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UNIFORM-TYPE STRUCTURES ON LATTICE-VALUED SPACES AND FRAMES

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ABSTRACT: By introducing lattice-valued covers of a set, we present a general framework for uniform structures on very general L-valued spaces (for L an integral commutative quantale). By showing, via an intermediate L-valued structure of uniformity, how filters of covers may describe the uniform operators of Hutton, we prove that, when restricted to Girard quantales, this general framework captures Hutton's uniform spaces.

The categories of L-valued uniform spaces and L-valued uniform frames here introduced provide (in the case L is a chain) the missing vertices in the commutative cube formed by the classical categories of topological and uniform spaces and their corresponding pointfree counterparts (forming the base of the cube) and the corresponding L-valued categories (forming the top of the cube).

KEYWORDS: Frame, locale, quantale, Girard quantale, Integral commutative quantale, L-valued space, L-valued frame, uniformity, cover, entourage, uniform operator, Galois connection, axiality, polarity.
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1. Introduction

In classical topology, uniform structures are usually approached in terms of covers (Tukey [30]), via double powersets of the form 2^{2^X}, or, equivalently, in terms of entourages (Weil [31]), via powersets of the form $2^{X\times X}$. Entourages, being binary relations, may be also easily described by polarities (that is, Galois connections between power sets) and axialities (that is, residuated pairs between power sets). This gives two more equivalent descriptions of uniform spaces.

One of the questions of interest in lattice-valued topology concerns well-founded definitions of uniform-type structures (see the introduction to [26]).

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The uniform structures of Hutton [15] have been regarded as the lattice-valued counterpart to the classical covering uniformities of Tukey (see [27]), in opposition to the fuzzy uniform spaces of Lowen [20], which are an extension of the entourage approach of Weil. It is well known that, opposite to the classical case, these two notions are not equivalent. In this paper we show that residuations rather than covers are the root of Hutton's approach, and how lattice-valued covers of a set may describe lattice-valued uniformities. This approach leads to a new category of L-valued uniform spaces in which Hutton uniform spaces fit nicely. It will be apparent that this category is the natural generalization of the classical Tukey's definition to the L-valued context. Some interesting observations about Hutton's definition will appear along the way. Namely, it will be clear why and when Hutton's axiomatization, based on residuated maps (axialities [4]), can be equivalently seen as an L-residuated uniformity. Also, an open question appears: what happens if Hutton's axiomatization is formulated in terms of Galois maps (polarities [4]), generalizing the meaning of being a reflexive relation (an entourage) in this contravariant case.

The motivation for this paper arose from an observation of Pultr and Rodabaugh in [26] that lattice-valued frames may be useful in the establishment of well-founded definitions of uniform-type structures. To put this in perspective, let us recall that a *frame* is a complete lattice M satisfying the infinite distributivity law

$$m \wedge \bigvee S = \bigvee \{m \wedge s \mid s \in S\}$$

for all $m \in M$ and $S \subseteq M$, and a *frame homomorphism* $h : M \to N$ is a map preserving finite meets (including the top 1) and arbitrary joins (including the bottom 0). The resulting category will be denoted by Frm. The two-element frame $\{0 < 1\}$ will be denoted by $\mathsf{2}$.

If X is a topological space, the lattice $\mathcal{O}(X)$ of its open sets is a frame, and if $f : X \to Y$ is a continuous map then $\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X)$ defined by $\mathcal{O}(f)(U) = f^{-1}(U)$ is a frame homomorphism. Thus we have a contravariant functor $\mathcal{O} : \mathsf{Top} \to \mathsf{Frm}$. Finally, recall the standard spectrum construction

$$\Sigma(M) = \big(\{p : M \to \mathsf{2} \mid p \in \mathsf{Frm}\}, \{\Sigma_m \mid m \in M\}\big)$$

for a frame M (where $\Sigma_m = \{p \mid p(m) = 1\}$). Defining, for each frame homomorphism $h : M \to N$, $\Sigma(h) : \Sigma(N) \to \Sigma(M)$ by $\Sigma(h)(p) = p \cdot h$, this constitutes a contravariant functor $\Sigma : \mathsf{Frm} \to \mathsf{Top}$. The following are well-known facts: Σ is a right adjoint for $\mathcal{O}$, each $\Sigma(M)$ is a sober space (i.e., a T_0 space X whose only meet irreducible elements are $X \setminus \overline{\{x\}}$), the unit map $X \to \Sigma\mathcal{O}(X)$ is a homeomorphism if and only if X is sober, and the unit morphism $\mathcal{O}\Sigma(M) \to M$ is an isomorphism if and only if M is spatial i.e., isomorphic to the frame of open sets of some topological space X. For more details about frames see, e.g., [17] or [24].

The above adjunction $\mathcal{O} \dashv \Sigma$ between the category of frames and the category of topological spaces can be easily adapted to the uniform setting, giving an adjunction between the category UFrm of uniform frames (introduced by Isbell [16], and studied in detail by Pultr [25] in terms of covers; for information about other different ways of describing them see [5] and [23]) and the category Unif of uniform spaces of Weil [31] and Tukey [30]. Then, denoting by F_1 and F_2, respectively, the forgetful functors $\mathsf{Unif} \to \mathsf{Top}$ and $\mathsf{UFrm} \to \mathsf{Frm}$ forgetting the uniform structure, the diagram

$$\begin{tikzcd}
\mathsf{Unif} \arrow[r, shift left, "\mathcal{O}"] \arrow[d, "F_1"'] & \mathsf{UFrm} \arrow[l, shift left, "\Sigma"] \arrow[d, "F_2"] \\
\mathsf{Top} \arrow[r, shift left, "\mathcal{O}"] & \mathsf{Frm} \arrow[l, shift left, "\Sigma"]
\end{tikzcd} \tag{1.1}$$

commutes.

In [26], the authors introduced L-valued frames, which relate to frames in a way parallel to that in which the ι_L functor (see [19, 18]) relates L-valued topological spaces to topological spaces. Moreover, when L is linearly ordered or, more generally, a spatial frame (see [13]), there is an adjunction between L-Top and L-Frm that shows that L-valued frames generalize L-valued topological spaces in a way parallel to frames generalizing topological spaces. Specifically, denoting by χ_L^{T} and χ_L^{F} the (characteristic) functors embedding the categories of $\mathsf{2}$-valued objects in question in the corresponding categories of L-valued objects, the diagram

$$\begin{tikzcd}
L\text{-}\mathsf{Top} \arrow[r, shift left, "\mathcal{O}"] & L\text{-}\mathsf{Frm} \arrow[l, shift left, "\Sigma"] \\
\mathsf{Top} \arrow[r, shift left, "\mathcal{O}"] \arrow[u, "\chi_L^{\mathsf{T}}"] & \mathsf{Frm} \arrow[l, shift left, "\Sigma"] \arrow[u, "\chi_L^{\mathsf{F}}"']
\end{tikzcd}$$

commutes. Putting it together with (1.1) we get an incomplete diagram:

$$
\begin{tikzcd}
? \arrow[rr, shift left] \arrow[dr] & & ? \arrow[ll, shift left] \arrow[dr] & \\
& L\text{-}\mathsf{Top} \arrow[rr, shift left, "\mathcal{O}"] & & L\text{-}\mathsf{Frm} \arrow[ll, shift left, "\Sigma"] \\
\mathsf{Unif} \arrow[uu] \arrow[rr, shift left, "\mathcal{O}"] \arrow[dr, "F_1"'] & & \mathsf{UFrm} \arrow[uu] \arrow[ll, shift left, "\Sigma"] \arrow[dr, "F_2"] & \\
& \mathsf{Top} \arrow[uu, "\chi_L^{\mathsf{T}}"] \arrow[rr, shift left, "\mathcal{O}"] & & \mathsf{Frm} \arrow[ll, shift left, "\Sigma"] \arrow[uu, "\chi_L^{\mathsf{F}}"']
\end{tikzcd}
$$

A natural question arises:

> *Does there exist two types of structure (i.e. appropriate notions of L-valued uniform spaces and frames) that would allow us to complete the cube (by filling in the two question marks) in such a way that the two new vertical arrows also represent embedding functors, that the two new diagonal arrows also represent forgetful functors, that the new horizontal arrows also establish an adjunction, and that the whole diagram commutes?*

The answer is not immediately obvious: as the authors of [26] point out, the direct approach through uniformizing the L-topology τ as a frame is not satisfactory. Indeed, a uniformity on τ induces a uniformity on the lattice L of values (as observed by Banaschewski – see [26]); so, when L is linear, we would stay within the crisp case, since the only linearly ordered frame admitting a uniformity is the two-point frame $\mathsf{2} = \{0 < 1\}$.

It is our purpose in this paper to show that all the points raised above can be addressed in a satisfactory way. We introduce categories L-Unif and L-UFrm, for general strictly two-sided commutative quantales L, that fill in the two question marks: they are related respectively to the categories L-Top and L-Frm in a parallel way and again, when L is linearly ordered or a spatial frame, L-UFrm generalizes L-Unif in a way parallel to uniform frames generalizing uniform spaces. Then we present an equivalent presentation for L-Unif in terms of residuated mappings that will encompass Hutton's original definition whenever L is a Girard quantale. It will be apparent that a slight change in one of the axioms of Hutton makes a big difference and allows the extension of the definition to more general contexts.

The paper is organized as follows. We begin, in Section 2, by establishing some notations and by recalling some background on L-topological spaces and uniform frames. Then, in Section 3, we introduce L-valued covers and use them to axiomatize L-valued uniform structures for very general lattices L (namely, strictly two-sided commutative quantales). In Section 4 we give an alternative, equivalent, formulation in terms of residuated mappings. In Section 5, we show that the latter approach, when restricted to Girard quantales, implies Hutton's axiomatization. Several arguments are presented in favor of our approach. After recalling, in Section 6, some basic facts about L-valued frames we relate, in Section 7, L-valued uniform spaces with uniform spaces and provide in Section 8 the missing vertex for the above cube: the notion of an L-valued uniform frame.

2. Preliminaries and notation

2.1. L-valued spaces. We recall that $(L, \leq, *)$ is a *quantale* if

(1) $(L, \leq)$ is a complete lattice (with top element $1 = \bigwedge \varnothing$ and bottom $0 = \bigvee \varnothing$).

(2) $*$ is an associative binary operation distributive over arbitrary joins:

$$\alpha * \Big(\bigvee_{i\in I} \beta_i\Big) = \bigvee_{i\in I}(\alpha * \beta_i).$$

Since the operator $\alpha * (-)$ preserves arbitrary joins, every quantale is residuated, i.e. there exists the corresponding right adjoint $\alpha \to (-)$ defined by the relation

$$\alpha * \beta \leq \gamma \iff \beta \leq \alpha \to \gamma.$$

In particular, $\to$ is given by $\alpha \to \gamma = \bigvee\{\beta \in L \mid \alpha * \beta \leq \gamma\}$.

A quantale $(L, \leq, *)$ is *strictly two-sided* (or simply *integral*) if

(3) $(L, *)$ is a monoid whose unit is the top element 1.

(Notice that an integral quantale is an integral *cl*-monoid in the sense of [11].)

We list here some of the basic properties of integral quantales needed in the sequel:

(Q1) $\beta \leq \gamma \Rightarrow \alpha * \beta \leq \alpha * \gamma$.
(Q2) $\alpha * 0 = 0 * \alpha = 0$ for all $\alpha \in L$.
(Q3) $\alpha \leq (\alpha \to 0) \to 0$ for all $\alpha \in L$.

Sometimes integral commutative quantales are referred to as *complete residuated lattices*. Any frame is automatically a commutative integral quantale (where the binary operation $*$ is given by the meet $\wedge$).

A commutative unital quantale (i.e. a commutative quantale with a unit ε satisfying $\alpha * \varepsilon = \varepsilon * \alpha = \alpha$ for all α) is called a *Girard quantale* if it satisfies the *law of double negation*

(Q4) $\alpha = (\alpha \to 0) \to 0$ for all $\alpha \in L$.

Note that a Girard quantale is necessarily integral since

$$\varepsilon \to 0 = \bigvee \{\alpha \in L \mid \varepsilon * \alpha \leq 0\} = 0$$

and, consequently, $\varepsilon = (\varepsilon \to 0) \to 0 = 0 \to 0 = 1$.

From now on, except when otherwise stated, we will assume that L is an integral commutative quantale. For any set X, the set L^X of mappings $X \to L$, with the partial order

$$a \leq b \;\equiv\; a(x) \leq b(x) \text{ for each } x \in X,$$

is also an integral quantale: joins, meets and the binary operation $*$ are just defined pointwisely. If $A \subset X$, then $1_A \in L^X$ denotes the characteristic function of A, hence we denote the top element of L^X by 1_X and the bottom by $1_\varnothing$. The constant member of L^X with value α is denoted α too. Given a map $f : X \to Y$, $a \in L^X$ and $b \in L^Y$, we define the usual (Zadeh) image and preimage operators:

$$f^{\to}(a) = \bigvee_{x \in X} a(x) \wedge 1_{\{f(x)\}},$$

and

$$f^{\leftarrow}(b) = b \cdot f$$

the composition of f and b. We shall need a number of properties of the operators just defined viz.:

Properties 2.1. *Let $f : X \to Y$, $g : Y \to Z$, $a \in L^X$, $b, b_1, b_2 \in L^Y$, $c \in L^Z$. Then:*

(1) $f^{\leftarrow}$ *preserves the existing sups and infs;* $f^{\rightarrow}$ *preserves the existing sups (in particular, they are both order-preserving),*
(2) $a \leq f^{\leftarrow}(f^{\rightarrow}(a))$,
(3) $f^{\rightarrow}(f^{\leftarrow}(b)) \leq b$, *and* $f^{\rightarrow}(f^{\leftarrow}(b)) = b$ *if and only if* f *is surjective,*
(4) $(g \cdot f)^{\rightarrow}(a) = g^{\rightarrow}(f^{\rightarrow}(a))$,
(5) $(g \cdot f)^{\leftarrow}(c) = g^{\leftarrow}(f^{\leftarrow}(c))$,
(6) $f^{\leftarrow}(b_1 * b_2) = f^{\leftarrow}(b_1) * f^{\leftarrow}(b_2)$,
(7) $f^{\rightarrow}(a) * b \neq 1_{\varnothing}$ *if and only if* $a * f^{\leftarrow}(b) \neq 1_{\varnothing}$.

An *L-valued topological space* ([3], [14]) (shortly, an *L-topological space*) is a pair (X, τ) consisting of a set X and a subset τ of L^X (the *L-valued topology* or *L-topology* on the set X), containing $1_{\varnothing}$ and 1_X and closed under finite meets and arbitrary joins.

Given two *L-valued topological spaces* $(X, \tau_1), (Y, \tau_2)$ a map $f : X \to Y$ is an *L-continuous map* if the correspondence $b \mapsto f^{\leftarrow}(b)$ maps τ_2 into τ_1. The resulting category will be denoted by L-Top.

Of course, when $L = 2$, an L-topological space is precisely a topological space and there is an isomorphism between Top and L-Top, via the characteristic functor (the one associating to each subset its characteristic function and leaving morphisms unchanged). If L is a frame then the L-topologies, being subframes of the frame L^X, are frames as well.

2.2. Uniform spaces. There are several equivalent axiomatizations of the notion of uniformity on a set X. We will refer to the one introduced by Tukey [30] in which the basic term is the one of *uniform cover* of X. A cover $\mathcal{U}$ *refines* a cover $\mathcal{V}$, and in this case one writes $\mathcal{U} \preccurlyeq \mathcal{V}$, if for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$. For each cover $\mathcal{U}$ of X and $A \subseteq X$, let

$$\mathrm{st}(A, \mathcal{U}) := \bigcup \{U \in \mathcal{U} \mid U \cap A \neq \varnothing\}$$

be the *star* of A in $\mathcal{U}$ and $\mathrm{st}(\mathcal{U}) := \{\mathrm{st}(V, \mathcal{U}) \mid V \in \mathcal{U}\}$, which is a cover too. A *uniformity* on X is a set μ of covers of X such that:

(U1) μ is a filter in the preordered set $(Cov(X), \preccurlyeq)$ of all covers of X.
(U2) for each $\mathcal{U} \in \mu$ there is some $\mathcal{V} \in \mu$ such that the cover $\mathrm{st}(\mathcal{V})$ refines $\mathcal{U}$.

A map $f : (X, \mu) \longrightarrow (Y, \nu)$ between uniform spaces is *uniformly continuous* if for every $\mathcal{V} \in \nu$, $f^{-1}[\mathcal{V}] = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ belongs to μ. We denote the resulting category of uniform spaces and uniformly continuous maps by Unif.

The *uniform topology* $\mathcal{T}_\mu$ induced by (X,μ) is the one generated by the neighborhood basis $\{\mathrm{st}(\{x\},\mathcal{U}) | \mathcal{U} \in \mu\}$ for each $x \in X$. The correspondence $(X,\mu) \to (X,\mathcal{T}_\mu)$ defines the forgetful functor $F_1 : \mathsf{Unif} \to \mathsf{Top}$.

2.3. Uniform frames. Tukey's approach to uniform spaces via covers was the first to be studied in the pointfree context of frames. In [16] Isbell introduced uniformities on frames, as the precise translation into frame terms of Tukey's notion, later developed in detail by Pultr [25].

Let M be a frame. A set $C \subseteq M$ is a *cover* of M if $\bigvee_{c\in C} c = 1$. The set of all covers of M, denoted as $Cov(M)$, can be preordered as follows: a cover C *refines* a cover D, written $C \preccurlyeq D$, if for each $c \in C$ there is some $d \in D$ with $c \leq d$. Thus $(Cov(M), \preccurlyeq)$ is a preordered set with meets and joins: take for $C \wedge D$ the cover $\{c \wedge d \mid c \in C, d \in D\}$ and for $C \vee D$ just the union $C \cup D$.

For each $m \in M$, the *star* of m in C is the element

$$\mathrm{st}(m, C) := \bigvee\{c \in C \mid c \wedge m \neq 0\}$$

and $\mathrm{st}(C) := \{\mathrm{st}(c, C) \mid c \in C\}$ which is also a cover of M. Further, for each family $\mathcal{C}$ of covers of M, let

$$n \overset{\mathcal{C}}{\lhd} m \text{ if there is } C \in \mathcal{C} \text{ such that } \mathrm{st}(n, C) \leq m.$$

A family $\mathcal{C}$ of covers of a frame M is a *uniformity* [25] provided that:

(U1) $\mathcal{C}$ is a filter in the preordered set $(Cov(M), \preccurlyeq)$.
(U2) For each $C \in \mathcal{C}$ there is a $D \in \mathcal{C}$ such that the cover $\mathrm{st}(D)$ refines C.
(U3) For every $m \in M$, $m = \bigvee\{n \in M \mid n \overset{\mathcal{C}}{\lhd} m\}$.

The pair $(M, \mathcal{C})$ is then called a *uniform frame*. Let $(M, \mathcal{C})$ and $(N, \mathcal{D})$ be uniform frames. A frame homomorphism $h : M \to N$ is a *uniform homomorphism* if, for every $C \in \mathcal{C}$, $h[C] = \{h(c) \mid c \in C\} \in \mathcal{D}$. We denote by UFrm the category of uniform frames and uniform homomorphisms. This category is related to the category Unif of uniform spaces and uniformly continuous maps by a dual adjunction via the *open* and *spectrum* (contravariant) functors:

The *open functor* $\mathcal{O} : \mathsf{Unif} \to \mathsf{UFrm}$ assigns to each uniform space (X,μ) the uniform frame $(\mathcal{T}_\mu, \mathcal{C}_{\mathcal{T}_\mu})$, where $\mathcal{T}_\mu$ is the topology induced by μ and $\mathcal{C}_{\mathcal{T}_\mu}$ is the collection of all $\mathcal{T}_\mu$-open covers of $\mathcal{T}_\mu$. If $f : (X,\mu) \to (Y,\nu)$ is uniformly continuous, then $\mathcal{O}(f) : \mathcal{O}(Y,\nu) \to \mathcal{O}(X,\mu)$ defined, for each $V \in \mathcal{T}_\nu$, by $\mathcal{O}(f)(V) = f^{-1}(V)$, is a uniform homomorphism.

On the other hand, the *spectrum functor* $\Sigma : \mathsf{UFrm} \to \mathsf{Unif}$ assigns to each uniform frame $(M,\mathcal{C})$ the uniform space $(\Sigma M, \mu_{\Sigma M})$, being $\Sigma M = \{p : M \to \mathsf{2} \mid p \in \mathsf{Frm}\}$ the set of points of M and $\mu_{\Sigma M}$ the filter of covers of ΣM generated by $(\{\Sigma_m \mid m \in C\})_{C\in\mathcal{C}}$, where $\Sigma_m = \{p \in \Sigma M \mid p(m) = 1\}$. If $h : (M,\mathcal{C}) \to (N,\mathcal{D})$ is a uniform homomorphism then $\Sigma(h) : \Sigma(N,\mathcal{D}) \to \Sigma(M,\mathcal{C})$, given by $\Sigma(h)(q) = q \cdot h$, is uniformly continuous.

This adjunction makes the diagram in (1.1) commutative.

3. Covering L-valued uniform spaces

Let L be an integral commutative quantale. We say that $\mathcal{A} \subseteq L^X$ is an *L-cover* of X if $\bigvee \mathcal{A} = 1_X$. For any $\mathcal{A}, \mathcal{B} \subseteq L^X$ we write

$$\mathcal{A} \preccurlyeq \mathcal{B} \text{ if for each } a \in \mathcal{A} \text{ there exists } b \in \mathcal{B} \text{ such that } a \leq b.$$

The relation just defined makes the set of all L-covers of X, noted as L-$Cov(X)$, a preordered set. Let $\mathcal{A} * \mathcal{B} = \{a * b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$, where $a * b : X \to L$ is defined pointwisely and $\mathcal{A} \wedge \mathcal{B} = \{a \wedge b \mid a \in \mathcal{A}, b \in \mathcal{B}\}$. Clearly, $\mathcal{A} * \mathcal{B} \preccurlyeq \mathcal{A} \wedge \mathcal{B}$, since $a * b \leq a * 1_X = a$ and $a * b \leq 1_X * b = b$.

Proposition 3.1. *For every L-covers $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}*\mathcal{B}$ and $\mathcal{A}\wedge\mathcal{B}$ are L-covers of X.*

Proof: Since $*$ distributes over arbitrary joins,

$$\bigvee(\mathcal{A} * \mathcal{B}) = \bigvee \mathcal{A} * \bigvee \mathcal{B} = 1_X * 1_X = 1_X,$$

and it follows that $\mathcal{A}*\mathcal{B}$ is an L-cover. Then $\mathcal{A}\wedge\mathcal{B}$ is also an L-cover because $\mathcal{A} * \mathcal{B} \preccurlyeq \mathcal{A} \wedge \mathcal{B}$. ■

Notice that, since $\mathcal{A}\wedge\mathcal{B}$ is an L-cover, then $\mathcal{A}\wedge\mathcal{B}$ is the infimum of $\mathcal{A}$ and $\mathcal{B}$ in the preordered set $(L\text{-}Cov(X), \preccurlyeq)$ of all L-covers of X.

For each $a \in L^X$ and $\mathcal{A} \subseteq L^X$, let

$$\mathrm{st}(a,\mathcal{A}) := \bigvee\{b \in \mathcal{A} \mid b * a \neq 1_\varnothing\}$$

and

$$\mathrm{st}(\mathcal{A}) := \{\mathrm{st}(a,\mathcal{A}) \mid a \in \mathcal{A}\},$$

which is an L-cover.

Proposition 3.2. *Let* $\mathcal{A}, \mathcal{B} \subseteq L^X$ *and* $a, b \in L^X$. *Then:*

(1) *If* $\mathcal{A}$ *is an* L*-cover then* $a \leq \mathrm{st}(a, \mathcal{A})$ *and, consequently,* $\mathcal{A} \preccurlyeq \mathrm{st}(\mathcal{A})$.
(2) *If* $a \leq b$ *then* $\mathrm{st}(a, \mathcal{A}) \leq \mathrm{st}(b, \mathcal{A})$.
(3) *If* $\mathcal{A} \preccurlyeq \mathcal{B}$ *then* $\mathrm{st}(a, \mathcal{A}) \leq \mathrm{st}(a, \mathcal{B})$.
(4) $\mathrm{st}(\bigvee \mathcal{B}, \mathcal{A}) = \bigvee_{b \in \mathcal{B}} \mathrm{st}(b, \mathcal{A})$.
(5) *If* $\mathcal{A}$ *is an* L*-cover then* $\mathrm{st}(\mathrm{st}(a, \mathcal{A}), \mathcal{A}) \leq \mathrm{st}(a, \mathrm{st}(\mathcal{A}))$.
(6) *Let* $f : X \to Y$, $\mathcal{B} \subseteq L^Y$, $f^{-1}[\mathcal{B}] := \{f^{\leftarrow}(b) \mid b \in \mathcal{B}\}$ *and* $c \in L^Y$. *Then,* $\mathrm{st}(f^{\leftarrow}(c), f^{-1}[\mathcal{B}]) \leq f^{\leftarrow}(\mathrm{st}(c, \mathcal{B}))$.

Proof: (1): $a = 1_X * a = \big(\bigvee_{b \in \mathcal{A}} b\big) * a = \bigvee\{b * a \mid b \in \mathcal{A}, b * a \neq 1_\varnothing\} \leq \mathrm{st}(a, \mathcal{A})$.

(2) and (3): These are obvious.

(4):
$$\begin{aligned}
\mathrm{st}(\bigvee \mathcal{B}, \mathcal{A}) &= \bigvee\{a \in \mathcal{A} \mid a * \bigvee \mathcal{B} \neq 1_\varnothing\} \\
&= \bigvee\{a \in \mathcal{A} \mid \bigvee_{b \in \mathcal{B}} (a * b) \neq 1_\varnothing\} \\
&= \bigvee\{a \in \mathcal{A} \mid a * b \neq 1_\varnothing \text{ for some } b \in \mathcal{B}\} \\
&= \bigvee_{b \in \mathcal{B}} \mathrm{st}(b, \mathcal{A}).
\end{aligned}$$

(5): For each $a \in L^X$ we have, using the previous property,

$$\mathrm{st}(\mathrm{st}(a, \mathcal{A}), \mathcal{A}) = \bigvee\{\mathrm{st}(b, \mathcal{A}) \mid b \in \mathcal{A}, b * a \neq 1_\varnothing\}.$$

Since, for each such b, $\mathrm{st}(b, \mathcal{A}) \in \mathrm{st}(\mathcal{A})$ and $\mathrm{st}(b, \mathcal{A}) * a \geq b * a \neq 1_\varnothing$, we have immediately $\mathrm{st}(\mathrm{st}(a, \mathcal{A}), \mathcal{A}) \leq \mathrm{st}(a, \mathrm{st}(\mathcal{A}))$.

(6) Since $f^{\leftarrow}$ preserves arbitrary sups and the binary operation $*$ (see Properties 2.1), we have:

$$\begin{aligned}
\mathrm{st}(f^{\leftarrow}(c), f^{-1}[\mathcal{B}]) &= \bigvee\{f^{\leftarrow}(b) \mid b \in \mathcal{B}, f^{\leftarrow}(b) * f^{\leftarrow}(c) \neq 1_\varnothing\} \\
&= f^{\leftarrow}\Big(\bigvee\{b \in \mathcal{B} \mid f^{\leftarrow}(b * c) \neq 1_\varnothing\}\Big) \\
&\leq f^{\leftarrow}\Big(\bigvee\{b \in \mathcal{B} \mid b * c \neq 1_\varnothing\}\Big) \\
&= f^{\leftarrow}(\mathrm{st}(c, \mathcal{B})).
\end{aligned}$$

■

Definition 3.3. We say that a pair $(X, \mathfrak{U})$ consisting of a set X and a non-empty family $\mathfrak{U}$ of L-covers of X is a *covering L-uniform space* whenever the following conditions are satisfied:

(C1) $\mathcal{A} \preccurlyeq \mathcal{B}, \mathcal{A} \in \mathfrak{U} \Rightarrow \mathcal{B} \in \mathfrak{U}$.
(C2) For every $\mathcal{A}, \mathcal{B} \in \mathfrak{U}$, $\mathcal{A} \wedge \mathcal{B} \in \mathfrak{U}$.
(C3) For each $\mathcal{A} \in \mathfrak{U}$ there exists $\mathcal{B} \in \mathfrak{U}$ such that $\mathrm{st}(\mathcal{B}) \preccurlyeq \mathcal{A}$.

A *base* for the covering L-uniformity $\mathfrak{U}$ is any subcollection of $\mathfrak{U}$ from which $\mathfrak{U}$ can be recovered by applying condition (C1).

A map $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ is a *uniform homomorphism* if, for every $\mathcal{B} \in \mathfrak{V}$, $f^{-1}[\mathcal{B}] \in \mathfrak{U}$. The resulting category will be denoted by L-Unif. Of course, for $L = \mathsf{2}$, this is precisely the category of (covering) uniform spaces of Tukey [30].

For each $(X, \mathfrak{U}) \in L$-Unif define

$$\tau_{\mathfrak{U}} := \left\{ a \in L^X \mid a = \bigvee \{ b \in L^X \mid \mathrm{st}(b, \mathcal{A}) \leq a \text{ for some } \mathcal{A} \in \mathfrak{U} \} \right\}.$$

Note that, when $L = \mathsf{2}$, $\tau_{\mathfrak{U}}$ is just the crisp topology induced by the (classical) uniformity $\mathfrak{U}$ on X.

Proposition 3.4. *$(X, \tau_{\mathfrak{U}})$ is an L-topological space whenever L is a frame.*

Proof: It suffices to check that $\mathrm{int} : L^X \to L^X$ defined by

$$\mathrm{int}(a) = \bigvee \{ b \in L^X \mid \mathrm{st}(b, \mathcal{A}) \leq a \text{ for some } \mathcal{A} \in \mathfrak{U} \}$$

is an L-interior operator [14], that is:

(I1) $\mathrm{int}(1_X) = 1_X$.
(I2) $\mathrm{int}(a) \leq a$ for every $a \in L^X$.
(I3) $\mathrm{int}(\mathrm{int}(a)) = \mathrm{int}(a)$ for every $a \in L^X$.
(I4) $\mathrm{int}(a \wedge b) = \mathrm{int}(a) \wedge \mathrm{int}(b)$ for every $a, b \in L^X$.

(I1) and (I2) are trivially satisfied.
(I3): Let $b \in L^X$ satisfying $\mathrm{st}(b, \mathcal{A}) \leq a$ for some $\mathcal{A} \in \mathfrak{U}$ and take $\mathcal{B} \in \mathfrak{U}$ such that $\mathrm{st}(\mathcal{B}) \preccurlyeq \mathcal{A}$. Then $\mathrm{st}(b, \mathcal{B}) \leq \mathrm{int}(a)$ since $\mathrm{st}(\mathrm{st}(b, \mathcal{B}), \mathcal{B}) \leq \mathrm{st}(b, \mathrm{st}(\mathcal{B})) \leq \mathrm{st}(b, \mathcal{A}) \leq a$ by Proposition 3.2. This shows that $\mathrm{int}(a) \leq \mathrm{int}(\mathrm{int}(a))$ and the equality follows from (I2).
(I4): Since L is a frame, meets distribute over arbitrary joins and then (I4) follows immediately from (C2). ∎

$\tau_{\mathfrak{U}}$-open L-covers of $\mathfrak{U}$ form a base for the covering L-uniformity $\mathfrak{U}$:

Proposition 3.5. *If* $\mathcal{A} \in \mathfrak{U}$ *then* $\mathrm{int}(\mathcal{A}) := \{\mathrm{int}(a) \mid a \in \mathcal{A}\} \in \mathfrak{U}$.

Proof: Let $\mathcal{B} \in \mathfrak{U}$ such that $\mathrm{st}(\mathcal{B}) \preccurlyeq \mathcal{A}$. Then $\mathcal{B} \preccurlyeq \mathrm{int}(\mathcal{A})$. Indeed, for each $b \in \mathcal{B}$, $b \le \mathrm{st}(b, \mathcal{B}) \le a$ for some $a \in \mathcal{A}$, which shows that $b \le \mathrm{int}(a)$. ■

For any $a, b \in L^X$ we write $b \overset{\mathfrak{U}}{\lhd} a$ whenever $\mathrm{st}(b, \mathcal{A}) \le a$ for some $\mathcal{A} \in \mathfrak{U}$. The following property of the L-topology $\tau_{\mathfrak{U}}$ follows immediately from the previous proposition.

Corollary 3.6. *For every* $a \in \tau_{\mathfrak{U}}$, $a = \bigvee\{b \in \tau_{\mathfrak{U}} \mid b \overset{\mathfrak{U}}{\lhd} a\}$. ■

Proposition 3.7. *For any* $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ *in* L-Unif, $f : (X, \tau_{\mathfrak{U}}) \to (Y, \tau_{\mathfrak{V}})$ *is a morphism of* L-Top.

Proof: Let $b \in \tau_{\mathfrak{V}}$. Then $b = \bigvee\{c \in L^Y \mid \mathrm{st}(c, \mathcal{B}) \le b \text{ for some } \mathcal{B} \in \mathfrak{V}\}$. Let us show that $f^{\leftarrow}(b) \in \tau_{\mathfrak{U}}$, by proving that

$$\begin{aligned} f^{\leftarrow}(b) &= \bigvee\{f^{\leftarrow}(c) \mid c \in L^Y, \mathrm{st}(c, \mathcal{B}) \le b \text{ for some } \mathcal{B} \in \mathfrak{V}\} \\ &\le \bigvee\{f^{\leftarrow}(c) \mid c \in L^Y, f^{\leftarrow}(\mathrm{st}(c, \mathcal{B})) \le f^{\leftarrow}(b) \text{ for some } \mathcal{B} \in \mathfrak{V}\}. \\ &\le \bigvee\{a \in L^X \mid \mathrm{st}(a, f^{-1}(\mathcal{B})) \le f^{\leftarrow}(b) \text{ for some } f^{-1}(\mathcal{B}) \in \mathfrak{U}\} \\ &\le \mathrm{int}(f^{\leftarrow}(b)). \end{aligned}$$

The first equality follows from Property 2.1(1) and the inequalities follow from Properties 2.1 and Proposition 3.2 (6). ■

Thus, the correspondence

$$(X, \mathfrak{U}) \in L\text{-}\mathsf{Unif} \longmapsto (X, \tau_{\mathfrak{U}}) \in L\text{-}\mathsf{Top}$$

is functorial and we have a functor $F_3 : L$-$\mathsf{Unif} \to L$-Top such that the diagram

$$\begin{tikzcd} L\text{-}\mathsf{Unif} \arrow[r, "F_3"] & L\text{-}\mathsf{Top} \\ \mathsf{Unif} \arrow[u, "\chi_L^{\mathsf{U}}"] \arrow[r, "F_1"'] & \mathsf{Top} \arrow[u, "\chi_L^{\mathsf{T}}"'] \end{tikzcd}$$

commutes (where χ_L^{U} denotes the uniform version of the embedding functor $\chi_L^{\mathsf{T}} : \mathsf{Top} \to L$-$\mathsf{Top}$). This shows that, for frames L, the notion of L-valued uniform space relates to a uniform space in a way similar to that in which an L-valued topological space is related to a topological space.

4. Residuated L-valued uniform spaces

The category of residuated L-valued uniform spaces that we introduce in this section is based on the notion of residuated pairs. It has nice features: it is equivalent to the category of covering L-valued uniform spaces on one hand and also, for a large class of lattices (more precisely, Girard quantales), captures Hutton uniformities.

Originally, Galois connections were expressed in a contravariant form with transformations that reverse order [2]. A *Galois connection* between partially ordered sets A and B is a pair (f, g) of order-reversing maps $f : A \to B$ and $g : B \to A$ such that

$$\mathrm{id}_A \leq g \cdot f, \qquad \mathrm{id}_B \leq f \cdot g \quad \text{equivalently,} \quad b \leq f(a) \text{ iff } a \leq g(b).$$

We denote by $Gal(A, B)$ the set of all order-reversing maps $f : A \to B$ for which there exists $f^+ : B \to A$ such that (f, f^+) is a Galois connection.

Nowadays many authors prefer to work with Galois connections in the covariant form by its convenience (the survey [4] contains a list of references to this form). We refer to this dualized form as a residuated pair: a *residuated pair* between the partially ordered sets A and B is a pair (f, g) of order-preserving maps $f : A \to B$ and $g : B \to A$ such that

$$f \cdot g \leq \mathrm{id}_B, \qquad \mathrm{id}_A \leq g \cdot f \quad \text{equivalently,} \quad f(a) \leq b \text{ iff } a \leq g(b).$$

The map $f : A \to B$ in a residuated pair (f, g) is called *residuated* and the map $g : B \to A$ is called *residual.* Thus f is residuated (resp. residual) if and only if $f \in Gal(A, B^{op})$ (resp. $f \in Gal(A^{op}, B)$). If A and B are complete lattices then $f : A \to B$ is residuated if and only if it is join-preserving. For any sets X and Y and any map $f : X \to Y$, $(f^{\leftarrow}, f^{\rightarrow})$ is a residuated pair (see [28]).

Now consider the image and preimage operators

$$f^{\Rightarrow} : (L^X)^{L^X} \to (L^Y)^{L^Y}$$

and

$$f^{\Leftarrow} : (L^Y)^{L^Y} \to (L^X)^{L^X},$$

defined by $f^{\Rightarrow}(\varphi) = f^{\rightarrow} \cdot \varphi \cdot f^{\leftarrow}$ and $f^{\Leftarrow}(\psi) = f^{\leftarrow} \cdot \psi \cdot f^{\rightarrow}$:

$$\begin{tikzcd}
L^X \arrow[r, "\varphi"] & L^X \arrow[d, "f^{\rightarrow}"] \\
L^Y \arrow[u, "f^{\leftarrow}"] \arrow[r, "f^{\Rightarrow}(\varphi)"'] & L^Y
\end{tikzcd}
\qquad
\begin{tikzcd}
L^Y \arrow[r, "\psi"] & L^Y \arrow[d, "f^{\leftarrow}"] \\
L^X \arrow[u, "f^{\rightarrow}"] \arrow[r, "f^{\Leftarrow}(\psi)"'] & L^X
\end{tikzcd}$$

Again, $(f^{\Leftarrow}, f^{\Rightarrow})$ is a residuated pair.

Let $\mathcal{H}(L, X)$ denote the collection of all join-preserving mappings $\phi : L^X \to L^X$ (in particular, $\phi(1_\varnothing) = 1_\varnothing$), partially ordered by

$$\phi_1 \le \phi_2 \ \equiv \ \phi_1(a) \le \phi_2(a) \text{ for every } a \in L^X.$$

Let $\phi \in \mathcal{H}(L, X)$. We say that $a \in L^X$ is *ϕ-small* if

$$a * b \neq 1_\varnothing \Rightarrow a \le \phi(b), \qquad \text{for any } b \in L^X.$$

Further, we say that ϕ is:

(1) *symmetric* whenever $\phi(a) * b = 1_\varnothing$ iff $a * \phi(b) = 1_\varnothing$ (or, equivalently, whenever $b \le \phi(a) \to 1_\varnothing$ iff $\phi(b) \le a \to 1_\varnothing$), for arbitrary $a, b \in L^X$.
(2) an *L-entourage* of X if $\{a \in L^X \mid a \text{ is } \phi\text{-small}\}$ is an L-cover of X.

Proposition 4.1. *Let $\phi \in \mathcal{H}(L, X)$.*

(1) *If ϕ is symmetric then, for any $a, b \in L^X$,*

$$b \to 1_\varnothing \le \phi(a) \to 1_\varnothing \iff \phi(b \to 1_\varnothing) \le a \to 1_\varnothing.$$

(2) *If L is a Girard quantale then the converse to (1) holds. In that case, ϕ is symmetric if and only if*

$$\phi(a) \le b \iff \phi(b \to 1_\varnothing) \le a \to 1_\varnothing.$$

(3) *If ϕ is an L-entourage then $\phi(a) \ge a$ for every $a \in L^X$.*
(4) *If L is atomic (i.e. every element of L is the join of all atoms below it) then the converse to (3) holds.*

Proof: (1) and (2): These are obvious.

(3): For each $a \in L^X$,

$$\begin{aligned} a = 1_X * a &= \left(\bigvee\{b \in L^X \mid b \text{ is } \phi\text{-small}\}\right) * a \\ &= \bigvee\{b * a \mid b * a \neq 1_\varnothing, b \text{ is } \phi\text{-small}\} \\ &\leq \bigvee\{b \mid b * a \neq 1_\varnothing, b \text{ is } \phi\text{-small}\} \\ &\leq \phi(a). \end{aligned}$$

(4): Let $\phi : L^X \to L^X$ be such that $\phi(a) \geq a$ for every $a \in L^X$. For each atom α of L, it follows that $\alpha \wedge 1_{\{x\}} \in L^X$ is an atom in L^X. Then

$$1_\varnothing \neq (\alpha \wedge 1_{\{x\}}) * b \leq \alpha \wedge 1_{\{x\}} \wedge b \leq \alpha \wedge 1_{\{x\}}$$

implies $\alpha \wedge 1_{\{x\}} \leq b$. In particular, this means that every $\alpha \wedge 1_{\{x\}}$ is ϕ-small. Hence, for every $x \in X$,

$$\begin{aligned} \left(\bigvee\{a \in L^X \mid a \text{ is } \phi\text{-small}\}\right)(x) &\geq \left(\bigvee\{\alpha \wedge 1_{\{x\}} \mid \alpha \text{ is an atom of } L\}\right)(x) \\ &= \bigvee\{\alpha \mid \alpha \text{ is an atom of } L\} = 1. \end{aligned}$$

■

Remark 4.2. If L fails to be atomic then the converse to Proposition 4.1 (3) is not true in general. The unit interval, with its usual order, gives rise to a couple of examples, when it is endowed with the following two well-known quantale structures (in fact a similar statement holds for any quantale structure defined on $[0, 1]$):

(1) $([0, 1], \leq, \wedge)$, (the unit interval considered as a Heyting algebra)
(2) $([0, 1], \leq, T_m)$, (the unit interval considered as an MV-algebra), where $T_m(\alpha, \beta) = \max\{\alpha + \beta - 1, 0\}$ for each $\alpha, \beta \in [0, 1]$ is the Luckasiewicz T-norm.

Indeed, the identity map $\mathrm{id}_X : L^X \to L^X$ belongs to $\mathcal{H}(L, X)$ and it is not difficult to check that, in the first case, $a \in L^X$ is id_X-small iff $a = 1_\varnothing$, while in the second $a \in L^X$ is id_X-small iff $a = \alpha \wedge 1_x$ for some $\alpha \leq \frac{1}{2}$ and $x \in X$. So, either $\bigvee\{a \in L^X | a \text{ is } \mathrm{id}_X\text{-small}\} = 1_\varnothing$ or $\bigvee\{a \in L^X | a \text{ is } \mathrm{id}_X\text{-small}\} = \frac{1}{2}$; in both cases id_X fails to be an L-entourage.

Definition 4.3. We say that a pair $(X, \mathcal{D})$ consisting of a set X and a non-empty subset $\mathcal{D}$ of $\mathcal{H}(L, X)$ is a *residuated L-uniform space* whenever the following conditions hold:

(R1) Each $\phi \in \mathcal{D}$ is an L-entourage of X.
(R2) $\phi_1 \in \mathcal{D}$ and $\phi_1 \leq \phi_2 \in \mathcal{H}(L, X)$ implies $\phi_2 \in \mathcal{D}$.
(R3) For each $\phi_1, \phi_2 \in \mathcal{D}$ there exists $\phi_3 \in \mathcal{D}$ such that $\phi_3 \leq \phi_1$ and $\phi_3 \leq \phi_2$.
(R4) For each $\phi_1 \in \mathcal{D}$ there exists $\phi_2 \in \mathcal{D}$ such that $\phi_2 \cdot \phi_2 \leq \phi_1$.
(R5) $\mathcal{D}$ has a base of symmetric entourages, that is, for each $\phi_1 \in \mathcal{D}$ there exists a symmetric $\phi_2 \in \mathcal{D}$ such that $\phi_2 \leq \phi_1$.

A *base* for the uniformity $\mathcal{D}$ is any subcollection of $\mathcal{D}$ from which $\mathcal{D}$ can be recovered by applying condition (R2).

The morphisms (*uniform morphisms*) of the resulting category L-Unif^r are the maps $f : (X, \mathcal{D}) \to (Y, \mathcal{E})$ satisfying $f^{\Leftarrow}(\phi) \in \mathcal{D}$ for every $\phi \in \mathcal{E}$.

When $L = 2$, these are precisely the classical uniformities, described in terms of axialities [4] (cf. also [5], Section 5). Since the symmetry condition (R5) is now explicitly axiomatized, by dropping it we have the notion of a (residuated) *L-quasi-uniform space*. When $L = 2$ this gives precisely the classical notion of a quasi-uniform space.

It is now our purpose to prove that the categories L-Unif^r and L-Unif are isomorphic.

Let $\mathfrak{U}$ be a covering L-uniformity on X and, for each $\mathcal{A} \in \mathfrak{U}$, let

$$\begin{array}{rccc} \phi_{\mathcal{A}} : & L^X & \to & L^X \\ & a & \mapsto & \mathrm{st}(a, \mathcal{A}). \end{array}$$

The next proposition with the exception of symmetry follows immediately from Proposition 3.2 and lists some basic properties of these maps.

Proposition 4.4.

(1) *For each $\mathcal{A} \in \mathcal{U}$, $\phi_{\mathcal{A}} \in \mathcal{H}(L, X)$. Also, since $st(\mathcal{A}) \in \mathcal{U}$, $\phi_{st(\mathcal{A})} \in \mathcal{H}(L, X)$ and $\mathcal{A} \preccurlyeq st(\mathcal{A})$.*
(2) *If $\mathcal{A} \preccurlyeq \mathcal{B}$ then $\phi_{\mathcal{A}} \leq \phi_{\mathcal{B}}$.*
(3) *For every $a \in L^X$, $a \leq \phi_{\mathcal{A}}(a)$.*
(4) *Each $a \in \mathcal{A}$ is $\phi_{\mathcal{A}}$-small.*
(5) *Each $\phi_{\mathcal{A}}$ is a symmetric L-entourage.*
(6) *If* $\mathrm{st}(\mathcal{A}) \preccurlyeq \mathcal{B}$ *then $\phi_{\mathcal{A}} \cdot \phi_{\mathcal{A}} \leq \phi_{\mathcal{B}}$.*

Proof: (5): It follows from the following equivalence:

$$\phi_{\mathcal{A}}(a) * b \neq 1_{\varnothing} \Leftrightarrow \exists c \in \mathcal{A} : c * a \neq 1_{\varnothing} \text{ and } c * b \neq 1_{\varnothing} \Leftrightarrow a * \phi_{\mathcal{A}}(b) \neq 1_{\varnothing}. \qquad \blacksquare$$

The next corollary is an immediate consequence of Proposition 4.4.

Corollary 4.5. *Let $\mathfrak{U}$ be a covering L-uniformity on X. Then $\{\phi_{\mathcal{A}} \mid \mathcal{A} \in \mathfrak{U}\}$ is a base for a residuated uniformity $\mathcal{D}_{\mathfrak{U}}$ on X.* ■

Proposition 4.6. *For every uniform homomorphism $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$, the map $f : (X, \mathcal{D}_{\mathfrak{U}}) \to (Y, \mathcal{D}_{\mathfrak{V}})$ is uniform.*

Proof: We need to prove that $f^{\Leftarrow}(\phi) = f^{\leftarrow} \cdot \phi \cdot f^{\rightarrow} \in \mathcal{D}_{\mathfrak{U}}$ for every $\phi \in \mathcal{D}_{\mathfrak{V}}$. So, let $\mathcal{B} \in \mathfrak{V}$ such that $\phi_{\mathcal{B}} \leq \phi$ and $\mathcal{A} \in \mathfrak{U}$ satisfying $\mathcal{A} \preccurlyeq f^{-1}[\mathcal{B}]$. Combining results in Proposition 3.2 (3) and (6), and Proposition 4.4 (2), for every $a \in L^X$ we may write:

$$\begin{aligned}\phi_{\mathcal{A}}(a) &\leq \phi_{f^{-1}[\mathcal{B}]}(a) \;=\; \mathrm{st}(a, f^{-1}[\mathcal{B}]) \;\leq\; \mathrm{st}(f^{\leftarrow}(f^{\rightarrow}(a)), f^{-1}[\mathcal{B}]) \\ &\leq f^{\leftarrow}\big(\mathrm{st}(f^{\rightarrow}(a), \mathcal{B})\big) \;=\; f^{\leftarrow}\big(\phi_{\mathcal{B}}(f^{\rightarrow}(a))\big) \;\leq\; f^{\leftarrow}\big(\phi(f^{\rightarrow}(a))\big) \\ &= (f^{\leftarrow} \cdot \phi \cdot f^{\rightarrow})(a).\end{aligned}$$ ■

Finally, we have:

Corollary 4.7. *The correspondences $(X, \mathfrak{U}) \mapsto (X, \mathcal{D}_{\mathfrak{U}})$ and $f \mapsto f$ given by Corollary 4.5 and Proposition 4.6 establish a functor $\Phi : L\text{-}\mathsf{Unif} \to L\text{-}\mathsf{Unif}^r$.*

Conversely, let $\mathcal{D}$ be a residuated uniformity on X. For each $\phi \in \mathcal{D}$, consider the L-cover

$$\mathcal{A}_{\phi} = \{a \in L^X \mid a \text{ is } \phi\text{-small}\}.$$

The following proposition lists some of the basic properties of these covers.

Proposition 4.8.
(1) *If $\phi_1 \leq \phi_2$ then $\mathcal{A}_{\phi_1} \preccurlyeq \mathcal{A}_{\phi_2}$.*
(2) *$\mathcal{A}_{\phi_1 \wedge \phi_2} \preccurlyeq \mathcal{A}_{\phi_1} \wedge \mathcal{A}_{\phi_2}$.*
(3) *If $\phi_1 \cdot \phi_1 \cdot \phi_1 \leq \phi_2$ and ϕ_1 is symmetric then $\mathrm{st}(\mathcal{A}_{\phi_1}) \preccurlyeq \mathcal{A}_{\phi_2}$.*

Proof: (1): It is obvious since any ϕ_1-small element is ϕ_2-small whenever $\phi_1 \leq \phi_2$.
(2): It is an immediate consequence of the previous property.
(3): Let $\mathrm{st}(a, \mathcal{A}_{\phi_1}) \in \mathrm{st}(\mathcal{A}_{\phi_1})$. Then $\mathrm{st}(a, \mathcal{A}_{\phi_1}) = \bigvee\{b \in \mathcal{A}_{\phi_1} \mid b * a \neq 1_{\varnothing}\} \leq \phi_1(a)$ (because each such b is ϕ_1-small). It suffices now to check that $\phi_1(a)$ is ϕ_2-small. By the symmetry of ϕ_1, $\phi_1(a) * b \neq 1_{\varnothing}$ if and only if $a * \phi_1(b) \neq 1_{\varnothing}$, so $\phi_1(a) * b \neq 1_{\varnothing}$ implies $a \leq (\phi_1 \cdot \phi_1)(b)$, and consequently, $\phi_1(a) \leq \phi_1^3(b) \leq \phi_2(b)$, which shows that $\phi_1(a)$ is indeed ϕ_2-small. ■

From this result it follows immediately that:

Corollary 4.9. *Let $\mathcal{D}$ be a residuated L-uniformity on X. Then $\{\mathcal{A}_\phi \mid \phi \in \mathcal{D}\}$ is a base for a covering L-uniformity $\mathfrak{U}_\mathcal{D}$ on X.* ∎

Lemma 4.10. *Let $f : X \to Y$, $a \in L^X$ and $\psi \in (L^Y)^{L^Y}$. If a is $f^\Leftarrow(\psi)$-small then $f^\to(a)$ is ψ-small.*

Proof: Let $f^\to(a) * b \neq 1_\varnothing$ equivalently (cf. Lemma 2.1 (8)) $a * f^\leftarrow(b) \neq 1_\varnothing$, which implies

$$a \leq f^\Leftarrow(\psi)(f^\leftarrow(b)) = (f^\leftarrow \cdot \psi \cdot f^\to \cdot f^\leftarrow)(b) \leq (f^\leftarrow \cdot \psi)(b),$$

since $f^\to \cdot f^\leftarrow \leq \mathrm{id}_{L^Y}$. Thus $a \leq (f^\leftarrow \cdot \psi)(b)$ and, consequently,

$$f^\to(a) \leq (f^\to \cdot f^\leftarrow \cdot \psi)(b) \leq \psi(b).$$ ∎

Proposition 4.11. *For every uniform homomorphism $f : (X, \mathcal{D}) \to (Y, \mathcal{E})$, the map $f : (X, \mathfrak{U}_\mathcal{D}) \to (Y, \mathfrak{U}_\mathcal{E})$ is uniform.*

Proof: For each $\mathcal{A} \in \mathfrak{U}_\mathcal{E}$ let $\psi \in \mathcal{E}$ such that $\mathcal{A}_\psi \preccurlyeq \mathcal{A}$. We need to prove that there exists $\phi \in \mathcal{D}$ for which $\mathcal{A}_\phi \preccurlyeq f^{-1}[\mathcal{A}]$. Since, by hypothesis, for each such ψ there exists $\phi \in \mathcal{D}$ satisfying $\phi \leq f^\Leftarrow(\psi)$, which implies $\mathcal{A}_\phi \preccurlyeq \mathcal{A}_{f^\Leftarrow(\psi)}$, it suffices to show that $\mathcal{A}_{f^\Leftarrow(\psi)} \preccurlyeq f^{-1}[\mathcal{A}]$. So let $b \in L^X$ be $f^\Leftarrow(\psi)$-small. Then $f^\to(b)$ is ψ-small by Lemma 4.10. But $\mathcal{A}_\psi \preccurlyeq \mathcal{A}$. Consequently, there exists $a \in \mathcal{A}$ such that $f^\to(b) \leq a$. Hence $b \leq f^\leftarrow(a)$. ∎

Corollary 4.12. *The correspondences $(X, \mathcal{D}) \mapsto (X, \mathfrak{U}_\mathcal{D})$ and $f \mapsto f$ given by Corollary 4.9 and Proposition 4.11 define a functor $\Psi : L\text{-}\mathsf{Unif}^r \to L\text{-}\mathsf{Unif}$.* ∎

Now let us show that $\Phi \cdot \Psi = \mathrm{id}_{L\text{-}\mathsf{Unif}^r}$ and $\Psi \cdot \Phi = \mathrm{id}_{L\text{-}\mathsf{Unif}}$.

Lemma 4.13. *For any L-covers $\mathcal{A}$ and $\mathcal{B}$ of X and any $\phi, \psi \in \mathcal{H}(L, X)$ we have:*

(1) $\mathcal{A} \preccurlyeq \mathcal{A}_{\phi_\mathcal{A}}$.
(2) *If* $\mathrm{st}(\mathcal{A}) \preccurlyeq \mathcal{B}$ *then* $\mathcal{A}_{\phi_\mathcal{A}} \preccurlyeq \mathcal{B}$.
(3) $\phi_{\mathcal{A}_\phi} \leq \phi$.
(4) *If* $\phi \cdot \phi \leq \psi$ *then* $\phi \leq \phi_{\mathcal{A}_\psi}$.

Proof: (1): For every $a \in \mathcal{A}$, $a * b \neq 1_\varnothing$ implies $a \leq \mathrm{st}(b, \mathcal{A}) = \phi_\mathcal{A}(b)$, so a is $\phi_\mathcal{A}$-small.
(2): Let $a \in L^X$ be $\phi_\mathcal{A}$-small. We need to show that $a \leq b$ for some $b \in \mathcal{B}$. The case $a = 1_\varnothing$ is trivial. If $a \neq 1_\varnothing$ then, since $a = a * \bigvee_{c \in \mathcal{A}} c$, there is some $c \in \mathcal{A}$ for which $a * c \neq 1_\varnothing$. Then $a \leq \phi_\mathcal{A}(c) = \mathrm{st}(c, \mathcal{A}) \leq b$ for some $b \in \mathcal{B}$.

(3): $\phi_{\mathcal{A}_\phi}(a) = \mathrm{st}(a, \mathcal{A}_\phi) = \bigvee\{b \in L^X \mid b \text{ is } \phi\text{-small}, \, b * a \neq 1_\varnothing\} \leq \phi(a)$.
(4): If $\phi \cdot \phi \leq \psi$ then

$$\begin{aligned}\phi(a) &= \phi(a) * \bigvee\{b \in L^X \mid b \text{ is } \phi\text{-small}\} \\ &= \bigvee\{\phi(a) * b \mid b \text{ is } \phi\text{-small}, \, \phi(a) * b \neq 1_\varnothing\} \\ &\leq (\phi \cdot \phi)(a) \\ &\leq \psi(a). \end{aligned}$$

■

Theorem 4.14. *The functors Φ and Ψ establish an isomorphism between the categories L-Unif and L-Unif^r.*

Proof: Properties (1) and (2) of Lemma 4.13 imply immediately that, for any covering uniformity $\mathfrak{U}$, $\mathfrak{U}_{\mathcal{D}_\mathfrak{U}} = \mathfrak{U}$. Similarly, properties (3) and (4) of Lemma 4.13 ensure us that, for any residuated uniformity $\mathcal{D}$, $\mathcal{D}_{\mathfrak{U}_\mathcal{D}} = \mathcal{D}$. Hence $\Phi \cdot \Psi = \mathrm{id}_{L\text{-}\mathsf{Unif}^r}$ and $\Psi \cdot \Phi = \mathrm{id}_{L\text{-}\mathsf{Unif}}$. ■

5. The relationship with Hutton uniformities

Let L be a Girard quantale. Each $\phi \in \mathcal{H}(L, X)$, being join-preserving, has a right adjoint $\phi^* : L^X \to L^X$. Let $\phi^{-1} : L^X \to L^X$ be defined by

$$\phi^{-1}(a) = \phi^*(a \to 1_\varnothing) \to 1_\varnothing = \bigwedge\{b \to 1_\varnothing \in L^X \mid \phi(b) \leq a \to 1_\varnothing\} \quad ([10], [15]).$$

It is easy to check that $(\phi^{-1})^{-1} = \phi$ and $\phi_1 \leq \phi_2 \Leftrightarrow \phi_2^* \leq \phi_1^* \Leftrightarrow \phi_1^{-1} \leq \phi_2^{-1}$.

In the original definition of Hutton [15], the involved lattice is a completely distributive one with an order reversing involution, $(L, \leq, ')$. Our context is a bit more general and complete distributivity is not needed (cf. [29]). We say that a non-empty subset $\mathcal{D}$ of $\mathcal{H}(L, X)$ is a Hutton L-uniformity on X if $\mathcal{D}$ satisfies the following axioms:

(H1) $\forall \phi \in \mathcal{D}$, $\phi(a) \geq a$ for all $a \in L^X$.
(H2) $\phi_1 \in \mathcal{D}$ and $\phi_1 \leq \phi_2 \in \mathcal{H}(L, X)$ implies $\phi_2 \in \mathcal{D}$.
(H3) For each $\phi_1, \phi_2 \in \mathcal{D}$ there exists $\phi_3 \in \mathcal{D}$ such that $\phi_3 \leq \phi_1$ and $\phi_3 \leq \phi_2$.
(H4) For each $\phi_1 \in \mathcal{D}$ there exists $\phi_2 \in \mathcal{D}$ such that $\phi_2 \cdot \phi_2 \leq \phi_1$.
(H5) $\forall \phi \in \mathcal{D}$, $\phi^{-1} \in \mathcal{D}$.

According to Hutton [15] an element $\phi \in \mathcal{D}$ is *symmetric* whenever $\phi = \phi^{-1}$. The following lemma shows that our definition of symmetry coincides with the original one of Hutton.

Lemma 5.1. *Let L be a Girard quantale. For each $\phi \in \mathcal{H}(L,X)$, ϕ is symmetric if and only if $\phi = \phi^{-1}$.*

Proof: If ϕ is symmetric then $\phi(b) \leq a \to 1_\varnothing \Leftrightarrow \phi(a) \leq b \to 1_\varnothing$, which implies

$$\begin{aligned}\phi^{-1}(a) &= \bigwedge\{b \to 1_\varnothing \in L^X \mid \phi(b) \leq a \to 1_\varnothing\}\\ &= \bigwedge\{b \to 1_\varnothing \in L^X \mid \phi(a) \leq b \to 1_\varnothing\}\\ &= \phi(a).\end{aligned}$$

Conversely, if $\phi = \phi^{-1}$ then

$$\begin{aligned}\phi(a) * b = 1_\varnothing &\Leftrightarrow \phi^{-1}(a) * b = 1_\varnothing\\ &\Leftrightarrow b \leq \phi^{-1}(a) \to 1_\varnothing = \phi^*(a \to 1_\varnothing)\\ &\Leftrightarrow \phi(b) \leq a \to 1_\varnothing\\ &\Leftrightarrow a * \phi(b) = 1_\varnothing,\end{aligned}$$

which shows that ϕ is symmetric. ■

Lemma 5.2. *For each $\phi \in \mathcal{H}(L,X)$, $\phi^{-1} \cdot \phi$ and $\phi \cdot \phi^{-1}$ are symmetric.*

Proof: We only prove the first assertion (the other may be proved similarly):

$$\begin{aligned}\phi^{-1}(\phi(a)) * b = 1_\varnothing &\Leftrightarrow b \leq \phi^{-1}(\phi(a)) \to 1_\varnothing = \phi^*(\phi(a) \to 1_\varnothing)\\ &\Leftrightarrow \phi(b) \leq \phi(a) \to 1_\varnothing\\ &\Leftrightarrow \phi(a) * \phi(b) = 1_\varnothing\\ &\Leftrightarrow \phi(a) \leq \phi(b) \to 1_\varnothing\\ &\Leftrightarrow a \leq \phi^*(\phi(b) \to 1_\varnothing) = \phi^{-1}(\phi(b)) \to 1_\varnothing\\ &\Leftrightarrow \phi^{-1}(\phi(b)) * a = 1_\varnothing\\ &\Leftrightarrow a * \phi^{-1}(\phi(b)) = 1_\varnothing.\end{aligned}$$ ■

Proposition 5.3. *Let L be a Girard quantale and $\mathcal{D} \subseteq \mathcal{H}(L,X)$. Then $\mathcal{D}$ satisfies* (H2)*,* (H3)*,* (H4) *and* (H5) *if and only if it satisfies* (R2)*,* (R3)*,* (R4) *and* (R5)*.*

Proof: Axioms (H2), (H3) and (H4) are precisely (R2), (R3) and (R4) respectively. Finally, in the presence of these axioms, (H5)⇔(R5):
⇒: Let $\phi \in \mathcal{D}$ and use (H4) to get $\psi \in \mathcal{D}$ such that $\psi \cdot \psi \leq \phi$. By (H5), $\phi^{-1} \in \mathcal{D}$. Applying (H3), consider $\xi \in \mathcal{D}$ such that $\xi \leq \psi$ and $\xi \leq \psi^{-1}$. Then $\xi^{-1} \leq (\psi^{-1})^{-1} = \psi$ and $\xi \cdot \xi^{-1} \leq \psi \cdot \psi \leq \phi$. Since $\xi \cdot \xi^{-1}$ is symmetric by Lemma 5.2, $\mathcal{D}$ satisfies (R5).
⇐: Let $\phi \in \mathcal{D}$. By (R5) there exists a symmetric $\psi \in \mathcal{D}$ such that $\psi \leq \phi$, that is, $\psi^{-1} \leq \phi^{-1}$. By Lemma 5.1, $\psi^{-1} = \psi \in \mathcal{D}$, thus $\phi^{-1} \in \mathcal{D}$ by (R2). ■

Therefore, since axiom (H1) is a consequence of (R1) by Proposition 4.1(3), we have immediately:

Corollary 5.4. *Let L be a Girard quantale. Every residuated L-uniformity on X is a Hutton uniformity on X.* ■

The converse is not true in general (recall Proposition 4.1(4)).

Remark 5.5. In the classical context, for a set X, binary relations $R \subseteq X \times X$ are described by Galois connections between power sets in two particularly simple ways [4]:

(1) A Galois connection between $\mathcal{P}(X)$ and $\mathcal{P}(X)$ is called a *polarity* [2] on $\mathcal{P}(X)$. Any relation $R \subseteq X \times X$ induces a polarity $(R_\forall, R^\forall)$ on $\mathcal{P}(X)$, defined by

$$\begin{aligned} R_\forall(A) &:= \{x \in X \mid \forall y \in A\ (x,y) \in R\} \quad \text{for } A \subseteq X, \\ R^\forall(B) &:= \{y \in X \mid \forall x \in B\ (x,y) \in R\} \quad \text{for } B \subseteq X. \end{aligned}$$

Conversely, every polarity (f, g) on $\mathcal{P}(X)$ induces a relation

$$(x,y) \in R_{(f,g)} \equiv y \in f(\{x\}) \text{ (or equivalently, } x \in g(\{y\})).$$

Since $\big((R_{(f,g)})_\forall, (R_{(f,g)})^\forall\big) = (f, g)$ and $R_{(R_\forall, R^\forall)} = R$, there is a bijection between relations $R \subseteq X \times X$ and polarities on $\mathcal{P}(X)$.

(2) The covariant case of Galois connections between power sets also describe all binary relations $R \subseteq X \times X$. Indeed, any relation $R \subseteq X \times X$ induces an *axiality* [4] $(R_\exists, R^\forall)$ on $\mathcal{P}(X)$, that is, a Galois connection between $\mathcal{P}(X)$ and $\mathcal{P}(X)^{op}$, defined by

$$\begin{aligned} R_\exists(A) &:= \{y \in X \mid \exists x \in A\ (x,y) \in R\} \quad \text{for } A \subseteq X, \\ R^\forall(B) &:= \{x \in X \mid \forall y \in X\ (x,y) \in R \Rightarrow y \in B)\} \quad \text{for } B \subseteq X. \end{aligned}$$

If we define, for each axiality (f, g) on $\mathcal{P}(X)$, the relation

$$(x, y) \in R_{(f,g)} \equiv y \in f(\{x\}),$$

then $\big((R_{(f,g)})_\exists, (R_{(f,g)})^\forall\big) = (f, g)$ and $R_{(R_\exists, R^\forall)} = R$.

Under bijection (1) (resp. (2)), entourages E, i.e. reflexive relations, correspond to polarities (resp. axialities) that are expansive on atoms, that is, $\{x\} \subseteq E_\forall(\{x\})$ (resp. $\{x\} \subseteq E_\exists(\{x\})$) for every $x \in X$. Nevertheless, to be expansive on atoms, have different meaning in polarities and axialities; in the case of polarities, this is equivalent to $\{A \subseteq X | A \subseteq E_\forall(A)\}$ being a cover of X while in the case of axialities, it is equivalent to $E_\exists$ being expansive on subsets (cf. axiom (H1)), and also to the collection of all $E_\exists$-small sets being a cover of X (cf. axiom (R1)).

This equivalence, in the case of axialities, relies on two facts: firstly, $\mathcal{P}(X)$ being atomic, its elements can be written as unions of points (atoms) and secondly, the involved maps $E_\exists$ are sup-preserving.

If the lattice 2 is replaced by a general L, even if the previous situation models a relation between polarities (axialites) from L^X to L^X and L-valued binary relations, i.e:, elements in $L^{X\times X}$ (see [10]), one cannot expect that, in this general case, either working with expansive maps or working with maps whose small elements form a cover might be still equivalent. And it is really the case, as examples in Remark 4.2 shows. Indeed, the equivalence established between L-valued binary relations and some family of sup-preserving maps from L^X to L^X, (see [10]) transforms reflexive L-valued binary relations (maps f from $X \times X \to L$, such that $f(x, x) = 1$) into expansive maps.

It would be interesting to investigate under which conditions the work in [5] (Sections 3 and 4; see also [23]) may be extended to our setting here, showing whether uniform structures stated in terms of residuated maps $L^X \to L^X$ (that is, elements of $\mathcal{H}(L, X)$) as we defined in Section 4, may be equivalently described in terms of Galois maps $L^X \to L^X$ (that is, residuated maps $L^X \to (L^X)^{op}$).

6. L-valued frames

For the motivation and justification for this notion see [26]. There the authors show that levels and level topologies may be interpreted as a *system of frame homomorphisms satisfying some categorical conditions.* Indeed, L-fuzzy and traditional structures, can be related via the functor ι_L and its levels $\{\iota_\alpha : \alpha \in L\}$ (see, among others [19, 33, 32, 34, 18] for topologies and [20, 21, 8, 9] for filters and uniformities).

Two relevant facts of the level (topological) functors $\{\iota_\alpha \mid \alpha \in L\}$ are:

(1) The collection $\{\iota_\alpha \mid \alpha \in L\}$ is nonincreasing (on functions).
(2) The collection $\{\iota_\alpha \mid \alpha \in L\}$ is a subbase for ι_L, that is: $\langle \bigcup_{\alpha\in L} \iota_\alpha \rangle = \iota_L$.

The categorical interpretation of these properties, lead Pultr and Rodabaugh [26] to introduce the notion of an L-valued frame. The main purpose is to have a general L-structure which relates to frames in the way L-topological spaces relates to topological spaces.

Let $L_1 = L \setminus \{1\}$. For each $\alpha \in L_1$ and $a \in L^X$, let

$$\iota_\alpha(a) = \{x \in X \mid a(x) \nleq \alpha\}. \tag{6.1}$$

This defines the *α-level mapping* $\iota_\alpha : L^X \to 2^X$. For each L-topological space (X, τ), the associated crisp topology $\iota_L^{\mathsf{T}}(\tau)$ is the topology on X with subbase: $\{\iota_\alpha(a) : \alpha \in L_1, a \in \tau\}$, that is:

$$\iota_L^{\mathsf{T}}(\tau) = \langle \bigcup \{\iota_\alpha(\tau) : \alpha \in L_1, \}\rangle$$

The correspondence $(X, \tau) \mapsto (X, \iota_L^{\mathsf{T}}(\tau))$ defines a functor $\iota_L^{\mathsf{T}} : L\text{-}\mathsf{Top} \to \mathsf{Top}$ (satisfying $\iota_L^{\mathsf{T}} \cdot \chi_L^{\mathsf{T}} = \mathrm{id}_{\mathsf{Top}}$): for each L-continuous map $f : (X, \tau_1) \to (Y, \tau_2)$, $f(X, \iota_L^{\mathsf{T}}(\tau_1)) \to (Y, \iota_L^{\mathsf{T}}(\tau_2))$ is continuous, since $b \cdot f \in \tau_1$ and $f^{-1}(\iota_\alpha(b)) = \iota_\alpha(b \cdot f)$ for every $b \in \tau_2$.

Recall also the (dual) adjunction between L-Top and Frm [12]: the contravariant functor $\mathcal{O}_L : L\text{-}\mathsf{Top} \to \mathsf{Frm}$ given by $\mathcal{O}_L(X, \tau) = \tau$ and $\mathcal{O}_L(f)(b) = b \cdot f$, has a right adjoint $\Sigma_L : \mathsf{Frm} \to L\text{-}\mathsf{Top}$ defined by

$$\Sigma_L(M) = (\{p : M \to L \mid p \in \mathsf{Frm}\}, \{\widehat{m} \mid m \in M\}),$$

where $\widehat{m}(p) = p(m)$ and $\Sigma_L(h)(p) = p \cdot h$ for every $h : M \to N$.

A family of morphisms $(f_i : A \to B_i)_{i\in I}$ in a category is said to be *jointly monic* (also *mono-source* in [1]) if $f_i \cdot g = f_i \cdot h$ for every $i \in I$ imply $g = h$. The family $(f_i : A_i \to B)_{i\in I}$ is said to be *jointly epic* (or an *epi-sink* [1]) if it satisfies the dual condition. A jointly monic family $(f_i : A \to B_i)_{i\in I}$ is *jointly extremally monic* if, moreover, $f_i = g_i \cdot e$, for every $i \in I$, with e epimorphic (that is, e is "right-cancellable" with respect to the composition), implies that e is an isomorphism. Dually, a jointly epic family $(f_i : A_i \to B)_{i\in I}$ is *jointly extremally epic* if $f_i = m \cdot g_i$ for every $i \in I$, with m monomorphic (that is, m is "left-cancellable" with respect to the composition), implies that m is an isomorphism.

An *L-valued frame* (shortly, *L-frame*) [26] is a system of frame homomorphisms

$$\mathcal{M} \equiv (\varphi_\alpha^{\mathcal{M}} : \mathcal{M}^{\mathfrak{u}} \to \mathcal{M}^{\mathfrak{l}})_{\alpha \in L_1}$$

satisfying the following axioms:

(1) For each non-empty $S \subseteq L_1$, $\varphi^{\mathcal{M}}_{\bigwedge S} = \bigvee_{\beta \in S} \varphi^{\mathcal{M}}_\beta$.
(2) $(\varphi_\alpha^{\mathcal{M}})_{\alpha \in L_1}$ is jointly extremally epic in Frm, that is, $\mathcal{M}^{\mathfrak{l}} = \langle \bigcup_\alpha \varphi_\alpha^{\mathcal{M}}[\mathcal{M}^{\mathfrak{u}}] \rangle$.
(3) $(\varphi_\alpha^{\mathcal{M}})_{\alpha \in L_1}$ is jointly monic in Frm.

An *L-frame homomorphism* [26] $h : \mathcal{M} \to \mathcal{N}$ is an ordered pair of frame homomorphisms

$$h \equiv (h^{\mathfrak{u}} : \mathcal{M}^{\mathfrak{u}} \to \mathcal{N}^{\mathfrak{u}}, h^{\mathfrak{l}} : \mathcal{M}^{\mathfrak{l}} \to \mathcal{N}^{\mathfrak{l}})$$

satisfying $h^{\mathfrak{l}} \cdot \varphi_\alpha^{\mathcal{M}} = \varphi_\alpha^{\mathcal{N}} \cdot h^{\mathfrak{u}}$ for every $\alpha \in L_1$. The resulting category is denoted by L-Frm.

It follows immediately from (2) and (3) that in any L-frame homomorphism $h \equiv (h^{\mathfrak{u}}, h^{\mathfrak{l}})$, each of the frame homomorphisms $h^{\mathfrak{u}}, h^{\mathfrak{l}}$ guarantees the uniqueness of the other.

If L is a linearly ordered complete lattice then, in the definition of $\iota_\alpha(a)$, we may replace $\not\leq$ by $>$ and the mappings $\iota_\alpha : \tau \to \iota_L^{\top}(\tau)$ ($\alpha \in L_1$) given by (6.1) are frame homomorphisms. Therefore, for each L-topological space (X, τ), the system

$$\mathcal{O}(X, \tau) = (\iota_\alpha^{\mathcal{O}} : \mathcal{O}^u \equiv \tau \to \mathcal{O}^l \equiv \iota_L^{\top}(\tau))_{\alpha \in L_1}$$

is an L-frame. This is the motivating example for the notion of L-valued frame (cf. [26]).

Remark 6.1. More generally, the linear L can be replaced by a spatial frame as far as L_1 is replaced by the meet-irreducible elements of L (cf. [13] for the details).

Further, for every L-continuous map $f : (X, \tau_1) \to (Y, \tau_2)$, the pair $\mathcal{O}(f) = (\mathcal{O}^{\mathfrak{u}}(f), \mathcal{O}^{\mathfrak{l}}(f))$, with $\mathcal{O}^{\mathfrak{u}}(f)(b) = f^{\leftarrow}(b)$ and $\mathcal{O}^{\mathfrak{l}}(f)(B) = f^{-1}(B)$, for all $b \in L^Y$ and $B \in 2^Y$, is an L-frame homomorphism. This defines a contravariant functor $\mathcal{O} : L$-$\mathsf{Top} \to L$-Frm. There are also the *L-valued spectrum* functor $\Sigma : L$-$\mathsf{Frm} \to L$-Top, right adjoint to $\mathcal{O}$, and the forgetful functor

$\iota_L^{\mathsf{F}} : L\text{-}\mathsf{Frm} \to \mathsf{Frm}$ (cf. [26] for the details), such that the diagram

$$\begin{tikzcd}
L\text{-}\mathsf{Top} \arrow[r, shift left, "\mathcal{O}"] \arrow[d, shift left, "\iota_L^{\mathsf{T}}"] & L\text{-}\mathsf{Frm} \arrow[l, shift left, "\Sigma"] \arrow[d, shift left, "\iota_L^{\mathsf{F}}"] \\
\mathsf{Top} \arrow[u, shift left, "\chi_L^{\mathsf{T}}"] \arrow[r, shift left, "\mathcal{O}"] & \mathsf{Frm} \arrow[u, shift left, "\chi_L^{\mathsf{F}}"] \arrow[l, shift left, "\Sigma"]
\end{tikzcd}$$

commutes.

When $L = 2$, an L-frame is just one frame homomorphism $\varphi_0^{\mathcal{M}}$ which, by conditions (2) and (3), must be an isomorphism. So, an L-valued frame stands for a pair of (possibly distinct) isomorphic frames $(\mathcal{M}^{\mathfrak{u}}, \mathcal{M}^{\mathfrak{l}})$ and each L-valued frame homomorphism is a pair of frame morphisms $(h^{\mathfrak{u}}, h^{\mathfrak{l}})$ such that each one factors through the other via an isomorphism. Therefore, the category 2-Frm is equivalent to Frm. Indeed, 2-Frm is the functor category $\mathsf{Frm}^{\mathbf{2}}$ (where $\mathbf{2}$ is the category with 2 objects $\{u, l\}$ and an isomorphism $u \to l$), and the latter category is clearly equivalent to Frm via functors $F : \mathsf{Frm} \to \mathsf{Frm}^{\mathbf{2}}$ and $G : \mathsf{Frm}^{\mathbf{2}} \to \mathsf{Frm}$ defined by $F(M) = (M, M)$ with $\varphi_0^M = \mathrm{id}_M$, $F(h) = (h, h)$, $G(M^{\mathfrak{u}}, M^{\mathfrak{l}}) = M^{\mathfrak{u}}$ and $G(h^{\mathfrak{u}}, h^{\mathfrak{l}}) = h^{\mathfrak{u}}$.

7. The uniform crisp modification of a covering L-valued uniform space

Let L be a linearly ordered complete lattice. This is an integral quantale with $* = \wedge$. Let $(X, \mathfrak{U})$ be a covering L-uniform space. For each $\mathcal{A} \in \mathfrak{U}$ and $\alpha \in L_1$ let

$$\iota_\alpha(\mathcal{A}) = \{\iota_\alpha(a) \mid a \in \mathcal{A}\}.$$

We state without proof some basic facts satisfied by the maps $\{\iota_\alpha : \alpha \in L_1\}$:

Remark 7.1. For every $\mathcal{A} \subset L^X$, $f : X \to Y$ and $\alpha, \beta \in L_1$, we have:

(1) $\iota_\alpha(\bigvee \mathcal{A}) = \bigcup_{a \in \mathcal{A}} \iota_\alpha(a)$.
(2) $\iota_\alpha(\bigwedge \mathcal{A}) = \bigcap_{a \in \mathcal{A}} \iota_\alpha(a)$ if $\mathcal{A}$ is finite.
(3) $\alpha \leq \beta \Rightarrow \iota_\beta(a) \subseteq \iota_\alpha(a)$.
(4) $f^{-1}(\iota_\alpha(b)) = \iota_\alpha(f^{\leftarrow}(b))$.

We have also the following:

Proposition 7.2. *Let $\mathcal{A}, \mathcal{B} \in \mathfrak{U}$ and $\alpha \in L_1$. Then:*

(1) $\iota_\alpha(\mathcal{A})$ *is a cover of* X.
(2) *If* $\mathcal{A} \preccurlyeq \mathcal{B}$ *then* $\iota_\alpha(\mathcal{A}) \preccurlyeq \iota_\alpha(\mathcal{B})$. *Hence* $\iota_\alpha(\mathcal{A} \wedge \mathcal{B}) \preccurlyeq \iota_\alpha(\mathcal{A})$ *and* $\iota_\alpha(\mathcal{A} \wedge \mathcal{B}) \preccurlyeq \iota_\alpha(\mathcal{B})$.
(3) $\mathrm{st}(\iota_\alpha(b), \iota_\alpha(\mathcal{A})) \subseteq \iota_\alpha(\mathrm{st}(b, \mathcal{A}))$.
(4) *If* $\mathrm{st}(\mathcal{A}) \preccurlyeq \mathcal{B}$ *then* $\mathrm{st}(\iota_\alpha(\mathcal{A})) \preccurlyeq \iota_\alpha(\mathcal{B})$.

Proof: (1): Since $\bigvee \mathcal{A} = 1_X$ then, for each $\alpha \in L_1$ and each $x \in X$ there exists $a \in \mathcal{A}$ such that $a(x) > \alpha$. Consequently, $\bigcup_{a \in \mathcal{A}} \{x \in X \mid a(x) > \alpha\} = X$, that is, $\bigcup_{a \in \mathcal{A}} \iota_\alpha(a) = X$.
(2): It is straightforward.
(3): It follows from 7.1 (1).
(4): For any $a \in \mathcal{A}$, let $\mathrm{st}(\iota_\alpha(a), \iota_\alpha(\mathcal{A})) \in \mathrm{st}(\iota_\alpha(\mathcal{A}))$ and $b \in \mathcal{B}$ satisfying $\mathrm{st}(a, \mathcal{A}) \leq b$. By property (3) of 7.1 and previous (3) one has $\mathrm{st}(\iota_\alpha(a), \iota_\alpha(\mathcal{A})) \subseteq \iota_\alpha(\mathrm{st}(a, \mathcal{A})) \subseteq \iota_\alpha(b)$. ■

It follows immediately from Proposition 7.2 that:

Corollary 7.3. *The family* $\{\iota_\alpha(\mathcal{A}) \mid \mathcal{A} \in \mathfrak{U}, \alpha \in L_1\}$ *is a base for a uniformity* $\iota_L^{\mathsf{U}}(\mathfrak{U})$ *on* X. ■

Proposition 7.4. *For any* $f : (X, \mathfrak{U}) \to (Y, \mathfrak{V})$ *in* L-Unif, $f : (X, \iota_L^{\mathsf{U}}(\mathfrak{U})) \to (Y, \iota_L^{\mathsf{U}}(\mathfrak{V}))$ *is a morphism of* Unif.

Proof: For each $\mathcal{B} \in \mathfrak{V}$ and $\alpha \in L_1$,

$$f^{-1}[\iota_\alpha(\mathcal{B})] = \{f^{-1}(\iota_\alpha(b)) \mid b \in \mathcal{B}\}$$

and $f^{-1}(\iota_\alpha(b)) = \iota_\alpha(f^{\leftarrow}(b))$ (Remark 7.1). By hypothesis, $f^{-1}[\mathcal{B}] = \{f^{\leftarrow}(b) \mid b \in \mathcal{B}\} \in \mathfrak{U}$, so

$$\iota_\alpha(f^{-1}[\mathcal{B}]) = \{\iota_\alpha(f^{\leftarrow}(b)) \mid b \in \mathcal{B}\} \in \iota_L^{\mathsf{U}}(\mathfrak{U}),$$

which shows that $f^{-1}[\iota_\alpha(\mathcal{B})] \in \iota_L^{\mathsf{U}}(\mathfrak{U})$ as required. ■

Thus we have a functor $\iota_L^{\mathsf{U}} : L\text{-}\mathsf{Unif} \to \mathsf{Unif}$ such that $\iota_L^{\mathsf{U}} \cdot \chi_L^{\mathsf{U}} = \mathrm{id}_{\mathsf{Unif}}$. Recall the forgetful functor $F_3 : L\text{-}\mathsf{Unif} \to L\text{-}\mathsf{Top}$ from Section 3. We have:

Proposition 7.5. $F_1 \cdot \iota_L^{\mathsf{U}} = \iota_L^{\mathsf{T}} \cdot F_3$.

Proof: For objects we need to show that $F_1(\iota_L^{\mathsf{U}}(X,\mathfrak{U})) = \iota_L^{\mathsf{T}}(F_3(X,\mathfrak{U}))$ for every L-uniform space $(X,\mathfrak{U})$, that is, $\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})$ is precisely the topology $\mathcal{T}_{\iota_L^{\mathsf{U}}(\mathfrak{U})}$ induced by the crisp uniformity $\iota_L^{\mathsf{U}}(\mathfrak{U})$.

$\underline{\mathcal{T}_{\iota_L^{\mathsf{U}}(\mathfrak{U})} \subseteq \iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})}$: Let $A \in \mathcal{T}_{\iota_L^{\mathsf{U}}(\mathfrak{U})}$. Then for every $x \in A$ there exists $\iota_{\alpha_x}(\mathcal{A}_x) \in \iota_L^{\mathsf{U}}(\mathfrak{U})$ such that $\mathrm{st}(\{x\}, \iota_{\alpha_x}(\mathcal{A}_x)) \subseteq A$. Therefore

$$A \supseteq \bigcup_{x\in A} \mathrm{st}(\{x\}, \iota_{\alpha_x}(\mathcal{A}_x)) = \bigcup_{x\in A} \Big(\bigcup \{\iota_{\alpha_x}(a) \mid a \in \mathcal{A}_x, x \in \iota_{\alpha_x}(a)\} \Big).$$

Since the reverse inclusion is obvious, we have

$$A = \bigcup_{x\in A} \mathrm{st}(\{x\}, \iota_{\alpha_x}(\mathcal{A}_x)) \in \iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}}).$$

$\underline{\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}}) \subseteq \mathcal{T}_{\iota_L^{\mathsf{U}}(\mathfrak{U})}}$: Now let $\iota_\alpha(a) \in \iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})$ and $x \in \iota_\alpha(a)$. Then $a(x) > \alpha$. Since $a \in \tau_{\mathfrak{U}}$, there exists $b \in L^X$ with $x \in \iota_\alpha(b)$ and $\mathrm{st}(b,\mathcal{A}) \leq a$, for some $\mathcal{A} \in \mathfrak{U}$. Hence $\iota_\alpha(\mathrm{st}(b,\mathcal{A})) \subseteq \iota_\alpha(a)$. Also, since $x \in \iota_\alpha(b)$, we have

$$x \in \mathrm{st}(\{x\}, \iota_\alpha(\mathcal{A})) \subseteq \mathrm{st}(\iota_\alpha(b), \iota_\alpha(\mathcal{A}) \subseteq \iota_\alpha \mathrm{st}(b,\mathcal{A}) \subseteq \iota_\alpha(a).$$

For morphisms the proof is straightforward. ∎

In conclusion, both squares in the following diagram commute:

$$\begin{tikzcd}
L\text{-}\mathsf{Unif} \arrow[r, "F_3"] \arrow[d, shift left, "\iota_L^{\mathsf{U}}"] & L\text{-}\mathsf{Top} \arrow[d, shift left, "\iota_L^{\mathsf{T}}"] \\
\mathsf{Unif} \arrow[r, "F_1"'] \arrow[u, shift left, "\chi_L^{\mathsf{U}}"] & \mathsf{Top} \arrow[u, shift left, "\chi_L^{\mathsf{T}}"]
\end{tikzcd}$$

8. The missing vertex: L-valued uniform frames

We say that $(\mathcal{M},\mathcal{C})$ is an *L-valued uniform frame* if

$$\mathcal{M} = (\varphi_\alpha^{\mathcal{M}} : \mathcal{M}^{\mathfrak{u}} \to \mathcal{M}^{\mathfrak{l}})_{\alpha \in L_1}$$

is an L-valued frame and $\mathcal{C}$ is a frame uniformity on the lower frame $\mathcal{M}^{\mathfrak{l}}$. An *$L$-valued uniform homomorphism* $h : (\mathcal{M},\mathcal{C}) \to (\mathcal{N},\mathcal{D})$ is a pair

$$(h^{\mathfrak{u}} : \mathcal{M}^{\mathfrak{u}} \to \mathcal{N}^{\mathfrak{u}}, h^{\mathfrak{l}} : (\mathcal{M}^{\mathfrak{l}},\mathcal{C}) \to (\mathcal{N}^{\mathfrak{l}},\mathcal{D}))$$

with $h^{\mathfrak{u}}$ a frame homomorphism and $h^{\mathfrak{l}}$ a uniform frame homomorphism satisfying $h^{\mathfrak{l}} \cdot \varphi_\alpha^{\mathcal{M}} = \varphi_\alpha^{\mathcal{N}} \cdot h^{\mathfrak{u}}$ for every $\alpha \in L_1$.

When $L = 2$, an L-valued uniform frame $(\mathcal{M}, \mathcal{C})$ is just a frame isomorphism $\varphi_0^{\mathcal{M}} : \mathcal{M}^{\mathfrak{u}} \to (\mathcal{M}^{\mathfrak{l}}, \mathcal{C})$ and an L-valued uniform homomorphism is a pair of frame morphisms $(h^{\mathfrak{u}}, h^{\mathfrak{l}})$, the latter one being uniform, such that each one factors through the other via an isomorphism. So, the category $\mathsf{2\text{-}UFrm}$ is clearly equivalent to UFrm via functors $F : \mathsf{UFrm} \to \mathsf{2\text{-}UFrm}$ and $G : \mathsf{2\text{-}UFrm} \to \mathsf{UFrm}$ defined by $F(\mathcal{M}, \mathcal{C}) = (\varphi_0^{\mathcal{M}} : \mathcal{M} \to (\mathcal{M}, \mathcal{C}))$ with $\varphi_0^{\mathcal{M}} = \mathrm{id}_{\mathcal{M}}$, $F(h) = (h, h)$, $G(\varphi_0^{\mathcal{M}} : \mathcal{M}^{\mathfrak{u}} \to (\mathcal{M}^{\mathfrak{l}}, \mathcal{C})) = (\mathcal{M}^{\mathfrak{l}}, \mathcal{C})$ and $G(h^{\mathfrak{u}}, h^{\mathfrak{l}}) = h^{\mathfrak{l}}$.

Let L be linearly ordered. Then, for each L-valued uniform space $(X, \mathfrak{U})$, we have

$$\mathcal{O}\iota_L^{\mathsf{U}}(X, \mathfrak{U}) = (\iota_\alpha : \tau_{\mathfrak{U}} \to \iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}}))_{\alpha \in L_1} \in L\text{-}\mathsf{Frm}.$$

In order to establish the open functor $\mathcal{O} : L\text{-}\mathsf{Unif} \to L\text{-}\mathsf{UFrm}$ we need to endow the crisp topology $\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})$ of $(X, \tau_{\mathfrak{U}})$ with a canonical frame uniformity. This is possible because the topology $\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})$ is always regular:

Proposition 8.1. *For every covering L-uniform space $(X, \mathfrak{U})$, the topology $\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})$ is regular.*

Proof: It follows from 7.5, since $\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}}) = \mathcal{T}_{\iota_L^{\mathsf{U}}(\mathfrak{U})}$. ■

Now, let $\mathfrak{U}_{\tau_{\mathfrak{U}}}$ be the collection of all $\tau_{\mathfrak{U}}$-open L-covers in $\mathfrak{U}$ (i.e. all $\mathcal{A} \in \mathfrak{U}$ such that $\mathcal{A} \subseteq \tau_{\mathfrak{U}}$). Each $\mathcal{A} \in \mathfrak{U}_{\tau_{\mathfrak{U}}}$ is a cover of the spatial frame $\tau_{\mathfrak{U}}$. For each $\mathcal{A} \in \mathfrak{U}_{\tau_{\mathfrak{U}}}$ and $\alpha \in L_1$ let $\iota_\alpha(\mathcal{A}) = \{\iota_\alpha(a) \mid a \in \mathcal{A}\}$. Then, by Remark 7.1 (1), $\iota_\alpha(\mathcal{A})$ is a cover of $\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})$.

Proposition 8.2. *For every covering L-uniform space $(X, \mathfrak{U})$, $\{\iota_\alpha(\mathcal{A}) \mid \alpha \in L_1, \mathcal{A} \in \mathfrak{U}_{\tau_{\mathfrak{U}}}\}$ is a base for a frame uniformity $\mathcal{C}_{\mathfrak{U}}$ on $\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}})$.*

Proof: It remains to show the star-refinement condition. This is a consequence of 7.2 (3), since $\mathrm{st}(\mathcal{B}) \preccurlyeq \mathcal{A}$ implies $\mathrm{st}(\iota_\alpha(\mathcal{B})) \preccurlyeq \iota_\alpha(\mathcal{A})$ for every $\alpha \in L_1$. ■

It is now easy to check that the correspondence

$$(X, \mathfrak{U}) \longmapsto \left(\iota_\alpha : \tau_{\mathfrak{U}} \to \left(\iota_L^{\mathsf{T}}(\tau_{\mathfrak{U}}), \mathcal{C}_{\mathfrak{U}}\right)\right)_{\alpha \in L_1}$$

is functorial and establishes a functor $\mathcal{O} : L\text{-}\mathsf{Unif} \to L\text{-}\mathsf{UFrm}$. Conversely, it is not hard to define the "spectrum functor" $\Sigma : L\text{-}\mathsf{UFrm} \to L\text{-}\mathsf{Unif}$, right adjoint to $\mathcal{O}$ and such that the diagram

$$\begin{tikzcd}
L\text{-}\mathsf{Unif} \arrow[r, shift left, "\mathcal{O}"] \arrow[d, shift right, "\iota_L^{\mathsf{U}}"'] & L\text{-}\mathsf{UFrm} \arrow[l, shift left, "\Sigma"] \arrow[d, shift right, "\iota_L^{\mathsf{UF}}"'] \\
\mathsf{Unif} \arrow[r, shift left, "\mathcal{O}"] \arrow[u, shift right, "\chi_L^{\mathsf{U}}"'] & \mathsf{UFrm} \arrow[l, shift left, "\Sigma"] \arrow[u, shift right, "\chi_L^{\mathsf{UF}}"']
\end{tikzcd}$$

commutes.

Finally, consider the forgetful functor $F_4 : L\text{-}\mathsf{UFrm} \to L\text{-}\mathsf{Frm}$ forgetting the uniform structure. Putting the functors here considered altogether we obtain the desired commutative cube:

$$\begin{tikzcd}
L\text{-}\mathsf{Unif} \arrow[rr, shift left] \arrow[dr, "F_3"'] \arrow[dd, shift left] & & L\text{-}\mathsf{UFrm} \arrow[ll, shift left] \arrow[dr, "F_4"] \arrow[dd, shift left] & \\
& L\text{-}\mathsf{Top} \arrow[rr, shift left] \arrow[dd, shift left] & & L\text{-}\mathsf{Frm} \arrow[ll, shift left] \arrow[dd, shift left] \\
\mathsf{Unif} \arrow[rr, shift left] \arrow[uu, shift left] \arrow[dr, "F_1"'] & & \mathsf{UFrm} \arrow[ll, shift left] \arrow[uu, shift left] \arrow[dr, "F_2"] & \\
& \mathsf{Top} \arrow[rr, shift left] \arrow[uu, shift left] & & \mathsf{Frm} \arrow[ll, shift left] \arrow[uu, shift left]
\end{tikzcd}$$

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