Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–14

ON COUNTABLE CHOICE AND SEQUENTIAL SPACES

GONÇALO GUTIERRES

Abstract: Under the axiom of choice, every first countable space is a Fréchet-Urysohn space. Although, in its absence even $\mathbb{R}$ may fail to be a sequential space.

Our goal in this paper is to discuss under which set-theoretic conditions some topological classes, such as the first countable spaces, the metric spaces or the subspaces of $\mathbb{R}$, are classes of Fréchet-Urysohn or sequential spaces.

In this context, it is seen that there are metric spaces which are not sequential spaces. This fact arises the question of knowing if the completion of a metric space exists and it is unique. The answer depends on the definition of completion.

Among other results it is shown that: every first countable space is a sequential space if and only if the axiom of countable choice holds; the sequential closure is idempotent in $\mathbb{R}$ if and only if the axiom of countable choice holds for families of subsets of $\mathbb{R}$; every metric space has a unique $\hat{\sigma}$-completion.

Keywords: axiom of (countable) choice, Fréchet-Urysohn space, sequential space, first countable space, completion of metric spaces.
AMS Subject Classification (2000): 03E25, 54D55, 54E50.

1. Introduction

It is well-known that every topological space is characterized by the limits of its filters (or nets). It is also of general knowledge that for a first countable space filters can be replaced by sequences. Although the results with filters are provable in the absence of the axiom of choice, the correspondent results with sequences heavily rely on the axiom of countable choice.

Definitions 1.1. Let A be a subspace of the topological space X. The *sequential closure* of A in X is the set:

$$\sigma_X(A) := \{x \in X : (\exists (x_n) \in A^{\mathbb{N}})[(x_n) \text{ converges to } x]\}.$$

A is *sequentially closed* in X if $\sigma_X(A) = A$.

With the some kind of notation, the *(usual) Kuratowski closure* of A in X is denoted by $k_X(A)$.

Received April 13, 2007.
The author acknowledges partial financial assistance by Centro de Matematica da Universidade de Coimbra/FCT.

Definitions 1.2. A topological space X is:

(a) a *sequential space* if, for all $A \subseteq X$, $\sigma_X(A) = A$ if and only if $k_X(A) = A$;
(b) a *Fréchet-Urysohn* space if, for all $A \subseteq X$, $k_X(A) = \sigma_X(A)$.

Note that a topological space X is sequential if and only if, for all $A \subseteq X$, $k_X(A) = \hat{\sigma}_X(A)$ where $\hat{\sigma}$ is the idempotent hull of σ, i.e. $\hat{\sigma}_X(A) := \bigcap\{B : A \subseteq B \subseteq X \text{ and } \sigma_X(B) = B\}$. Immediately we have that, for all $A \subseteq X$, $\sigma_X(A) \subseteq \hat{\sigma}_X(A) \subseteq k_X(A)$.

Proposition 1.3. *A topological space X is a Fréchet-Urysohn space if and only if it is sequential and the sequential closure σ_X is idempotent.*

This Proposition just say that $k_X = \sigma_X$ if and only if $k_X = \hat{\sigma}_X$ and $\hat{\sigma}_X = \sigma_X$.

The usual topology in $\mathbb{R}$ is not necessary sequential (see [5]), which implies that the theorem *every first countable space is a Fréchet-Urysohn space* is not a theorem of ZF (*Zermelo-Fraenkel set theory without the axiom of choice*). Between $\mathbb{R}$ and the class of first countable spaces, there are other classes for which is interesting to study under which conditions each of them is contained in the class of the Fréchet-Urysohn spaces, and in the class of the sequential spaces. Among those classes, we will consider the following ones:

(a) the topological space $\mathbb{R}$,
(b) subspaces of $\mathbb{R}$,
(c) second countable T_0–spaces,
(d) second countable spaces,
(e) metric spaces,
(f) first countable spaces,

Definition 1.4. A metric space X is *complete* if every Cauchy sequence in X converges.

Proposition 1.5. *Let X be a complete metric space. A subspace A of X is complete if and only if $\sigma_X(A) = A$.*

Corollary 1.6. *If A is a closed subspace of a complete metric space then it is complete.*

In the Corollary 2.2, we will see that, when the axiom of choice fails, the reverse implication of the Corollary 1.6 may not be true. Motivated by this, in the last section we will focus our attention in the uniqueness, and existence, of metric completions. The main problem is, in fact, to define what is a metric completion in a choice free context.

Lemma 1.7. *If X is a second countable T_0–space, then $|X| \leq |\mathbb{R}| = 2^{\aleph_0}$.*

Now, we introduce the definition of the two choice principles which will be used in this paper. Their definitions, as everything else, take place in the setting of ZF.

Definition 1.8. The *axiom of countable choice* (**CC**) states that every countable family of non-empty sets has a choice function.

Definition 1.9. **CC**($\mathbb{R}$) is the axiom of countable choice restricted to families of sets of real numbers.

Proposition 1.10. ([3, p.76], [6]) *Equivalent are:*

(i) **CC** *(respectively* **CC**($\mathbb{R}$)*);*
(ii) *every countable family of non-empty sets (resp. subsets of $\mathbb{R}$) has an infinite subfamily with a choice function;*
(iii) *for every countable family $\{X_n : n \in \mathbb{N}\}$ of non-empty sets (resp. subsets of $\mathbb{R}$), there is a sequence which meets infinitely many of the sets X_n.*

2. Sequential spaces

In the context of *choice free topology* there are some results about Fréchet-Urysohn spaces (see [2] and [6]), but very little is known about sequential spaces. In this paper, we will try to narrow this gap.

In view of Proposition 1.3, it is also interesting to see when the sequential closure is idempotent, in each of the classes (a)-(f) above. In particular, we will see when is $\sigma_{\mathbb{R}}$ idempotent. Let us point out that the idempotency of $\sigma_{\mathbb{R}}$ played an important role in the study of the sequential compactness for subspaces of $\mathbb{R}$ in ZF made by W. Felschner [2, p.128] (see also [5]).

Theorem 2.1. *Every metric space is a sequential space if and only if the axiom of countable choice holds.*

Proof: ($\Leftarrow$) In the usual proof, the only choice principle used to prove that every first countable space is Fréchet-Urysohn is the axiom of countable choice. See, for instance, [1, 1.6.14].

($\Rightarrow$) Let $\{X_n : n \in \mathbb{N}\}$ be a countable family of non-empty sets. By Proposition 1.10, it is enough to prove that there is a sequence which meets infinitely many of the X_n's.

We define $Y := \bigcup_n (X_n \times \{n\}) \cup \{(*, \infty)\}$ and we consider $\frac{1}{\infty} = 0$. The function $d : Y \times Y \longrightarrow \mathbb{R}$ defined by

$$d((x,n),(y,m)) := \begin{cases} 0 & \text{if } (x,n) = (y,m) \\ \frac{1}{n} + \frac{1}{m} & \text{if } (x,n) \neq (y,m), \end{cases}$$

is a metric on Y. By hypothesis Y is a sequential space, and $X := Y \backslash \{(*, \infty)\}$ is not closed, then X is also not sequentially closed, what means that there is a sequence in X converging to $(*, \infty)$. Clearly, this fact completes the proof. ■

Corollary 2.2. *The following conditions are equivalent to* **CC***:*

(i) *every complete metric space is a sequential space;*
(ii) *every complete subspace of a metric space is closed.*

Corollary 2.3. *The following conditions are equivalent to* **CC***:*

(i) *every metric space is a Fréchet-Urysohn space;*
(ii) *every first countable space is a Fréchet-Urysohn space;*
(iii) *every first countable space is a sequential space.*

Remark. The equivalence between **CC** and the condition (i) is in Proposition 5 of [6]. The proof of Theorem 2.1 is based on the proof of that proposition.

Theorem 2.4. *Every second countable space is a sequential space if and only if the axiom of countable choice holds.*

Proof: One direction is clear. To check the other one, we consider $\{X_n : n \in \mathbb{N}\}$ and Y as in the proof of Theorem 2.1, and one defines a topology on Y by giving a countable base

$$\mathcal{B} = \{X_n \times \{n\} : n \in \mathbb{N}\} \cup \{\bigcup_{k=n}^{\infty} (X_k \times \{k\}) \cup \{(*, \infty)\} : n \in \mathbb{N}\}.$$

Once again $Y \setminus \{(*, \infty)\}$ is not closed. The proof proceeds as in the proof of Theorem 2.1. ■

Corollary 2.5. *Every second countable space is a Fréchet-Urysohn space if and only if the axiom of countable choice holds.*

The results above show the equivalence between some results concerning Fréchet-Urysohn and sequential spaces, and the axiom of countable choice. Looking at Proposition 1.3, it is not surprising that there are also similar results for the class of spaces with idempotent sequential closure. However, the proofs seem not so easy as before.

Theorem 2.6. *The sequential closure is idempotent in every metric space if and only if the axiom of countable choice holds.*

Proof: Since, in every Fréchet-Urysohn space, the sequential closure is idempotent, one implication is clear.

Let $\{X_n : n \in \mathbb{N}\}$ be a countable family of non-empty sets, and we take $*, \infty \notin \bigcup_n X_n$ and $\frac{1}{\infty} = 0$. We define also

$$Y := \bigcup_n [(X_n \times \mathbb{N} \times \{n\}) \cup \{(*, \infty, n)\}] \cup \{(*, \infty, \infty)\}.$$

We put a metric on Y in the following way,

$$d((x,m,n),(x',m',n')) := \begin{cases} 0 & \text{if } (x,m,n) = (x',m',n') \\ \frac{1}{m} + \frac{1}{m'} + |\frac{1}{n} - \frac{1}{n'}| & \text{if } (x,m,n) \neq (x',m',n'). \end{cases}$$

Since for a fixed $x \in X_n$, the sequence $((x,k,n))_{k\in\mathbb{N}}$ converges to $(*,\infty,n)$, the point $(*,\infty,n)$ is in $\sigma_Y(X_n \times \mathbb{N} \times \{n\})$. It is now clear that for $X := \bigcup_n X_n \times \mathbb{N} \times \{n\}$, $\{(*,\infty,n) : n \in \mathbb{N}\} \subseteq \sigma_Y(X)$. It is also true that the sequence $((*,\infty,n))_n$ converges to $(*,\infty,\infty)$, and then $(*,\infty,\infty) \in \sigma_Y(\sigma_Y(X))$.

Finally, by hypothesis $\sigma_Y^2 = \sigma_Y$, which implies that there is a sequence in X converging to $(*,\infty,\infty)$. This easily implies that there is a sequence which meets infinitely many of the sets X_n, which together with Proposition 1.10 completes the proof. ∎

Corollary 2.7. *The sequential closure is idempotent in every first countable space if and only if the axiom of countable choice holds.*

Theorem 2.8. *The sequential closure is idempotent in every second countable space if and only if the axiom of countable choice holds.*

Proof: $(\Leftarrow)$ Clear.

($\Rightarrow$) Let $\{X_n : n \in \mathbb{N}\}$ be a countable family of non-empty sets, and $Y_n := X_n \cup \{*\}$, $Y := \bigcup_n (Y_n \times \{n\}) \cup \{(*, \infty)\}$ with $*, \infty \notin \bigcup_n X_n$. The set

$$\mathcal{B} = \{Y_n \times \{n\} : n \in \mathbb{N}\} \cup \{\bigcup_{k=n}^{\infty} (Y_k \times \{k\}) \cup \{(*, \infty)\} : n \in \mathbb{N}\}$$

is a countable base for a topology on Y.

For a fixed x in X_n, the element $(*, n)$ is a limit of the constant sequence $((x, n))_k$, and $((*, n))_n$ converges to $(*, \infty)$. This means that, for $X := \bigcup_n X_n \times \{n\}$, $(*, \infty) \in \sigma_Y^2(X)$. The proof is concluded as the proof of Theorem 2.6. ■

In ZFC, the sequential closure σ_X is idempotent if and only if the following property is satisfied in X (see, e.g., [1, p.64]):

> ($\star$) if $(x_n)_{n\in\mathbb{N}}$ converges to x and, for each $n \in \mathbb{N}$, $(x_m^n)_{m\in\mathbb{N}}$ converges to x_n, then there is a sequence in $\{x_m^n : m, n \in \mathbb{N}\}$ converging to x.

If the axiom of countable choice fails, the topological spaces Y in the proofs of the theorems 2.8 and 2.6 have the property ($\star$), but σ_Y are not idempotent.

Proposition 2.9. *For every topological space X, the idempotency of σ_X is equivalent to ($\star$) if and only if the axiom of countable choice holds.*

Proof: ($\Leftarrow$) It is a result of ZF that if σ_X is idempotent, then ($\star$) is satisfied in X.

Suppose now that ($\star$) is satisfied in X and that $x \in \sigma_X^2(A)$. There is a sequence $(x_n)_n$ in $\sigma_X(A)$ which converges to x and, for every $n \in \mathbb{N}$, there is a sequence in X converging to x_n. By the axiom of countable choice, one can choose a sequence of sequences $(x_m^n)_{m,n}$ in X such that, for each $n \in \mathbb{N}$, $(x_m^n)_m$ converges to x_n. Then, by ($\star$), there a sequence in $\{x_m^n : m, n \in \mathbb{N}\} \subseteq X$ which converges to x.

($\Rightarrow$) Suppose that **CC** fails. By Proposition 1.10, there is a countable family of non-empty sets $\{X_n : n \in \mathbb{N}\}$ such that there is no sequence which meets infinitely many of the sets X_n. Define the topological space Y as in the proof of the Theorem 2.8. The space Y satisfies ($\star$), since there is no sequence of sequences with values in more than a finite number of the sets X_n. But $\sigma_Y^2 \neq \sigma_Y$ as it was seen in the proof of 2.8. ■

It is already known that $\mathbb{R}$ is a Frećhet-Urysohn space if and only if $\mathbf{CC}(\mathbb{R})$ holds (see [2, p.124] and [7]).

The question *When is* $\mathbb{R}$ *a sequential space?* has no definitive answer. However, we do know that there are models of ZF where $\mathbb{R}$ is not sequential (e.g., basic Cohen model – $\mathcal{M}1$ in [8]) and that $\mathbb{R}$ might be sequential, but fails to be Fréchet-Urysohn (Feferman/Levy model – $\mathcal{M}9$ in [8]). For details see [5].

Theorem 2.10. *The sequential closure is idempotent in* $\mathbb{R}$ *if and only if the axiom of countable choice holds for families of sets of reals.*

Proof: It remains only to prove that the idempotency of $\sigma_{\mathbb{R}}$ implies $\mathbf{CC}(\mathbb{R})$.

Let $\{X_n : n \in \mathbb{N}\}$ be a countable family of non-empty subsets of $\mathbb{R}$, and define $Y_n := \{x + p : x \in X_n,\ p \in \mathbb{N}\}$. For every n in $\mathbb{N}$, one can define an increasing bijection ϕ_n from $\mathbb{R}$ to $(\frac{1}{n+1}, \frac{1}{n})$ and $Z_n := \phi_n(Y_n)$.

By the way how Y_n and ϕ_n were defined, in each Z_n there is a sequence converging to $\frac{1}{n}$. This implies that for $Z := \bigcup_n Z_n$, $\{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \sigma_{\mathbb{R}}(Z)$ and then $0 \in \sigma^2_{\mathbb{R}}(Z)$. Since $\sigma_{\mathbb{R}}$ is idempotent, $0 \in \sigma_{\mathbb{R}}(Z)$, i.e. there is a sequence $(a_k)_k$ in Z converging to 0.

The sets Z_n are disjoint, hence for all $k \in \mathbb{N}$ there is a unique $n(k) \in \mathbb{N}$ such that $a_k \in Z_{n(k)}$. It is now possible to define $b_k := \phi^{-1}_{n(k)}(a_k)$ and $B_k := \{x \in X_{n(k)} : (\exists p \in \mathbb{N})\, x + p = b_k\}$. If x_1, x_2 are two elements of B_k, then $|x_1 - x_2| \in \mathbb{N}$. The set B_k is non-empty, it is discrete and it has an upper bound b_k, hence it has also a maximum c_k.

The fact that $\{c_k : k \in \mathbb{N}\}$ intersects infinitely many of the sets X_n is a consequence from the convergence of $(a_k)_k$ to 0. The version of the Proposition 1.10 restricted to the real numbers finish the proof. ∎

The proof that a first countable space X is Fréchet-Urysohn only uses the axiom of countable choice for subsets of X. This fact together with Lemma 1.7 makes the next corollary clear.

Corollary 2.11. *The following conditions are equivalent to* $\mathbf{CC}(\mathbb{R})$*:*

(i) *every second countable* T_0*–space is a Fréchet-Urysohn space;*
(ii) *the sequential closure is idempotent in every second countable* T_0*–space;*
(iii) *every subspace of* $\mathbb{R}$ *is a Fréchet-Urysohn space;*
(iv) *the sequential closure is idempotent in every subspace of* $\mathbb{R}$;
(v) $\mathbb{R}$ *is a Fréchet-Urysohn space.*

Theorem 2.12. *Every subspace of* $\mathbb{R}$ *is sequential if and only if the axiom of countable choice holds for families of sets of reals.*

Proof: ($\Rightarrow$) Let $\{X_n : n \in \mathbb{N}\}$ be a countable family of non-empty subsets of $\mathbb{R}$. Without loss of generality, we consider $X_n \subseteq (\frac{1}{n+1}, \frac{1}{n})$ and we define $X := \bigcup_n X_n \cup \{0\}$. The set $X \setminus \{0\}$ is not closed in X, and then also not sequentially closed in X by hypothesis. That means that 0 is the limit of a sequence in $X \setminus \{0\}$. Such a sequence meets infinitely many of the sets X_n. ∎

Corollary 2.13. *Every second countable T_0–space is a sequential space if and only if* $\mathbf{CC}(\mathbb{R})$ *holds.*

3. Completions

The results presented in this section are motivated by the Corollary 2.2. That corollary says that *every complete metric space is a sequential space if and only if the axiom of countable choice holds.* In other words, in the absence of the axiom of countable choice there is a sequentially closed not closed subspace of a complete metric space. Such a space has two non-isometric completions, because the usual definition of completion says that Y is a completion of X if it is complete and X is dense in Y.

Also the existence of metric completion might be problematic in ZF. For instance, one of the most usual proofs for the existence of a completion of a metric space is done using Cauchy sequences. To show that the metric space constructed in that way is complete, one needs the axiom of countable choice.

Let (X, d) be a metric space and $\mathcal{H}$ the set of all Cauchy sequences in X. Define a metric d' in $\mathcal{H}$ with $d((x_n)_n, (y_n)_n) = \lim\limits_n d(x_n, y_n)$. The pseudometric space $(\mathcal{H}, d')$ is complete in ZF. But in order to prove that its metric reflection is complete, and then the completion of (X, d), it is necessary to use the axiom of countable choice. In fact, we have this more general result.

Proposition 3.1. *The following conditions are equivalent:*

(i) *the axiom of countable choice;*
(ii) *a pseudometric space is complete if and only if its metric reflection is complete.*

Proof: (i)$\Rightarrow$(ii) Let (X, d) be a pseudometric space. We define an equivalence relation in X by saying that x is equivalent to y if $d(x, y) = 0$ and the metric reflection of X is $X' = \{[x] : x \in X\}$. If (X', d) is complete then it is choice free to prove that (X, d) is complete.

Suppose now that (X, d) is complete and $(x_n)_n$ is a Cauchy sequence in X'. By the axiom of countable choice there is $(y_n)_n$ such that $y_n \in x_n$ for every

$n \in \mathbb{N}$. Since X is complete, there is $y \in X$ which is a limit of $(y_n)_n$. The sequence $(x_n)_n$ converge to $[y]$.

(ii)⇒(i) Let $\{X_n : n \in \mathbb{N}\}$ be a countable family of non-empty sets. We define $X := \bigcup_n (X_n \times \mathbb{N} \times \{n\})$ and the metric d in X with $d((x,n),(y,m)) = |\frac{1}{n} - \frac{1}{m}|$ The metric reflection (X', d) is isometric to the subspace of $\mathbb{R}$, $\{\frac{1}{n} : n \in \mathbb{N}\}$, and then it is not complete. By (ii), X is also not complete which means that there is a sequence which meets infinitely many of the sets X_n. This, together with Proposition 1.10, completes the proof. ■

One should remark that it is not immediate that an iteration of the process of building a complete metric space, with Cauchy sequences, gives a complete space. In fact, M. Gitik [4] constructed a model where all the limit ordinals are the limit of a sequence of smaller ordinals. This means that constructions of this kind might be done indefinitely (see also [8, Forme 182]).

One can conclude that the uniqueness of the completion is a consequence from the equality $\hat{\sigma} = k$, and, at least, one construction of completion is a consequence of the equality $\sigma = \hat{\sigma}$. This is exactly the reason why in ZFC the completion exists and it is unique, since the metric spaces are Fréchet-Urysohn spaces, that is $\sigma = k$.

At this point, we introduce three definitions of completion. The completions should be considered up to a isometry.

Definitions 3.2. Let X be a complete metric space and A one of its subspaces. One says that X is a:

(a) σ-completion of A if $\sigma_X(A) = X$;
(b) $\hat{\sigma}$-completion of A if $\hat{\sigma}_X(A) = X$;
(c) k-completion of A if $k_X(A) = X$.

The definition of k-completion is the usual.

A σ-completion exists only when the construction that was done with Cauchy sequences produces a complete space.

Since the sequentially closed subspaces of a complete metric space X are exactly its complete subspaces, if $\hat{\sigma}_X(A) = X$, then there is no complete space between A and X. One concludes that a $\hat{\sigma}$-completion is minimal.

Proposition 3.3. *Every metric space has a $\hat{\sigma}$-completion.*

Proof: Every metric space X can be seen, up to isometry, as a subspace of the space of the bounded real functions in X and the suprem metric, $\mathcal{B}(X, \mathbb{R})$

(for details see, e.g., [1]). This metric space is complete and then $\hat{\sigma}_{\mathcal{B}(X,\mathbb{R})}(X)$ is a $\hat{\sigma}$-completion of X. ■

Corollary 3.4. *Every metric space has a k-completion.*

To prove the uniqueness of the $\hat{\sigma}$-completion, we use an iteration process as it is suggested by the construction with Cauchy sequences.

Definition 3.5. Let A be a subspace of a topological space X. For any ordinal α, define:

$$\sigma_X^\alpha(A) := \begin{cases} \sigma_X(A) & \text{if } \alpha = 1 \\ \sigma_X(\sigma_X^\beta(A)) & \text{if } \alpha = \beta + 1 \\ \bigcup\{\sigma_X^\beta(A) : \beta < \alpha\} & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

Proposition 3.6. *Let A be a subspace of a topological space X.*

$$\hat{\sigma}_X(A) = \sigma_X^\alpha(A), \textit{ for } \alpha := \min\{\beta : \sigma_X^{\beta+1}(A) = \sigma_X^\beta(A)\}.$$

Proof: Since X is a set, it is clear that for some ordinal α, $\sigma_X^\alpha(A)$ is sequentially closed. The set $\hat{\sigma}_X(A)$ is just the union of all $\sigma_X^\alpha(A)$, and then the result is straightforward. ■

Theorem 3.7. *The $\hat{\sigma}$-completion of a metric space is unique.*

Proof: Let A be a metric space with two $\hat{\sigma}$-completions (X, d) and (Y, d'). We want to show that they are isometric.

From the proposition above, there are ordinals α and β such that $X = \sigma_X^\alpha(A)$ and $Y = \sigma_Y^\beta(A)$.

With transfinite induction, one can prove that for each ordinal γ, $\sigma_X^\gamma(A)$ is isometric to $\sigma_Y^\gamma(A)$.

If $\alpha < \beta$, then $\sigma_Y^\alpha(A)$ is complete because it is isometric to $\sigma_X^\alpha(A)$. This just means that $Y \neq \hat{\sigma}_Y(A)$, which contradict the hypothesis. So, α must be equal to β, and then X is isometric to Y. ■

Corollary 3.8. *Let A be a metric space and X its $\hat{\sigma}$-completion.*

If f is a non-expansive map from A to a complete metric space B, then there is a unique non-expansive function $\hat{f}$ from X to B such that its restriction to A is f.*

*$f : (A, d) \longrightarrow (B, d')$ is a non-expansive map if $d'(f(x), f(y)) \leq d(x, y)$.

In other words, the complete metric spaces is a reflexive subcategory of the category of metric spaces and non-expansive maps.

The result is still valid when the morphisms are the uniformly continuous functions.

Proposition 3.9. *Every metric space has a unique k-completion if and only if the axiom of countable choice holds.*

Proof: If **CC** holds, then from Corollary 2.2 every complete metric space is sequential. This just mens that the k-completion and the $\hat{\sigma}$-completion coincide.

If **CC** does not hold, the space X defined in the proof of the Theorem 2.1 has two non-isometric completions. ■

Proposition 3.10. *Every metric space has a σ-completion if and only if the axiom of countable choice holds. If the σ-completion exists, then it is unique.*

Proof: Since $\sigma \leq \hat{\sigma}$, if the σ-completion exists, then must be equal to the $\hat{\sigma}$-completion, and as we have seen already it is unique.

If **CC** holds, then $\sigma = \hat{\sigma}$, which means that the σ-completion exists.

If **CC** fails, the metric space X defined in the proof of the Theorem 2.6 has no σ-completion. ■

As it was already said, the limits of filters characterized the topological spaces. Also for complete metric spaces and completions, filters convergence and Cauchy filters might be used to avoid the difficulties caused by the absence of the axiom of choice. This leads us to the class of metric spaces *complete for filters* or f-complete, and to the definition of f-completion. It is not hard to prove that there is a unique f-completion, but it will not be done here.

In the last theorem, it is shown that the approach with filters is still related with countability.

Definitions 3.11. A metric space X is:

(a) *f-complete* if every *Cauchy filter*[†] converges.

(b) *Cantor-complete* if, for every family $(F_n)_{n\in\mathbb{N}}$ of non-empty closed subsets of X such that $F_{n+1} \subseteq F_n$ and $\lim_n \mathsf{diam}(F_n) = 0$, $\bigcap_n F_n \neq \emptyset$.

[†] $\mathcal{F}$ is a Cauchy filter if, for every n, there is $F \in \mathcal{F}$ such that $\mathsf{diam}(F) < \frac{1}{n}$.

Theorem 3.12. *For a metric space X, the following conditions are equivalent:*

(i) *X is f-complete;*
(ii) *every Cauchy filter in X with a countable base converges;*
(iii) *X is Cantor-complete.*

Proof: That (i) implies (ii) is clear.

(ii)$\Rightarrow$(iii) A family $(F_n)_n$ in the conditions of the definition of Cantor-complete space is a base for a Cauchy filter. By hypothesis, this filter converges. This limit is an element of the set $\bigcap_n F_n$.

(iii)$\Rightarrow$(i) Let $\mathcal{F}$ be a Cauchy filter in a Cantor-complete space X. For each $n \in \mathbb{N}$, one defines $A(n) := \bigcup\{F \in \mathcal{F} : \mathsf{diam}(F) < \frac{1}{n}\}$. The sets $A(n)$ are non-empty because $\mathcal{F}$ is a Cauchy filter.

Each of the sets $F_n := k_X(A(n))$ has diameter at most $\frac{2}{n}$ and $F_{n+1} \subseteq F_n$. Since by (iii), $\bigcap F_n \neq \emptyset$, there is $x \in \bigcap F_n$. The filter $\mathcal{F}$ converges to x and then X is f-complete.

■

Note that, in general, $(A(n))_{n\in\mathbb{N}}$ is not a base for the filter $\mathcal{F}$.

As a last note, we just say that every complete metric space is f-complete if and only if the axiom of countable choice holds.

References

[1] R. Engelking, *General Topology*, Revised and completed edition, Heldermann Verlag Berlin, 1987.

[2] W. Felschner, *Naive Mengen und abstrakte Zahlen III – Transfinite Methoden*, Bibliographisches Institut, 1979.

[3] A. A. Fraenkel, Y. Bar-Hillel and A. Levy, *Foundations of set theory*, Second revised edition, North-Holland Publ. Co., 1973.

[4] M.Gitik, All uncountable cardinals can be singular, *Israel Journal of Mathematics* **35** (1980), 61–88.

[5] G. Gutierres, Sequential topological conditions in $\mathbb{R}$ in the absence of the axiom of choice, *Math. Log. Quart.* **49** (2003), 293–298.

[6] H. Herrlich and J. Steprāns, Maximal filters, continuity and choice principles, *Quaestiones Mathematicae* **20** (1997), 697–705.

[7] H. Herrlich and G. E. Strecker, When $\mathbb{N}$ is Lindelöf?, *Comment. Math. Univ. Carolinae* **38** (1997), 553–556.

[8] P. Howard and J. E. Rubin, *Consequences of the Axiom of Choice*, AMS Math. Surveys and Monographs **59**, 1998.

Gonçalo Gutierres
CMUC, Department of Mathematics, University of Coimbra, P-3001 454 Coimbra, Portugal

E-mail address: ggutc@mat.uc.pt
URL: http://www.mat.uc.pt/~ggutc