

$$kN^* \mathrm{d}_A \operatorname{Tr} N = \mathrm{d}_A \operatorname{Tr} N^k, \quad k \in \mathbb{N}. \tag{6}$$

Now simply observe that

$$\begin{aligned} N^k X_{(N,\pi)} &= N^k \mathrm{d}_\pi(\operatorname{Tr} N) = N^k [\pi, \operatorname{Tr} N] \\ &= -N^i \pi^\sharp(N^{*\,k-i} \mathrm{d}_A \operatorname{Tr} N) = -\frac{1}{k-i+1} (N^i\pi)^\sharp \left(\mathrm{d}_A \operatorname{Tr} N^{k-i+1} \right) \\ &= \frac{1}{k-i+1} \mathrm{d}_{N^i\pi} \left(\operatorname{Tr} N^{k-i+1} \right) = \frac{1}{k-i+1} X_{(N^{k-i+1},\, N^i\pi)}. \end{aligned}$$

■

In case N is non-degenerated, we obtain the PN-algebroid generalization of a result in [8] (which corresponds to the case $A = TM$; see examples below):

Theorem 9. *Let (A, π, N) be a Poisson-Nijenhuis Lie algebroid with N a non-degenerated Nijenhuis operator. Then the modular vector field $X_{(N,\pi)}$ is a $\mathrm{d}_{N\pi}$-coboundary and determines a hierarchy of vector fields*

$$X^{i+j}_{(N,\pi)} = N^{i+j} X_{(N,\pi)} = \mathrm{d}_{N^i\pi} h_j = \mathrm{d}_{N^j\pi} h_i, \quad (i, j \in \mathbb{Z}) \tag{7}$$

where

$$h_0 = \ln(\det N) \text{ and } h_i = \frac{1}{i} \operatorname{Tr} N^i, \quad (i \neq 0). \tag{8}$$

Proof: For any integer k, N^k is a non-degenerated Nijenhuis operator and satisfies the identity

$$kN^{*\,k} \mathrm{d}_A(\ln \det N) = \mathrm{d}_A(\operatorname{Tr} N^k). \tag{9}$$

It follows that

$$X_{(N,\pi)} = -\pi^\sharp \left(\mathrm{d}_A \operatorname{Tr} N \right) = -\pi^\sharp\, N^* \mathrm{d}_A \left(\ln \det N \right) = \mathrm{d}_{N\pi}(\ln \det N).$$

We also have

$$\begin{aligned} N^{-1} X_{(N,\pi)} &= X_{(N, N^{-1}\pi)} = -N^{-1}\pi^\sharp(\mathrm{d}_A \operatorname{Tr} N) = -\pi^\sharp(\mathrm{d}_A \ln \det N) \\ &= -\pi^\sharp \left(N^* \mathrm{d}_A \operatorname{Tr} N^{-1} \right) = \mathrm{d}_{N\pi} \operatorname{Tr} N^{-1} = X_{(N^{-1}, N\pi)}. \end{aligned}$$

The expressions for the hierarchy now follow from identity (9) and Proposition 8 applied to N and to N^{-1}. ■

4. Integrable hierarchies and PN-Algebroids

4.1. Flows of A-vector fields. A vector field $X \in \mathfrak{X}(A)$ on a Lie algebroid has a *flow*, generalizing the usual picture which corresponds to the case $A = TM$.

Given a vector field X on M, we denote by Φ^t_X its flow:

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi^t_X(x) = X(\Phi^t_X(x)), \quad \Phi^0_X(x) = x \ .$$

We have that Φ^t is a 1-parameter group of diffeomorphisms and differentiating, we obtain the infinitesimal flow of X:

$$\phi^t_X(x) \equiv (\mathrm{d}\Phi^t_X)_x : T_xM \to T_{\Phi^t_X(x)}M.$$

It is easy to see that ϕ^t_X is a 1-parameter group of automorphisms of the Lie algebroid TM:

$$\phi^t_X([Y,Z]) = [\phi^t_X(Y), \phi^t_X(Z)],$$

for every pair of vector fields $Y, Z \in \mathfrak{X}(M)$.

We can generalize this to any Lie algebroid A: to any vector field $X \in \mathfrak{X}(A)$ (i.e., a section of A) one associates a 1-parameter group ϕ^t_X of automorphisms of A. For this we can use the following general construction of (infinitesimal) flows. Let us assume that E is a vector bundle over M. A *derivation* on E is a pair (D, X) where $D : \Gamma(E) \to \Gamma(E)$ is a differential operator, X is a vector field on M, satisfying the Leibniz rule

$$D(f\alpha) = fD(\alpha) + X(f)\alpha, \quad \forall\ f \in C^\infty(M), \alpha \in \Gamma(E).$$

Now, any derivation (D, X) on E has an associated (infinitesimal) flow: a standard argument shows that there is a unique 1-parameter family of linear isomorphisms

$$\phi^t_D(x) : E_x \to E_{\Phi^t_X(x)},$$

which is characterized uniquely by the property:

$$\frac{\mathrm{d}}{\mathrm{d}t}(\phi^t_D)^* s = D(s), \tag{10}$$

for all sections $s \in \Gamma(E)$. Here $(\phi^t_D)^* s := \phi^{-t}_D \circ s \circ \Phi^t_X$.

The canonical lift of the vector field X to E associated with D is the vector field X^E_D on E (section of the tangent bundle $TE \to E$) with flow ϕ^t_D:

$$X^E_D = \frac{\mathrm{d}}{\mathrm{d}t}\ \phi^t_D.$$

Denote by D^* be the dual derivation of D, i.e. the derivation on E^* defined by

$$\langle D^*(\alpha), s\rangle + \langle \alpha, D(s)\rangle = X\langle \alpha, s\rangle, \quad s \in \Gamma(E),\ \alpha \in \Gamma(E^*),$$

and by $f_s : E^* \to \mathbb{R}$ the linear function on the fibers defined by evaluation of the section s of E:

$$f_s(\alpha(x)) = \langle \alpha(x), s(x)\rangle, \quad \alpha \in \Gamma(E^*).$$

Notice that

$$\begin{aligned} X_D^E(f_\alpha) &= \frac{\mathrm{d}}{\mathrm{d}t} f_\alpha \circ \phi_D^t = f_{D^*\alpha}, \quad \alpha \in \Gamma(E^*) \\ X_D^E(g \circ p) &= X(g) \circ p, \quad g \in C^\infty(M), \end{aligned}$$

where $f_\alpha : E^* \to \mathbb{R}$ is the function defined by evaluation of the section α of E^*. and $p : E \to M$ is the projection of the vector bundle.

We can apply the previous construction to a vector field $X \in \mathfrak{X}(A)$ where we consider the derivation $(D_X, \rho(X))$ with $D_X = [X, -]$. The resulting flow, called the *flow* of X,

$$\phi_X^t(x) : A_x \to A_{\Phi^t_{\rho(X)}(x)}$$

is uniquely determined by the formula (10) above.

An alternative description can be obtained using the fiberwise linear Poisson structure $\{\ ,\ \}_A$ on the dual bundle A^*: denote by X_{f_X} the Hamiltonian vector field associated with the function $f_X : A^* \to \mathbb{R}$. It is easy to check (see [7]) that:

(a) The assignment $X \mapsto f_X$ defines a Lie algebra homomorphism

$$(\mathfrak{X}(A), [\ ,\ ]) \to (C^\infty(A^*), \{\ ,\ \}_A);$$

(b) Denoting by $q : A^* \to M$ the projection, X_{f_X} is q-related to $\rho(X)$:

$$q_* X_{f_X} = \rho(X).$$

So X_{f_X} is the canonical lift of the vector field $\rho(X)$ to A^* (associated with the derivation D_X^*). For each t, the flow $\Phi^t_{X_{f_X}}$ defines a Poisson automorphism of A^* (wherever defined), which maps linearly fibers to fibers of A^*. So, in fact, we have a bundle map

$$\Phi^t_{X_{f_X}} : A^* \to A^*$$

and, from (b), it covers $\Phi^t_{\rho(X)}$, the flow of $\rho(X)$. By transposition we obtain the flow of X:

$$\phi^t_X(x) : A_x \to A_{\Phi^t_{\rho(X)}(x)}.$$

4.2. Hamiltonian flows on Lie algebroids. Let us assume now that $\pi \in \mathfrak{X}^2(A)$ is a Poisson structure on the Lie algebroid A. This bivector field determines a bundle map $\pi^\sharp : A^* \to A$, and for the Lie algebroid structure on A^*, we have that $\pi^\sharp$ is a Lie algebroid morphism.

Definition 10. *Let $f \in C^\infty(M)$. Its **Hamiltonian vector field** $X_f \in \mathfrak{X}(A)$ is the vector field:*

$$X_f := \pi^\sharp \mathrm{d}_A f.$$

*The corresponding flow $\phi^t_{X_f}$ will be called the **Hamiltonian flow** associated with the Hamiltonian function f.*

The Poisson structure π on A covers a (ordinary) Poisson structure π_M on the base manifold M which is defined by $\pi^\sharp_M = \rho \circ \pi^\sharp \circ \rho^*$, i.e.

$$\{f, g\}_{\pi_M} = \langle \mathrm{d}_A f, \mathrm{d}_\pi g\rangle = \pi(\mathrm{d}_A f, \mathrm{d}_A g), \quad f, g \in C^\infty(M).$$

We have that:

Proposition 11. *For any $f \in C^\infty(M)$ the Hamiltonian vector field X_f on A covers the (ordinary) Hamiltonian vector field on M associated with f. The Hamiltonian flow $\phi^t_{X_f}$ on A is dual to the flow of the section $\mathrm{d}_A f$ of A^*.*

Proof: By definition, the flow of X_f covers the flow of $\rho(X_f)$. This vector field is the Hamiltonian vector field on M associated with f

$$\rho(X_f) = \rho \circ \pi^\sharp(\mathrm{d}_A f) = \rho \circ \pi^\sharp \circ \rho^*(\mathrm{d} f) = \pi^\sharp_M(\mathrm{d} f).$$

Simply observe that $\mathrm{d}_A f$ and X_f, they both cover the same vector field on M and the associated derivations are dual

$$D_{\mathrm{d}_A f}(\alpha) = [\mathrm{d}_A f, \alpha]_\pi = \mathcal{L}_{X_f}\alpha = D^*_{X_f}(\alpha), \quad \alpha \in \Gamma(A^*),$$

so, the flows must be dual. ∎

Another way of seeing this duality is considering $\{\ ,\ \}_{A^*}$, the linear Poisson bracket on A defined by the Lie algebroid $(A^*, [\ ,\ ]_\pi, \rho \circ \pi^\sharp)$: the Hamiltonian vector field $X_{f_{\mathrm{d}_A f}}$ is the canonical lift of $\rho(X_f)$ to A, associated with the derivation $D_{X_f} = [X_f,\ ]$ and X_{f_X} is the canonical lift of the vector field $\rho(X)$ to A^*, associated with the derivation $D_{\mathrm{d}_A f} = D^*_{X_f}$.

4.3. Integrable Hierarchies on Lie algebroids.

Definition 12. *Let X be a vector field on A. A* ***first integral*** *of X is a function $f : A \to \mathbb{R}$ which is constant along each integral curve of X:*

$$\frac{\mathrm{d}}{\mathrm{d}t} f \circ \phi_X^t = 0.$$

Note that a first integral of X is just a first integral of X^A, the canonical lift of $\rho(X)$ to A.

Observe that, on one hand, given $\alpha \in \Gamma(A^*)$, f_α is a first integral of X if and only if $D_X^*(\alpha) = \mathcal{L}_X \alpha = 0$. On the other hand, since the flow of X covers the flow of $\rho(X)$, pull-backs of first integrals of $\rho(X)$ are first integrals of X.

Definition 13. *Given π a Poisson structure on A, we say that two functions $g, f \in C^\infty(A)$* ***π-commute*** *if $\{f, g\}_{A^*} = 0$. A vector field X is said to be* ***integrable*** *if X^A, the canonical lift of $\rho(X)$ to A, is Liouville integrable.*

Two first integrals of the vector field $\rho(X)$ may not π_M-commute but their pull-backs always π-commute, because basic functions always commute with respect to the linear Poisson bracket defined on the dual of a Lie algebroid. In particular, we have:

Proposition 14. *Let $f \in C^\infty(M)$. The first integrals of the Hamiltonian vector field*

$$X_{f_{\mathrm{d}_A f}} = \pi^\sharp \mathrm{d}_A f,$$

are the functions which π-commute with $f_{\mathrm{d}_A f}$. In particular, evaluations of sections of A^* which commute with $\mathrm{d}_A f$ and pull-backs of functions which π_M-commute with f are first integrals of X_f.

Let N be a Nijenhuis operator on A compatible with π. The sequence of Poisson structures $\pi_k = N^k \pi$ covers a sequence of Poisson structure on M:

$$\pi_M^{k\ \sharp} = \rho \circ N^k \pi \circ \rho^*,$$

but, as an example below shows, these Poisson tensors may not be related by a Nijenhuis operator on M. We also have a sequence of linear Poisson brackets on A:

$$\{\ ,\ \}_{A^*}^k$$

and, given a section α on A^*, a sequence of Hamiltonian vector fields

$$X_{f_\alpha}^k = \{f_\alpha, -\}_{A^*}^k.$$

A bi-Hamiltonian vector field

$$X = \pi_0^\sharp(\mathrm{d}_A h_1) = \pi_1^\sharp(\mathrm{d}_A h_0)$$

defines a hierarchy of multi-Hamiltonian vector fields on A:

$$X_{k+i} = \pi_k^\sharp(\mathrm{d}_A h_i) = \pi_i^\sharp(\mathrm{d}_A h_k).$$

This hierarchy, on one hand, covers a hierarchy of Hamiltonian vector fields on M

$$\rho(X_{i+k}) = {\pi_M^k}^\sharp \mathrm{d}h_i = {\pi_M^i}^\sharp \mathrm{d}h_k$$

and, on the other hand, is associated with the hierarchy of canonical lifts

$$\rho(X_{i+k})^A = X^0_{f_{\mathrm{d}_A h_{i+k}}} = X^i_{f_{\mathrm{d}_A h_k}} = X^k_{f_{\mathrm{d}_A h_i}}.$$

4.4. Covering Integrable Hierarchies. We can try to apply our main result (Theorem 9) to obtain an integrable hierarchy on a Lie algebroid. However, one observes that in Theorem 9 the Hamiltonian functions, which are first integrals of the vector fields in the hierarchy, are all basic functions, i.e., are pull-backs of functions on the base. Hence, in general, they will not provide a complete set of first integrals. However, there is yet another connection with (classical) integrable systems, due to the following theorem:

Theorem 15. *Let (A, π, N) be a Poisson-Nijenhuis Lie algebroid with N a non-degenerated Nijenhuis operator. Then the the modular vector field $X_{(N,\pi)}$ covers a bi-Hamiltonian vector field on M, and the associated hierarchy (7) of A-vector fields covers a (classical) hierarchy of flows on M. This hierarchy is given by:*

$$X_{i+j} = -\pi_i^\sharp \mathrm{d}h_j = -\pi_j^\sharp \mathrm{d}h_i \quad (i, j \in \mathbb{Z}) \tag{11}$$

where π_j are Poisson structures on M and h_i are the functions given by (8).

Although we have a hierarchy of modular vector fields $X_{(N^k,\pi)}$ generated by the Nijenhuis operator N, generally the covered hierarchy of bi-Hamiltonian vector fields on M is not generated by any Nijenhuis operator. This is illustrated by the next example.

4.5. The classical Toda lattice. The classical Toda lattice was already considered in [1], using specific properties of this system. We use our general approach to show how one can recover those results and explain some of those formulas.

4.5.1. *Toda lattice in physical coordinates.* The Hamiltonian defining the Toda lattice is given in canonical coordinates (p_i, q_i) of $\mathbb{R}^{2n}$ by

$$h_2(q_1, \dots, q_n, p_1, \dots, p_n) = \sum_{i=1}^{n} \frac{1}{2} p_i^2 + \sum_{i=1}^{n-1} e^{q_i - q_{i+1}}. \tag{12}$$

For the integrability of the system we refer to the classical paper of Flaschka [10].

Let us recall the bi-Hamiltonian structure given in [6]. The first Poisson tensor in the hierarchy is the standard canonical symplectic tensor, which we denote by $\widetilde{\pi}_0$, so that

$$\{q_i, p_j\}_0 = \delta_{ij},$$

while the second Poisson tensor $\widetilde{\pi}_1$ is determined by the relations

$$\begin{aligned} \{q_i, q_j\}_1 &= -1, \qquad (i < j) \\ \{q_i, p_j\}_1 &= p_i \delta_{ij}, \\ \{p_j, p_i\}_1 &= e^{q_i - q_{i+1}} \delta_{j,i+1}. \end{aligned}$$

Then setting $h_1 = p_1 + p_2 + \dots + p_n$, we obtain the bi-Hamiltonian formulation:

$$\widetilde{\pi}_0^\sharp \mathrm{d}h_2 = \widetilde{\pi}_1^\sharp \mathrm{d}h_1.$$

If we set, as usual,

$$N := \widetilde{\pi}_1^\sharp \circ (\widetilde{\pi}_0^\sharp)^{-1},$$

then a small computation gives the following multi-Hamiltonian formulation:

Proposition 16. *The Toda hierarchy admits the multi-Hamiltonian formulation:*

$$\widetilde{\pi}_j^\sharp \mathrm{d}h_2 = \widetilde{\pi}_{j+2}^\sharp \mathrm{d}h_0,$$

where $h_0 = \frac{1}{2} \log(\det N)$ *and* h_2 *is the original Hamiltonian (12).*

4.5.2. *Toda lattice in Flaschka coordinates.* Let us recall the Flaschka coordinates $(a_1, \dots, a_{n-1}, b_1, \dots, b_n)$ where:

$$\begin{aligned} b_i &= p_i, \quad (i = 1, \dots, n) \\ a_i &= e^{q_i - q_{i+1}}, \quad (i = 1, \dots, n-1) \end{aligned}$$

In these new coordinates there is no recursion operator anymore (this is a singular change of coordinates, where we loose one degree of freedom). Nevertheless, the multi-Hamiltonian structure does reduce ([6]). One can then compute the modular vector fields of the reduced Poisson tensors $\overline{\pi}_j$ relative to the standard volume form:

$$\mu = \mathrm{d}a_1 \wedge \cdots \wedge \mathrm{d}a_{n-1} \wedge \mathrm{d}b_1 \wedge \cdots \wedge \mathrm{d}b_n.$$

It turns out that the modular vector fields X_μ^j are Hamiltonian vector fields with Hamiltonian function

$$h = \log(a_1 \cdots a_{n-1}) + j \log(\det(L)),$$

where L is the Lax matrix. This is observed in [1], where one also finds the multi-Hamiltonian formulation:

$$\overline{\pi}_j^\sharp \mathrm{d}h_{2-j} = \overline{\pi}_{j-1}^\sharp \mathrm{d}h_{3-j}, \quad k \in \mathbf{Z}$$

with $h_j = \frac{1}{j} \operatorname{Tr} L^j$ for $j \neq 0$ and $h_0 = \ln(\det(L))$.

We would like to give now an intrinsic explanation for these formulas, similar to the one given above for the Toda chain in physical coordinates.

4.5.3. *Toda lattice in extended Flaschka coordinates.* Let us extend the Flaschka coordinates by considering a variable a_n defined by:

$$a_n := q_n.$$

Then the transformation $(q_i, p_i) \mapsto (a_i, b_i)$ is a honest change of coordinates. In these extended Flaschka coordinates, the first Poisson tensor $\widetilde{\pi}_0$ is determined by:

$$\begin{aligned} \{a_i, b_i\}_0 &= a_i, & (i = 1, \ldots, n-1) \\ \{a_i, b_{i+1}\}_0 &= -a_i, & (i = 1, \ldots, n-1) \\ \{a_n, b_n\}_0 &= 1. \end{aligned}$$

while the second Poisson $\widetilde{\pi}_1$ structure is given by:

$$\begin{aligned} \{a_i, a_{i+1}\}_1 &= -a_i a_{i+1}, & (i = 1, \ldots, n-1) \\ \{a_i, b_i\}_1 &= a_i b_i, & (i = 1, \ldots, n-1) \\ \{a_n, b_n\}_1 &= b_n & \\ \{a_i, b_{i+1}\}_1 &= -a_i b_{i+1}, & (i = 1, \ldots, n-1) \\ \{b_i, b_{i+1}\}_1 &= -a_i, & (i = 1, \ldots, n-1). \end{aligned}$$

In these coordinates, we still have the Nijenhuis tensor, relating the various Poisson tensors in the hierarchy.

The submanifold $\mathbb{R}^{2n-1} \subset \mathbb{R}^{2n}$ defined by $a_n = 0$ is a Poisson submanifold for all Poisson tensors in the hierarchy, so that the bi-Hamiltonian structure reduces to this submanifold, and yields the bi-Hamiltonian formulation for the Toda lattice in Flaschka coordinates. However, the tangent space to this submanifold is not left invariant by the Nijenhuis operator N, and on $\mathbb{R}^{2n-1}$ we do not have an induced PN-structure.

Another way of expressing these facts is to observe that the involutive diffeomorphism $\phi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by:

$$\phi(a_1, \dots, a_n, b_1, \dots, b_n) = (a_1, \dots, -a_n, b_1, \dots, b_n),$$

is a Poisson diffeomorphism for all Poisson structures. Hence, the group $\mathbb{Z}_2 = \{I, \phi\}$ acts by Poisson diffeomorphisms on $\mathbb{R}^{2n}$, for all Poisson structures. It follows that its fix point set, which is just $\mathbb{R}^{2n-1}$, has induced Poisson brackets (see the Poisson Involution Theorem in [9, 19]), and these form the hierarchy in Flaschka coordinates.

4.5.4. *Toda lattice on a Lie algebroid.* We now consider the following bi-Hamiltonian formulation on a Lie algebroid.

We let $A = \mathbb{R}^{2n-1} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n-1}$ be the trivial vector bundle with fiber $\mathbb{R}^{2n}$. We denote by $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ a basis of global sections and we let $(a_1, \dots, a_{n-1}, b_1, \dots, b_n)$ be global coordinates on the base. Now we define a Lie algebroid structure by declaring that the bracket satisfies:

$$[e_i, e_j]_A = [f_i, f_j]_A = [e_i, f_j]_A = 0,$$

and that the anchor is given by:

$$\rho_A(e_i) = \frac{\partial}{\partial a_i} \quad (i = 1, \dots, n-1) \qquad\qquad \rho_A(e_n) = 0$$
$$\rho_A(f_i) = \frac{\partial}{\partial b_i} \quad (i = 1, \dots, n).$$

Notice that $(A, [\ ,\ ]_A, \rho_A)$ is just the trivial extension of Lie algebroids:

$$0 \longrightarrow L_{\mathbb{R}} \longrightarrow A \longrightarrow T\mathbb{R}^{2n-1} \longrightarrow 0$$

where $L_{\mathbb{R}}$ denotes the trivial line bundle over $\mathbb{R}^{2n-1}$.

Now on A we can define the following Poisson tensors:

$$\pi_0 = \sum_{i=1}^{n-1} a_i e_i \wedge (f_i - f_{i+1}) + e_n \wedge f_n$$

$$\pi_1 = -\sum_{i=1}^{n-2} a_i a_{i+1} e_i \wedge e_{i+1} - a_{n-1} e_{n-1} \wedge e_n + \sum_{i=1}^{n-1} a_i e_i \wedge (b_i f_i - b_{i+1} f_{i+1})$$
$$+ b_n e_n \wedge f_n - \sum_{i=1}^{n-1} a_i f_i \wedge f_{i+1}.$$

These Poisson structures on A cover ordinary Poisson structures on the base $\mathbb{R}^{2n-1}$, which are just the Poisson structures $\overline{\pi}_0$ and $\overline{\pi}_1$ of the Toda lattice, in Flaschka coordinates.

Since π_0 is symplectic, the Poisson tensors on A are associated with a PN-algebroid structure. By our main theorem, they give rise to an integrable hierarchy on A

$$\pi_j^\sharp \mathrm{d}h_{2-j} = \pi_{j-1}^\sharp \mathrm{d}h_{3-j}, \quad k \in \mathbf{Z}$$

with $h_j = \frac{1}{j} \operatorname{Tr} N^j$ for $j \neq 0$ and $h_0 = \ln(\det(N))$, covering an integrable hierarchy on the base

$$\overline{\pi}_j^\sharp \mathrm{d}h_{2-j} = \overline{\pi}_{j-1}^\sharp \mathrm{d}h_{3-j}. \quad k \in \mathbf{Z}$$

In this hierarchy the Hamiltonians differ by a factor of 2 relative to the Hamiltonians in the multi-Hamiltonian formulation of the Toda lattice given by Proposition 16. Although the hierarchy in the Lie algebroid is generated by a Nijenhuis operator, it is well known that this is not the case with the Toda lattice in the base manifold.

5. Examples

In this section, we consider 3 examples that illustrates the results above. The first two examples are the two extreme cases of (i) a Lie algebra over a point $A = \mathfrak{g}$ and (ii) the tangent bundle of a manifold $A = TM$. The third example, is the Lie algebroid connected with dynamical R-matrices, which is a product of a tangent Lie algebroid and a Lie algebra, therefore combining aspects of both.

5.1. Lie algebras. Let $\mathfrak{g}$ be a finite dimensional Lie algebra, considered as a Lie algebroid over a one point space, and let N be a Nijenhuis operator compatible with a Poisson tensor $\pi \in \mathfrak{X}^2(\mathfrak{g})$.

The modular class of $\mathfrak{g}$ is given by the adjoint character (see, e.g.,[17]):

$$\xi_{\mathfrak{g}}(X) = \mathrm{Tr}(\mathrm{ad}_X), \quad X \in \mathfrak{g}.$$

where ad denotes the adjoint representation of $\mathfrak{g}$.

Now we consider a Nijenhuis tensor $N : \mathfrak{g} \to \mathfrak{g}$, i.e., a linear map such that:

$$N[NX, Y] + N[X, NY] - N^2[X, Y] - [NX, NY] = 0, \quad X, Y \in \mathfrak{g}. \tag{13}$$

This allows us to deform the Lie bracket on $\mathfrak{g}$ to the new Lie bracket:

$$[X, Y]_N := [NX, Y] + [X, NY] - N[X, Y], \quad X, Y \in \mathfrak{g}.$$

The new Lie algebra $\mathfrak{g}_N = (\mathfrak{g}, [\ ,\ ]_N)$ (again viewed as Lie algebroid over a point) has modular class:

$$\xi_{\mathfrak{g}_N}(X) = \mathrm{Tr}(\mathrm{ad}^N_X), \quad X \in \mathfrak{g}.$$

where ad^N denotes the adjoint representation of $\mathfrak{g}_N$. Since the base manifold is a single point, Proposition 4 says simply that:

$$\xi_{\mathfrak{g}_N} = N^* \xi_{\mathfrak{g}},$$

which of course maybe checked directly. Similarly, by iteration, we have:

$$\xi_{\mathfrak{g}_{N^k}} = N^k\, \xi_{\mathfrak{g}}.$$

Let us denote by $r : \mathfrak{g}^* \to \mathfrak{g}$ the skew-symmetric linear transformation determined by the bivector $\pi \in \mathfrak{X}^2(\mathfrak{g})$. The condition that π is Poisson is just the condition that r is a solution of the *Classical Yang-Baxter Equation*:

$$[r, r] = 0 \tag{14}$$

For this reason, in this example we suppress any mention to the Poisson structure π, and we use instead the notation r, a solution of (14). We have also a Lie bialgebra structure $(\mathfrak{g}, \mathfrak{g}^*)$, and the bracket on $\mathfrak{g}^*$ is given by:

$$[\alpha, \beta]_* = [\alpha, \beta]_r = \mathrm{ad}^*(r(\alpha)) \cdot \beta - \mathrm{ad}^*(r(\beta)) \cdot \alpha, \quad \alpha, \beta \in \mathfrak{g}^*.$$

where ad^* denotes the coadjoint representation of $\mathfrak{g}$. The modular class of $\mathfrak{g}^*$ is given by

$$\xi_{\mathfrak{g}^*}(\alpha) = \mathrm{Tr}(\mathrm{ad}^r_\alpha), \quad \alpha \in \mathfrak{g}^*,$$

where ad^r is the adjoint representation of $(\mathfrak{g}^*, [\ ,\ ]_r)$.

Let us assume now that we have a triple $(\mathfrak{g}, r, N)$. Under the compatibility assumptions,

$$N \circ r = r \circ N^* \quad \text{and} \quad \left[\mathrm{ad}^*_{r\alpha}, N^*\right](\beta) = \left[\mathrm{ad}^*_{r\beta}, N^*\right](\alpha), \quad \alpha, \beta \in \mathfrak{g}^*, \tag{15}$$

where $[\ ,\ ]$ denotes the usual commutator of operators, we obtain a PN-algebroid $(\mathfrak{g}, r, N)$, with zero anchor (since the base manifold is a single point). Then N defines a sequence of modular forms on $\mathfrak{g}^*$, which are associated with the higher brackets on $\mathfrak{g}^*$, and are just related by:

$$\xi_{\mathfrak{g}^*_{N^{k*}}} = N^k \, \xi_{\mathfrak{g}^*_{N^*}}.$$

This is just an algebraic relation which must be satisfied by any pair of linear maps N and r which satisfy the algebraic equations (13), (14) and (15).

Obviously, in this case Theorem 9 says nothing, since every function is constant.

5.2. Tangent bundles. Let us consider now the case of $A = TM$, the tangent bundle of some manifold M. Since the anchor on A is the identity map, Proposition 4 yields the following result of [8]:

Proposition 17. *The modular class of* $(TM, [\ ,\ ]_N, \rho_N)$ *is the cohomology class represented by the 1-form* $\mathrm{d}(\mathrm{Tr}\, N)$*.*

Now a Poisson-Nijenhuis structure on A is just an ordinary Poisson-Nijenhuis manifold (M, π, N). Note, however, that the modular vector field associated with a Poisson manifold, as originally defined by Weinstein in [17], differs from $\xi_{T^*M,\pi}$ by a factor of $1/2$. For this reason, the modular class of a Poisson-Nijenhuis manifold was defined in [8] as

$$X_{(N,\pi)} = \frac{1}{2}\xi_{T^*M_{N^*}} - \frac{1}{2}N\xi_{T^*M} = \frac{1}{2}\mathrm{d}_\pi(\mathrm{Tr}\, N).$$

Then Theorem 9 immediately yields the main result of [8]:

Theorem 18. *Let* (M, π, N) *be a Poisson-Nijenhuis manifold and assume that* N *is non-degenerate. Then the modular vector field* X_N *is bi-Hamiltonian and hence determines a hierarchy of flows which are given by:*

$$X_{i+j} = \pi_i^\sharp \mathrm{d}h_j = \pi_j^\sharp \mathrm{d}h_i \quad (i, j \in \mathbf{Z})$$

where

$$h_0 = -\frac{1}{2}\log(\det N), \quad h_i = -\frac{1}{2i}\,\mathrm{Tr}\, N^i \quad (i \neq 0).$$

In [8] the authors show that many known hierarchies of integrable systems can be obtained in this manner, therefore providing a new approach to the integrability of those systems.

5.3. Dynamical Lie Algebroid. Let us start by describing the Lie bialgebroids that arise in connection with the dynamical Yang-Baxter equation.

Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an abelian Lie subalgebra of dimension l. We let $A = T\mathfrak{h}^* \times \mathfrak{g}$ be the direct product of Lie algebroids, where $T\mathfrak{h}^*$ is the tangent algebroid of the manifold $\mathfrak{h}^*$ and $\mathfrak{g}$ is viewed as a Lie algebroid over a point. More explicitly, the anchor $\rho : T\mathfrak{h}^* \times \mathfrak{g} \to T\mathfrak{h}^*$ is the projection on the first factor, while the bracket between two sections of A is given by:

$$[(v, f), (w, g)]_A = ([v, w], v(g) - w(f) + [f, g]_{\mathfrak{g}}),$$

where $v, w \in \mathfrak{X}(\mathfrak{h}^*)$ are vector fields in $\mathfrak{h}^*$ and $f, g : \mathfrak{h}^* \to \mathfrak{g}$ are smooth $\mathfrak{g}$-valued functions on $\mathfrak{h}^*$.

Let $\{h_1, \dots h_l\}$ be a basis of $\mathfrak{h}$ and $(q_1, \dots, q_l)$ the dual system of linear coordinates on $\mathfrak{h}^*$. Then we have a constant bivector field $\pi_0 \in \mathfrak{X}^2(A)$ defined by:

$$\pi_0 = \sum_{i=1}^{l} h_i \wedge \frac{\partial}{\partial q_i}.$$

Since $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$, it follows that π_0 is in fact a Poisson structure on A. Now we recall (see [4]):

Definition 19. *A* ***triangular dynamical r-matrix*** *on $\mathfrak{g}$ is an $\mathfrak{h}$-equivariant function $r : \mathfrak{h}^* \to \mathfrak{g} \wedge \mathfrak{g}$ satisfying the classical dynamical Yang-Baxter equation*

$$\sum_{i=1}^{l} h_i \wedge \frac{\partial r(q)}{\partial q_i} + \frac{1}{2}[r, r](q) = 0. \tag{16}$$

Its well known (see [2, 18]) that any triangular dynamical r-matrix on $\mathfrak{g}$, defines a new Poisson bivector on the Lie algebroid A by:

$$\pi_r(q) = \sum_{i=1}^{l} h_i \wedge \frac{\partial}{\partial q_i} + r(q).$$

Thus r determines a Lie bialgebroid structure on (A, A^*). The Lie bracket on the sections of A^* is given by

$$\begin{aligned}\left[h_q, h'_q\right]_* &= 0\\ [\varepsilon_q, h_q]_* &= \operatorname{ad}^*_{h_q}\varepsilon_q\\ \left[\varepsilon_q, \varepsilon'_q\right]_* &= -\operatorname{ad}^*_{r\varepsilon_q}\varepsilon'_q + \operatorname{ad}^*_{r\varepsilon'_q}\varepsilon_q + \mathrm{d}r(\varepsilon_q, \varepsilon'_q),\end{aligned}$$

for $h_q, h'_q \in \mathfrak{h}$ and $\varepsilon_q, \varepsilon'_q \in \mathfrak{g}^*$, where ad^* is the co-adjoint representation of $\mathfrak{g}$

$$\langle \operatorname{ad}^*_X \varepsilon, Y\rangle = \langle \varepsilon, [X, Y]\rangle, \quad X, Y \in \mathfrak{g},\ \varepsilon \in \mathfrak{g}^*.$$

The modular class of the Lie algebroid A^* is represented by

$$\xi_{A^*}(h_q, \varepsilon_q) = \operatorname{Tr}\operatorname{ad}_{h_q} - \operatorname{Tr}\left[(\delta r(q))^*\varepsilon_q\right], \tag{17}$$

where $\delta r(q)$ is the 1-cocycle on $\mathfrak{g}$ defined by the anti-symmetric bivector $r(q)$.

Now, if N is a Nijenhuis operator on A compatible with π_r, then the modular vector field of this Poisson-Nijenhuis Lie algebroid is

$$\mathrm{d}_{\pi_r}(\operatorname{Tr} N) = \mathrm{d}_{\pi_0}(\operatorname{Tr} N) = \sum_{i=1}^{l} \frac{\partial \operatorname{Tr} N}{\partial q_i}\frac{\partial}{\partial q_i}.$$

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Raquel Caseiro

Universidade de Coimbra, Departamento de Matemática, 3000 Coimbra, Portugal

E-mail address: raquel@mat.uc.pt