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# GEOMETRY OF THE NUMERICAL RANGE OF KREIN SPACE OPERATORS

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Abstract: The characteristic polynomial of the pencil generated by two $J$-Hermitian matrices is studied in connection with the numerical range. Geometric properties of the numerical range of linear operators on an indefinite inner product space are investigated. The point equation of the associated curve of the numerical range is derived, following Fiedler's approach for definite inner product spaces. The classification of the associated curve in the $3 \times 3$ case is presented, using Newton's classification of cubic curves. As a consequence, the respective numerical ranges are characterized. Illustrative examples of all the different possibilities are given.

Keywords: numerical range, generalized numerical range, indefinite inner product space, plane algebraic curve.
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## 1. Introduction

Let $J = I_r \oplus -I_{n-r}$ $(0 \leq r \leq n)$, where $I_m$ denotes de identity matrix of order $m$. If $r \neq 0, n$, the matrix $J$ endows $\mathbb{C}^n$ with the Krein structure defined by the indefinite inner product $\langle x, y\rangle_J = y^* J x,\ x, y \in \mathbb{C}^n$. For $A \in M_n$, the algebra of $n \times n$ complex matrices, let $A^{[*]} = JA^*J$ and consider the *$J$-Cartesian decomposition* $A = H^J + iK^J$, where $H^J = (A + A^{[*]})/2$ and $K^J = (A - A^{[*]})/(2i)$ are *$J$-Hermitian* matrices, that is, $H^J = (H^J)^{[*]}$ and $K^J = (K^J)^{[*]}$. If $J = \pm I_n$, we obtain the well known *Cartesian decomposition* of $A$, where $H = H^J$ and $K = K^J$ are Hermitian matrices.

Let $F_A^J(u, v, w) = \det(uH^J + vK^J + wI_n)$ be the characteristic polynomial of the pencil $uH^J + vK^J$. Our aim is to discuss the connection between $F_A^J(u, v, w)$ and the *$J$-numerical range* of $A$ denoted and defined by

$$W_J(A) = \left\{ \frac{x^* J A x}{x^* J x} : x \in \mathbb{C}^n,\ x^* J x \neq 0 \right\}.$$

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If $J = \pm I_n$, then $W_J(A)$ reduces to the well-known *classical numerical range* or *field of values* of $A$, usually denoted by $W(A)$.

The equation $F_A^J(u, v, w) = 0$, with $u$, $v$, $w$ viewed as homogeneous line coordinates, defines an algebraic curve of class $n$. The real part of this curve is the *associated curve* of $W_J(A)$, denoted by $C_J(A)$.

This paper is organized as follows. Section 2 consists of definitions and preliminary results. In Section 3, some properties of $C_J(A)$ are studied. In Section 4, the point equation of $C_J(A)$ is derived, following the approach of Fiedler [5, 6] in the particular case of $J$ being a positive definite Hermitian matrix. In Section 5, the associated curves $C_J(A)$ are classified in the $2\times 2$ case. In particular, a new proof of the Hyperbolical Range Theorem is given. In Section 6, the associated curves $C_J(A)$ are classified in the $3 \times 3$ case, using Newton's classification of cubic curves. These results extend for Krein spaces results of Kippenhahn [8] on the classical numerical range. In section 7, illustrative examples are presented.

## 2. Definitions and preliminaries

We briefly recall some known properties of $W_J(A)$. For any $A \in M_n$, $W(A)$ contains the spectrum of $A$, denoted by $\sigma(A)$, while for the $J$-numerical range the following inclusion holds: $\sigma_J(A) \subseteq W_J(A)$, $\sigma_J(A)$ denoting the set of the eigenvalues of $A$ that have *anisotropic* eigenvectors, that is, vectors $x$ for which $x^*Jx \neq 0$. We denote by $\sigma_+(A)$ and $\sigma_-(A)$ the sets of eigenvalues of $A$ with associated eigenvectors having positive and negative $J$-norms, respectively. By $\sigma_0(A)$ we denote the set of the eigenvalues of $A$ with isotropic eigenvectors, i.e., vectors $x$ such that $x^*Jx = 0$. If $\lambda \in \sigma_J(A) = \sigma_+(A) \cup \sigma_-(A)$, then $\lambda \in W_J(A)$. However, if $\lambda \in \sigma_0(A)$, it may not belong to $W_J(A)$.

The field of values $W(A)$ is a compact and convex set for $A \in M_n$ [7]. In contrast with the classical case, $W_J(A)$ may not be closed and is either unbounded or a singleton [11, 10]. For $\lambda \in \mathbb{C}$, $W_J(A) = \{\lambda\}$ if and only if $A = \lambda I_n$. On the other hand, $W_J(A)$ is not usually convex. However, it is the union of the convex sets

$$W_J(A) = W_J^+(A) \cup W_{-J}^+(A) \tag{1}$$

where

$$W_J^{\pm}(A) = \{x^*JAx \colon x \in \mathbb{C}^n,\ x^*Jx = \pm 1\},$$

being $W_{-J}^+(A) = -W_J^-(A)$ [11]. Moreover, $W_J(A)$ is *pseudo-convex* [11]; that is, for any pair of distinct points $x, y \in W_J(A)$, either $W_J(A)$ contains the closed line segment joining $x$ and $y$ if $x, y$ belong to the same convex set in (1) ($W_J^+(A)$ or $W_{-J}^+(A)$), otherwise $W_J(A)$ contains the two closed half-lines of the line defined by $x$ and $y$ with endpoints $x$ and $y$.

A matrix $U \in M_n$ is called *$J$-unitary* of signature $(r, n-r)$, $0 \leq r \leq n$, if $U^{-1} = JU^*J$. These matrices form a group denoted by $\mathcal{U}_{r,n-r}$. For any $U \in \mathcal{U}_{r,n-r}$, we have $W_J(A) = W_J(U^*AU)$ and the following holds

$$W_J(\alpha I_n + \beta A) = \alpha + \beta\, W_J(A), \quad \alpha, \beta \in \mathbb{C}. \tag{2}$$

For $A \in M_2$, Murnaghan [12] proved that $W(A)$ is an elliptical disc (possibly degenerate) whose foci are the eigenvalues of $A$, $\alpha_1$ and $\alpha_2$. The major and minor axis are of length

$$\sqrt{\operatorname{Tr}(A^*A) - 2\operatorname{Re}(\overline{\alpha_1}\alpha_2)} \quad \text{and} \quad \sqrt{\operatorname{Tr}(A^*A) - |\alpha_1|^2 - |\alpha_2|^2},$$

respectively. This result is the well-known Elliptical Range Theorem. In the indefinite case, for $A \in M_2$ and $J = \operatorname{diag}(1, -1)$, the Hyperbolical Range Theorem [2] states that $W_J(A)$ is bounded by the hyperbola (possibly degenerate), with foci at $\alpha_1$ and $\alpha_2$, and transverse and non-transverse axis of length

$$\sqrt{\operatorname{Tr}(A^{[*]}A) - 2\operatorname{Re}(\overline{\alpha_1}\alpha_2)} \quad \text{and} \quad \sqrt{|\alpha_1|^2 + |\alpha_2|^2 - \operatorname{Tr}(A^{[*]}A)}, \tag{3}$$

respectively. For the degenerate cases, $W_J(A)$ may be a singleton, a line, a subset of a line, the whole complex plane, or the complex plane except a line.

The description of $W_J(A)$, when $A \in M_n$ and $n > 2$, is not an easy task. Some authors have characterized (generalized) numerical ranges for some special classes of matrices. In certain cases, $W_J(A)$ still is an hyperbola and its "interior", independently of the size of $A$.

A *supporting line* of a convex set $K \subseteq \mathbb{C}$ is a line that intersects $K$ at least in one point and that defines two half-planes, such that one of them does not contain any point of $K$. For each point on the boundary of $K$, supporting lines may not exist or they may not be unique. The *supporting lines* of $W_J(A)$ are the supporting lines of the convex sets $W_J^+(A)$ and $W_{-J}^+(A)$. As proved in [2, Theorem 2.2], if $ux + vy + w = 0$ is the equation of a supporting line of $W_J^+(A)$ ($W_J^-(A)$), then

$$\det(uH^J + vK^J + wI_n) = 0, \tag{4}$$

and $-w$ is the maximum, or the minimum, eigenvalue of the pencil $uH^J + vK^J$ in $\sigma_+(A)$ ($\sigma_-(A)$), according to $ux + vy + w \leq 0$, or $ux + vy + w \geq 0$, for all points in $W_J^+(A)$ ($W_J^-(A)$). We recall that the converse is not always true, that is a solution of (4) may not be a supporting line of $W_J(A)$.

Since $F_A^J(u,v,w) = \det(uH^J + vK^J + wI_n)$ is a homogeneous polynomial of degree $n$, (4) may be considered the line equation of an algebraic curve on the complex projective plane $P_2(\mathbb{C})$. We consider the complex projective curve $\Gamma$ defined by

$$\Gamma = \{(u : v : w) \in P_2(\mathbb{C})\colon\ \det(uH^J + vK^J + wI_n) = 0\},$$

and its dual curve

$$\Gamma^\wedge = \{(x : y : z) \in P_2(\mathbb{C})\colon xu + yv + zw = 0 \text{ is a tangent of } \Gamma\}.$$

The real affine part of $\Gamma^\wedge$,

$$\Gamma^* = \{(x,y) \in \mathbb{R}^2\colon (x : y : 1) \in \Gamma^\wedge\},$$

is the *associated curve* of $W_J(A)$. Some authors use the designation *boundary generating curve*, following the German terminology ("randerzeugende" curve). The curve $F_A(u,v,w) = 0$ has class $n$, that is, through a general point in the plane there are $n$ lines (may be complex) tangent to the curve. (For details on plane algebraic curves, we refer to [4].)

If $J = \pm I_n$, $C_J(A)$ is simply denoted by $C(A)$. Kippenhahn proved that the curve $C(A)$ generates $W(A)$ as its convex hull [8]. For $C_J(A)$ the following holds.

**Theorem 1** ([3]). *Let $A = H^J + iK^J \in M_n$. Assume that there exists a real interval $[\theta_1, \theta_2]$ such that the $n$ eigenvalues $\lambda_1(\theta), \ldots, \lambda_n(\theta)$ of $\cos\theta H^J + \sin\theta K^J$ ($\theta \in [\theta_1,\theta_2]$) are real and have an associated basis of anisotropic eigenvectors. Let $u_k(\theta)$ be an eigenvector of $\cos\theta H^J + \sin\theta K^J$ associated with $\lambda_k(\theta)$, $k = 1,\ldots,n$. Then $C_J(A)$ is given by*

$$\left\{ z_k(\theta) = \frac{u_k^*(\theta) JA\, u_k(\theta)}{u_k^*(\theta) J u_k(\theta)}\colon \theta \in [\theta_1,\theta_2],\ k = 1,\ldots,n \right\}$$

*and $W_J(A)$ is the pseudo-convex hull of $C_J(A)$.*

Since every complex matrix $A$ can be uniquely expressed as $A = H^J + iK^J$, where $H^J$ and $K^J$ are $J$-Hermitian, we have

$$\langle Ax, x\rangle_J = \langle H^J x, x\rangle_J + i\langle K^J x, x\rangle_J.$$

Denoting by Re $S$ and Im $S$ the projection of $S \subseteq \mathbb{C}$ on the real and imaginary axis, respectively, and having in mind that $\langle H^J x, x\rangle_J$, $\langle K^J x, x\rangle_J$ are real, it follows that Re $W_J(A) = W_J(H^J)$ and Im $W_J(A) = W_J(K^J)$. More generally, the projection of $W_J(A)$ on the line passing through the origin with argument $\theta$ is given by $W_J(\cos\theta H^J + \sin\theta K^J)$.

Let $A$ be a $J$-Hermitian matrix such that $\sigma_+(A) = \{\alpha_1 \geq \cdots \geq \alpha_r\}$ and $\sigma_-(A) = \{\alpha_{r+1} \geq \cdots \geq \alpha_n\}$. If $\sigma_0(A) = \emptyset$, the eigenvalues of $A$ are said to *interlace* if $[\alpha_1, \alpha_r] \cap [\alpha_{r+1}, \alpha_n] \neq \emptyset$ and $\alpha_r \neq \alpha_{r+1}$, $\alpha_1 \neq \alpha_n$. The following result was obtained in [13] for $J$-Hermitian matrices of size $n$ and it is very useful in our discussions.

**Theorem 2.** *Let $A \in M_3$ be a $J$-Hermitian matrix. If one of the conditions (a)-(d) holds, then $W_J(A)$ is the whole real line:*

(a) *$A$ has complex (conjugate) eigenvalues;*
(b) *$A$ has real spectrum and $A$ is nilpotent with nilpotency index (that is, the smallest natural $k$ such that $A^k = 0$) equal to 3;*
(c) *$A$ has three anisotropic eigenvalues and they interlace;*
(d) *$A$ has one eigenvalue in $\sigma_+(A)$ and another in $\sigma_0(A)$ with multiplicity 2, and 0 belongs to the interior of $W(JA + iJ)$.*
(e) *If $A$ has one eigenvalue in $\sigma_+(A)$ and $\lambda \in \sigma_0(A)$ has multiplicity 2, and if 0 does not belong to the interior of $W(JA + iJ)$, then $W_J(A) = \mathbb{R}$ or $W_J(A) = \mathbb{R}\backslash\{\lambda\}$.*

## 3. Properties of $\mathbf{C_J(A)}$

**Proposition 1.** *The coefficients of the algebraic curve $F_A^J(u, v, w) = 0$ are real.*

*Proof*: We easily find

$$\begin{aligned} F_A^J(u, v, w) &= \det(uH^J + vK^J + wI_n) \\ &= \det(uJ(H^J)^* J + vJ(K^J)^* J + wI_n) \\ &= \overline{F_A^J}(u, v, w), \end{aligned}$$

where $\overline{F_A^J}(u, v, w)$ denotes the polynomial obtained from $F_A^J(u, v, w)$ taking the conjugates of its coefficients. ∎

Next, we characterize the associated curve of the numerical range of a $J$-Hermitian matrix.

**Proposition 2.** *Let $A$ be a $J$-Hermitian matrix of order $n$. Then, $C_J(A)$ is the set of points $(\lambda, 0)$ of the affine plane, for $\lambda$ a real eigenvalue of $A$. If the* *eigenvalues of $A$ are all complex (and so $A$ has even order), then $C_J(A)$ is the real axis.*

*Proof*: We have $F_A^J(u,v,w) = \det(uA + wI_n)$, because $A$ is a $J$-Hermitian matrix. If $u = 0$, then the unique solution of the equation $F_A^J(u,v,w) = 0$ is given in line coordinates by $(0:1:0)$, i.e., the real axis. Suppose now that $u \neq 0$. Since we are considering homogeneous coordinates, the solutions of the polynomial equation $F_A^J(u,v,w) = 0$ are of the form $(1 : v : w_0/u_0)$, for each pair $(u_0, w_0)$ satisfying that equation. Taking $u = 1$, we obtain the polynomial in $w$ with real coefficients, $\det(A + wI)$. Obviously, its real solutions are $(1 : v : -\lambda)$, for $\lambda$ a real eigenvalue of $A$. But $(1 : v : -\lambda)$ is the pencil of lines passing through the point $(\lambda : 0 : 1)$ (the real axis is included) and as usual, we identify the pencil with this point. If the matrix does not have real eigenvalues, then the real axis is the unique possible solution. ■

The following statements generalize results of Kippenhahn for the classical numerical range [8].

**Theorem 3.** *Let $A = H^J + iK^J \in M_n$. Assume that there exists a real interval $[\theta_1, \theta_2]$ such that the $n$ eigenvalues $\lambda_1(\theta), \ldots, \lambda_n(\theta)$ of $\cos\theta H^J + \sin\theta K^J$ ( $\theta \in [\theta_1, \theta_2]$) are real and the corresponding eigenvectors are anisotropic. Then, for each $\theta \in [\theta_1, \theta_2]$, there are $n$ real tangents to $C_J(A)$ with the direction $-\theta + \pi/2$.*

*Proof*: Consider the real direction given by $u = \cos\theta$, $v = \sin\theta$, for $\theta$ according to the hypothesis. Then the equation

$$F_A^J(u,v,w) = \det(uH^J + vK^J + wI_n) = 0$$

has $n$ real solutions in $w$, since by the hypothesis the $J$-Hermitian matrix $uH^J + vK^J$ has $n$ real eigenvalues (counting the multiplicities). These $n$ real solutions are the tangent lines to $C_J(A)$ with argument $-\theta + \pi/2$. ■

**Corollary 1.** *The arcs of $C_J(A)$ with tangent lines perpendicular to the direction $\theta \in [\theta_1, \theta_2]$ in Theorem 3 do not have flexional tangents.*

**Remark 1.** The real interval $[\theta_1, \theta_2]$ may reduce to a singleton $\{\theta'\}$. Then the supporting lines of $W_J(A)$, if they exist, are perpendicular to $\theta'$. In this case, the boundaries of $W_J^+(A)$ and $W_{-J}^+(A)$ are lines with argument $-\theta' + \pi/2$.

We recall that a point $P$, not equal to the circular points at infinity $(1 : i : 0)$ and $(1 : -i : 0)$, is called a *focus* of a curve $C$ if the line $l_1$ through $P$ and

$(1 : i : 0)$ and the line $l_2$ through $P$ and $(1 : -i : 0)$ are tangent to $C$ at points other than the circular points at infinity.

A curve of class $n$ with real coefficients has $n$ real foci, counting multiplicities, and $n^2 - n$ foci which are not real [14]. Murnaghan [12] and Kippenhahn [8] independently proved that the real foci of the algebraic curve defined by $\det(uH + vK + I_n) = 0$ are the eigenvalues of $A = H + iK$.

**Proposition 3.** *The $n$ real foci of the algebraic curve defined by the equation $F_A^J(u, v, w) = \det(uH^J + vK^J + wI_n) = 0$ are the eigenvalues $\alpha_1, \ldots, \alpha_n$ of the matrix $A = H^J + iK^J$.*

*Proof*: Taking $u = 1$ and $v = i$ in $F_A^J(u, v, w) = 0$, we obtain $\det(H^J + iK^J + wI_n) = \det(A + wI_n) = 0$, that is, the roots $w$ are the eigenvalues of $A$ multiplied by $-1$. Repeating the above procedure for $u = 1$ and $v = -i$, we find $\det(H^J - iK^J + wI_n) = \det(A^* + wI_n) = 0$, and so $w$ are the eigenvalues of $A^*$ multiplied by $-1$. Thus, $w$ coincides with $-\overline{\alpha_j}$, $j = 1, \ldots, n$. For $k, l = 1, \ldots, n$, let $g_k$ and $\overline{g_l}$ be the lines given in line coordinates by $(1 : i : -\alpha_k)$ and $(1 : -i : -\overline{\alpha_l})$, respectively. We observe that these lines pass through the circular points at infinity. More precisely, the lines $g_k$ pass through $(1 : i : 0)$, while the lines $\overline{g_l}$ pass through $(1 : -i : 0)$. Since $g_k$ and $\overline{g_l}$ are solutions of the equation

$$\det(uH^J + vK^J + wI_n) = 0,$$

they are the unique tangent lines to the curve that pass through the circular points at infinity. Easy calculations show that the intersections of $g_k$ with $\overline{g_k}$, $k = 1, \ldots, n$, are given by $(\operatorname{Re}\alpha_k : \operatorname{Im}\alpha_k : 1)$, $k = 1, \ldots, n$. ■

## 4. The point equation of $\mathbf{C_J(A)}$

We recall that an usual procedure to find the point equation of the associated curve is to eliminate one of the indeterminates, say $u$, from (4) and $ux + vy + w = 0$, dehomogenize the result by setting $w = 1$, and eliminate the remaining parameter $v$ from the resulting equation $F_A(v, x, y) = 0$ and $\partial F_A(v, x, y)/\partial v = 0$. In this Section, we present an alternative procedure to find the point equation of $C_J(A)$. The *second mixed compound* of two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ of the same size $m \times n$, denoted by $\langle A, B\rangle$, is the $\binom{m}{2} \times \binom{n}{2}$ matrix with entries

$$\langle A, B\rangle_{PQ} = \frac{1}{2}(a_{ir}b_{js} + a_{js}b_{ir} - a_{is}b_{jr} + a_{jr}b_{is}),$$

where $P = (i, j)$, $1 \leq i < j \leq m$, and $Q = (r, s)$, $1 \leq r < s \leq n$. In particular, the *second compound* of a matrix $A$ is denoted and defined by

$$A^{(2)} = \langle A, A \rangle.$$

It follows from the definition that

$$(A^{(2)})_{PQ} = \det \begin{pmatrix} a_{ir} & a_{is} \\ a_{jr} & a_{js} \end{pmatrix},$$

for $P = (i, j)$, $1 \leq i < j \leq m$, and $Q = (r, s)$, $1 \leq r < s \leq n$. Therefore, $A^{(2)}$ is the array of all minors of $A$ of second order. Next, we list some properties of the second compound and of the second mixed compound, obtained in [5].

**Lemma 1.** *For any matrices $A, B, A_1, A_2$ of the same size, and for any complex numbers $\alpha_1, \alpha_2$, the following holds:*

(a) $\langle \alpha_1 A_1 + \alpha_2 A_2, B \rangle = \alpha_1 \langle A_1, B \rangle + \alpha_2 \langle A_2, B \rangle$;
(b) $\langle A, B \rangle = \langle B, A \rangle$.

**Lemma 2.** *For $A, B \in M_n$, the form*

$$\det(xA + yB)$$

*(which is a product of linear complex factors $\alpha_i x + \beta_j y$) is either identically zero, or has a multiple linear factor if and only if*

$$\det \begin{pmatrix} A^{(2)} & \langle A, B \rangle \\ \langle A, B \rangle & B^{(2)} \end{pmatrix} = 0.$$

Next Theorem is not practical for large matrices, but it can be used for small size matrices and for theoretical purposes.

**Theorem 4.** *Let $A = H^J + iK^J \in M_n$, where $H^J$ and $K^J$ are $J$-Hermitian. If the associated curve $C_J(A)$, given in line coordinates by the equation (4), is irreducible, then its point equation is given by the non-linear part of the equation*

$$\det \begin{pmatrix} (H^J - xI_n)^{(2)} & \langle H^J - xI_n, K^J - yI_n \rangle \\ \langle H^J - xI_n, K^J - yI_n \rangle & (K^J - yI_n)^{(2)} \end{pmatrix} = 0, \qquad (5)$$

*or, equivalently, by the non-linear part of the equation*

$$\det\begin{pmatrix} (H^J)^{(2)} & \langle H^J, K^J\rangle & \langle H^J, I_n\rangle & xI_{\binom{n}{2}} \\ \langle H^J, K^J\rangle & (K^J)^{(2)} & \langle K^J, I_n\rangle & yI_{\binom{n}{2}} \\ \langle H^J, I_n\rangle & \langle K^J, I_n\rangle & I_{\binom{n}{2}} & I_{\binom{n}{2}} \\ xI_{\binom{n}{2}} & yI_{\binom{n}{2}} & I_{\binom{n}{2}} & 0 \end{pmatrix} = 0. \tag{6}$$

*The linear factors of the above equations correspond to multiple tangents, in the case of their existence. If $C_J(A)$ contains a line, they also correspond to that line.*

*Proof*: Performing some elementary operations with the blocks of the matrix in (6), using Laplace Theorem and Lemma 1, we can easily see that the first members of equations (5) and (6) are equal if $\binom{n}{2}$ is even, or symmetric if $\binom{n}{2}$ is odd. Therefore, the equations (5) and (6) are equivalent. On the other hand, (5) is invariant under the replacement $A \mapsto A \pm (x+iy)I_n$. Therefore, it is sufficient to prove that the result holds for the origin $(0,0)$. Moreover, the origin satisfies (6) if and only if

$$\det\begin{pmatrix} (H^J)^{(2)} & \langle H^J, K^J\rangle \\ \langle H^J, K^J\rangle & (K^J)^{(2)} \end{pmatrix} = 0.$$

By Lemma 2, this means that the form $\det(uH^J + vK^J)$ is identically zero or has a multiple linear factor. However, the first case may not hold, otherwise that would mean that $\det(uH^J + vK^J + wI_n)$ is divisible by $w$ and, consequently, $C_J(A)$ is reducible. Then, only one possibility remains: the form $\det(uH^J + vK^J)$ has a multiple linear factor. The proof follows by dual considerations. We eliminate $w$ from $\det(uH^J + vK^J + wI_n)$ and $ux + vy + w = 0$ for $x = y = 0$. Therefore, there is a multiple tangent to $C_J(A)$ passing through the origin. Obviously, the origin lies either on that tangent or on the associated curve itself. The non-linear factor of (5) or (6) corresponds to $C_J(A)$ and the linear factors to multiple tangents, in case of their existence, or to a line, if that line belongs to $C_J(A)$. ■

## 5. Characterization of $\mathbf{C_J(A)}$ for $\mathbf{A \in M_2}$

Let $A$ be an arbitrary matrix in $M_2$ and $J = \mathrm{diag}(1, -1)$. By (2), without loss of generality we may consider $A$ with zero trace and real determinant. Writing $A = H^J + iK^J$, where $H^J$ and $K^J$ are $J$-Hermitian matrices, by easy computations we get

$$F_A^J(u,v,w) = \det(H^J)u^2 + \det(K^J)v^2 + w^2 \\ + \operatorname{Re}\operatorname{Tr}(A)uw + \operatorname{Im}\operatorname{Tr}(A)vw + \operatorname{Im}\det(A)uv. \tag{7}$$

Having in mind the conditions on $A$, this polynomial reduces to

$$F_A^J(u,v,w) = \det(H^J)u^2 + \det(K^J)v^2 + w^2. \tag{8}$$

**Proposition 4.** *Let $C$ be the real part of the algebraic curve defined in line coordinates by the equation $G(u,v,w) = c_u u^2 + c_v v^2 + w^2 = 0$, where $c_u$, $c_v \in \mathbb{R}$.*

(a) *If $c_u = c_v = 0$, then $C = \{0\}$;*
(b) *If $c_u < 0$ and $c_v = 0$, then $C = \{(-\sqrt{|c_u|}, 0), (\sqrt{|c_u|}, 0)\}$;*
(c) *If $c_u > 0$ and $c_v = 0$, then $C = \mathbb{R}$;*
(d) *If $c_u = 0$ and $c_v < 0$, then $C = \{(0, -\sqrt{|c_v|}), (0, \sqrt{|c_v|})\}$;*
(e) *If $c_u = 0$ and $c_v > 0$, then $C = i\mathbb{R}$;*
(f) *If $c_u > 0$ and $c_v > 0$, then $C = \emptyset$ (an imaginary ellipse);*
(g) *If $c_u > 0$ and $c_v < 0$, then $C$ is the hyperbola centered at the origin with transverse and non-transverse axis on the imaginary and real axis of length $2\sqrt{|c_v|}$ and $2\sqrt{|c_u|}$, respectively;*
(h) *If $c_u < 0$ and $c_v > 0$, then $C$ is the hyperbola centered at the origin with transverse and non-transverse axis on the real and imaginary axis of length $2\sqrt{|c_u|}$ and $2\sqrt{|c_v|}$, respectively;*
(i) *If $c_u < 0$ and $c_v < 0$, then $C$ is the ellipse centered at the origin with major and minor axis on the coordinate axis, of lengths $2\sqrt{\max\{|c_u|, |c_v|\}}$ and $2\sqrt{\min\{|c_u|, |c_v|\}}$, respectively.*

*Proof*: (a) If $c_u = c_v = 0$, then $G(u,v,w) = 0$ if and only if $w = 0$, and the solution of this polynomial equation is the pencil of lines $(u : v : 0)$ passing through the point $(0 : 0 : 1)$. As usual, we identify the pencil with the point $(0 : 0 : 1)$, and so $C$ is the origin of the affine plane.

(b) If $c_v = 0$, it follows that $G(u,v,w) = 0$ if and only if $w = \pm\sqrt{-c_u}\,u$. If $c_u < 0$, then we get the pencils of lines $(u : v : \sqrt{|c_u|}u)$ and $(u : v : -\sqrt{|c_u|}u)$, which pass through the points $(-\sqrt{|c_u|} : 0 : 1)$ and $(\sqrt{|c_u|} : 0 : 1)$, respectively. Identifying the pencils with the respective points of the real axis, the result follows.

(c) If $c_v = 0$ and $c_u > 0$, then the unique real solution of the equation $G(u, v, w) = 0$ is obtained for $u = 0$ and $w = 0$. This solution is the line with coordinates $(0 : 1 : 0)$, i.e., the real axis.

The proof of (d) and (e) follows analogously to (b) and (c), respectively.

(f) If $c_u > 0$ and $c_v > 0$, then $G(u, v, w) = 0$ if and only if $|c_u|u^2 + |c_v|v^2 + w^2 = 0$ and this equation does not have real solutions. Since $(0 : 0 : 0)$ does not represent any line in the projective plane, $C$ is the empty set.

(g) Suppose that $c_u > 0$ and $c_v < 0$. Thus, $G(u, v, w) = |c_u|u^2 - |c_v|v^2 + w^2$. To obtain the equation of the algebraic curve in point coordinates, let $w = 1$ and $u = (-1 - vy)/x$. We easily get

$$\frac{\partial G}{\partial v} = |c_u| \, \frac{2y^2 v + 2y}{x^2} - 2|c_v|v.$$

Solving the equation $\frac{\partial G}{\partial v} = 0$ with respect to $v$, we obtain

$$v = \frac{y|c_u|}{x^2|c_v| - y^2|c_u|}.$$

Obviously,

$$u = -\frac{x|c_v|}{x^2|c_v| - y^2|c_u|}.$$

Substituting the expressions of $u$ and $v$ in $G(u, v, w) = 0$, we get

$$\frac{y^2}{|c_v|} - \frac{x^2}{|c_u|} = 1.$$

The proofs of (h) and (i) follow by analogous arguments, and the Proposition is proved. ■

As seen in the proof of Proposition 4, the polynomial $F_A^J(u, v, w)$ is irreducible if and only if neither $\det(H^J)$ nor $\det(K^J)$ is equal to zero.

The Elliptical Range Theorem is easily obtained from Proposition 4. If $H$ and $K$ are Hermitian and $\mathrm{Tr}(H) = \mathrm{Tr}(K) = 0$, then $\det(H) \leq 0$ and $\det(K) \leq 0$. Recalling that $W(A)$ is the closed convex hull of $C(A)$, the Elliptical Range Theorem follows from Proposition 4 (a), (b), (d) and (i). To prove the Hyperbolical Range Theorem we need the two following Lemmas.

**Lemma 3.** *Let $0 \neq A = [a_{ij}] \in M_2$ be a $J$-Hermitian matrix such that $\mathrm{Tr}(A) = 0$. The following holds:*

(a) *If $\det(A) < 0$, then $C_J(A) = \{(-\sqrt{|\det(A)|}, 0), (\sqrt{|\det(A)|}, 0)\}$ and $W_J(A) = \mathbb{R} \setminus ] - \sqrt{|\det(A)|}, \sqrt{|\det(A)|}[$;*

(b) *If* $\det(A) = 0$*, then* $C_J(A) = \{0\}$ *and* $W_J(A) = \mathbb{R}\backslash\{0\}$*;*
(c) *If* $\det(A) > 0$*, then* $C_J(A) = \mathbb{R}$ *and* $W_J(A) = \mathbb{R}$*.*

*Proof*: Recalling (8), $C_J(A)$ is given in line coordinates by the equation $\det(A)u^2 + w^2 = 0$, and it is characterized by Proposition 4. Without loss of generality, we may consider

$$A = \begin{bmatrix} k & a \\ -a & -k \end{bmatrix},$$

being $a \geq 0$ and $k = 1$ if $a_{11} \neq 0$, and $k = 0$, otherwise. We observe that $k = 0$ only if $\det(A) > 0$.

Let $x = [\cosh t \ (e^{i\alpha} \sinh t)]^T$, $t \in \mathbb{R}$, $\alpha \in [0, 2\pi]$. If $k = 1$, by easy calculations we find $x^*Jx = 1$ and $x^*JAx = \cosh 2t + a \cos\alpha \sinh 2t$ . For $t \in \mathbb{R}$,

$$\frac{e^{2|t|}(1-a) + e^{-2|t|}(1+a)}{2} \leq x^*JAx \leq \frac{e^{2|t|}(1+a) + e^{-2|t|}(1-a)}{2}. \tag{9}$$

If $k = 0$, then $x^*JAx = a \cos\alpha \sinh 2t$, and for $t \in \mathbb{R}$ we have

$$-a \sinh(2|t|) \leq x^*JAx \leq a \sinh(2|t|). \tag{10}$$

(a) If $\det(A) < 0$, then $0 \leq a < 1$ and $k = 1$. By (9), $x^*JAx$ is a positive real number. Moreover,

$$\frac{e^{2|t|}(1-a) + e^{-2|t|}(1+a)}{2} \geq \sqrt{1-a^2}.$$

On the other hand, $\frac{e^{2|t|}(1+a)+e^{-2|t|}(1-a)}{2} \mapsto +\infty$, when $|t| \mapsto +\infty$. Thus, $W_J^+(A) = [\sqrt{|\det(A)|}, +\infty[$, $W_{-J}^+(A) = ]-\infty, -\sqrt{|\det(A)|}]$, and so $W_J(A) = \mathbb{R}\backslash] - \sqrt{|\det(A)|}, \sqrt{|\det(A)|}[$.

(b) If $\det(A) = 0$, then $a = 1$ and $k = 1$, and by (9),

$$e^{-2|t|} \leq x^*JAx \leq e^{2|t|},$$

for $t \in \mathbb{R}$. Then $W_J^+(A) = \mathbb{R}^+$ and $W_{-J}^+(A) = \mathbb{R}^-$. Therefore, $W_J(A) = \mathbb{R}\backslash\{0\}$.

(c) If $\det(A) > 0$, either $a > 1$ and $k = 1$ or $a \neq 0$ and $k = 0$. If $|t| \mapsto +\infty$, from (9) and (10) it follows that $W_J^+(A) = \mathbb{R}$, $W_{-J}^+(A) = \mathbb{R}$, respectively, and so $W_J(A) = \mathbb{R}$. ∎

**Lemma 4.** *Let* $A = [a_{ij}] = H^J + iK^J \in M_2$*, where* $H^J$ *and* $K^J$ *are* $J$*-Hermitian matrices, being* $\mathrm{Tr}(A) = 0$ *and* $\mathrm{Im}\det(A) = 0$*.*

(a) *The inequalities* $\det(H^J) < 0$ *and* $\det(K^J) < 0$ *may not occur simultaneously;*
(b) *If* $\det(H^J) < 0$ *and* $\det(K^J) = 0$*, then* $K^J = 0$*;*
(c) *If* $\det(H^J) = 0$ *and* $\det(K^J) < 0$*, then* $H^J = 0$*.*

*Proof*: (a) Since $\mathrm{Tr}(A) = 0$, it follows that $a_{22} = -a_{11}$. From the inequalities $\det(H^J) < 0$ and $\det(K^J) < 0$, we easily get

$$\frac{1}{4}(a_{12} - \overline{a}_{21})(\overline{a}_{12} - a_{21}) < \frac{1}{4}(a_{11} + \overline{a}_{11})^2;$$
$$\frac{1}{4}(a_{12} + \overline{a}_{21})(\overline{a}_{12} + a_{21}) < -\frac{1}{4}(a_{11} - \overline{a}_{11})^2.$$

The members of the above inequalities are non-negative, and by multiplication we find

$$\frac{1}{16}(a_{12}^2 - \overline{a}_{21}^2)(\overline{a}_{12}^2 - a_{21}^2) < -\frac{1}{16}(a_{11}^2 - \overline{a}_{11}^2)^2. \tag{11}$$

Since $\det(A)$ is a real number, $-a_{11}^2 - a_{12}a_{21} = -\overline{a}_{11}^2 - \overline{a}_{12}\overline{a}_{21}$ and so

$$(a_{12}^2 - \overline{a}_{21}^2)(\overline{a}_{12}^2 - a_{21}^2) < -(a_{12}a_{21} - \overline{a}_{12}\overline{a}_{21})^2.$$

Then $|a_{12}|^4 + |a_{21}|^4 - 2|a_{12}|^2|a_{21}|^2 < 0$, a contradiction.

(b) If $\det(H^J) < 0$ and $\det(K^J) = 0$, having in mind (11), $a_{11}$ is a real number and $a_{12} = -\overline{a_{21}}$. It follows that $K^J = 0$.

(c) The proof of (c) is analogous to (b). ∎

**Theorem 5.** *For $A$ under the assumptions of Lemma 4, $h = \det(H^J)$ and $k = \det(K^J)$, the following holds:*

(a) *If $h = k = 0$, then $C_J(A) = \{0\}$. If $A = 0$, then $W_J(A) = \{0\}$; otherwise, $W_J(A)$ is the line defined by $a_{11}$ and $-a_{11}$, except the origin;*
(b) *If $h < 0$, $k = 0$, then $C_J(A) = \{(-\sqrt{|h|}, 0), (\sqrt{|h|}, 0)\}$, and $W_J(A) = \mathbb{R}\backslash ] - \sqrt{|h|}, \sqrt{|h|}[$;*
(c) *If $h > 0$, $k = 0$, then $C_J(A) = \mathbb{R}$. If $K^J = 0$, then $W_J(A) = \mathbb{R}$; otherwise, $W_J(A) = \mathbb{C}\backslash\mathbb{R}$;*
(d) *If $h = 0$, $k < 0$, then $C_J(A) = \{(0, -\sqrt{|k|}), (0, \sqrt{|k|})\}$, and $W_J(A) = i\mathbb{R}\backslash ] - \sqrt{|k|}, \sqrt{|k|}[$;*
(e) *If $h = 0$, $k > 0$, then $C_J(A) = i\mathbb{R}$. If $H^J = 0$, then $W_J(A) = i\mathbb{R}$; otherwise, $W_J(A) = \mathbb{C}\backslash i\mathbb{R}$;*
(f) *If $h > 0$, $k > 0$, then $C_J(A) = \emptyset$, and $W_J(A) = \mathbb{C}$;*

(g) *If $h>0$, $k<0$, then $C_J(A)$ is the hyperbola centered at the origin with transverse and non-transverse axis on the imaginary and on the real axis, of length $2\sqrt{|k|}$ and $2\sqrt{|h|}$, respectively; $W_J(A)$ is the hyperbola and its "interior";*

(h) *If $h<0$, $k>0$, then $C_J(A)$ is the hyperbola centered at the origin with transverse and non-transverse axis on the real and on the imaginary axis, of length $2\sqrt{|h|}$ and $2\sqrt{|k|}$, respectively; $W_J(A)$ is the hyperbola and its "interior".*

*Proof*: The characterization of $C_J(A)$ follows from Proposition 4 and Lemma 4(a).

(a) Suppose that $h=k=0$.

(i) If $H^J=K^J=0$, then $A=0$ and $W_J(A)=\{0\}$.

(ii) If $H^J=0$ and $K^J\neq 0$, then $a_{11}$ is imaginary, $A=iK^J$ and by Lemma 3, $W_J(K^J)=\mathbb{R}\backslash\{0\}$.

(iii) If $H^J\neq 0$ and $K^J=0$, then $a_{11}$ is real, $A=H^J$ and Lemma 3 implies that $W_J(A)=\mathbb{R}\backslash\{0\}$.

(iv) If $H^J\neq 0$ and $K^J\neq 0$, let $B=e^{-i\theta}A\neq 0$, where $\theta=\arg a_{11}$. From the conditions $h=0$ and $k=0$, we get $(\text{Re } a_{11})^2=1/4|a_{12}-\overline{a_{21}}|^2$ and $(\text{Im } a_{11})^2=1/4|a_{12}+\overline{a_{21}}|^2$. It follows that

$$\text{Re}(a_{11}^2)=(\text{Re } a_{11})^2-(\text{Im } a_{11})^2=-\text{Re}(a_{12}a_{21}). \tag{12}$$

Since $\det(A)\in\mathbb{R}$, $-a_{11}^2-a_{12}a_{21}$ is real. Then (12) implies that $\det(A)=0$. It is also easily seen that $|a_{11}|=|a_{12}|=|a_{21}|$. Then $-a_{11}^2-a_{12}a_{21}=0$ implies that

$$e^{i\arg a_{11}}e^{i\arg a_{11}}+e^{i\arg a_{12}}e^{i\arg a_{21}}=0,$$

and so $a_{11}\overline{a_{21}}+\overline{a_{11}}a_{12}=0$. It follows that $B$ is $J$-Hermitian and $\det(B)=0$. By Lemma 3, $W_J(B)=\mathbb{R}\backslash\{0\}$. Consequently, $W^J(A)=e^{i\theta}\mathbb{R}\backslash\{0\}$.

(b) Suppose that $h<0$ and $k=0$. By Lemma 4(b), $K^J=0$. Thus, $A=H^J$ and by Lemma 3 the result follows.

(c) Consider $h>0$ and $k=0$. Then $C_J(A)$ is the real axis. If $K^J=0$, then $A=H^J$ and by Lemma 3 $W_J(A)=\mathbb{R}$. If $K^J\neq 0$, by Lemma 3 $W_J(K^J)=\mathbb{R}\backslash\{0\}$. Then, the projection of $W_J(A)$ on the imaginary axis is the imaginary axis with the origin deleted. Thus, $W_J(A)$ is the whole complex plane without the real axis.

The proofs of (d) and (e) are analogous to (b) and (c), respectively.

(f) Suppose $h>0$ and $k>0$. Lemma 3 ensures that $W_J(H^J)=\mathbb{R}$ and $W_J(K^J)=\mathbb{R}$. By Proposition 4, $C_J(A)=\emptyset$, and so $W_J(A)=\mathbb{C}$.

(g) Let $h > 0$ and $k < 0$. By Lemma 3, $W_J(H^J) = \mathbb{R}$ and $W_J(K^J) = \mathbb{R}\backslash] - \sqrt{|k|}, \sqrt{|k|}[$. By Proposition 4, $C_J(A)$ is the hyperbola centered at the origin with transverse axis on the imaginary axis of length $2\sqrt{|k|}$, and non-transverse axis on the real axis of length $2\sqrt{|h|}$. Then, $W_J(A)$ is the hyperbola and its "interior".

(h) The proof is analogous to (g). ∎

If $A \in M_2$ has zero trace and real determinant, then its eigenvalues are symmetric, and both real or both pure imaginary. Therefore, easy calculations show that the expression (3) for these matrices may be obtained from Theorem 5 having in mind that

$$\mathrm{Re}\ \det(A) = \det(H^J) - \det(K^J) \tag{13}$$

holds for an arbitrary matrix $A = H^J + iK^J \in M_2$. The proof of (13) follows by Lemma 1 and recalling that the second compound of $A = H^J + iK^J \in M_2$ coincides with its determinant.

## 6. Characterization of $\mathbf{C_J(A)}$ for $\mathbf{A \in M_3}$

For $A \in M_3$, we characterize the associated curve $C_J(A)$ based on the factorability of the polynomial $F_A^J(u, v, w)$. Kippenhahn in [8] classified $C(A)$ following Newton's classification of algebraic curves of order 3 [1], and our approach is similar. Without loss of generality, we consider $J = \mathrm{diag}(1, 1, -1)$. We denote by $A_{kk}$ the submatrix of $A \in M_3$ obtained deleting its $k$-th lines, $k = 1, 2, 3$.

**1.st Case** Suppose that $A \in M_3$ is $J$-decomposable, i.e., there is a $J$-unitary matrix $U \in M_3$ such that

$$(i) \quad U^{-1}AU = \begin{bmatrix} a_{11} & 0 \\ 0 & A_{11} \end{bmatrix} \qquad \text{or} \qquad (ii) \quad U^{-1}AU = \begin{bmatrix} A_{33} & 0 \\ 0 & a_{33} \end{bmatrix}, \tag{14}$$

where $A_{11}$ and $A_{33}$ may be diagonal matrices. Since $W_J(A) = W_J(U^{-1}AU)$, we may consider $A$ of the form (i) or (ii). Suppose that $A$ is of the form (14)(i). The polynomial $F_A^J(u, v, w)$ is reducible and can be written as follows

$$F_A^J(u, v, w) = (\mathrm{Re}\, a_{11}u + \mathrm{Im}\, a_{11}v + w)\, F_{A_{11}}^J(u, v, w).$$

Each linear factor of $F_A^J(u, v, w)$ corresponds to an eigenvalue of $A$. The equation in line coordinates

$$\mathrm{Re}\, a_{11}u + \mathrm{Im}\, a_{11}v + w = 0 \tag{15}$$

gives the family of lines $(u : v : -\operatorname{Re} a_{11}u - \operatorname{Im} a_{11}v)$ that pass through the point $(\operatorname{Re} a_{11} : \operatorname{Im} a_{11} : 1)$, and as usual, we identify the family of lines with this point. Therefore, the point $a_{11}$ of the affine plane is the solution of the equation (15) and $C_J(A) = \{a_{11}\} \cup C_{J_1}(A_{11})$, where $J_1 = \operatorname{diag}(1, -1)$. By the Hyperbolical Range Theorem ([2] or Theorem 5) we conclude that $C_J(A)$ is a point and a hyperbola (possibly degenerate).

Suppose that $A$ is of the form (14)(ii). Clearly, $C_J(A) = \{a_{33}\} \cup C(A_{33})$ and by the Elliptical Range Theorem [9] $C_J(A)$ is a point and an ellipse (possible degenerate).

**2.$^{\text{nd}}$ Case** The matrix $A$ is $J$-indecomposable and the polynomial $F_A^J(u, v, w)$ factorizes into linear factors or into a linear factor and a quadratic factor. Therefore, $C_J(A)$ consists of the eigenvalues of $A$ corresponding to the linear factors and to a conic (may be degenerate) corresponding to the quadratic factor. We note that the conic is a hyperbola or an ellipse. In fact, the conic may not be a parabola because a parabola has one focus in the line of infinity of the projective plane, contradicting Proposition 3.

**3.$^{\text{rd}}$ Case** Suppose that the polynomial $F_A^J(u, v, w)$ is irreducible and so it is of degree 3. By standard arguments and using projective transformations, it can be shown that the line equation of $C_J(A)$ may be written in the form

$$v^2 w = \alpha(u - \beta w)(u - \gamma w)(u - \delta w).$$

Viewing the coordinates $u, v, w$ as point coordinates, we may use Newton's classification for algebraic curves of order 3 and apply the duality principle. The following possibilities may occur:

- **(i):** $\beta < \gamma < \delta$: the dual curve consists of an oval and an infinite branch with 3 real flex points. $C_J(A)$ has oval components and components with cusps (see Examples 1 and 2);
- **(ii):** $\beta < \gamma = \delta$: the dual curve has a node as singular point. Then $C_J(A)$ has a double tangent (see Examples 3 and 4);
- **(iii):** $\beta = \gamma < \delta$: the dual curve has an isolated point. Thus, $C_J(A)$ has a line (cfr. Examples 5 and 6);
- **(iv):** $\beta = \gamma = \delta$: the dual curve has a cusp as singular point and $C_J(A)$ also has a cusp as singular point (see Example 7);
- **(v):** Two of the three roots are complex conjugate: the dual curve has only an infinite branch and $C_J(A)$ is a deltoid-like curve of order 6 (cfr. Example 8).

Kippenhahn [8] proved that the possibilities (iii), (iv) and (v) can not occur for $C(A)$. In contrast with the classical case, the examples in Section 7 show that all the above mentioned curves may in fact appear as associated curves of $W_J(A)$.

The knowledge of the associated curve allows the complete characterization of the $J$-numerical range. In Case 1, when $A$ is of the form (14) (ii), it is easily seen that $W(A_{33}) \subseteq W_J^+(A)$ and $a_{33} \in W_{-J}^+(A)$. Therefore, to obtain $W_J(A)$ we take the line connecting each $z \in W(A_{33})$ and $a_{33}$ deleting the open line segment joining these two points. On the other hand, when $A$ is of the form (14)(i), we easily conclude that $W_{J_1}^+(A_{11}) \subseteq W_J^+(A)$ and $W_{-J_1}^+(A_{11}) \subseteq W_{-J}^+(A)$. Moreover, $a_{11} \in W_J^+(A)$. Hence, to obtain $W_J(A)$ we consider the closed line segment connecting each $z \in W_{J_1}^+(A_{11})$ and $a_{11}$, and for each $z \in W_{-J_1}^+(A_{11})$, we take the line connecting $z$ and $a_{11}$ removing the open line segment joining these two points.

In the Case 2 and Case 3, the following analysis takes place. If there exists a real interval $[\theta_1, \theta_2]$ such that the $n$ eigenvalues $\lambda_1(\theta), \ldots, \lambda_n(\theta)$ of the pencil $\cos\theta H^J + \sin\theta K^J$ $(\theta \in [\theta_1, \theta_2])$ are real and there exists an associated basis of anisotropic eigenvectors, then we use Theorem 1. Otherwise, we use Theorem 2, which characterize the projection of the $J$-numerical range of $A$ on the lines passing through the origin.

## 7. Examples

In the examples presented in this Section the point equation of $C_J(A)$, $A \in M_3$, is obtained using Theorem 4 (and eventually Proposition 2 for $J$-Hermitian matrices). The line equation of $C_J(A)$ is given by

$$\begin{aligned} F_A^J(u,v,w) = {} & w^3 + \det(H^J)\,u^3 + \det(K^J)\,v^3 + \operatorname{Re}\operatorname{Tr}(A)\,uw^2 \\ & + \operatorname{Im}\operatorname{Tr}(A)\,vw^2 + \operatorname{Im}\zeta_A\,uvw + \zeta_{H^J}\,u^2w + \zeta_{K^J}\,v^2w \\ & + [\det(H^J) - \operatorname{Re}\det(A)]\,uv^2 + [\det(K^J) + \operatorname{Im}\det(A)]\,u^2v, \end{aligned} \tag{16}$$

where $A = H^J + iK^J$ is the $J$-Cartesian decomposition of $A$ and $\zeta_B$ denotes the sum of the $2 \times 2$ principal minors of the matrix $B$. In the figures, the eigenvalues of $A$ are represented by dark dots and the curves were plot using *Mathematica 5.1.*

**Example 1** (2.$^{\mathbf{nd}}$ Case)**.** It can be easily shown that the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

is $J$-indecomposable and $F_A^J(u,v,w) = -(u-w)(u+w)^2$. Thus, $C_J(A) = \{-1, 1\}$. Now, we determine the $J$-numerical range of the pencil $\cos\theta\, H^J + \sin\theta\, K^J$ for each real direction $\theta$. If $\theta = \pi/2$, we use Theorem 2 (b), and so the $J$-numerical range of the pencil is $\mathbb{R}$. Analysing all the real directions $\theta \neq \pi/2$ and using Theorem 2 (e), we may conclude that $W_J(A) = \mathbb{C}$.

**Example 2** (3.$^{\mathbf{rd}}$ Case (i))**.** Consider the matrix

$$A = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & 1/2 & \sqrt{2} \end{bmatrix}.$$

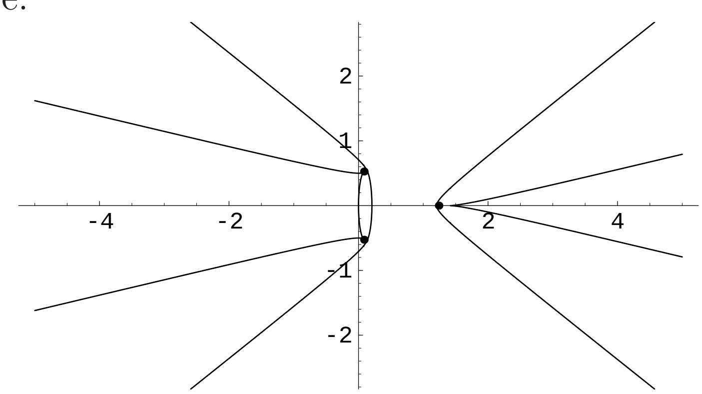
The line equation of $C_J(A)$ is

$$1 + \sqrt{2}u + 1/4u^2 - 1/4v^2 - \sqrt{2}/4uv^2 = 0,$$

being the point equation

$$32\sqrt{2}x^5 - 8x^6 - y^2 + 6y^4 - 8y^6 + 2\sqrt{2}x^3(35 - 124y^2) + 2x^4(-49 + 76y^2) + \\ 2\sqrt{2}x(2 - 11y^2 + 12y^4) - 4x^2(11 - 57y^2 + 54y^4) = 0.$$

Hence, the associated curve is a "broken" ovular curve and a "broken" deltoid-like curve.




The eigenvalues of $H^J$ are $0 \in \sigma_+(H^J)$, $1/2(-1+\sqrt{2}) \in \sigma_+(H^J)$ and $1/2(1+\sqrt{2}) \in \sigma_-(H^J)$. Then $W_J^+(H^J) = ]-\infty, 1/2(-1+\sqrt{2})]$ and $W_{-J}^+(H^J) = [1/2(1+\sqrt{2}), +\infty[$. Analysing directions $\theta \neq 0$ and using Theorem 1, we conclude that $W_J^+(A)$ is contained in the half-plane $x \leq 1/2(-1+\sqrt{2})$ and that $W^+_{-J}(A)$ is contained in the half-plane $x \geq 1/2(1+\sqrt{2})$. Moreover, $W^+_{-J}(A)$ is bounded by the outer curve in this half-plane, while $W^+_J(A)$ is bounded by the outer curve in $x \leq 1/2(-1+\sqrt{2})$.

**Example 3** (3.$^{\mathbf{rd}}$ Case (i))**.** Consider the matrix

$$B = \begin{bmatrix} 0 & 29/10 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

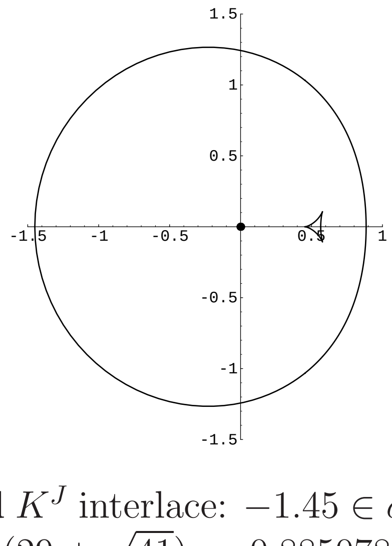
The line equation of the associated curve is

$$1 - 29/40u(u^2+v^2) - 641/400(u^2+v^2) = 0,$$

being the point equation:

$$(-290+641x)^3(290-641x+400x^3) + 2(-34555092100+ \\ x(108857138180+x(-39443756161+600x(-238310980+187684721x))))y^2+ \\ (25245963839+1200x(-119155490+111994721x))y^4+14521888400y^6 = 0.$$

The associated curve is an ovular curve and a deltoid-like curve.

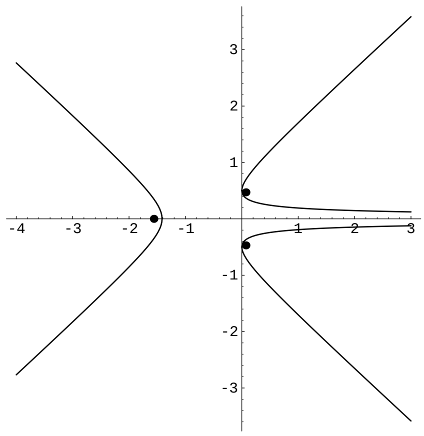



The eigenvalues of $H^J$ and $K^J$ interlace: $-1.45 \in \sigma_+(H^J)$, $1/40(29-\sqrt{41}) \approx 0.564922 \in \sigma_-(H^J)$, $1/40(29+\sqrt{41}) \approx 0.885078 \in \sigma_+(H^J)$, $-\sqrt{641}/20 \approx -1.2659 \in \sigma_+(K^J)$, $0 \in \sigma_-(K^J)$, $\sqrt{641}/20 \approx 1.2659 \in \sigma_+(K^J)$. In the directions $\theta \neq 0, \pi/2$ an analogous situation occurs with the pencil $\cos\theta H^J + \sin\theta K^J$. By Theorem 2, we conclude that $W^+_J(A) = W^+_{-J}(A) = W_J(A) = \mathbb{C}$.

**Example 4** (3.$^{\text{rd}}$ Case (ii))**.** Let

$$C = \begin{bmatrix} 0 & -1/2 & 0 \\ 1/2 & 0 & -1/2 \\ 0 & -1/2 & -\sqrt{2} \end{bmatrix}.$$

The line equation of the associated curve is $1-\sqrt{2}u+\sqrt{2}/4uv^2=0$, and the point equation is $27x^2y^2-2(1-4y^2)^2+\sqrt{2}x(-1+36y^2)=0$, being $x=0$ a double tangent. The associated curve is a "broken" cardioid.




The eigenvalues of $H^J$ are $-\sqrt{2}\in\sigma_-(H^J)$ and $0\in\sigma_+(H^J)$, being 0 a double eigenvalue with 2 linearly independent eigenvectors of positive norm. Thus, $W^+_{-J}(H^J)=]-\infty,-\sqrt{2}]$ and $W^+_J(H^J)=[0,+\infty[$. The projection of $W_J(A)$ on other directions, within a certain interval, are also two closed half-rays. By Theorem 1, we conclude that $W^+_J(A)$ is contained in the half-plane $x\geq 0$ and it is the convex hull of the branches of $C_J(A)$ in the right closed half-plane. Analogously, $W^+_{-J}(A)$ is contained in the half-plane $x\leq -\sqrt{2}$, being the convex hull of the associated curve contained in this region.

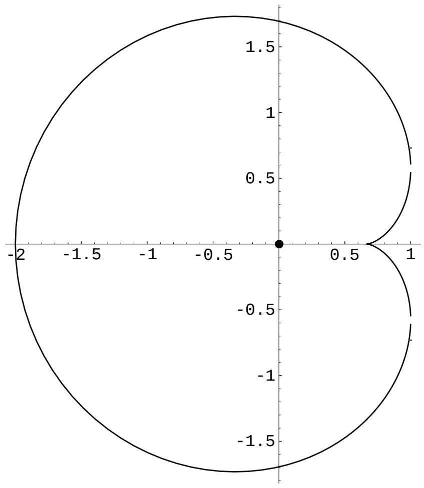
**Example 5** (3.$^{\text{rd}}$ Case (ii))**.** Consider the matrix

$$D = \begin{bmatrix} 0 & 4 & 1-i \\ 0 & 0 & 1-i \\ 0 & 0 & 0 \end{bmatrix}.$$

The line equation of $C_J(A)$ is $1-2u(u^2+v^2)-3(u^2+v^2)=0$, the point equation is

$$(2+x)(-2+3x)^3+18(-4+3x^2)y^2+27y^4=0,$$

and $x=1$ is a double tangent. The associated curve is a cardioid.




The eigenvalues of $H^J$ are $-2 \in \sigma_+(H^J)$, $1 \in \sigma_0(H^J)$, being 1 a double eigenvalue with an associated isotropic eigenvector. Thus, $W_J^+(H^J)$ and $W_{-J}^+(H^J)$ are the half-planes $x > 1$ and $x < 1$, respectively (in this case, it can be easily shown that $1 \notin W_J(H^J)$). The pencil $\cos\theta H^J + \sin\theta K^J$ has interlacing eigenvalues in all real directions except for the real axis, and so $W_{-J}^+(A)$ and $W_J^+(A)$ are the half-planes $x < 1$ and $x > 1$, respectively.

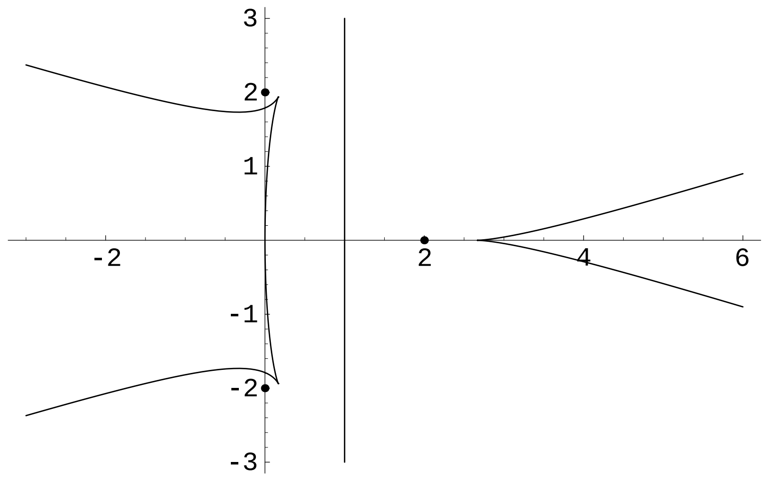
**Example 6** (3.$^{\mathbf{rd}}$ Case (iii))**.** Let

$$E = \begin{bmatrix} 0 & 2\sqrt{2} & 0 \\ -2\sqrt{2} & 1 & \sqrt{5} \\ 0 & \sqrt{5} & 1 \end{bmatrix}.$$

The line equation of $C_J(A)$ is $1 + 2u + u^2 - 3v^2 - 8uv^2 = 0$, and the point equation is

$$(-1+x)^2(x(-8+3x)^3 - 2(-8+3x(-28+39x))y^2 - 5y^4) = 0.$$

The associated curve is a "broken" deltoid-like curve and a line.




The eigenvalues of $H^J$ are $0 \in \sigma_+(H^J)$ and $1 \in \sigma_+(H^J) \cap \sigma_-(H^J)$. Hence, $W_J(H^J) = \mathbb{R}$. For $\theta \neq 0$, we use Theorem 2 (a) and (c) to characterize the $J$-numerical range of the pencil $\cos\theta H^J + \sin\theta K^J$. Thus, $W_J^+(A)$ is the half plane $x \leq 1$, and $W_{-J}^+(A)$ is the half plane $x \geq 1$, being $W_J(A) = \mathbb{C}$.

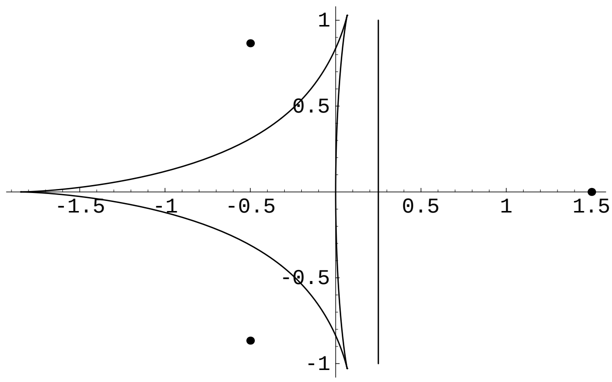
**Example 7** (3.$^{\mathbf{rd}}$ Case (iii))**.** Consider the matrix

$$G = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & -3/2 & 1/2 \end{bmatrix}.$$

The line equation of $C_J(A)$ is $u^2 + 9v^2 + 16 - 8u(v^2 - 1) = 0$, and the point equation is

$$(1 - 4x)^2(-x(8 + 9x)^3 - 2(-8 + 27x(-28 + 179x))y^2 - 41y^4) = 0.$$

The associated curve is a deltoid-like curve and a line.




The eigenvalues of $H^J$ are $0 \in \sigma_+(H^J)$ and $1/4 \in \sigma_0(H^J)$, where $1/4$ has algebraic multiplicity 2 and an isotropic eigenvector. In this case, $W_J(H^J) = \mathbb{R}\backslash\{1/4\}$. For $\theta \neq 0$, the pencil $\cos\theta H^J + \sin\theta K^J$ has complex eigenvalues.

Then, $W^+_{-J}(A)$ is the half-plane $x > 1/4$ and $W^+_J(A)$ the half-plane $x < 1/4$. Finally,

$$W_J(A) = \mathbb{C}\backslash\{z : \operatorname{Re} z = 1/4\}.$$

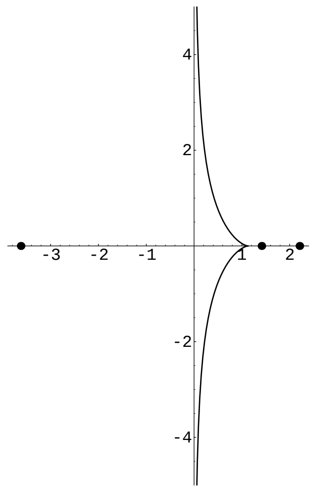
**Example 8** (3.$^{\mathbf{rd}}$ Case (iv))**.** Consider the matrix

$$M = \begin{bmatrix} \sqrt{2} & 0 & 2 \\ 0 & -\sqrt{2} & 4 \\ 0 & 2 & 0 \end{bmatrix}.$$

The line equation of $C_J(A)$ is $1 + 10v^2 + 8\sqrt{2}uv^2 = 0$, and the point equation is

$$64\sqrt{2} + 150\sqrt{2}x^2 - 125/2x^3 - 6x(40 + 9y^2) = 0,$$

being $x = 0$ the equation of its asymptote (the asymptotes are tangent lines of the curve at points at infinity). The associated curve is a Cissoid of Diocles.




The matrix $H^J$ has the (triple) 0 eigenvalue, which is isotropic. Since $H^J$ is nilpotent with nilpotency index equal to 3, we get $W_J(H^J) = \mathbb{R}$. Since the eigenvalues of $K^J$ are $\pm i\sqrt{10}$ and 0, we find that $W_J(K^J) = \mathbb{R}$. For $\theta \neq 0, \pi/2$, the pencil $\cos\theta H^J + \sin\theta K^J$ has complex eigenvalues. Hence,

$$W_J(A) = W^+_J(A) = W^+_{-J}(A) = \mathbb{C}.$$

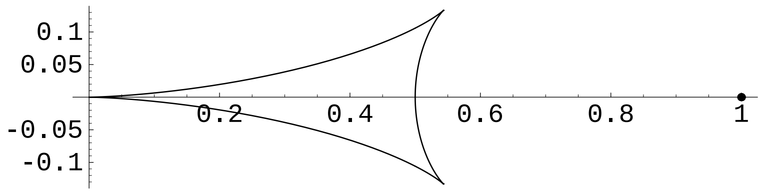
**Example 9** (3.$^{\mathbf{rd}}$ Case (v))**.** Let

$$N = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The line equation of the associated curve is $1+u^3+3u+13/4u^2+1/4v^2=0$, and the point equation is

$$x^3(-1+2x)(4+x(-5+2x))+2(54+x(-234+x(349+12x(-19+5x))))y^2+ (253+12x(-37+19x))y^4+112y^6=0.$$

The associated curve is a deltoid-like curve of order 6.




The eigenvalues of $H^J$ are $1/4(5\pm i\sqrt{7})$ and $1/2$; so $W_J(H^J)=\mathbb{R}$. The eigenvalues of $K^J$ are $\pm i/2$ and 0, and so $W_J(K^J)=\mathbb{R}$. For $\theta\neq 0,\pi/2$, the pencil $\cos\theta H^J+\sin\theta K^J$ has complex eigenvalues. Hence,

$$W_J(A)=W_J^+(A)=W_{-J}^+(A)=\mathbb{C}.$$

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