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ODD-QUADRATIC LIE SUPERALGEBRAS

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ABSTRACT: An odd-quadratic Lie superalgebra is a Lie superalgebra with a non-degenerate supersymmetric odd invariant bilinear form. In this paper we give examples and present some properties of odd-quadratic Lie superalgebras, introduce the notions of double extension and generalized double extension of odd-quadratic Lie superalgebras and give an inductive classification of odd-quadratic Lie superalgebras such that the even part is a reductive Lie algebra.

KEYWORDS: Lie superalgebras; Reductive Lie algebras; Quadratic forms.
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1. Introduction

A Lie superalgebra $\mathfrak{g}$ is called *quadratic* (respectively *odd-quadratic*) if there is a bilinear form B on $\mathfrak{g}$ such that B is non-degenerate, supersymmetric, even (respectively odd), and $\mathfrak{g}$-invariant. In [8], V. Kac has classified finite dimensional simple Lie superalgebras over an algebraically closed field of characteristic 0, proving that they have reductive even part if and only if the action of the even part on the odd part is completely reducible. About the behaviour of invariant bilinear forms defined in these Lie superalgebras, he has concluded that we have two possibilities for each one: either all invariant bilinear forms are even (for example $spl(m, n), m \neq n$), or all invariant bilinear forms are odd (for example $d(n)/KI_{2n}, n \geq 3$) [11]. Besides, contrary to what happens in Lie theory where all semisimple Lie algebras are quadratic, in Lie superalgebra theory this affirmation is not true. Other examples of quadratic Lie algebras that are not semisimple, were presented by A. Medina and P. Revoy in [9]. These authors introduced the notion of double extension to give an inductive classification of quadratic Lie algebras. Motivated by these results, and since the notion of quadratic Lie superalgebra is a generalization of quadratic Lie algebra, H. Benamor and S. Benayadi, in [3], began research into finite dimensional quadratic Lie superalgebras over a field of

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characteristic zero. The two authors extended the notion of double extension to quadratic Lie superalgebras and in [3], it is presented an inductive classification of quadratic Lie superalgebras $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ such that dim $\mathfrak{g}_{\bar{1}} = 2$. In [4], S. Benayadi obtained an inductive classification of quadratic Lie superalgebras such that the even part is a reductive Lie algebra and the odd part is a completely reducible module on the even part after had introduced the notion of elementary double extension. We recall that Lie superalgebras with semisimple even part were studied by F. A. Berezin and V. S. Retakh in [6], and more generally in [7], A. Elduque has obtained very interesting results about the classification of Lie superalgebras with reductive even part and the action of the even part on the odd part is completely reducible. In [1] using the notions of double extension and generalized double extension presented in [2], we gave some examples of solvable and non solvable quadratic Lie superalgebras that have a reductive even part and the action of the even part on the odd part is not completely reducible and we have classified inductively this class of Lie superalgebras, improving the theory of S. Benayadi ([4]).

Motivated by mathematical and physical applications [10], in this paper we will analyse the class of odd-quadratic Lie superalgebras studying some properties and looking at non simple examples. We shall give an inductive classification of odd-quadratic Lie superalgebras such that the even part is a reductive Lie algebra. We would like to remark that contrary to what we know in quadratic Lie superalgebras, in odd-quadratic Lie superalgebras with reductive even part, the odd part must be necessarily a completely reducible module on the even part. We begin this work presenting a characterization of odd-quadratic Lie superalgebras. More precisely, we will show that a Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is odd-quadratic if and only if there exists an isomorphism of $\mathfrak{g}_{\bar{0}}$-modules $\mathfrak{g}_{\bar{0}} \longrightarrow (\mathfrak{g}_{\bar{1}})^*$ with certain properties. In this case, dimension of the even part is equal to the dimension of the odd part (in quadratic case, we know that the dimension of the odd part is even). All non-abelian odd-quadratic Lie superalgebras of dimension two are isomorphic. We will show that, as in case of quadratic Lie superalgebras, we can reduce the study of odd-quadratic Lie superalgebras $(\mathfrak{g}, B)$, to the B-irreducible case. In the third section we will present the construction of odd double extension of odd-quadratic Lie superalgebras by the one-dimensional Lie algebra. Further, we will prove that if a B-irreducible odd-quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ $(\dim \mathfrak{g} > 1)$ has non-zero odd elements in the centre then $\mathfrak{g}$ is an odd double extension of an odd-quadratic Lie superalgebra $\mathfrak{h}$ ($\dim \mathfrak{h} = \dim \mathfrak{g} - 2$) by a one-dimensional Lie algebra. In the fourth section we will introduce the concept of generalized odd double extension of an odd-quadratic Lie superalgebra by the one-dimensional Lie superalgebra with even part zero. For an odd-quadratic Lie superalgebra $(\mathfrak{g}, B)$, we can use an odd skew-supersymmetric superderivation of $(\mathfrak{g}, B)$ with some properties to construct a new odd-quadratic Lie superalgebra, namely the generalized odd double extension of $(\mathfrak{g}, B)$ by the one-dimensional Lie superalgebra with even part zero. Conversely, we will show that a B-irreducible odd-quadratic Lie superalgebra $(\mathfrak{g}, B)$ submitted to certain conditions is a generalized odd double extension of an odd-quadratic Lie superalgebra by the one-dimensional Lie superalgebra with even part zero. Introduced the notion of generalized double extension we will characterize some classes of irreducible odd-quadratic Lie superalgebras that will be very useful to deduce an inductive classification of odd-quadratic Lie superalgebras with reductive even part (and consequently with the action of the even part on the odd part completely reducible). Section five analyses the particular case of odd-quadratic Lie superalgebras such that the even part is a reductive Lie algebra. We will prove that for odd-quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ we have $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$. If the even part is a reductive Lie algebra, then the centre of the even part is zero if and only if the centre of $\mathfrak{g}$ is zero. Furthermore, if we add the B-irreducibility assumption we will conclude that the centre of $\mathfrak{g}$ coincide with the centre of the even part, and it only has even elements. Contrarily to quadratic case, for odd-quadratic Lie superalgebra such that the even part is a reductive Lie algebra the action of the even part on the odd part is completely reducible. We will characterize the minimal graded ideals of a B-irreducible odd-quadratic Lie superalgebra such that the even part is a reductive Lie algebra. For a B-irreducible odd-quadratic Lie superalgebra $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, we will prove that a graded ideal I of $\mathfrak{g}$ is minimal if and only if $I \subseteq \mathfrak{z}(\mathfrak{g})$ and $\dim I = 1$ or I is a non trivial irreducible s-submodule of $\mathfrak{g}_{\bar{1}}$ such that $[\mathfrak{g}_{\bar{1}}, I] = \{0\}$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. The B-irreducible odd-quadratic Lie superalgebras with reductive even part, that it is neither zero nor simple, with the centre zero, are the trivial odd double extensions of a simple Lie algebra. We will use the results of the preceeding sections to obtain an inductive classification of odd-quadratic Lie superalgebras such that the even part is a reductive Lie algebra. Although we will show that the action of the even part on the odd part is completely reducible, to give the classification we will have to use the tool of odd double extension and the generalized odd double extension of odd-quadratic Lie superalgebras. More precisely, we will prove that a non-zero odd-quadratic Lie superalgebra of our type are the simple Lie superalgebras $d(n)/\mathbb{K}I_{2n}$ $(n \geq 3)$, and/or is obtained by a sequence of generalized odd double extensions by the one-dimensional Lie superalgebra with even part zero, and/or by trivial odd double extensions of simple Lie algebra, and/or by orthogonal direct sums of odd-quadratic Lie superalgebras from a finite number of algebras of the former type. Finally, in the remaining section we will generalize a result of classical Lie superalgebras, namely, we will prove that a quadratic Lie superalgebra with the even part a reductive Lie algebra, with certain conditions, does not admit an odd-invariant scalar product, and vice versa. Although the results are similar to the correspondent results for quadratic Lie superalgebras, the techniques used to obtain them are very different.

2. Basic properties

We will refer to the work due to Scheunert [11] for an exhaustive treatment of the Lie superalgebra theory. In this paper, we shall consider finite dimensional Lie superalgebras over an algebraically closed commutative field $\mathbb{K}$ of characteristic zero.

Definition 2.1. Let $\mathfrak{g}$ be a Lie superalgebra. A bilinear form B on $\mathfrak{g}$ is called

- *supersymmetric* if $B(X,Y) = (-1)^{xy}B(Y,X)$, for all $X \in \mathfrak{g}_x$ and $Y \in \mathfrak{g}_y$.
- *skew-supersymmetric* if $B(X,Y) = -(-1)^{xy}B(Y,X)$, for all $X \in \mathfrak{g}_x$ and $Y \in \mathfrak{g}_y$.
- *non-degenerate* if $X \in \mathfrak{g}$ satisfies $B(X,Y) = 0$, for all $Y \in \mathfrak{g}$, then $X = 0$.
- *invariant* if $B([X,Y],Z) = B(X,[Y,Z])$, for all $X,Y,Z \in \mathfrak{g}$.
- *even* if $B(X,Y) = 0$, for all $(X,Y) \in \mathfrak{g}_\alpha \times \mathfrak{g}_\beta$, where $(\alpha,\beta) \in \mathbb{Z}_2 \times \mathbb{Z}_2$, (with $\alpha \neq \beta$).
- *odd* if $B(X,Y) = 0$, for all $(X,Y) \in \mathfrak{g}_\alpha \times \mathfrak{g}_\alpha$, where $\alpha \in \mathbb{Z}_2$.
- 2-*cocycle* on $\mathfrak{g}$ if B is skew-supersymmetric and B verifies:

$$(-1)^{xz}B(X,[Y,Z]) + (-1)^{xy}B(Y,[Z,X]) + (-1)^{yz}B(Z,[X,Y]) = 0,$$

  whenever $X \in \mathfrak{g}_x$, $Y \in \mathfrak{g}_y$, and $Z \in \mathfrak{g}_z$. We denote by $Z^2(\mathfrak{g},\mathbb{K})$ the set of all 2-cocycles on $\mathfrak{g}$.

We will need the orthogonal direct sum of odd-quadratic Lie superalgebras to formalize an inductive classification. We recall

Definition 2.2. Let $(\mathfrak{g}, B)$ and $(\mathfrak{h}, T)$ be two odd-quadratic Lie superalgebras. It is easy to see that $(\mathfrak{g} \oplus \mathfrak{h}, \gamma)$ is an odd-quadratic Lie superalgebra where the odd-invariant scalar product on $\mathfrak{g} \oplus \mathfrak{h}$ is defined by

$$\gamma\mid_{(\mathfrak{g}\times\mathfrak{g})} = B, \quad \gamma\mid_{(\mathfrak{h}\times\mathfrak{h})} = T, \quad \gamma\mid_{(\mathfrak{g}\times\mathfrak{h})} = 0.$$

It is called the *orthogonal direct sum* of $(\mathfrak{g}, B)$ and $(\mathfrak{h}, T)$.

Definition 2.3. Let $(\mathfrak{g}, B)$ be an odd-quadratic Lie superalgebra. A homogeneous superderivation D of $(\mathfrak{g}, B)$ of degree α (where $\alpha \in \mathbb{Z}_2$) is called *skew-supersymmetric* if

$$B(D(X), Y) \;=\; -(-1)^{\alpha x} B(X, D(Y)), \qquad \forall_{X\in\mathfrak{g}_x, Y\in\mathfrak{g}}.$$

It is proved that the vector subspace of $Der(\mathfrak{g})$ generated by the set of all homogeneous skew-supersymmetric superderivations of $(\mathfrak{g}, B)$ is a Lie subsuperalgebra of $\mathrm{Der}(\mathfrak{g})$ and it is denoted by $Der_a(\mathfrak{g}, B)$.

Definition 2.4. Let $(\mathfrak{g}, B)$ be an odd-quadratic Lie superalgebra.

- A graded ideal I of $\mathfrak{g}$ is called *non-degenerate* if the restriction of B to $I \times I$ is a non-degenerate bilinear form on I. Otherwise, we said that I is *degenerate*.
- $\mathfrak{g}$ is called *B-irreducible* if $\mathfrak{g}$ does not have non-zero non-degenerate graded ideals.
- A graded ideal I of $\mathfrak{g}$ is called *B-irreducible* if I is non-degenerate and I does not have non-zero non-degenerate graded ideals of $\mathfrak{g}$.

Definition 2.5. Let $\mathfrak{g}$ be a Lie superalgebra. A graded ideal I of $\mathfrak{g}$ is called

- *minimal* if $I \notin \{\{0\}, \mathfrak{g}\}$ and if J is a graded ideal of $\mathfrak{g}$ such that $J \subseteq I$ then $J \in \{\{0\}, I\}$.
- *maximal* if $I \notin \{\{0\}, \mathfrak{g}\}$ and if J is a graded ideal of $\mathfrak{g}$ such that $I \subseteq J$ then $J \in \{I, \mathfrak{g}\}$.

Definition 2.6. Let $(\mathfrak{g}, B)$ be a quadratic (or an odd-quadratic) Lie superalgebra. If I is a graded ideal of $\mathfrak{g}$ then we denote by $I^{\perp}$ (or $I^{\perp B}$, in case of ambiguity) the *orthogonal* of I with respect to B, which means that

$$I^{\perp} \;=\; \{X \in \mathfrak{g} : B(X, Y) = 0, \forall_{Y\in I}\}.$$

Then I is called *isotropic* if $I \subseteq I^{\perp}$.

Definition 2.7. Let $\mathfrak{g}$ be a Lie superalgebra. The *socle* of $\mathfrak{g}$ is the sum of all minimal graded ideals of $\mathfrak{g}$ and it is denoted by $soc(\mathfrak{g})$. By convention, we will suppose that if $\mathfrak{g}$ is a one-dimensional Lie superalgebra or a simple Lie superalgebra then $soc(\mathfrak{g}) = \{0\}$.

Proposition 2.8. *The Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is odd-quadratic if and only if there exists an isomorphism of $\mathfrak{g}_{\bar{0}}$-modules $\varphi : \mathfrak{g}_{\bar{0}} \longrightarrow (\mathfrak{g}_{\bar{1}})^*$ such that*

$$\varphi([X,Y])(Z) \;=\; \varphi([Y,Z])(X), \qquad \forall_{X,Y,Z\in\mathfrak{g}_{\bar{1}}}.$$

In this case, $\dim \mathfrak{g}_{\bar{0}} = \dim \mathfrak{g}_{\bar{1}}$ *and dimension of* $\mathfrak{g}$ *is even.*

Proof: First we assume that $(\mathfrak{g}, B)$ is an odd-quadratic Lie superalgebra. Define the linear map $\varphi : \mathfrak{g}_{\bar{0}} \longrightarrow (\mathfrak{g}_{\bar{1}})^*$ by $\varphi(X) = B(X, .)$, for all $X \in \mathfrak{g}_{\bar{0}}$. As B is an odd non-degenerate bilinear form, it is obvious that φ is an isomorphism of vector spaces. We consider the adjoint representation of $\mathfrak{g}_{\bar{0}}$ and the representation of $\mathfrak{g}_{\bar{0}}$ denoted by $\rho : \mathfrak{g}_{\bar{0}} \longrightarrow gl((\mathfrak{g}_{\bar{1}})^*)$ and defined by

$$\rho(X)(f)(Y) \;=\; -f([X,Y]), \qquad \forall_{X\in\mathfrak{g}_{\bar{0}}, Y\in\mathfrak{g}_{\bar{1}}, f\in(\mathfrak{g}_{\bar{1}})^*}.$$

We can easily see that $\varphi([X,Y]) = \rho(X)(\varphi(Y))$, whenever $X, Y \in \mathfrak{g}_{\bar{0}}$, which means that φ is a homomorphism of $\mathfrak{g}_{\bar{0}}$-modules. By invariance and supersymmetry of B we infer that $\varphi([X,Y])(Z) = \varphi([Y,Z])(X)$, for every $X, Y, Z \in \mathfrak{g}_{\bar{1}}$.
Conversely, we suppose that there exists an isomorphism of $\mathfrak{g}_{\bar{0}}$-modules $\varphi : \mathfrak{g}_{\bar{0}} \longrightarrow (\mathfrak{g}_{\bar{1}})^*$ such that $\varphi([X,Y])(Z) = \varphi([Y,Z])(X)$, for all $X, Y, Z \in \mathfrak{g}_{\bar{1}}$. Let $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{K}$ be the supersymmetric bilinear form defined by $B(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}) = B(\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}) = \{0\}$ and $B(X,Y) = \varphi(X)(Y)$, for all $X \in \mathfrak{g}_{\bar{0}}, Y \in \mathfrak{g}_{\bar{1}}$. Since φ is an isomorphism of vector spaces we infer that B is odd and non-degenerate. Now we ensure the invariance of B. For $X, Y \in \mathfrak{g}_{\bar{0}}, Z \in \mathfrak{g}_{\bar{1}}$ we get

$$\begin{aligned} B([X,Y],Z) &= \varphi([X,Y])(Z) \;=\; -\varphi([Y,X])(Z) \;=\; -\rho(Y)(\varphi(X))(Z) \\ &= \varphi(X)([Y,Z]) \;=\; B(X,[Y,Z]). \end{aligned}$$

On the other hand, for $X, Y, Z \in \mathfrak{g}_{\bar{1}}$ we have

$$\begin{aligned} B([X,Y],Z) &= \varphi([X,Y])(Z) \;=\; \varphi([Y,Z])(X) \;=\; B([Y,Z],X) \\ &= B(X,[Y,Z]). \end{aligned}$$

Therefore $(\mathfrak{g}, B)$ is an odd-quadratic Lie superalgebra as required. ∎

Remark 2.9. As we will see, we can characterize all two-dimensional odd-quadratic Lie superalgebras. Consider $N = N_{\bar{0}} \oplus N_{\bar{1}}$ an odd-quadratic Lie superalgebra of dimension 2. By the last result we know that $N_{\bar{0}} = \mathbb{K}e_0$ and $N_{\bar{1}} = \mathbb{K}e_1$, where e_0 and e_1 are non zero elements of N. We also have the product defined by $[e_0, e_0] = 0$, $[e_0, e_1] = \alpha e_1$, and $[e_1, e_1] = \beta e_0$, for $\alpha, \beta \in \mathbb{K}$. Define an odd invariant scalar product B on N by

$$B(e_0, e_0) = 0,\ B(e_1, e_1) = 0,\ B(e_0, e_1) = 1.$$

Using invariance of B we show that $\alpha = 0$. If $\beta = 0$ then N is abelian. If $\beta = 1$, we denote the odd-quadratic Lie superalgebra N of dimension 2 by $\mathfrak{N}$. Remark that $\mathfrak{z}(\mathfrak{N}) = \mathfrak{z}(\mathfrak{N}_{\bar{0}})$. It is easy to show that every non-abelian odd-quadratic Lie superalgebra of dimension 2 is isomorphic to $\mathfrak{N}$. Therefore the non-abelian odd-quadratic Lie superalgebra of dimension 2 are all isomorphic.

Another example of odd-quadratic Lie superalgebra

Example 2.10. *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g}^*$ it dual. Consider $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ a Lie superalgebra, where $\mathfrak{h}_{\bar{0}} = \mathfrak{g}$, $\mathfrak{h}_{\bar{1}} = \mathfrak{g}^*$, and $B : \mathfrak{h} \times \mathfrak{h} \longrightarrow \mathbb{K}$ defined by*

$$B(X + f, Y + h) \ = \ f(Y) + (-1)^{xy} h(X), \qquad \forall_{(X+f) \in \mathfrak{h}_x, (Y+h) \in \mathfrak{h}_y}.$$

It is easy to see that $(\mathfrak{h}, B)$ is an odd-quadratic Lie superalgebra.

We will need the following lemma whose proof is straightforward,

Lemma 2.11. *Let $(\mathfrak{g}, B)$ be a B- irreducible odd-quadratic Lie superalgebra not simple. If I is a minimal graded ideal of $\mathfrak{g}$ then I is isotropic.*

As in quadratic case, the next proposition reduces the study of odd-quadratic Lie superalgebra $(\mathfrak{g}, B)$ to the B-irreducible case.

Proposition 2.12. *Let $(\mathfrak{g}, B)$ be an odd-quadratic Lie superalgebra. Then $\mathfrak{g} = \bigoplus_{k=1}^{m} \mathfrak{g}_k$, where each $\mathfrak{g}_k$ is a B-irreducible graded ideal of $\mathfrak{g}$ such that $B(\mathfrak{g}_k, \mathfrak{g}_{k'}) = \{0\}$, when $k, k' \in \{1, \ldots, m\}$ and $k \neq k'$.*

Proof: We proceed by induction on the dimension d of $\mathfrak{g}$. If $d = 2$ we have the result with $m = 1$. Indeed, we know that $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with $\dim \mathfrak{g}_{\bar{0}} = 1$ and $\dim \mathfrak{g}_{\bar{1}} = 1$. Consider a proper graded ideal I of $\mathfrak{g}$, then $I = \mathfrak{g}_{\bar{0}}$ or $I = \mathfrak{g}_{\bar{1}}$. Without loss of generality, we can assume that $I = \mathfrak{g}_{\bar{0}}$. As B is odd it follows that $B(I, I) = \{0\}$, and so $B \mid_{I \times I}$ is degenerate. Therefore the assertion is true. To prove the induction step we act as in the Proof of the correspondent result in quadratic case ([5, Theorem 1.1.]) ∎

Occasionally, we have to change the gradation of Lie superalgebras as we will describe. Let $\mathfrak{h} = \mathfrak{h}_{\bar{0}} \oplus \mathfrak{h}_{\bar{1}}$ be a Lie superalgebra. Denote by $P(\mathfrak{h}) = V_{\bar{0}} \oplus V_{\bar{1}}$ the $\mathbb{Z}_2$-graded vector space obtained from $\mathfrak{h}$ with gradation defined by

$$V_{\bar{0}} = \mathfrak{h}_{\bar{1}} \quad \text{and} \quad V_{\bar{1}} = \mathfrak{h}_{\bar{0}}.$$

Clearly, the associative superalgebras $\mathrm{Hom}(\mathfrak{h})$ and $\mathrm{Hom}(P(\mathfrak{h}))$ coincide. Therefore the representations of a Lie superalgebra $\mathfrak{g}$ in $\mathfrak{h}$ coincide with representations of $\mathfrak{g}$ in $P(\mathfrak{h})$. Note that the dual space $P(\mathfrak{h}^*)$ and $\mathfrak{h}^*$ are equal as $\mathbb{Z}_2$-graded vector spaces, however

$$V^*_{\bar{0}} = \mathfrak{h}^*_{\bar{1}} \quad \text{and} \quad V^*_{\bar{1}} = \mathfrak{h}^*_{\bar{0}}.$$

Denote by $\pi_{\mathfrak{h}} : \mathfrak{h} \longrightarrow Hom(P(\mathfrak{h}^*))$ the linear map defined for homogeneous elements as follows

$$\pi_{\mathfrak{h}}(Z)(f)(Y) \;=\; -(-1)^{z\delta} f([Z,Y]), \qquad \forall_{f \in (P(\mathfrak{h}^*))_\delta, Z \in \mathfrak{h}_z, Y \in \mathfrak{h}}.$$

It is quite easy to show that $\pi_{\mathfrak{h}}$ is a representation of $\mathfrak{h}$ in $P(\mathfrak{h}^*)$, but it is not the co-adjoint representation of $\mathfrak{h}$. Now, we will adapt the [3, Lemma 3.6.] to the odd-quadratic Lie superalgebras using the new graded vector space defined just above

Lemma 2.13. *Let $(\mathfrak{g}, B)$ be an odd-quadratic Lie superalgebra.*

(i) *Let D be a homogeneous skew-supersymmetric superderivation of $(\mathfrak{g}, B)$ of degree $\alpha \in Z_2$. Then $\omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow P(\mathbb{K})$ defined by*

$$\omega(X,Y) \;=\; B(D(X),Y), \qquad \forall_{X,Y \in \mathfrak{g}},$$

is a homogeneous skew-supersymmetric 2-cocycle on $(\mathfrak{g}, B)$ of degree α.

(ii) *If $\omega : \mathfrak{g} \times \mathfrak{g} \longrightarrow P(\mathbb{K})$ is a homogeneous skew-supersymmetric 2-cocycle on $\mathfrak{g}$ of degree α then there exists a unique homogeneous skew-supersymmetric superderivation D of $\mathfrak{g}$ of degree α such that*

$$\omega(X,Y) \;=\; B(D(X),Y), \qquad \forall_{X \in \mathfrak{g}, Y \in \mathfrak{g}}.$$

Proof: The proof is analogous to that of [3, Lemma 3.6.]. ∎

3. Odd double extension of odd-quadratic Lie superalgebras

In this section we will introduce the notion of odd double extension of odd-quadratic Lie superalgebras.

Proposition 3.1. *Let $(\mathfrak{g}, B)$ be an odd-quadratic Lie superalgebra and $\mathfrak{h}$ a Lie superalgebra. Suppose that $\psi : \mathfrak{h} \longrightarrow Der_a(\mathfrak{g}, B)$ is a homomorphism of Lie superalgebras. Define a bilinear map $\varphi : \mathfrak{g}\times\mathfrak{g} \longrightarrow P(\mathfrak{h}^*)$ for homogeneous elements by*

$$\varphi(X,Y)(Z) \;=\; (-1)^{(x+y)z}B(\psi(Z)(X),Y), \qquad \forall X\in\mathfrak{g}_x, Y\in\mathfrak{g}_y, Z\in\mathfrak{h}_z.$$

Then the vector space $\mathfrak{g}\oplus P(\mathfrak{h}^*)$ endowed with the multiplication defined by

$$[X+f, Y+h] \;=\; [X,Y]_{\mathfrak{g}} + \varphi(X,Y), \qquad \forall_{(X+f),(Y+h)\in(\mathfrak{g}\oplus P(\mathfrak{h}^*))}, \tag{3.1}$$

is a Lie superalgebra. It is called the central extension of $P(\mathfrak{h}^*)$ by $\mathfrak{g}$ (by means of φ).

Proof: We start the proof by showing that $\mathfrak{g}\oplus P(\mathfrak{h}^*)$ equipped with the multiplication (3.1) is a Lie superalgebra. We choose the gradation

$$(\mathfrak{g}\oplus P(\mathfrak{h}^*))_\alpha \;=\; \mathfrak{g}_\alpha \oplus P(\mathfrak{h}^*)_\alpha, \qquad \forall_{\alpha\in\mathbb{Z}_2}.$$

Set $X\in(\mathfrak{g})_x, Y\in(\mathfrak{g})_y$, and $W\in\mathfrak{h}_w$. Since B is supersymmetric and $\psi(W)$ is a skew-supersymmetric superderivation of $(\mathfrak{g}, B)$ of degree w, we obtain

$$\varphi(X,Y)(W) \;=\; -(-1)^{xy}\varphi(Y,X)(W),$$

therefore

$$\varphi(X,Y) \;=\; -(-1)^{xy}\varphi(Y,X), \qquad \forall_{X\in\mathfrak{g}_x, Y\in\mathfrak{g}_y}.$$

We also have that

$$(-1)^{xz}\varphi(X,[Y,Z]_{\mathfrak{g}}) + (-1)^{xy}\varphi(Y,[Z,X]_{\mathfrak{g}}) + (-1)^{yz}\varphi(Z,[X,Y]_{\mathfrak{g}}) \;=\; 0,$$

for all $X\in\mathfrak{g}_x$, $Y\in\mathfrak{g}_y$, and $Z\in\mathfrak{g}_z$. Meaning that φ is a 2-cocycle of $\mathfrak{g}$ in the trivial $\mathfrak{g}$-module $P(\mathfrak{h}^*)$. Then $\mathfrak{g}\oplus P(\mathfrak{h}^*)$ with the multiplication (3.1) is a Lie superalgebra called the central extension of $P(\mathfrak{h}^*)$ by $\mathfrak{g}$ (by means of φ) as required. ∎

Now, we will formalize the notion of odd double extension of an odd-quadratic Lie superalgebra

Theorem 3.2. *Let $(\mathfrak{g}, B)$ be an odd-quadratic Lie superalgebra and $\mathfrak{h}$ a Lie superalgebra. Consider $\psi : \mathfrak{h} \longrightarrow Der_a(\mathfrak{g}, B)$ a homomorphism of Lie superalgebras. Define the linear map $\widetilde{\psi} : \mathfrak{h} \longrightarrow Hom(\mathfrak{g} \oplus P(\mathfrak{h}^*))$ by*

$$\widetilde{\psi}(Z)(X+f) \;=\; \psi(Z)(X) + \pi_{\mathfrak{h}}(Z)(f), \qquad \forall_{(X+f)\in(\mathfrak{g}\oplus P(\mathfrak{h}^*)), Z\in\mathfrak{h}},$$

where $\pi_{\mathfrak{h}} : \mathfrak{h} \longrightarrow Hom(P(\mathfrak{h}^*))$ is the linear map defined for homogeneous elements as follows

$$\pi_{\mathfrak{h}}(Z)(f)(Y) \;=\; -(-1)^{z\delta} f([Z,Y]), \qquad \forall_{f\in(P(\mathfrak{h}^*))_\delta, Z\in\mathfrak{h}_z, Y\in\mathfrak{h}}.$$

Then $\widetilde{\psi}(Z) \in \Big(Der(\mathfrak{g} \oplus P(\mathfrak{h}^*))\Big)_z$, for all $Z \in \mathfrak{h}_z$ where $\mathfrak{g} \oplus P(\mathfrak{h}^*)$ is the central extension of $P(\mathfrak{h}^*)$ by $\mathfrak{g}$ (by means of φ). Moreover, $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g} \oplus P(\mathfrak{h}^*)$ with the multiplication defined by

$$\begin{aligned}[Z+X+f, W+Y+h] \;=\; & [Z,W]_{\mathfrak{h}} + [X,Y]_{\mathfrak{g}} + \psi(Z)(Y) - (-1)^{xy}\psi(W)(X) \\ & +\pi_{\mathfrak{h}}(Z)(h) - (-1)^{xy}\pi_{\mathfrak{h}}(W)(f) + \varphi(X,Y), \quad (3.2)\end{aligned}$$

for all $(Z+X+f) \in \mathfrak{k}_x, (W+Y+h) \in \mathfrak{k}_y$, where φ is defined in Proposition 3.1, is a Lie superalgebra, more precisely, $\mathfrak{k}$ is the semi-direct product of $\mathfrak{g} \oplus P(\mathfrak{h}^*)$ by $\mathfrak{h}$ by means of $\widetilde{\psi}$. Furthermore, let γ be an odd supersymmetric invariant bilinear form on $\mathfrak{h}$ (not necessarily non-degenerate). Then the bilinear form $\widetilde{B} : \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathbb{K}$ defined by

$$\widetilde{B}(Z+X+f, W+Y+h) \;=\; B(X,Y) + \gamma(Z,W) + f(W) + (-1)^{xy}h(Z),$$

whenever $(Z+X+f) \in \mathfrak{k}_x, (W+Y+h) \in \mathfrak{k}_y$, is an odd invariant scalar product on $\mathfrak{k}$ and $(\mathfrak{k}, \widetilde{B})$ is an odd-quadratic Lie superalgebra. We say that the odd-quadratic Lie superalgebra $(\mathfrak{k}, \widetilde{B})$ is an odd double extension of $(\mathfrak{g}, B)$ by $\mathfrak{h}$ (by means of ψ).

Proof: First we ensure that $\widetilde{\psi}(Z)$ is a superderivation of $\mathfrak{g} \oplus P(\mathfrak{h}^*)$, when $Z \in \mathfrak{h}_z$. Since $\psi(Z)$ is a superderivation of $\mathfrak{g}$ of degree z, we can easily show that, for $(X+f) \in (\mathfrak{g} \oplus P(\mathfrak{h}^*))_x, (Y+h) \in (\mathfrak{g} \oplus P(\mathfrak{h}^*))_y$,

$$\begin{aligned}\widetilde{\psi}(Z)([X+f, Y+h]) \;=\; & [\psi(Z)(X), Y]_{\mathfrak{g}} + (-1)^{xz}[X, \psi(Z)(Y)]_{\mathfrak{g}} \\ =\; & +\pi_{\mathfrak{h}}(Z)(\varphi(X,Y)).\end{aligned}$$

On the other hand,

$$[\widetilde{\psi}(Z)(X+f), Y+h] \;=\; [\psi(Z)(X), Y]_{\mathfrak{g}} + \varphi(\psi(Z)(X), Y),$$

and

$$[X+f,\widetilde{\psi}(Z)(Y+h)] \;=\; [X,\psi(Z)(Y)]_{\mathfrak{g}}+\varphi(X,\psi(Z)(Y)).$$

We note that

$$\begin{aligned}\widetilde{\psi}(Z)([X+f,Y+h]) \;=\;& [\widetilde{\psi}(Z)(X+f),Y+h]\\ &+(-1)^{xz}[X+f,\widetilde{\psi}(Z)(Y+h)]+\pi_{\mathfrak{h}}(Z)(\varphi(X,Y))\\ &-\varphi(\psi(Z)(X),Y)-(-1)^{xz}\varphi(X,\psi(Z)(Y)).\end{aligned}$$

We obtain

$$\pi_{\mathfrak{h}}(Z)(\varphi(X,Y)) \;=\; \varphi(\psi(Z)(X),Y)+(-1)^{xz}\varphi(X,\psi(Z)(Y)),$$

therefore $\widetilde{\psi}(Z)\in\Big(Der(\mathfrak{g}\oplus P(\mathfrak{h}^*))\Big)_z$. Since $\widetilde{\psi}$ is a homomorphism of Lie superalgebra, it follows that $\mathfrak{k}$ is a semi-direct product of $\mathfrak{h}$ and $\mathfrak{g}\oplus P(\mathfrak{h}^*)$ by means of $\widetilde{\psi}$. Consequently $\mathfrak{k}$ with the multiplication (3.2) is a Lie superalgebra (cf. in [2]) Now we will guarantee that $\widetilde{B}$ is an odd invariant scalar product of $\mathfrak{k}$. By definition it comes easily that $\widetilde{B}$ is odd and supersymmetric. Let us prove that $\widetilde{B}$ is non-degenerate. Set $(Z+X+f)\in(\mathfrak{h}\oplus\mathfrak{g}\oplus P(\mathfrak{h}^*))$ such that

$$\widetilde{B}(Z+X+f,W+Y+h) \;=\; 0, \qquad \forall_{(W+Y+h)\in(\mathfrak{h}\oplus\mathfrak{g}\oplus P(\mathfrak{h}^*))}.$$

In particular, for $Y\in\mathfrak{g}$ we have $\widetilde{B}(Z+X+f,Y)=0$. Since B is non-degenerate we conclude that $X=0$. On the other hand, $\widetilde{B}(Z+f,h)=0$, for all $h\in P(\mathfrak{h}^*)$, implies that $Z=0$. Similarly, $\widetilde{B}(f,W)=0$, for all $W\in\mathfrak{h}$, leads to $f=0$. The invariance of $\widetilde{B}$ comes easily by straightforward calculations. We conclude that $\widetilde{B}$ is an odd invariant scalar product as desired. ■

Our next goal is to prove the Theorem 3.6 below, where a non-simple B-irreducible odd-quadratic Lie superalgebra $(\mathfrak{g},B)$ is described as an odd double extension of an odd-quadratic Lie superalgebra. The result is established by the next sequence of lemmas. We will prove the lemmas in the following context. Let $(\mathfrak{g},B)$ be a B-irreducible odd-quadratic Lie superalgebra. Consider that I is a maximal graded ideal of $\mathfrak{g}$ such that there exists a sub-superalgebra V of $\mathfrak{g}$ which satisfies $\mathfrak{g}=I\oplus V$.

Lemma 3.3. *Let $(\mathfrak{g},B)$ be a non-simple B-irreducible odd-quadratic Lie superalgebra. If I is a maximal graded ideal of $\mathfrak{g}$ then $I/I^{\perp}$ equipped with the structure inherited from $\mathfrak{g}$ is an odd-quadratic Lie superalgebra.*

Proof: By assumption I is a maximal graded ideal of $\mathfrak{g}$. Then $I^\perp$ is a minimal graded ideal of $\mathfrak{g}$. Due to the B-irreducibility of $\mathfrak{g}$ we have that $I^\perp \subseteq I$. It is immediate that $I/I^\perp$ with the natural multiplication

$$[X + I^\perp, Y + I^\perp] \;=\; [X, Y] + I^\perp, \qquad \forall_{X,Y \in I},$$

is a Lie superalgebra. Define a bilinear form $\widetilde{B} : I/I^\perp \times I/I^\perp \longrightarrow \mathbb{K}$ by

$$\widetilde{B}(X + I^\perp, Y + I^\perp) \;=\; B(X, Y), \qquad \forall_{X,Y \in I}.$$

It is clear that $\widetilde{B}$ is well defined. And we can easily prove the non degeneracy of $\widetilde{B}$. The bilinear form $\widetilde{B}$ is an odd invariant scalar product of $I/I^\perp$. ■

Lemma 3.4. *Let $(\mathfrak{g}, B)$ be a non-simple B-irreducible odd-quadratic Lie superalgebra. Consider I a maximal graded ideal of $\mathfrak{g}$ such that there exists a subsuperalgebra V of $\mathfrak{g}$ which satisfies $\mathfrak{g} = I \oplus V$. Then $W = I^\perp \oplus V$ is a vector subspace of $\mathfrak{g}$ and $W^\perp$ is an odd-quadratic Lie superalgebra. Moreover, the map $\theta : W^\perp \longrightarrow I/I^\perp$ defined by*

$$\theta(X) \;=\; X + I^\perp, \qquad \forall_{X \in W^\perp}, \tag{3.3}$$

is an isomorphism of Lie superalgebras.

Proof: As in the proof of Lemma 3.3 we see that $I^\perp \subseteq I$. From $\mathfrak{g} = I \oplus V$ we get that $W = I^\perp \oplus V$ is a sub-vector space of $\mathfrak{g}$. We can easily show that $B\mid_{W \times W}$ is non-degenerate. Furthermore, $\mathfrak{g} = W \oplus W^\perp = V \oplus W^\perp \oplus I^\perp$ and the restriction of B to $W^\perp$ is non-degenerate. In order to provided $W^\perp$ with a structure of Lie superalgebra, we proceed as follows: choose $X, Y \in W^\perp$. Since $W^\perp$ is a subset of the graded ideal I of $\mathfrak{g}$ then $[X, Y] \in I = W^\perp \oplus I^\perp$. We can write $[X, Y] = \alpha(X, Y) + \beta(X, Y)$, with $\alpha(X, Y) \in W^\perp$ and $\beta(X, Y) \in I^\perp$. It is clear that $W^\perp$ with the multiplication α is a Lie superalgebra. Denote $[,]_{W^\perp} = \alpha$. The restriction $B_{W^\perp} = B\mid_{W^\perp \times W^\perp}$ is $W^\perp$-invariant. Consequently $B_{W^\perp}$ is an odd invariant scalar product of $W^\perp$ and $(W^\perp, B_{W^\perp})$ is an odd-quadratic Lie superalgebra. Finally, it is easy to prove that the linear map θ defined by 3.3 is an isomorphism of Lie superalgebras, completing the Proof. ■

Lemma 3.5. *Let $(\mathfrak{g}, B)$ be a non-simple B-irreducible odd-quadratic Lie superalgebra. Consider I a maximal graded ideal of $\mathfrak{g}$ such that there exists a sub-superalgebra V of $\mathfrak{g}$ which satisfies $\mathfrak{g} = I \oplus V$. Then $\mathfrak{g}$ is isomorphic to the odd double extension of $I/I^\perp$ by V.*

Proof: Define the map $\nu : I^{\perp} \longrightarrow P(V^{*})$ by $\nu(X) = B(X, .)$, for all $X \in I^{\perp}$, that is linear. The non degeneracy of B entails that ν is injective. Similarly, it is easy to prove that the linear map $\delta : P(V) \longrightarrow (I^{\perp})^{*}$ defined by $\delta(X) = B(X, .)$, for all $X \in P(V)$, is injective. Therefore $\dim I^{\perp} = \dim P(V^{*})$ and ν is an isomorphism of $\mathbb{Z}_2$-graded vector spaces. Now consider the linear map $\psi : V \longrightarrow Der(I/I^{\perp})$ defined by

$$\psi(Y)(X + I^{\perp}) \;=\; [Y, X] + I^{\perp}, \qquad \forall_{Y \in V, X \in I}.$$

Using invariance of B and graded Jacobi identity we have that ψ is a homomorphism of Lie superalgebras and $\psi(Y) \in Der_a(I/I^{\perp}, \widetilde{B})$, for all $Y \in V$. According to Theorem 3.2 we conclude that $\mathfrak{h} = V \oplus I/I^{\perp} \oplus P(V^{*})$ is the odd double extension of $I/I^{\perp}$ by V by means of ψ. Finally, we define $\tau : \mathfrak{g} = V \oplus W^{\perp} \oplus I^{\perp} \longrightarrow \mathfrak{h} = V \oplus I/I^{\perp} \oplus P(V^{*})$ by

$$\tau(X + Y + Z) \;=\; X + \theta(Y) + \nu(Z), \qquad \forall_{(X+Y+Z) \in \mathfrak{g} = (V \oplus W^{\perp} \oplus I^{\perp})},$$

where the map θ is defined in Lemma 3.4. Since θ and ν are bijectives, then so is τ. It remains to prove that τ is a homomorphism of Lie superalgebras. Using the structure of V-module of $P(V^{*})$, $\rho : V \longrightarrow Hom(P(V^{*}))$ defined by

$$\rho(X)(f)(Y) \;=\; -(-1)^{x\gamma} f([X, Y]), \qquad \forall_{X \in V_x, f \in (P(V^{*}))_{\gamma}, Y \in P(V)},$$

we easily see that τ preserves the structure of Lie superalgebra, which means that τ is a homomorphism of Lie superalgebras. ∎

By the results of Lemma 3.3 to Lemma 3.5 we can conclude

Theorem 3.6. *Let $(\mathfrak{g}, B)$ be a B-irreducible odd-quadratic Lie superalgebra. Suppose that I is a maximal graded ideal of $\mathfrak{g}$ such that there exists a subsuperalgebra V of $\mathfrak{g}$ which satisfies $\mathfrak{g} = I \oplus V$. Then the odd-quadratic Lie superalgebra $(\mathfrak{g}, B)$ is an odd double extension of the odd-quadratic Liesuperalgebra $(I/I^{\perp}, \widetilde{B})$ by V.*

The next fundamental result, which is the converse of Theorem 3.2, is a immediate consequence of the previous theorem

Corollary 3.7. *Suppose that $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ is a B-irreducible odd-quadratic Lie superalgebra. If $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$ then $\mathfrak{g}$ is an odd double extension of an odd-quadratic Lie superalgebra $\mathfrak{h}$ (such that $\dim \mathfrak{h} = \dim \mathfrak{g} - 2$) by the one-dimensional Lie algebra $(\mathbb{K}e)_{\bar{0}}$.*

Proof: Since $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} \neq \{0\}$ we can choose a non-zero element $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}}$. Then $I = \mathbb{K}X$ is a minimal graded ideal of $\mathfrak{g}$ and $I^{\perp}$ is a maximal graded ideal of $\mathfrak{g}$. Since $X \in \mathfrak{g}_{\bar{1}}$ and B is odd it follows that $\mathfrak{g}_{\bar{1}} \subseteq I^{\perp}$. As B is non-degenerate there exists $Y \in \mathfrak{g}_{\bar{0}}$ such that $B(X, Y) \neq 0$. Therefore we have that $\mathfrak{g} = I^{\perp} \oplus \mathbb{K}Y$, where the vector subspace $\mathbb{K}Y$ is a subsuperalgebra of $\mathfrak{g}$. Define $\psi : \mathbb{K}Y \longrightarrow \mathrm{Der}(I^{\perp}/I)$ by

$$\psi(Y)(X + I) \;=\; [Y, X] + I, \qquad \forall_{X \in I^{\perp}},$$

and $\widetilde{B} : I^{\perp}/I \times I^{\perp}/I \longrightarrow \mathbb{K}$ by

$$\widetilde{B}(X + I, Y + I) \;=\; B(X, Y), \qquad \forall_{X,Y \in I^{\perp}}.$$

Invoking Theorem 3.6 we conclude that the odd-quadratic Lie superalgebra $(\mathfrak{g}, B)$ is an odd double extension of the odd-quadratic Lie superalgebra $(I^{\perp}/I, \widetilde{B})$ by $\mathbb{K}Y$ (by means of ψ). ■

Definition 3.8. Consider $\mathfrak{h}$ a Lie superalgebra. By a *trivial odd double extension* of $\mathfrak{h}$ we mean an odd double extension $(\mathfrak{k} = \mathfrak{h} \oplus P(\mathfrak{h}^*), \widetilde{B})$ of $\{0\}$ by $\mathfrak{h}$.

Lemma 3.9. *Let $\mathfrak{h}$ be a Lie superalgebra and $(\mathfrak{k} = \mathfrak{h} \oplus P(\mathfrak{h}^*), \widetilde{B})$ a trivial odd double extension of $\mathfrak{h}$. Then*

$$\mathfrak{z}(\mathfrak{k}) \;=\; \mathfrak{z}(\mathfrak{h}) \oplus \Big\{ f \in P(\mathfrak{h}^*) : \; f([\mathfrak{h}, \mathfrak{h}]) = \{0\} \Big\}. \tag{3.4}$$

In particular, if $\mathfrak{h}$ verifies $\mathfrak{z}(\mathfrak{h}) = \{0\}$ and $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$ then $\mathfrak{z}(\mathfrak{k}) = \{0\}$.

Proof: Let $(X + f) \in (\mathfrak{h} \oplus P(\mathfrak{h}^*))_x$, which means that,

$$[X + f, Y + h] \;=\; 0, \qquad \forall_{(Y+h) \in (\mathfrak{h} \oplus P(\mathfrak{h}^*))_y},$$

more explicitly,

$$[X, Y]_{\mathfrak{h}} + \pi_{\mathfrak{h}}(X)(h) - (-1)^{xy} \pi_{\mathfrak{h}}(Y)(f) \;=\; 0, \qquad \forall_{(Y+h) \in (\mathfrak{h} \oplus P(\mathfrak{h}^*))_y}.$$

Therefore $X \in \mathfrak{z}(\mathfrak{h})$ and $f([\mathfrak{h}, \mathfrak{h}]) = \{0\}$, which implies the assertion 3.4. Moreover, if $\mathfrak{z}(\mathfrak{h}) = \{0\}$ and $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, clearly $\mathfrak{z}(\mathfrak{k}) = \{0\}$ as required. ■

Example 3.10. *We easily see that the abelian two-dimensional odd-quadratic Lie superalgebra is a trivial odd double extension of the one-dimensional Lie algebra.*

Example 3.11. *Let* $sl\,(2)$ *be the special Lie algebra with its standard basis* $\{h, e, f\}$ *such that*

$$[h, e] = e \qquad [h, f] = -f \qquad [e, f] = 2h,$$

and $\{h^*, e^*, f^*\}$ *is the correspondent basis of the dual* $sl\,(2)^*$. *It is easy to show that on the trivial odd double extension* $(\mathfrak{g} = sl\,(2) \oplus P(sl\,(2)^*), B)$ *of the simple Lie algebra* $sl\,(2)$, *the graded skew-symmetric multiplication is calculated for the elements of the basis by*

$$\begin{array}{lll} [h,e] = e & [h,e^*] = -e^* & [e,e^*] = h^* \\ [h,f] = -f & [h,f^*] = f^* & [f,h^*] = 2e^* \\ [e,f] = 2h & [e,h^*] = -2f^* & [f,f^*] = -h^*, \end{array}$$

and the others are zero. And the supersymmetric bilinear form B *is given by*

$$B(h, h^*) = B(e, e^*) = B(f, f^*) = 1,$$

and the rest are zero.

4. Generalized odd double extension of odd-quadratic Lie superalgebras

Now we will present the concept of generalized odd double extension of an odd-quadratic Lie superalgebras by the one-dimensional Lie superalgebra $\mathbb{K}e = (\mathbb{K}e)_{\bar{1}}$. This notion will have an important role in an inductive classification of odd-quadratic Lie superalgebras with even part a reductive Lie algebra. In the first step, we will obtain the central extension of $P(\mathbb{K}e^*)$ by an odd-quadratic Lie superalgebra (we consider e^* the dual basis element of $\mathbb{K}e$).

Proposition 4.1. *Let* $(\mathfrak{g}, B)$ *be an odd-quadratic Lie superalgebra. Let* D *be an odd skew-supersymmetric superderivation of* $(\mathfrak{g}, B)$. *Define a bilinear map* $\varphi : \mathfrak{g} \times \mathfrak{g} \longrightarrow P(\mathbb{K})$ *by*

$$\varphi(X, Y) \;=\; B(D(X), Y), \qquad \forall_{X,Y\in\mathfrak{g}}.$$

Then the vector space $\mathfrak{g} \oplus P(\mathbb{K}e^*)$ *endowed with the multiplication defined by*

$$[X + \alpha e^*, Y + \beta e^*] \;=\; [X, Y]_{\mathfrak{g}} + \varphi(X, Y)e^*, \tag{4.5}$$

whenever $(X + \alpha e^*), (Y + \beta e^*) \in (\mathfrak{g} \oplus P(\mathbb{K}e^*))$, *is a Lie superalgebra. It is called the central extension of* $P(\mathbb{K}e^*)$ *by* $\mathfrak{g}$ *(by means of* φ*).*

Proof: We will show that $\mathfrak{g}\oplus P(\mathbb{K}e^*)$ endowed with the multiplication defined by (4.5) is a Lie superalgebra. Since D is odd and skew-supersymmetric we infer that

$$\varphi(X,Y) \;=\; -(-1)^{xy}\varphi(Y,X), \qquad \forall_{X\in\mathfrak{g}_x, Y\in\mathfrak{g}_y}.$$

As D is an odd skew-supersymmetric superderivation of $(\mathfrak{g},B)$ and applying Lemma 2.13 we have that

$$(-1)^{xz}\varphi(X,[Y,Z]_{\mathfrak{g}}) + (-1)^{xy}\varphi(Y,[Z,X]_{\mathfrak{g}}) + (-1)^{yz}\varphi(Z,[X,Y]_{\mathfrak{g}}) \;=\; 0,$$

for all $X\in\mathfrak{g}_x$, $Y\in\mathfrak{g}_y$, and $Z\in\mathfrak{g}_z$. Meaning that φ is a 2-cocycle of $\mathfrak{g}$ on $P(\mathbb{K})$. Then $\mathfrak{g}\oplus P(\mathbb{K}e^*)$ with the multiplication (4.5) is a Lie superalgebra called the central extension of $P(\mathbb{K}e^*)$ by $\mathfrak{g}$ (by means of φ) as desired. ■

For an odd-quadratic Lie superalgebra $(\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}},B)$ we can use an odd skew-supersymmetric superderivation of $(\mathfrak{g},B)$ with certain properties to construct a new odd-quadratic Lie superalgebra, namely the generalized odd double extension of $(\mathfrak{g},B)$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$.

Theorem 4.2. *Let $(\mathfrak{g}=\mathfrak{g}_{\bar{0}}\oplus\mathfrak{g}_{\bar{1}},B)$ be an odd-quadratic Lie superalgebra. Consider D an odd skew-supersymmetric superderivation of $(\mathfrak{g},B)$, $X_0\in\mathfrak{g}_{\bar{0}}$, and $\lambda_0\in\mathbb{K}$ such that*

$$D(X_0) \;=\; 0, \tag{4.6}$$

$$D^2 \;=\; \frac{1}{2}[X_0,.]_{\mathfrak{g}}. \tag{4.7}$$

Define a map $\Omega:(\mathbb{K}e)_{\bar{1}}\longrightarrow Der(\mathfrak{g}\oplus P(\mathbb{K}e^*))$ by $\Omega(e)=\widetilde{D}$, where $\widetilde{D}:\mathfrak{g}\oplus P(\mathbb{K}e^*)\longrightarrow\mathfrak{g}\oplus P(\mathbb{K}e^*)$ satisfies $\widetilde{D}(e^*)=0$ and

$$\widetilde{D}(X) \;=\; D(X)-(-1)^{x}B(X,X_0)e^*, \qquad \forall_{X\in\mathfrak{g}_x}.$$

Consider the map $\zeta:\mathbb{K}e\times\mathbb{K}e\longrightarrow\mathfrak{g}\oplus P(\mathbb{K}e^*)$ defined by $\zeta(e,e)=X_0+\lambda_0e^*$. Then $\mathfrak{k}=\mathbb{K}e\oplus\mathfrak{g}\oplus P(\mathbb{K}e^*)$ endowed with the even skew-symmetric bilinear map $[,]:\mathfrak{k}\times\mathfrak{k}\longrightarrow\mathfrak{k}$ defined by

$$\begin{aligned}
[e,e] &= X_0+\lambda_0e^*, \\
[e,X] &= D(X)-(-1)^{x}B(X,X_0)e^*, \qquad \forall_{X\in\mathfrak{g}_x}, \\
[X,Y] &= [X,Y]_{\mathfrak{g}}+B(D(X),Y)e^*, \qquad \forall_{X,Y\in\mathfrak{g}}, \\
[e^*,\mathfrak{k}] &= \{0\},
\end{aligned} \tag{4.8}$$

is a Lie superalgebra. More precisely is the generalized semi-direct product of $\mathfrak{g}\oplus P(\mathbb{K}e^*)$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of Ω

and ζ). Moreover, the supersymmetric bilinear form $\widetilde{B} : \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathbb{K}$ defined by

$$\begin{aligned}\widetilde{B}\mid_{\mathfrak{g}\times\mathfrak{g}} &= B,\\ \widetilde{B}(e,e^*) &= 1,\\ \widetilde{B}(\mathfrak{g},e) = \widetilde{B}(\mathfrak{g},e^*) &= \{0\},\end{aligned}$$

is an odd-invariant scalar product on $\mathfrak{k}$. In this case, we say that $(\mathfrak{k},\widetilde{B})$ is the generalized odd double extension of $(\mathfrak{g}, B)$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of D, X_0, and λ_0).

Proof: We start by proving that $\widetilde{D} \in D(\mathfrak{g}\oplus P(\mathbb{K}e^*))_{\bar{1}}$. First, for every $X \in \mathfrak{g}_x$ and $Y \in \mathfrak{g}_y$, we have to show that

$$\widetilde{D}([X,Y]_{\mathfrak{g}\oplus P(\mathbb{K}e^*)}) = [\widetilde{D}(X),Y]_{\mathfrak{g}\oplus P(\mathbb{K}e^*)} + (-1)^x[X,\widetilde{D}(Y)]_{\mathfrak{g}\oplus P(\mathbb{K}e^*)},$$

which is equivalent to

$$\begin{aligned}D([X,Y]_{\mathfrak{g}}) - (-1)^{x+y}B([X,Y]_{\mathfrak{g}},X_0)e^* &=\\ &= [D(X),Y]_{\mathfrak{g}} + (-1)^x[X,D(Y)]_{\mathfrak{g}}\\ &+\Big\{B(D^2(X),Y) + (-1)^x B(D(X),D(Y))\Big\}e^*.\end{aligned}$$

Since D is an odd superderivation of $\mathfrak{g}$ it remains to see that

$$-(-1)^{x+y}B([X,Y]_{\mathfrak{g}},X_0) = B(D^2(X),Y) + (-1)^x B(D(X),D(Y)).$$

¿From (4.7), as B is supersymmetric, D is an odd skew-supersymmetric superderivation of $(\mathfrak{g}, B)$, and B is invariant, we obtain that

$$(1 + (-1)^{x+y})B([X,Y]_{\mathfrak{g}},X_0) = 0, \qquad \forall_{X\in\mathfrak{g}_x, Y\in\mathfrak{g}_y}. \tag{4.9}$$

As B is odd then condition (4.9) is verified for every $X \in \mathfrak{g}_x$ and $Y \in \mathfrak{g}_y$. The others cases are trivial. So we conclude that $\widetilde{D}$ is an odd superderivation of $\mathfrak{g}\oplus P(\mathbb{K}e^*)$. We easily show that ζ is a skew-supersymmetric even bilinear map. Doing some calculations, it is elementary to see that conditions (4.6) and (4.7) imply that $\mathfrak{k} = \mathbb{K}e \oplus \mathfrak{g} \oplus P(\mathbb{K}e^*)$ equipped with multiplication defined by (4.8) is the generalized semi-direct product of $\mathfrak{g}\oplus P(\mathbb{K}e^*)$ by the one dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of Ω and ζ). ∎

Now, we will show the converse of Theorem 4.2

Proposition 4.3. *Suppose that* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *is a* B*-irreducible odd-quadratic Lie superalgebra such that* $\dim \mathfrak{g} > 1$*. If* $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$ *then* $(\mathfrak{g}, B)$ *is a generalized odd double extension of an odd-quadratic Lie superalgebra* $(\mathfrak{h}, \widetilde{B})$ *(such that* $\dim \mathfrak{h} = \dim \mathfrak{g} - 2$*) by the one-dimensional Lie superalgebra* $(\mathbb{K}e)_{\bar{1}}$*.*

We will prove the result by showing the following sequence of claims. We will present a broad outline of the proof: first, we will determine the odd-quadratic Lie superalgebra $(\mathfrak{h}, \widetilde{B})$; then we will show that the odd-quadratic Lie superalgebra $(\mathfrak{g}, B)$ is the generalized odd-double extension of $(\mathfrak{h}, \widetilde{B})$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$. To undertake the proof, let us assume that $(\mathfrak{g}, B)$ is a B-irreducible odd-quadratic Lie superalgebra such that $\dim \mathfrak{g} > 1$ and $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$. We set e^* a non-zero element of $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$ and denote $I = \mathbb{K}e^*$. As B is odd we have $\mathfrak{g}_{\bar{0}} \subseteq J$, where J is the orthogonal of I with respect to B. Since B is non-degenerate and odd then there exists $e \in \mathfrak{g}_{\bar{1}}$ such that $B(e^*, e) \neq 0$. We may assume that $B(e^*, e) = 1$. As $e \notin J$ and $\dim J = \dim \mathfrak{g} - 1$ we infer that $\mathfrak{g} = J \oplus \mathbb{K}e$. Consider the two-dimensional vector subspace $A = \mathbb{K}e^* \oplus \mathbb{K}e$ of $\mathfrak{g}$. Since $B\mid_{A\times A}$ is non-degenerate we have $\mathfrak{g} = A \oplus \mathfrak{h}$, where $\mathfrak{h}$ is the orthogonal of A with respect to B. It comes that $\widetilde{B} = B\mid_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate. As B is odd we have $\mathbb{K}e^* \subseteq J$, and so $\mathbb{K}e^* \oplus \mathfrak{h} \subseteq J$. ¿From $\dim(\mathbb{K}e^* \oplus \mathfrak{h}) = \dim \mathfrak{g} - 1 = \dim J$ it comes that $J = \mathbb{K}e^* \oplus \mathfrak{h}$. So $\mathfrak{h}$ is a graded vector subspace of $\mathfrak{g}$ contained in the graded ideal $J = \mathfrak{h} \oplus \mathbb{K}e^*$ of $\mathfrak{g}$. Then we have

$$[X, Y] = \alpha(X, Y) + \varphi(X, Y)e^*, \qquad \forall_{X,Y\in\mathfrak{h}},$$

where $\alpha(X, Y) \in \mathfrak{h}$ and $\varphi(X, Y) \in \mathbb{K}$. Further,

$$[e, X] = D(X) + \psi(X)e^*, \qquad \forall_{X\in\mathfrak{h}},$$

where $D(X) \in \mathfrak{h}$ and $\psi(X) \in \mathbb{K}$.

Claim 4.4. *The Lie superalgebra* $(\mathfrak{h}, \alpha = [\,,\,]_{\mathfrak{h}}, \widetilde{B})$ *is odd-quadratic.*

Proof: ¿From graded skew-symmetry on $\mathfrak{g}$ it comes that

$$\begin{aligned} \alpha(X, Y) &= -(-1)^{xy}\alpha(Y, X), \qquad \forall_{X\in\mathfrak{h}_x, Y\in\mathfrak{h}_y}, \\ \varphi(X, Y) &= -(-1)^{xy}\varphi(Y, X), \qquad \forall_{X\in\mathfrak{h}_x, Y\in\mathfrak{h}_y}. \end{aligned} \tag{4.10}$$

Moreover, by the graded Jacobi identity on $\mathfrak{g}$ it follows that

$$(-1)^{xz}\alpha\big(X,\alpha(Y,Z)\big)+(-1)^{xy}\alpha\big(Y,\alpha(Z,X)\big)+(-1)^{yz}\alpha\big(Z,\alpha(X,Y)\big) = 0,\quad (4.11)$$
$$(-1)^{xz}\varphi\big(X,\varphi(Y,Z)\big)+(-1)^{xy}\varphi\big(Y,\varphi(Z,X)\big)+(-1)^{yz}\varphi\big(Z,\varphi(X,Y)\big) = 0,$$

for $X\in\mathfrak{h}_x, Y\in\mathfrak{h}_y$, and $Z\in\mathfrak{h}_z$. By (4.10) and (4.11) we conclude that $(\mathfrak{h},\alpha)$ is a Lie superalgebra. By invariance of B in $\mathfrak{g}$ it comes straightforward that

$$\widetilde{B}\big(\alpha(X,Y),Z\big) = \widetilde{B}\big(X,\alpha(Y,Z)\big),\qquad \forall_{X,Y,Z\in\mathfrak{h}},$$

which means that $\widetilde{B}$ is invariant in $\mathfrak{h}$. Therefore it is immediate that $\widetilde{B}$ is an odd-invariant scalar product on $\mathfrak{h}$ as desired. ■

Since $e\in\mathfrak{g}_{\bar 1}$ then $[e,e]$ is not necessarily zero, and we can write

$$[e,e] = X_0+\lambda_0 e^*,$$

where $X_0\in\mathfrak{g}_{\bar 0}$ and $\lambda_0\in\mathbb{K}$.

Claim 4.5. *Then D is an odd skew-supersymmetric superderivation of $(\mathfrak{h},\widetilde{B})$ such that*

$$D(X_0) = 0,$$
$$D^2 = \frac{1}{2}[X_0,.]_{\mathfrak{h}}.$$

Moreover, $(\mathfrak{g},B)$ is the generalized odd double extension of $(\mathfrak{h},\widetilde{B})$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar 1}$ (by means of D, X_0, and λ_0).

Proof: We start by proving that D is an odd skew-supersymmetric superderivation of $(\mathfrak{h},\widetilde{B})$. It is immediate that D is a homogeneous linear map of degree $\bar 1$. Using graded Jacobi identity

$$(-1)^y[e,[X,Y]]+(-1)^x[X,[Y,e]]+(-1)^{xy}[Y,[e,X]] = 0,\qquad \forall_{X\in\mathfrak{h}_x,Y\in\mathfrak{h}_y},$$

it comes that

$$D\big([X,Y]_{\mathfrak{h}}\big) = [D(X),Y]_{\mathfrak{h}}+(-1)^x[X,D(Y)]_{\mathfrak{h}},\qquad \forall_{X\in\mathfrak{h}_x,Y\in\mathfrak{h}_y},\quad (4.12)$$
$$\psi\big([X,Y]_{\mathfrak{h}}\big) = \varphi\big(D(X),Y\big)+(-1)^x\varphi\big(X,D(Y)\big),\qquad \forall_{X\in\mathfrak{h}_x,Y\in\mathfrak{h}_y}.$$

¿From (4.12) we say that $D\in\Big(Der(\mathfrak{h})\Big)_{\bar 1}$. Using successively the invariance of B, from

$$B([e,X],Y) = -(-1)^x B(X,[e,Y]),\qquad \forall_{X\in\mathfrak{h}_x,Y\in\mathfrak{h}},$$

we obtain

$$\widetilde{B}\big(D(X), Y\big) \ = \ -(-1)^x \widetilde{B}\big(X, D(Y)\big), \qquad \forall_{X\in\mathfrak{h}_x, Y\in\mathfrak{h}},$$

which means that $D \in \Big(Der_a(\mathfrak{h}, \widetilde{B})\Big)_{\bar{1}}$. From

$$B([e, X], Y) \ = \ B(e, [X, Y]), \qquad \forall_{X,Y\in\mathfrak{h}},$$

we infer that

$$\varphi(X, Y) \ = \ B(D(X), Y), \qquad \forall_{X,Y\in\mathfrak{h}}.$$

And, using

$$B([e, X], e) \ = \ -(-1)^x B(X, [e, e]), \qquad \forall_{X\in\mathfrak{h}_x},$$

we conclude that

$$\psi(X) \ = \ -(-1)^x B(X, X_0), \qquad \forall_{X\in\mathfrak{h}_x}.$$

¿From graded Jacobi identity $[e, [e, e]] = 0$ we have $D(X_0) = 0$. Furthermore

$$(-1)^x[e, [e, X]] - [e, [X, e]] + (-1)^x[X, [e, e]] \ = \ 0, \qquad \forall_{X\in\mathfrak{h}_x},$$

leads to

$$D^2(X) \ = \ \frac{1}{2}[X_0, X]_{\mathfrak{h}}, \qquad \forall_{X\in\mathfrak{h}_x},$$

which we extend by linearity to all $\mathfrak{h}$, completing the proof. ■

Example 4.6. *The two-dimensional odd-quadratic Lie superalgebra $\mathfrak{N}$ defined in Remark 2.9 is a generalized odd double extension of $\{0\}$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$.*

To the inductive classification of odd-quadratic Lie superalgebras with reductive even part, it is important to present a generalized odd double extension of the trivial odd double extension of a simple Lie algebra by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ which it is not a direct sum of two non-degenerate graded ideals. Let $(\mathfrak{g} = s\oplus P(s^*), B)$ be the trivial odd double extension of a simple Lie algebra s. We seek for an odd skew-supersymmetric superderivation of $(\mathfrak{g}, B)$ such that

$$D(s) \ = \ \{0\},$$
$$D(P(s^*)) \ \subseteq \ s.$$

Suppose that there exists a superderivation D of $\mathfrak{g}$ with the conditions mentioned above. Define a linear map $\phi : s \longrightarrow s$ by

$$\phi(X) \;=\; D\big(\kappa(X, .)\big), \qquad \forall_{X\in s},$$

where κ is the Killing form on the simple Lie algebra s. Considering the s-modules s and $P(s^*)$, since κ is invariant, we have

$$\kappa(X.Y, Z) \;=\; \big(X.\kappa(Y, .)\big)(Z), \qquad \forall_{X,Y,Z\in s},$$

or equivalently,

$$\kappa\big([X, Y], .\big) \;=\; \big[X, \kappa(Y, .)\big], \qquad \forall_{X,Y\in s}.$$

As D is an odd superderivation on $\mathfrak{g}$ and $D(s) = \{0\}$, we obtain

$$D\Big(\kappa\big([X, Y], .\big)\Big) \;=\; \Big[X, D\big(\kappa(Y, .)\big)\Big], \qquad \forall_{X,Y\in s},$$

which means that

$$\phi\big([X, Y]\big) \;=\; \big[X, \phi(Y)\big], \qquad \forall_{X,Y\in s}.$$

Invoking Schur's Lemma, since ϕ is an even s-invariant linear map of s, we conclude that there exists $\lambda \in \mathbb{K}$ such that $\phi(X) = \lambda I(X)$, for all $X \in s$, or equivalently,

$$D\big(\kappa(X, .)\big) \;=\; \lambda X, \qquad \forall_{X\in s}.$$

We can consider, for example, $\lambda = 1$. Define a linear map $\psi : s \longrightarrow P(s^*)$ by

$$\psi(X) \;=\; \kappa(X, .), \qquad \forall_{X\in s}.$$

Since s is a simple Lie algebra then the Killing form κ on s is non-degenerate and ψ is bijective. We can make an identification of $f \in P(s^*)$ and $X_f \in s$ by the relation $f = \kappa(X_f, .)$, and define a superderivation in the following auxiliary result

Lemma 4.7. *Consider the trivial odd double extension* $(\mathfrak{g} = s \oplus P(s^*), B)$ *of a simple Lie algebra* s*. Define a linear map* $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ *by*

$$\begin{aligned} D(s) \;&=\; \{0\}, \\ D(f) \;&=\; X_f, \qquad \forall_{f\in P(s^*)}, \end{aligned}$$

where $X_f \in s$ *is determined by the relation* $f = \kappa(X_f, .)$*. Then* D *is an odd skew-supersymmetric superderivation on* $(\mathfrak{g}, B)$ *such that it is not inner and* $D^2 = 0$.

Proof: It is immediate that $D^2 = 0$. On the trivial odd double extension $\mathfrak{g} = s \oplus P(s^*)$ the multiplication is defined for homogeneous elements by

$$\begin{aligned} [X,Y] &= [X,Y]_s, \qquad \forall_{X,Y\in s} \\ [X,f] &= \pi_s(X)(f), \qquad \forall_{X\in s, f\in P(s^*)} \\ [f,h] &= 0, \qquad \forall_{f,h\in P(s^*)}, \end{aligned}$$

where $\pi_s : \mathfrak{h} \longrightarrow Hom(P(s^*))$ is defined for homogeneous elements as follows

$$\pi_s(Z)(f)(Y) \;=\; -(-1)^{z\delta} f([Z,Y]), \qquad \forall_{f\in(P(s^*))_\delta, Z\in s_z, Y\in s},$$

and extended by linearity. We start by proving that D is an odd superderivation of $\mathfrak{g}$. From $D(s) = \{0\}$ it is obvious that

$$D\big([X,Y]\big) \;=\; \big[D(X),Y\big] + \big[X,D(Y)\big], \qquad \forall_{X,Y\in s}.$$

Let $X \in s$ and $f \in P(s^*)$. As the Killing form is s-invariant we obtain

$$[X,f](Y) \;=\; \kappa\big([X,X_f]\big)(Y), \qquad \forall_{Y\in s}.$$

Since $D(s) = \{0\}$, we conclude that

$$D\big([X,f]\big) \;=\; \big[D(X),f\big] + \big[X,D(f)\big], \qquad \forall_{X\in s, f\in P(s^*)}.$$

Finally, as the Killing form on s is invariant and $D(P(s^*)) \subseteq s$, we infer that

$$\big[D(f),h\big](X) \;=\; \big[f,D(h)\big](X), \qquad \forall_{X\in s, f,h\in P(s^*)},$$

which yields that

$$\big[D(f),h\big] - \big[f,D(h)\big] \;=\; D\big([f,h]\big), \qquad \forall_{f,h\in P(s^*)}.$$

Now we ensure that the odd superderivation D is skew-supersymmetric on $(\mathfrak{g}, B)$. As $D(s) = \{0\}$ we obtain

$$B\big(D(X),Y\big) \;=\; -B\big(X,D(Y)\big), \qquad \forall_{X,Y\in s}.$$

Since $D(s) = \{0\}$, $D(P(s^*)) \subseteq s$, and B is odd it comes

$$B\big(D(X),f\big) \;=\; -B\big(X,D(f)\big), \qquad \forall_{X\in s, f\in P(s^*)}.$$

By definition of the odd invariant scalar product on $\mathfrak{g}$, since the Killing form is symmetric, and B is supersymmetric we infer that

$$B\big(D(f),h\big) \;=\; B\big(f,D(h)\big), \qquad \forall_{f,h\in P(s^*)}.$$

Finally, we suppose that D is a inner derivation, meaning that, there exist $X \in s$ and $f \in P(s^*)$ such that $D = \mathrm{ad}\ (X + f)$. From $D(s) = \{0\}$, we get

$$[X,Y] - \pi_s(Y)(f) \;=\; 0, \qquad \forall_{Y\in s},$$

then

$$[X,Y] \;=\; 0 \text{ and } \pi_s(Y)(f) \;=\; 0, \qquad \forall_{Y\in s}.$$

As s is simple then $\mathfrak{z}(s) = \{0\}$. From $[X,Y] = 0$, for every $Y \in s$, we conclude that $X = 0$. On the other hand, since s is simple then $[s,s] = s$. Due to $f([Y,Z]) = 0$, whenever $Y, Z \in s$, we state that $f = 0$. Therefore the superderivation D is not inner and the proof is complete. ■

Proposition 4.8. *Let $(\mathfrak{g} = s \oplus P(s^*), B)$ be the trivial odd double extension of a simple Lie algebra s. Consider $\lambda_0 \in \mathbb{K}$ and the odd derivation $D : \mathfrak{g} \longrightarrow \mathfrak{g}$ of $\mathfrak{g}$ defined by*

$$\begin{aligned} D(s) \;&=\; \{0\},\\ D(f) \;&=\; X_f, \qquad \forall_{f\in P(s^*)}, \end{aligned}$$

where X_f is the element in s determined by the relation $f = \kappa(X_f, .)$. In this case, $\mathfrak{k} = \mathbb{K}e \oplus \mathfrak{g} \oplus P(\mathbb{K}e^*)$ endowed with the even skew-symmetric bilinear map $[,] : \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathfrak{k}$ given by

$$\begin{aligned} [e,e] \;&=\; \lambda_0 e^*,\\ [e,s] \;&=\; \{0\},\\ [e,f] \;&=\; X_f, \qquad \forall_{f\in P(s^*)},\\ [X,Y] \;&=\; [X,Y]_s, \qquad \forall_{X,Y\in s},\\ [X,f] \;&=\; \pi_s(X)(f), \qquad \forall_{X\in s, f\in P(s^*)},\\ [f,g] \;&=\; B(X_f, g)e^*, \qquad \forall_{f,g\in P(s^*)},\\ [e^*,\mathfrak{k}] \;&=\; \{0\}, \end{aligned}$$

and with the supersymmetric bilinear form $\widetilde{B} : \mathfrak{k} \times \mathfrak{k} \longrightarrow \mathbb{K}$ defined by

$$\begin{aligned} &\widetilde{B}(s,s) \;=\; \widetilde{B}(P(s^*), P(s^*)) \;=\; \{0\},\\ &\widetilde{B}\,|_{s\times P(s^*)} = \; B,\\ &\widetilde{B}(e,e^*) \;=\; 1,\\ &\widetilde{B}(\mathfrak{g},e) = \widetilde{B}(\mathfrak{g},e^*) \;=\; \{0\}, \end{aligned}$$

is the generalized odd double extension of $(\mathfrak{g} = s \oplus P(s^*), B)$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of D, $X_0 = 0$, and λ_0). Moreover, $\mathfrak{z}(\mathfrak{k}) = \mathbb{K}e^*$ and $\mathfrak{k}$ is $\widetilde{B}$-irreducible.

Proof: Applying Lemma 4.7, we infer that D is an odd skew-supersymmetric superderivation on $(\mathfrak{g}, B)$ such that $D^2 = 0$. Therefore the conditions of Theorem 4.2 are satisfied. Doing straightforward calculations we conclude that $(\mathfrak{k} = \mathbb{K}e \oplus \mathfrak{g} \oplus P(\mathbb{K}e^*), \widetilde{B})$ is the generalized odd double extension of $(\mathfrak{g} = s \oplus P(s^*), B)$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of D, $X_0 = 0$, and λ_0). Now let us show that $\mathfrak{z}(\mathfrak{k}) = \mathbb{K}e^*$. Let

$$X \;=\; x + f + \lambda e + \beta e^* \in \mathfrak{z}(\mathfrak{k}),$$

with $x \in s$, $f \in P(s^*)$, and $\lambda, \beta \in \mathbb{K}$. We have $[X, y] = 0$, for all $y \in s$, meaning that $[x, y] - \pi_s(y)(f) = 0$, for all $y \in s$, then

$$[x, y] = 0, \qquad \forall_{y \in s}, \tag{4.13}$$

$$f([y, z]) = 0, \qquad \forall_{y,z \in s}. \tag{4.14}$$

As s is simple, we have that $\mathfrak{z}(s) = \{0\}$ and from (4.13) it comes that $x = 0$. We also have $[s, s] = s$ and using (4.14) we obtain that $f = 0$. Therefore $X = \lambda e + \beta e^* \in \mathfrak{z}(\mathfrak{k})$. Moreover, from $[X, h] = 0$, for all $h \in P(s^*)$, we have that $\lambda X_h = 0$, for all $h \in P(s^*)$. Since $X_h \neq 0$, for some $h \in P(s^*)$, then $\lambda = 0$. Consequently, $\mathfrak{z}(\mathfrak{k}) \subseteq \mathbb{K}e^*$. As $[e^*, \mathfrak{k}] = \{0\}$ we infer that $\mathfrak{z}(\mathfrak{k}) = \mathbb{K}e^*$. Finally, we prove that $\mathfrak{k}$ is $\widetilde{B}$-irreducible. Let us assume that $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, where $\mathfrak{k}_1, \mathfrak{k}_2$ are ideals of $\mathfrak{k}$. Since $\widetilde{B}\,|_{\mathfrak{k}_1 \times \mathfrak{k}_1}$ and $\widetilde{B}\,|_{\mathfrak{k}_2 \times \mathfrak{k}_2}$ are non-degenerate we have that $(\mathfrak{k}_1)_{\bar{0}} \neq \{0\}$ and $(\mathfrak{k}_2)_{\bar{0}} \neq \{0\}$. From $(\mathfrak{k}) = (\mathfrak{k}_1)_{\bar{0}} \oplus (\mathfrak{k}_2)_{\bar{0}} = s \oplus \mathbb{K}e^*$, we infer that $(\mathfrak{k}_1)_{\bar{0}} = s$ and $(\mathfrak{k}_2)_{\bar{0}} = \mathbb{K}e^*$, or $(\mathfrak{k}_1)_{\bar{0}} = \mathbb{K}e^*$ and $(\mathfrak{k}_2)_{\bar{0}} = s$. We suppose the former case (the second case is proved in a similar way). Since $\mathfrak{k}_1$ is an ideal, $(\mathfrak{k}_1)_{\bar{0}} = s$, and $[\mathfrak{k}_1, \mathfrak{k}_2] = \{0\}$ then

$$[s, \mathfrak{k}] \;=\; [s, \mathfrak{k}_1] \oplus [s, \mathfrak{k}_2] \;\subseteq\; \mathfrak{k}_1.$$

As s is simple then $\mathfrak{z}(s) = \{0\}$ and $[s, s] = s$, by Lemma 3.9 we infer that $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and so $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Since $[s, s] = s$ and $[P(s^*), P(s^*)] = \{0\}$, from

$$\mathfrak{g} \;=\; [\mathfrak{g}, \mathfrak{g}] \;=\; [s, s] + [s, P(s^*)] + [P(s^*), P(s^*)],$$

it comes that $[s, P(s^*)] = P(s^*)$. Due to $(\mathfrak{k}_1)_{\bar{0}} = s$ and $P(s^*) \subseteq (\mathfrak{k})_{\bar{1}}$ we get

$$P(s^*) \;=\; [s, P(s^*)] \;\subseteq\; (\mathfrak{k}_1)_{\bar{1}}.$$

So $s \oplus P(s^*) \subseteq \mathfrak{k}_1$. Consequently $\dim \mathfrak{k}_2 \leq 2$. Since $\mathfrak{k}_2$ is odd-quadratic is comes that $\dim \mathfrak{k}_2 = 2$. We write $\mathfrak{k}_2 = \mathbb{K}e^* \oplus \mathbb{K}X$, meaning that e^*, X are linearly independent vectors. Let

$$X \;=\; x + f + \lambda e + \beta e^*,$$

with $x \in s$, $f \in P(s^*)$, and $\lambda, \beta \in \mathbb{K}$. In view of $[\mathfrak{k}_1, \mathfrak{k}_2] = \{0\}$ and $(\mathfrak{k}_1)_{\bar{0}} = s$ we have that $[X, y] = 0$, for all $y \in s$, meaning that $[x, y]_s - \pi_s(y)(f) = 0$, for all $y \in s$, then

$$[x, y]_s = 0, \qquad \forall_{y \in s}, \tag{4.15}$$

$$f([y, z]) = 0, \qquad \forall_{y,z \in s}. \tag{4.16}$$

Again, as s is simple, from (4.15) we have that $x = 0$ and due to (4.16) it comes that $f = 0$. Then $X = \lambda e + \beta e^*$. Moreover, as $\mathfrak{k}_2 = \mathbb{K}e^* \oplus \mathbb{K}X$ and $[\mathfrak{k}_1, \mathfrak{k}_2] = \{0\}$ we conclude that $\lambda[e, h] = \{0\}$, for all $h \in P(s^*)$. Let $h \in P(s^*) \setminus \{0\}$ then $[e, h] = X_h \neq 0$ and $\lambda = 0$. Consequently $X = \beta e^*$, contradicting the fact that e^*, X are linearly independent vectors. Therefore $\mathfrak{k}$ is $\widetilde{B}$-irreducible as required. ■

Example 4.9. *Consider the trivial odd double extension ($\mathfrak{g} = sl(2) \oplus P(sl(2)^*), B$) of the simple Lie algebra $sl(2)$, where $sl(2)$ is equipped with its standard basis $\{h, e, f\}$ and the dual basis $\{h^*, e^*, f^*\}$ (see Example 3.11). It is easy to see that the Killing form on $sl(2)$ is given in the elements of the basis by*

$$\kappa(h, h) = 2 \qquad \kappa(e, f) = \kappa(f, e) = 4,$$

and the rest are zero. Moreover, the odd skew-supersymmetric superderivation D on ($\mathfrak{g} = sl(2) \oplus P(sl(2)^*), B$) defined in Proposition 4.8 is determined by

$$D(h^*) = \tfrac{1}{2}h \qquad D(e^*) = \tfrac{1}{4}f \qquad D(f^*) = \tfrac{1}{4}e.$$

Consider the one-dimensional abelian Lie superalgebra $(\mathbb{K}i)_{\bar{1}}$ with even part zero. From Proposition 4.8, the generalized odd double extension of ($\mathfrak{g} = sl(2) \oplus P(sl(2)^*), B$) by the one-dimensional Lie superalgebra $(\mathbb{K}i)_{\bar{1}}$ (by means of D, $X_0 = 0$, and $\lambda_0 \in \mathbb{K}$) is an odd-quadratic Lie superalgebra ($\mathfrak{k} = \mathbb{K}i \oplus sl(2) \oplus P(sl(2)^*) \oplus P(\mathbb{K}i^*), \widetilde{B}$) with even part generated by $\langle i^*, h, e, f\rangle$ and the odd part by $\langle i, h^*, e^*, f^*\rangle$ equipped with the structures defined explicitly as follows: the graded skew-symmetric multiplication is given for the elements of the basis by

$$\begin{array}{lll}
[h, e] = e & [h, e^*] = -e^* & [h^*, h^*] = \tfrac{1}{2}i^* \\
[h, f] = -f & [h, f^*] = f^* & [h^*, i] = \tfrac{1}{2}h \\
[e, f] = 2h & [e, h^*] = -2f^* & [e^*, f^*] = \tfrac{1}{4}i^* \\
 & [f, h^*] = 2e^* & [e^*, i] = -\tfrac{1}{4}f \\
 & [e, e^*] = -[f, f^*] = h^* & [f^*, i] = -\tfrac{1}{4}e,
\end{array}$$

and the others are zero. Moreover, the non-zero values of the supersymmetric bilinear form $\widetilde{B}$ are

$$\widetilde{B}(i^*, i) = \widetilde{B}(h, h^*) = \widetilde{B}(e, e^*) = \widetilde{B}(f, f^*) = 1.$$

5. Odd-quadratic Lie superalgebras with reductive even part

This section analyses the particular case of odd-quadratic Lie superalgebras such that the even part is a reductive Lie algebra. We will give examples of this class of Lie superalgebras.

Examples 5.1.

(1) *The superalgebra $d(n)/\mathbb{K}I_{2n}$ (with $n \geq 3$), isomorphic to the (f, d) algebra of Gell-Mann, Michel, and Radicati, is the unique odd-quadratic Lie superalgebra with even part reductive that is simple* [11].
(2) *The Lie superalgebra $\mathfrak{N}$ of the Remark 2.9 ($\mathfrak{N}$ is abelian).*
(3) *The trivial odd double extension $\mathfrak{g} = s \oplus P(s^*)$ of a simple Lie algebra s. This Lie superalgebra is perfect, however since $P(s^*)$ is an abelian ideal, $\mathfrak{g}$ is not semisimple.*
(4) *Examples constructed in Proposition 4.8.*

Proposition 5.2. *If $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ is an odd-quadratic Lie superalgebra then $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$. Moreover, if $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, then $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \{0\}$ if and only if $\mathfrak{z}(\mathfrak{g}) = \{0\}$.*

Proof: It is clear that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \subseteq \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. Let us prove the opposite inclusion. From invariance of B we get $B([\mathfrak{g}_{\bar{1}}, \mathfrak{z}(\mathfrak{g}_{\bar{0}})], \mathfrak{g}_{\bar{0}}) = \{0\}$. On the other hand, since B is odd we infer that $B([\mathfrak{g}_{\bar{1}}, \mathfrak{z}(\mathfrak{g}_{\bar{0}})], \mathfrak{g}_{\bar{1}}) = \{0\}$. As B is non-degenerate we get $[\mathfrak{g}_{\bar{1}}, \mathfrak{z}(\mathfrak{g}_{\bar{0}})] = \{0\}$. Due to $[\mathfrak{g}_{\bar{0}}, \mathfrak{z}(\mathfrak{g}_{\bar{0}})] = \{0\}$ we infer that $[\mathfrak{g}, \mathfrak{z}(\mathfrak{g}_{\bar{0}})] = \{0\}$, which means that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subseteq \mathfrak{z}(\mathfrak{g})$, and so $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subseteq \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$. Furthermore, consider that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. From the above if $\mathfrak{z}(\mathfrak{g}) = \{0\}$ it is obvious that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \{0\}$. Conversely, we assume that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \{0\}$. Since $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} = \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ we obtain $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} = \mathfrak{z}(\mathfrak{g})$. Choose $X \in \mathfrak{z}(\mathfrak{g})$. As $X \in \mathfrak{g}_{\bar{1}}$ and B is odd it follows that $B(\mathfrak{g}_{\bar{1}}, X) = \{0\}$. By assumption $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, so we can have $\mathfrak{g}_{\bar{0}} = s \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}})$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. Using invariance of B and as $s = [s, s]$ we get that $B(\mathfrak{g}_{\bar{0}}, X) = \{0\}$. Since B is non-degenerate we conclude that $X = 0$, and consequently $\mathfrak{z}(\mathfrak{g}) = \{0\}$. ■

Corollary 5.3. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a B-irreducible odd-quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Suppose that $\mathfrak{g}$ is neither* *simple with* $\dim \mathfrak{g} \neq 2$ *nor abelian of dimension* 2. *Then* $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$.

Proof: From Proposition 5.2 we have immediately that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}}$ and $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subseteq \mathfrak{z}(\mathfrak{g})$. It remains to see that $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. Let us suppose that there exists a non-zero element $X \in \mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}}$. Then $I = \mathbb{K}X$ is a graded ideal of $\mathfrak{g}$. Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra then $\mathfrak{g}_{\bar{0}} = s \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}})$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. As s is semisimple we have $s = [s, s]$. By invariance of B, $s = [s, s]$, and $X \in \mathfrak{z}(\mathfrak{g})$ we infer that $B(X, s) = \{0\}$. On the other hand, as B is odd and $X \in \mathfrak{g}_{\bar{1}}$ we get that $B(X, \mathfrak{g}_{\bar{1}}) = \{0\}$. By the non degeneracy of B we conclude that there exists $Y \in \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ such that $B(X, Y) \neq 0$. Since $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subseteq \mathfrak{z}(\mathfrak{g})$ then $J = \mathbb{K}X \oplus \mathbb{K}Y$ is an abelian graded ideal of $\mathfrak{g}$. As $\mathfrak{g}$ is B-irreducible and $B\mid_{J \times J}$ is non-degenerate then $\mathfrak{g} = J$. By assumption $\dim \mathfrak{g} \neq 2$, therefore we obtain a contradiction. We conclude that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{1}} = \{0\}$ and consequently $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ as required. ∎

Corollary 5.4. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be an odd-quadratic Lie superalgebra such that* $\mathfrak{g}_{\bar{0}}$ *is a reductive Lie algebra. Then* $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ *if and only if* $\mathfrak{g}$ *does not contain any proper non-degenerate abelian graded ideal of dimension* 2.

Proof: To start we assume that $\mathfrak{g}$ does not contain any proper non-degenerate abelian graded ideal of dimension 2. By Proposition 2.12, $\mathfrak{g} = \bigoplus_{k=1}^{n} \mathfrak{g}_k$, where $\mathfrak{g}_k$ is a B-irreducible non-degenerate graded ideal of $\mathfrak{g}$, for all $k \in \{1, \dots, n\}$. Then $\mathfrak{g}_k$ is not abelian of dimension 2, for all $k \in \{1, \dots, n\}$. We have to consider two cases. If $\mathfrak{g}_k$ is simple, then $\mathfrak{z}(\mathfrak{g}_k) = \{0\}$. While, if $\mathfrak{g}_k$ is not simple, since $\mathfrak{g}_k$ is B-irreducible with $(\mathfrak{g}_k)_{\bar{0}}$ reductive Lie algebra then, by Corollary 5.3, we infer that $\mathfrak{z}(\mathfrak{g}_k) = \mathfrak{z}((\mathfrak{g}_k)_{\bar{0}})$. Therefore

$$\mathfrak{z}(\mathfrak{g}) \;=\; \mathfrak{z}(\mathfrak{g}_1) \oplus \ldots \oplus \mathfrak{z}(\mathfrak{g}_n) \;=\; \mathfrak{z}(\mathfrak{g}_{\bar{0}}).$$

Suppose now that $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. Let us assume that $\mathfrak{g}$ contains a proper non-degenerate abelian graded ideal $I = I_{\bar{0}} \oplus I_{\bar{1}}$ of dimenion 2. Then $\dim I_{\bar{0}} = \dim I_{\bar{1}} = 1$ and $I_{\bar{1}} \subseteq \mathfrak{z}(I)$. As $\mathfrak{g} = I \oplus I^{\perp}$, we have

$$\mathfrak{z}(\mathfrak{g}) \;=\; \mathfrak{z}(I) \oplus \mathfrak{z}(I^{\perp}).$$

Consequently $\mathfrak{z}(\mathfrak{g}) \nsubseteq \mathfrak{g}_{\bar{0}}$, which contradicts the hypothesis. ∎

As we will show, it is remarkable that, for odd-quadratic Lie superalgebra with even part a reductive Lie algebra, the action of the even part in the odd part is completely reducible. Notice that in quadratic case the situation is very different. In fact, if $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ is a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra the action of $\mathfrak{g}_{\bar{0}}$ in $\mathfrak{g}_{\bar{1}}$ is not necessarily completely reducible (see examples in [1]).

Lemma 5.5. *Consider $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ an odd-quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Then $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module completely reducible.*

Proof: We consider $\rho : \mathfrak{g}_{\bar{0}} \longrightarrow Hom(\mathfrak{g}_{\bar{1}})$ the representation of $\mathfrak{g}_{\bar{0}}$ defined by $\rho(X) = [X, .] \mid_{\mathfrak{g}_{\bar{1}}}$, for all $X \in \mathfrak{g}_{\bar{0}}$. According to Proposition 5.2, it is clear that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) \subseteq \mathrm{Ker}\rho$. To show the other inclusion, we fix $X \in \mathrm{Ker}\rho$. By invariance of B we get $B([X, \mathfrak{g}_{\bar{0}}], \mathfrak{g}_{\bar{1}}) = \{0\}$. On the other hand, as B is odd it leads to $B([X, \mathfrak{g}_{\bar{0}}], \mathfrak{g}_{\bar{0}}) = \{0\}$. Since B is non-degenerate we infer that $X \in \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. Therefore we show that $\mathrm{Ker}\rho = \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. By assumption $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, so we ca have $\mathfrak{g}_{\bar{0}} = s \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}})$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. Consequently $\rho \mid_s : s \longrightarrow Hom(\mathfrak{g}_{\bar{1}})$ is also a representation of s. As s is semisimple, we invoke Weyl's Theorem to conclude that $\rho \mid_s$ is completely reducible, therefore $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module completely reducible. ∎

Now we will characterize the minimal graded ideals of the B-irreducible odd-quadratic Lie superalgebras such that the even part is a reductive Lie algebra.

Proposition 5.6. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a non-simple B-irreducible odd-quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Then a graded ideal I of $\mathfrak{g}$ is minimal if and only if $I \subseteq \mathfrak{z}(\mathfrak{g})$ and $\dim I = 1$ or I is a non trivial irreducible s-submodule of $\mathfrak{g}_{\bar{1}}$ such that $[\mathfrak{g}_{\bar{1}}, I] = \{0\}$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$.*

Proof: Assume that $I \subseteq \mathfrak{z}(\mathfrak{g})$ and $\dim I = 1$. It is obvious that I is a minimal graded ideal of $\mathfrak{g}$. Now we suppose that I is a non trivial irreducible s-submodule of $\mathfrak{g}_{\bar{1}}$. Let J be a graded ideal of $\mathfrak{g}$ such that $J \subseteq I$. It follows that $[s, J] \subseteq J$, and so J is a s-submodule of I. Since I is an irreducible s-submodule we have that $J = \{0\}$ or $J = I$, consequently I is minimal. Conversely, Let I be a minimal graded ideal of $\mathfrak{g}$. We know that I is isotropic, hence $[I, I] = \{0\}$. So the ideal I is abelian. As I is graded we can write $I = I_{\bar{0}} \oplus I_{\bar{1}}$, where $I_{\bar{0}} = I \cap \mathfrak{g}_{\bar{0}}$ and $I_{\bar{1}} = I \cap \mathfrak{g}_{\bar{1}}$, and so $I_{\bar{0}}$ is abelian. As we are supposing that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra, we have $\mathfrak{g}_{\bar{0}} = s \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}})$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. Therefore $I_{\bar{0}} = I \cap \mathfrak{z}(\mathfrak{g}_{\bar{0}})$, and so applying Proposition 5.2 we have that $I_{\bar{0}} \subseteq \mathfrak{z}(\mathfrak{g})$. We have to consider two cases. If $I_{\bar{0}} \neq \{0\}$. Then I is minimal and $I_{\bar{0}}$ is a non zero graded ideal of $\mathfrak{g}$ contained in I, we conclude that $I = I_{\bar{0}}$. Consequently $I \subseteq \mathfrak{z}(\mathfrak{g})$ and $\dim I = 1$. If $I_{\bar{0}} = \{0\}$, is clear that $I = I \cap \mathfrak{g}_{\bar{1}}$ and $[\mathfrak{g}_{\bar{1}}, I] = \{0\}$. On the other hand, using Proposition 5.2 we have $[\mathfrak{z}(\mathfrak{g}_{\bar{0}}), I] = \{0\}$. If $[s, I] = \{0\}$ we have $I \subseteq \mathfrak{z}(\mathfrak{g})$ and $\dim I = 1$. While if $[s, I] \neq \{0\}$, it is elementary to prove that $[s, I]$ is a graded ideal of $\mathfrak{g}$. Indeed, $[[s, I], \mathfrak{g}_{\bar{1}}] = \{0\}$, $[[s, I], s] \subseteq [s, I]$ and $[[s, I], \mathfrak{z}(\mathfrak{g}_{\bar{0}})] = \{0\}$. Then $[s, I]$ is a non zero graded ideal of $\mathfrak{g}$ contained in I. Using minimality of I we conclude that $[s, I] = I$. Now according to Lemma 5.5, $\mathfrak{g}_{\bar{1}}$ is a $\mathfrak{g}_{\bar{0}}$-module completely reducible. Since $I_{\bar{0}} = \{0\}$ and $[s, I] = I$ it follows that I is a s-submodule of $\mathfrak{g}_{\bar{1}}$. Finally, let us prove that I is irreducible. We assume that there exists I' a S-submodule of I. Hence $[s, I'] \subseteq I'$. In view of Proposition 5.2 we get $[\mathfrak{z}(\mathfrak{g}_{\bar{0}}), I'] = \{0\}$ and we also have $[\mathfrak{g}_{\bar{1}}, I'] = \{0\}$. Then I' is a graded ideal of $\mathfrak{g}$ contained in I. From minimality of I we have $I' = \{0\}$ or $I' = I$, which guarantees the irreducibility of I. ■

The next auxiliary lemma will help us to prove the following proposition.

Lemma 5.7. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a non-simple B-irreducible odd-quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra. Suppose that I is a non trivial irreducible $\mathfrak{g}_{\bar{0}}$-submodule of $\mathfrak{g}_{\bar{1}}$ such that $[\mathfrak{g}_{\bar{1}}, I] = \{0\}$. If we denote $s' = \mathfrak{g}_{\bar{0}} \cap I^{\perp}$ then $\mathfrak{g}_{\bar{0}} = s' \oplus s$, where s is a semisimple ideal of $\mathfrak{g}_{\bar{0}}$. Moreover, $[s, I] = I$ and* $\dim I = \dim s = \dim\, [s, \mathfrak{g}_{\bar{1}}]$.

Proof: Set $s' = \mathfrak{g}_{\bar{0}} \cap I^{\perp} = \{X \in \mathfrak{g}_{\bar{0}} : B(X, I) = \{0\}\}$. Since B is invariant and I a graded ideal of $\mathfrak{g}$ we infer that s' is a graded ideal of $\mathfrak{g}_{\bar{0}}$. Therefore $\mathfrak{g}_{\bar{0}} = s' \oplus s$, where s is a semisimple ideal of $\mathfrak{g}_{\bar{0}}$. Furthermore, since $\mathfrak{g}_{\bar{0}}$ is semisimple so is s. Consequently $I^{\perp} = s' \oplus \mathfrak{g}_{\bar{1}}$. To proceed the Proof, we define the linear map $\varphi : [s, \mathfrak{g}_{\bar{1}}] \longrightarrow s^*$ by $\varphi(X) = B(X, .)$, for all $X \in [s, \mathfrak{g}_{\bar{1}}]$. Let us show that φ is injective. As B is odd we have $B([s, \mathfrak{g}_{\bar{1}}], \mathfrak{g}_{\bar{1}}) = \{0\}$. From $[s, s'] = \{0\}$ and invariance of B we obtain $B([s, \mathfrak{g}_{\bar{1}}], s') = \{0\}$. Since B is non-degenerate we infer the injectivity of φ. Therefore $\dim\, [s, \mathfrak{g}_{\bar{1}}] \leq \dim s$. We also have that $[s, I] = I$. Indeed, since B is odd we get $B([s', I], \mathfrak{g}_{\bar{1}}) = \{0\}$. As s' is an ideal of $\mathfrak{g}_{\bar{0}}$ and by the invariance of B we obtain $B([s', I], \mathfrak{g}_{\bar{0}}) = \{0\}$. Due to the non degeneracy of B we infer that $[s', I] = \{0\}$. As I is irreducible and $[\mathfrak{g}_{\bar{0}}, I]$ is a non trivial $\mathfrak{g}_{\bar{0}}$-submodule of I then $[\mathfrak{g}_{\bar{0}}, I] = I$, and so $[s, I] = I$. Consequently $\dim I \leq \dim [s, \mathfrak{g}_{\bar{1}}]$. From $\mathfrak{g} = I^{\perp} \oplus s$ we get that $\dim s = \dim \mathfrak{g} - \dim I^{\perp} = \dim I$, hence $\dim I = \dim s = \dim\, [s, \mathfrak{g}_{\bar{1}}]$. ■

We will see that the B-irreducible odd-quadratic Lie superalgebras with reductive even part, that it is neither zero nor simple, with the centre zero, are precisely the trivial odd double extensions of a simple Lie algebra.

Proposition 5.8. *Consider* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *a* B*-irreducible odd-quadratic Lie superalgebra such that* $\mathfrak{g}_{\bar{0}}$ *is a reductive Lie algebra, that it is neither zero nor simple. Then the following assertions are equivalent:*

(i) $\mathfrak{z}(\mathfrak{g}) = \{0\}$;
(ii) $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \{0\}$;
(iii) $\mathfrak{g} = s \oplus P(s^*)$, *where* s *is a simple Lie algebra.*

Proof: According to Proposition 5.2 we have that (i) is equivalent to (ii). Now we suppose that (i) and (ii) are true. As $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \{0\}$ and we know that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra it follows that $\mathfrak{g}_{\bar{0}}$ is a semisimple Lie algebra. Applying Proposition 5.6 and due to $\mathfrak{z}(\mathfrak{g}) = \{0\}$ there exists a minimal graded ideal I of $\mathfrak{g}$ such that it is a non trivial irreducible $\mathfrak{g}_{\bar{0}}$-submodule of $\mathfrak{g}_{\bar{1}}$ and $[\mathfrak{g}_{\bar{1}}, I] = \{0\}$. Set $s' = \{X \in \mathfrak{g}_{\bar{0}} : B(X, I) = \{0\}\}$. By Lemma 5.7, $\mathfrak{g}_{\bar{0}} = s' \oplus s$, where s is a semi-simple ideal of $\mathfrak{g}_{\bar{0}}$. Invoking Weyl's Theorem we show that $\mathfrak{g}_{\bar{1}}$ is a s-module completely reducible. We know that $[s, I] = I$ then I is a s-submodule of $\mathfrak{g}_{\bar{1}}$. Therefore $\mathfrak{g}_{\bar{1}} = I \oplus M$, where M is a s-submodule of $\mathfrak{g}_{\bar{1}}$. We also have that $M = M^s \oplus [s, M]$, where $M^s = \{X \in M : [s, X] = \{0\}\}$. Using Lemma 5.7 and $[s, \mathfrak{g}_{\bar{1}}] = [s, I] \oplus [s, M]$ we obtain that $\dim\,[s, M] = \dim\,[s, \mathfrak{g}_{\bar{1}}] - \dim I = 0$. Therefore $\mathfrak{g}_{\bar{1}} = I \oplus M^s$, and so

$$\mathfrak{g} \;=\; \mathfrak{g}_{\bar{0}} \oplus I \oplus M^s \;=\; (s \oplus I) \oplus (s' \oplus M^s). \tag{5.17}$$

Our next aim is to prove that $s \oplus I$ is a non-degenerate graded ideal of $\mathfrak{g}$. As s is semisimple, $[s', I] = \{0\}$, and $[s, I] = I$ we obtain $[\mathfrak{g}_{\bar{0}}, s \oplus I] = s \oplus I$. On the other hand, $[\mathfrak{g}_{\bar{1}}, s \oplus I] = [I \oplus M^s, s \oplus I] \subseteq I$. Then $s \oplus I$ is a graded ideal of $\mathfrak{g}$. We also have that $s \oplus I$ is non-degenerate. In fact, since s is semisimple, $[s, I] = I$ and by invariance of B we obtain that $B(s \oplus I, s' \oplus M^s) = \{0\}$. Due to non degeneracy of B we conclude that $s \oplus I$ is non-degenerate. From (5.17) we get that $s \oplus I$ is a non trivial non-degenerate graded ideal of $\mathfrak{g}$. Since $\mathfrak{g}$ is B-irreducible we infer that $\mathfrak{g} = s \oplus I$. Now it is clear that $I = I^{\perp}$, and so I is a maximal ideal of $\mathfrak{g}$. Since $s \simeq \mathfrak{g}/I$ with I maximal ideal of $\mathfrak{g}$ then s is semisimple. Invoking Theorem 3.6 we conclude that $(\mathfrak{g}, B)$ is an odd double extension of $I/I^{\perp} = \{0\}$ by s, which means that $\mathfrak{g} = s \oplus P(s^*)$, where s is a simple Lie algebra as desired. ■

6. Inductive classification of odd-quadratic Lie superalgebras with reductive even part

We are now ready to formalize our must important result about odd-quadratic Lie superalgebras, namely an inductive classification of odd-quadratic Lie superalgebras such that the even part is a reductive Lie algebra. Let $\mathfrak{V}$ be the set consisting by $\{0\}$ and the simple Lie superalgebras $d(n)/\mathbb{K}I_{2n}$ $(n \geq 3)$.

Theorem 6.1. *Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be an odd-quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Then either $\mathfrak{g}$ is an element of $\mathfrak{V}$, and/or $\mathfrak{g}$ is obtained by a sequence of generalized odd double extensions by the one-dimensional Lie superalgebra, and/or by trivial odd double extensions of simple Lie algebra, and/or by orthogonal direct sums of odd-quadratic Lie superalgebras from a finite number of element of $\mathfrak{V}$.*

Proof: We proceed by induction on the even dimension of $\mathfrak{g}$. If $\dim \mathfrak{g} = 0$ then $\mathfrak{g} = \{0\} \in \mathfrak{U}$. If $\dim \mathfrak{g} = 2$ by Remark 2.9, either $\mathfrak{g}$ is the abelian two-dimensional odd-quadratic Lie superalgebra which is a generalized odd double extension of $\{0\}$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$ (by means of $(0,0)$) or the two-dimensional odd-quadratic Lie superalgebra $\mathfrak{N}$ which is a generalized odd double extension of $\{0\}$ by the one-dimensional Lie superalgebra $(\mathbb{K}e)_{\bar{1}}$. Suppose that the theorem is true for $\dim \mathfrak{g} < n$, with $n \geq 4$. We consider $\dim \mathfrak{g} = n$. We have to analyse two cases.

First case: Suppose that $\mathfrak{g}$ is B-irreducible. If $\mathfrak{z}(\mathfrak{g}) = \{0\}$ we apply Proposition 5.8 to infer that $\mathfrak{g}$ is a trivial odd double extension of simple Lie algebras, so an element of $\mathfrak{V}$. If $\mathfrak{z}(\mathfrak{g}) \neq \{0\}$, by Corollary 5.3 we infer that $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{g}_{\bar{0}} \neq \{0\}$. Applying Proposition 4.3 we get that $\mathfrak{g}$ is a generalized odd double extension of an odd-quadratic Lie superalgebra $\mathfrak{h}$ by the one-dimensional Lie superalgebra. In this case, $\dim \mathfrak{h} = \dim \mathfrak{g} - 2 < n$, and applying the induction hypothesis to $\mathfrak{h}$ we infer the theorem for $\mathfrak{g}$.

Second case: Now we assume that $\mathfrak{g}$ is not B-irreducible. In view of Proposition 2.12, $\mathfrak{g} = \bigoplus_{k=1}^{m} \mathfrak{g}_k$, where $\{\mathfrak{g}_k | 1 \leq k \leq m\}$ is a set of B-irreducible graded ideals of $\mathfrak{g}$ such that $B(\mathfrak{g}_k, \mathfrak{g}_{k'}) = \{0\}$, for all $k, k' \in \{1, \ldots, m\}$ and $k \neq k'$, and $(\mathfrak{g}_k)_{\bar{0}}$ is a reductive Lie algebra, whenever $k \in \{1, \ldots, m\}$. We apply the result to $\mathfrak{g}_k$, for all $k \in \{1, \ldots, m\}$. ■

7. Incompatibility of quadratic and odd-quadratic structures

It is fairly known the following fact for classical Lie superalgebras (we recall that by a classical Lie superalgebra we mean a simple Lie superalgebra such that the even part is a reductive Lie algebra)[11]

Proposition 7.1. *The invariant bilinear forms on a classical Lie superalgebra are either all even or else all odd.*

Our next goal is the generalization of this result, in the sense that our Lie superalgebras are not necessarily simple. First, we will prove the auxiliary lemma.

Lemma 7.2. *Let $\mathfrak{h}$ be a Lie algebra not abelian. Then the trivial odd double extension of $\mathfrak{h}$ is not quadratic.*

Proof: Set the Lie superalgebra $\mathfrak{g} = \mathfrak{h} \oplus P(\mathfrak{h}^*)$. Since $\mathfrak{h}$ is a Lie algebra then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \mathfrak{h}$ and $\mathfrak{g}_{\bar{1}} = P(\mathfrak{h}^*)$. As $\mathfrak{g}$ is the trivial odd double extension of $\mathfrak{h}$ then $P(\mathfrak{h}^*)$ is an ideal of $\mathfrak{g}$ such that $[P(\mathfrak{h}^*), P(\mathfrak{h}^*)] = \{0\}$. In terms of the odd part of $\mathfrak{g}$, this means that $\mathfrak{g}_{\bar{1}}$ is an ideal of $\mathfrak{g}$ such that $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}] = \{0\}$. Let us suppose that there exists a bilinear form B on $\mathfrak{g}$ that is an invariant scalar product on $\mathfrak{g}$. Since B is even and non-degenerate then B restricted to $\mathfrak{g}_{\bar{1}} \times \mathfrak{g}_{\bar{1}}$ is non-degenerate and so $\mathfrak{g}_{\bar{1}}$ is a B non-degenerate ideal of $\mathfrak{g}$. It follows that $\mathfrak{g} = \mathfrak{g}_{\bar{1}} \oplus \mathfrak{g}_{\bar{1}}^{\perp}$, where $\mathfrak{g}_{\bar{1}}^{\perp}$ is the orthogonal of $\mathfrak{g}_{\bar{1}}$ with respect to B. Since B is even then $\mathfrak{g}_{\bar{0}} \subseteq \mathfrak{g}_{\bar{1}}^{\perp}$, therefore $\mathfrak{g}_{\bar{0}} = \mathfrak{g}_{\bar{1}}^{\perp}$ and so $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}] = \{0\}$. Which means that

$$[X, f] \;=\; 0, \qquad \forall_{X\in\mathfrak{h}, f\in(P(\mathfrak{h}^*))}.$$

Using the expression of multiplication on the trivial odd double extension of $\mathfrak{h}$, we obtain

$$f([X, Y]) \;=\; 0, \qquad \forall_{X\in\mathfrak{h}, Y\in\mathfrak{h}, f\in(P(\mathfrak{h}^*))},$$

and so

$$[X, Y] \;=\; 0, \qquad \forall_{X\in\mathfrak{h}_x, Y\in\mathfrak{h}}.$$

So $\mathfrak{h}$ is abelian, which is not true by assumption. Therefore the Lie superalgebra $\mathfrak{g}$ is not quadratic as required. ∎

We will prove that, with certain conditions, a quadratic Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ with reductive Lie algebra $\mathfrak{g}_{\bar{0}}$ does not admit an odd-invariant scalar product, and vice-versa.

Proposition 7.3. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ *be a quadratic (respectively, odd-quadratic) Lie superalgebra such that* $\mathfrak{g}_{\bar{0}}$ *is a reductive Lie algebra. If* $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ *(i.e.* $\mathfrak{g}$ *is perfect) then there is not any odd-invariant (respectively, invariant) scalar product on* $\mathfrak{g}$*.*

Proof: Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B)$ be a quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Let us assume that $(\mathfrak{g}, B')$ is an odd-quadratic Lie superalgebra. By Proposition 2.12, $\mathfrak{g} = \bigoplus_{k=1}^{n} \mathfrak{g}_k$, where $\mathfrak{g}_k$ is a B'-irreducible B'-non-degenerate graded ideal of $\mathfrak{g}$, for all $k \in \{1, \ldots, n\}$. Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra then $(\mathfrak{g}_k)_{\bar{0}}$ is a reductive Lie algebra, and so $(\mathfrak{g}_k)_{\bar{0}} = s_k \oplus \mathfrak{z}((\mathfrak{g}_k)_{\bar{0}})$, where s_k is the greatest semisimple ideal of $(\mathfrak{g}_k)_{\bar{0}}$. As $[\mathfrak{g}_k, \mathfrak{g}_k] = \mathfrak{g}_k$ and $[\mathfrak{g}_k, \mathfrak{g}'_k] = \{0\}$ if $k \neq k'$ then $B(\mathfrak{g}_k, \mathfrak{g}'_k) = \{0\}$, for all distinct elements $k, k' \in \{1, \ldots, n\}$. Consequently,if $k \in \{1, \ldots, n\}$ then $\mathfrak{g}_k$ is B-non-degenerate. On the other hand, if $\mathfrak{g}_k$ is simple, since $(\mathfrak{g}_k)_{\bar{0}}$ is a reductive Lie algebra and $\mathfrak{g}_k$ is B'-non-degenerate, then $\mathfrak{g}_k$ is not quadratic, which is false. Further, if $\mathfrak{g}_k$ is not simple, since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then $\mathfrak{z}(\mathfrak{g}) = \{0\}$. From

$$\mathfrak{z}(\mathfrak{g}) \;=\; \mathfrak{z}(\mathfrak{g}_1) \oplus \ldots \oplus \mathfrak{z}(\mathfrak{g}_n)$$

we infer that $\mathfrak{z}(\mathfrak{g}_k) = \{0\}$. By Proposition 5.8, since $\mathfrak{g}_k$ is B'-irreducible, $(\mathfrak{g}_k)_{\bar{0}}$ is a reductive Lie algebra, and $\mathfrak{z}(\mathfrak{g}_k) = \{0\}$, then $\mathfrak{g}_k = s_k \oplus (P(s_k^*))$, where s_k is a simple Lie algebra. So s_k is non-abelian. Applying Lemma 7.2, $\mathfrak{g}_k$ is not quadratic, which is false. Moreover, if $(\mathfrak{g}, B)$ is an odd-quadratic Lie superalgebra, in a similar way we prove that there is not any invariant scalar product on $\mathfrak{g}$. ■

Proposition 7.4. *Let* $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B')$ *be an odd-quadratic (respectively, quadratic) Lie superalgebra such that* $\mathfrak{g}_{\bar{0}}$ *is a reductive Lie algebra. If* $\mathfrak{g}$ *does not contain any proper non-degenerate abelian graded ideal of dimension* 2 *then there is not any invariant (respectively, odd-invariant) scalar product on* $\mathfrak{g}$*.*

Proof: Let $(\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}, B')$ be an odd-quadratic Lie superalgebra such that $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra. Let us assume that $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra. Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra then $\mathfrak{g}_{\bar{0}} = s \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}})$, where s is the greatest semisimple ideal of $\mathfrak{g}_{\bar{0}}$. As $\mathfrak{z}(\mathfrak{g}_{\bar{0}})$ is a B-non-degenerate ideal of $\mathfrak{g}$ then

$$\mathfrak{g} \;=\; \Big[\mathfrak{z}(\mathfrak{g}_{\bar{0}})\Big]^{\perp B} \oplus \mathfrak{z}(\mathfrak{g}_{\bar{0}}).$$

Denote $J = \Big[\mathfrak{z}(\mathfrak{g}_{\bar{0}})\Big]^{\perp B} = s \oplus \mathfrak{g}_{\bar{1}}$. Applying Corollary 5.4 we infer that $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_{\bar{0}})$. Then $[J, \mathfrak{z}(\mathfrak{g}_{\bar{0}})] = \{0\}$ and $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ imply that $\mathfrak{z}(J) = \{0\}$. Therefore $[J, J] = J$, because J is B-non-degenerate ideal of $\mathfrak{g}$. From $\mathfrak{z}(\mathfrak{g}) = \mathfrak{z}(\mathfrak{g}_{\bar{0}})$ and $[J, J] = J$ it comes that $B'(J, \mathfrak{z}(\mathfrak{g}_{\bar{0}})) = \{0\}$. Since B' is odd then $B'(\mathfrak{z}(\mathfrak{g}_{\bar{0}}), \mathfrak{z}(\mathfrak{g}_{\bar{0}})) = \{0\}$, therefore $B'(\mathfrak{g}, \mathfrak{z}(\mathfrak{g}_{\bar{0}})) = \{0\}$. Now, by B'-non-degeneracy we infer that $\mathfrak{z}(\mathfrak{g}_{\bar{0}}) = \{0\}$, in conclusion $\mathfrak{z}(\mathfrak{g}) = \{0\}$. By Proposition 2.12, $\mathfrak{g} = \bigoplus_{k=1}^{n} \mathfrak{g}_k$, where $\mathfrak{g}_k$ is a B'-irreducible B'-non-degenerate graded ideal of $\mathfrak{g}$, for all $k \in \{1, \ldots, n\}$. Since $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra then $(\mathfrak{g}_k)_{\bar{0}}$ is a reductive Lie algebra, for all $k \in \{1, \ldots, n\}$. Moreover, from $\mathfrak{z}(\mathfrak{g}) = \{0\}$ and

$$\mathfrak{z}(\mathfrak{g}) \;=\; \mathfrak{z}(\mathfrak{g}_1) \oplus \ldots \oplus \mathfrak{z}(\mathfrak{g}_n)$$

we infer that $\mathfrak{z}(\mathfrak{g}_k) = \{0\}$, and so $[\mathfrak{g}_k, \mathfrak{g}_k] = \mathfrak{g}_k$, for all $k \in \{1, \ldots, n\}$. Consequently, $B(\mathfrak{g}_k, \mathfrak{g}_{k'}) = \{0\}$, for all distinct elements $k, k' \in \{1, \ldots, n\}$, and so $B\mid_{\mathfrak{g}_k \times \mathfrak{g}_k}$ is non-degenerate, for $k \in \{1, \ldots, n\}$. This contradicts, by Proposition 7.3, the fact that $B'\mid_{\mathfrak{g}_k \times \mathfrak{g}_k}$ is non-degenerate, for $k \in \{1, \ldots, n\}$. Finally, if $(\mathfrak{g}, B)$ is a quadratic Lie superalgebra, similarly we prove that there is not any odd-invariant scalar product on $\mathfrak{g}$, and the Proof is complete. ■

Remark 7.5. Notice that there are Lie superalgebras with reductive even part that admit simultaneously a quadratic and an odd-quadratic structures. For example, let us consider the abelian Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where $\mathfrak{g}_{\bar{0}} = \langle X, Z \rangle$ and $\mathfrak{g}_{\bar{1}} = \langle Y, T \rangle$. Then the supersymmetric bilinear form $B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{K}$ such that the non-zeros values are $B(X, Z) = B(Y, T) = 1$, is an invariant scalar product on $\mathfrak{g}$. On the other hand, the symmetric bilinear form $B' : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{K}$ such that the non-zeros values are $B'(X, Y) = B'(Z, T) = 1$, is an odd-invariant scalar product on $\mathfrak{g}$. So, to generalize the Proposition 7.1 for Lie superalgebras with reductive Lie algebra not necessarily simple, we have to impose always certain conditions to the superalgebra.

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