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A CONVERGENCE RESULT IN THE STUDY OF BONE REMODELING CONTACT PROBLEMS

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ABSTRACT: We consider the approximation of a bone remodeling model with the Signorini contact conditions by a contact problem with normal compliant obstacle, when the obstacle's deformability coefficient converges to zero (that is, the obstacle's stiffness tends to infinity). The variational problem is a coupled system composed of a nonlinear variational equation (in the case of normal compliance contact conditions) or a variational inequality (for the case of Signorini's contact conditions), for the mechanical displacement field, and a first-order ordinary differential equation for the bone remodeling function. A theoretical result, which states the convergence of the contact problem with normal compliance contact law to the Signorini problem, is then proved. Some numerical simulations, involving examples in one and two dimensions, are reported and show this behaviour.

KEYWORDS: Bone remodeling, Signorini conditions, normal compliance, weak solutions, convergence, numerical simulations.

Introduction

In this work, two contact problems between a body, made of a bone remodeling material, and an obstacle are considered from the variational point of view. The bone remodeling model, derived by Cowin and Hegedus (see [2, 11]), is a generalization of the nonlinear elasticity, and it is based on the fact that the "living bone is continuously adapting itself to external stimuli". The ability of these models to predict the bone remodeling is of great importance because this process has an enormous effect on the overall behaviour and health of the entire body.

During the last ten years, some papers dealt with mathematical issues of these models as the existence and uniqueness of weak solutions under some quite strong assumptions (see, e.g., [15, 16, 17, 18, 20]), the analysis of an asymptotic rod model ([5, 6, 7, 8]) or the numerical stability of finite element models ([10]). This paper concludes somehow the results presented in [4] and [7], providing a theoretical result which states the convergence of the solution

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to the normal compliance contact problem to the solution to the Signorini contact problem, when the obstacle's deformability coefficient converges to zero (i.e., when the stiffness coefficient tends to infinity). Moreover, some numerical results are also shown to demonstrate this convergence numerically.

The paper is outlined as follows. In Section 1 we describe briefly the mechanical problems and we derive their variational formulation. Our main theoretical result is then proved in Theorem 3. Some numerical simulations, involving examples in one and two dimensions, are then provided in Section 2, which demonstrate the accuracy and the behaviour of this convergence.

1. Mechanical and variational problems

In this section we present a brief description of the models (see [4] for further details of the models and also [7] for the description of an asymptotic bone remodeling rod model with Signorini contact conditions).

Let us denote by $\Omega \subset \mathbb{R}^d$, $d = 1, 2, 3$, an open bounded domain and let $\Gamma = \partial\Omega$ be its outer surface which is assumed to be Lipschitz continuous and it is divided into three disjoint parts Γ_D, Γ_N and Γ_C. The body is being acted upon by a volume force of density $\boldsymbol{f}$, it is clamped on Γ_D and surface tractions with density $\boldsymbol{g}$ act on Γ_N. Finally, we assume that the body may come in contact with an obstacle, which can be deformable or rigid, on the boundary part Γ_C which is located at a distance s, measured along the outward unit normal vector $\boldsymbol{\nu}$ (see Fig. 1).

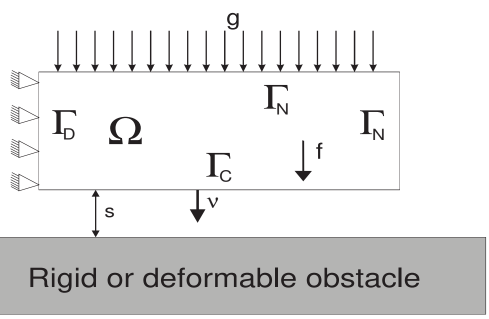

Figure 1. Contact problem for a bone remodeling body Ω.

Let $\boldsymbol{u}$ be the mechanical displacement field, $\boldsymbol{\sigma}$ the stress field, $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d$ the linearized strain field given by

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and e the so-called bone remodeling function, which measures the change in volume fraction of elastic material (present in the bone, which is a porous material), from a reference volume fraction of the elastic material denoted in the sequel by ξ_0.

According to [2, 11], the constitutive law for the body Ω is the following,

$$\boldsymbol{\sigma} = (\xi_0 + e)\mathcal{C}(e)\boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where the fourth-order tensor $\mathcal{C}(e) = (C_{ijkl}(e))_{i,j,k,l=1}^d$ is a constitutive function whose properties will be described below in formula (1). We notice that the classical Hooke's law is derived if $\xi_0 = 1$ and $e = 0$.

We turn now to the description of the contact conditions. First, we assume that the contact is produced with a deformable obstacle, and the well-known normal compliance contact condition is employed (see [12, 14]); that is, the normal stress $\sigma_\nu = \boldsymbol{\sigma\nu} \cdot \boldsymbol{\nu}$ on Γ_C is given by

$$-\sigma_\nu = \frac{1}{\mu}(u_\nu - s)_+,$$

where $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$ denotes the normal displacement in such a way that, when $u_\nu > s$, the difference $u_\nu - s$ represents the interpenetration of the body's asperities into those of the obstacle, $\mu > 0$ is a deformability coefficient (thus, $1/\mu$ represents somehow the stiffness of the obstacle) and $(u_\nu - s)_+ = \max\{0, u_\nu - s\}$.

Secondly, we assume now that the contact is produced with a rigid obstacle, and the classical Signorini contact conditions are employed (see [13]); that is,

$$u_\nu \leq s, \quad \sigma_\nu \leq 0, \quad (u_\nu - s)\sigma_\nu = 0.$$

We remark the Signorini contact conditions can be understood as the limit of the normal compliance contact condition when $\mu \to 0$.

We also assume that the contact is frictionless for both problems, i.e. the tangential component of the stress field, denoted $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu\boldsymbol{\nu}$, vanishes on the contact surface.

Let us denote by $\cdot$ the inner product in $\mathbb{R}^d$ and by $|\cdot|$ its corresponding norm. Let $\mathbb{S}^d$ be the space of second order symmetric tensors on $\mathbb{R}^d$, or equivalently, the space of symmetric matrices of order d, and let $:$ be its inner product and $|\cdot|$ its norm.

The evolution of the bone remodeling function is obtained from the following first-order ordinary differential equation (see [2, 11]),

$$\dot{e} = a(e) + \mathcal{A}(e) : \boldsymbol{\varepsilon}(\boldsymbol{u}),$$

where $a(e)$ and the second order tensor $\mathcal{A}(e) = (A_{ij}(e))_{i,j=1}^d$ are constitutive material coefficients depending upon the bone remodeling function e. Again, their properties will be described below in formulas (2). Moreover, a dot above a variable represents the time derivative.

We emphasize that for the functions $\mathcal{C}(e)$, $a(e)$ and $\mathcal{A}(e)$, which characterize the material properties, there are few experimental data concerning their form. In some papers, polynomial approximations have been employed (see, for instance, [11]).

Let us define the following truncation operator $\Phi_L : \mathbb{R} \to [-L, L]$ by

$$\Phi_L(r) = \begin{cases} r & \text{if} \quad |r| \leq L, \\ L & \text{otherwise.} \end{cases}$$

Finally, the model is assumed quasistatic and therefore, the inertia effects are neglected. Moreover, let $e_\mu^0 = e_\mu(t = 0)$ denote the initial bone remodeling function at time $t = 0$ for the normal compliance problem and $e_0 = e(t = 0)$ for the Signorini's problem.

If we suppose that the obstacle is deformable, then the strong formulation of the contact problem, within the framework of adaptive elasticity and small strains, is the following (see [4]).

Problem P_μ. *Find the mechanical displacement field $\boldsymbol{u}_\mu : \Omega \times (0, T) \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma}_\mu : \Omega \times (0, T) \to \mathbb{S}^d$ and the bone remodeling function $e_\mu : \Omega \times (0, T) \to \mathbb{R}$ such that $e_\mu(0) = e_\mu^0$ and for a.e. $t \in (0, T)$,*

$$\begin{aligned}
&\boldsymbol{\sigma}_\mu(t) = (\xi_0 + e_\mu(t))\mathcal{C}(e_\mu(t))\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) \quad \text{in} \quad \Omega, \\
&\dot{e}_\mu(t) = a(e_\mu(t)) + \mathcal{A}(e_\mu(t)) : \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)) \quad \text{in} \quad \Omega, \\
&-\mathrm{Div}\, \boldsymbol{\sigma}_\mu(t) = \gamma(\xi_0 + \Phi_L(e_\mu(t)))\boldsymbol{f}(t) \quad \text{in} \quad \Omega, \\
&\boldsymbol{u}_\mu(t) = \boldsymbol{0} \quad \text{on} \quad \Gamma_D, \\
&\boldsymbol{\sigma}_\mu(t)\boldsymbol{\nu} = \boldsymbol{g}(t) \quad \text{on} \quad \Gamma_N, \\
&(\boldsymbol{\sigma}_\mu)_\tau(t) = \boldsymbol{0}, \quad (\sigma_\mu)_\nu(t) = -\frac{1}{\mu}((u_\mu)_\nu(t) - s)_+ \quad \text{on} \quad \Gamma_C.
\end{aligned}$$

Here, $\gamma > 0$ is assumed to be constant, for the sake of simplicity, and it represents the density of the full elastic material present in the bone.

If we assume that the obstacle is rigid and within exactly the same framework of the previous problem, the strong formulation of the problem with Signorini contact law is the following (compare with the asymptotic contact rod model of [7]).

Problem P. *Find the mechanical displacement field* $\boldsymbol{u} : \Omega \times (0, T) \to \mathbb{R}^d$, *the stress field* $\boldsymbol{\sigma} : \Omega \times (0, T) \to \mathbb{S}^d$ *and the bone remodeling function* $e : \Omega \times (0, T) \to \mathbb{R}$ *such that* $e(0) = e_0$ *and for a.e.* $t \in (0, T)$,

$$
\begin{aligned}
&\boldsymbol{\sigma}(t) = (\xi_0 + e(t))\mathcal{C}(e(t))\boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad in \quad \Omega,\\
&\dot{e}(t) = a(e(t)) + \mathcal{A}(e(t)) : \boldsymbol{\varepsilon}(\boldsymbol{u}(t)) \quad in \quad \Omega,\\
&-\mathrm{Div}\,\boldsymbol{\sigma}(t) = \gamma(\xi_0 + \Phi_L(e(t)))\boldsymbol{f}(t) \quad in \quad \Omega,\\
&\boldsymbol{u}(t) = \mathbf{0} \quad on \quad \Gamma_D,\\
&\boldsymbol{\sigma}(t)\boldsymbol{\nu} = \boldsymbol{g}(t) \quad on \quad \Gamma_N,\\
&\boldsymbol{\sigma}_\tau(t) = \mathbf{0}, \quad u_\nu(t) \le s, \quad \sigma_\nu(t) \le 0, \quad (u_\nu - s)\sigma_\nu = 0 \quad on \quad \Gamma_C.
\end{aligned}
$$

We obtain now a variational formulation of both problems P_μ and P. First, let us denote by $H = [L^2(\Omega)]^d$, $Y = L^2(\Omega)$, and define the following variational spaces,

$$
\begin{aligned}
&V = \{\boldsymbol{w} \in [H^1(\Omega)]^d \,;\, \boldsymbol{w} = \mathbf{0} \quad \text{on} \quad \Gamma_D\},\\
&Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d} \,;\, \tau_{ij} = \tau_{ji}, \quad 1 \le i, j \le d\},
\end{aligned}
$$

and let U be the admissible mechanical displacement convex set given by

$$U = \{\boldsymbol{w} \in V \,;\, \boldsymbol{w} \cdot \boldsymbol{\nu} = w_\nu \le s \quad \text{on} \quad \Gamma_C\}.$$

The following assumptions are done on the given data.

The elasticity coefficients $C_{ijkl}(e)$ are assumed to satisfy the following properties:

$$
\left.
\begin{array}{l}
\text{(a) There exists } L_C > 0 \text{ such that for all } e_1, e_2 \in \mathbb{R}\\
\qquad |(\xi_0 + e_1)C_{ijkl}(e_1) - (\xi_0 + e_2)C_{ijkl}(e_2)| \le L_C|e_1 - e_2|.\\
\text{(b) There exists } M_C > 0 \text{ such that}\\
\qquad |(\xi_0 + e)C_{ijkl}(e)| \le M_C, \quad \forall e \in \mathbb{R}.\\
\text{(c) } C_{ijkl}(e) = C_{jikl}(e) = C_{klij}(e) \text{ for } i, j, k, l = 1, \dots, d.\\
\text{(d) There exists } m_{\mathcal{C}} > 0 \text{ such that } (\xi_0 + e)\mathcal{C}(e)\boldsymbol{\tau} : \boldsymbol{\tau} \ge m_C|\boldsymbol{\tau}|^2,\\
\qquad \forall \boldsymbol{\tau} \in \mathbb{S}^d.
\end{array}
\right\} \quad (1)
$$

The constitutive function $a(e)$ and the bone remodeling rate coefficients $A_{ij}(e)$ are assumed Lipschitz and bounded functions. Therefore, there exist

L_a, $L_{\mathcal{A}}$, M_a and $M_{\mathcal{A}}$ such that,

$$\left.\begin{array}{l}\text{(a) } |a(e_1)-a(e_2)| \le L_a|e_1-e_2|, \quad |a(e)| \le M_a, \quad \forall e_1, e_2, e \in \mathbb{R}, \\ \text{(b) } |\mathcal{A}(e_1)-\mathcal{A}(e_2)| \le L_{\mathcal{A}}|e_1-e_2|, \quad |\mathcal{A}(e)| \le M_{\mathcal{A}}, \quad \forall e_1, e_2, e \in \mathbb{R}.\end{array}\right\} \tag{2}$$

The reference volume fraction ξ_0 satisfies the following conditions for some $\xi_0^m < 1$,

$$\xi_0 \in C(\overline{\Omega}), \quad 0 < \xi_0^m \le \xi_0(\boldsymbol{x}) \le 1 \text{ for all } \boldsymbol{x} \in \overline{\Omega}. \tag{3}$$

The density forces have the regularity,

$$\boldsymbol{f} \in C([0,T];[C(\overline{\Omega})]^d), \quad \boldsymbol{g} \in C([0,T];[C(\overline{\Gamma_N})]^d), \tag{4}$$

and the initial values of the bone remodeling functions e_0 and e_μ^0 verify that

$$e_0,\, e_\mu^0 \in C(\overline{\Omega}). \tag{5}$$

For every $e \in L^\infty(\Omega)$, let us define the following bilinear form $c(e;\cdot,\cdot) : V \times V \to \mathbb{R}$,

$$c(e;\boldsymbol{u},\boldsymbol{v}) = \int_\Omega (\xi_0 + e)\mathcal{C}(e)\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})\, d\boldsymbol{x} \quad \forall \boldsymbol{u},\boldsymbol{v} \in V,$$

and the linear form $L(e;\cdot) : V \to \mathbb{R}$ given by

$$L(e;\boldsymbol{v}) = \int_\Omega \gamma(\xi_0 + \Phi_L(e))\boldsymbol{f}\cdot\boldsymbol{v}\, d\boldsymbol{x} + \int_{\Gamma_N} \boldsymbol{g}\cdot\boldsymbol{v}\, da \quad \forall \boldsymbol{v} \in V.$$

Let us define the contact functional $j : V \times V \to \mathbb{R}$ as,

$$j(\boldsymbol{u},\boldsymbol{v}) = \frac{1}{\mu}\int_{\Gamma_C} (u_\nu - s)_+\, v_\nu\, da \qquad \forall \boldsymbol{u},\boldsymbol{v} \in V.$$

Applying Green's formula, we then derive the following variational formulations of Problems P_μ and P.

Problem VP_μ**.** *Find the mechanical displacement field* $\boldsymbol{u}_\mu : [0,T] \to V$ *and the bone remodeling function* $e_\mu : [0,T] \to L^\infty(\Omega)$ *such that* $e_\mu(0) = e_\mu^0$ *and for a.e.* $t \in (0,T)$,

$$\dot{e}_\mu(t) = a(e_\mu(t)) + \mathcal{A}(e_\mu(t)) : \boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(t)), \tag{6}$$

$$c(e_\mu(t);\boldsymbol{u}_\mu(t),\boldsymbol{v}) + j(\boldsymbol{u}_\mu(t),\boldsymbol{v}) = L(e_\mu(t);\boldsymbol{v}) \quad \forall \boldsymbol{v} \in V. \tag{7}$$

Problem VP**.** *Find the mechanical displacement field* $\boldsymbol{u} : [0,T] \to U$ *and the bone remodeling function* $e : [0,T] \to L^\infty(\Omega)$ *such that* $e(0) = e_0$ *and for a.e.* $t \in (0,T)$*,*

$$\dot{e}(t) = a(e(t)) + \mathcal{A}(e(t)) : \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \tag{8}$$

$$c(e(t); \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \geq L(e(t); \boldsymbol{v} - \boldsymbol{u}(t)) \quad \forall \boldsymbol{v} \in U. \tag{9}$$

The following result states the existence of a unique solution to Problem VP. It can be proved by using similar arguments to those employed in [7] for the case of an asymptotic bone remodeling rod model with Signorini contact conditions (full details were provided recently in [19]).

Theorem 1. *Let the assumptions (1)-(5) hold. Assume that, for any given function* $e \in C^1([0,T]; C(\overline{\Omega}))$*, the unique solution to the following problem:*

$$\boldsymbol{u}(t) \in U, \quad c(e(t); \boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) \geq L(e(t); \boldsymbol{v} - \boldsymbol{u}(t)) \quad \forall \boldsymbol{v} \in U,$$

has the regularity $\boldsymbol{u} \in C([0,T]; [H^3(\Omega)]^d)$ *for* $d = 2,3$ *or the regularity* $u \in C([0,T]; H^2(\Omega))$ *for* $d = 1$*. Then, there exists a unique solution to Problem* VP *with the following regularity:*

$$\boldsymbol{u} \in C([0,T]; [C^1(\overline{\Omega})]^d \cap U), \quad e \in C^1([0,T]; C(\overline{\Omega})).$$

Arguing in an analogous way, we also have.

Theorem 2. *Let the assumptions (1)-(5) hold. Assume that, for any given function* $e_\mu \in C^1([0,T]; C(\overline{\Omega}))$*, the unique solution to the following problem:*

$$\boldsymbol{u}_\mu(t) \in V, \quad c(e_\mu(t); \boldsymbol{u}_\mu(t), \boldsymbol{v}) + j(\boldsymbol{u}_\mu(t), \boldsymbol{v}) = L(e_\mu(t); \boldsymbol{v}) \quad \forall \boldsymbol{v} \in V,$$

has the regularity $\boldsymbol{u}_\mu \in C([0,T]; [H^3(\Omega)]^d)$ *for* $d = 2,3$ *or the regularity* $u \in C([0,T]; H^2(\Omega))$ *for* $d = 1$*. Then, there exists a unique solution to Problem* VP_μ *with the following regularity:*

$$\boldsymbol{u}_\mu \in C([0,T]; [C^1(\overline{\Omega})]^d), \quad e_\mu \in C^1([0,T]; C(\overline{\Omega})).$$

The aim of this section, and of the paper, is to prove the convergence of the solution to Problem VP_μ to the solution to Problem VP, when the deformability coefficient μ tends to zero. This is established in the following.

Theorem 3. *Let the assumptions (1)-(5) hold. Then, the solution* $(\boldsymbol{u}_\mu, e_\mu)$ *to Problem* VP_μ *converges to the solution* $(\boldsymbol{u}, e)$ *to Problem* VP *in the space*

$C([0,T];V\times Y)$ when the deformability coefficient μ tends to the zero; that is,

$$\max_{0\le t\le T}\{\|\boldsymbol{u}_\mu(t)-\boldsymbol{u}(t)\|_V+\|e_\mu(t)-e(t)\|_Y\}\to 0 \quad \textit{when} \quad \mu\to 0. \tag{10}$$

Remark 1. *In addition to the mathematical importance of this result, it is interesting to remark that, in applications, the solution to the contact problem with a rigid obstacle may be then approached by the solution to the contact problem with a deformable foundation, for small obstacle's deformability coefficients.*

Proof: In order to simplify the writing and the calculus we assume that $s=0$ and that the initial conditions coincide (i.e. $e_\mu^0=e_0$ for all $\mu>0$). Clearly, it is straightforward to extend the results presented below to more general situations.

First, let us estimate the error on the bone remodeling function. Integrating in time both differential equations (6) and (8) we have,

$$\begin{aligned} e(t)&=\int_0^t\Big[a(e(s))+\mathcal{A}(e(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\Big]\,ds+e_0,\\ e_\mu(t)&=\int_0^t\Big[a(e_\mu(s))+\mathcal{A}(e_\mu(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\Big]\,ds+e_0. \end{aligned}$$

Subtracting both expressions we find

$$\begin{aligned} \|e(t)-e_\mu(t)\|_Y\le\int_0^t\Big(&\|a(e(s))-a(e_\mu(s))\|_Y\\ &+\|\mathcal{A}(e(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))-\mathcal{A}(e_\mu(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\|_Y\Big)\,ds. \end{aligned}$$

Using now properties (2) it follows that

$$\begin{aligned} &\|a(e(s))-a(e_\mu(s))\|_Y\le L_a\|e(s)-e_\mu(s)\|_Y,\\ &\|\mathcal{A}(e(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))-\mathcal{A}(e_\mu(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\|_Y\\ &\qquad\le\|\mathcal{A}(e_\mu(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))-\mathcal{A}(e_\mu(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}_\mu(s))\|_Y\\ &\qquad\qquad+\|\mathcal{A}(e(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))-\mathcal{A}(e_\mu(s))\,:\,\boldsymbol{\varepsilon}(\boldsymbol{u}(s))\|_Y\\ &\qquad\le C(\|\boldsymbol{u}(s)-\boldsymbol{u}_\mu(s)\|_V+\|e_\mu(s)-e(s)\|_Y), \end{aligned}$$

where C, in what follows, is a generic positive constant which depends on the problem data and, here, it is linearly dependent on the norm $\|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{[C(\overline{\Omega})]^{d\times d}}$. Moreover, the regularity provided in Theorem 1 has been used.

Thus, we obtain that

$$\|e(t) - e_\mu(t)\|_Y \leq C \int_0^t \Big(\|\boldsymbol{u}(s) - \boldsymbol{u}_\mu(s)\|_V + \|e_\mu(s) - e(s)\|_Y \Big)\, ds, \qquad (11)$$

where C is independent of μ, t and e.

We proceed now with the mechanical displacement fields. In what follows, we suppress the dependence on time to simplify the writing. Taking $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_\mu \in V$ in the nonlinear variational equation (7) we have

$$c(e_\mu; \boldsymbol{u}_\mu, \boldsymbol{u} - \boldsymbol{u}_\mu) + j(\boldsymbol{u}_\mu, \boldsymbol{u} - \boldsymbol{u}_\mu) = L(e_\mu; \boldsymbol{u} - \boldsymbol{u}_\mu),$$

and taking $c(e; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_\mu + \boldsymbol{u}_\mu - \boldsymbol{u})$ instead of $c(e; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u})$ in (9), it follows that

$$c(e; \boldsymbol{u}, \boldsymbol{u}_\mu - \boldsymbol{u}) \geq L(e; \boldsymbol{v} - \boldsymbol{u}) - c(e; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_\mu) \quad \forall \boldsymbol{v} \in U.$$

We observe that we can not take $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{u}_\mu \in U$ in (9) because, in general, we can not guarantee that $\boldsymbol{u}_\mu \in U$.

Since $u_\nu \leq 0$ on Γ_C it is easy to check that

$$j(\boldsymbol{u}_\mu, \boldsymbol{u} - \boldsymbol{u}_\mu) = j(\boldsymbol{u}_\mu, \boldsymbol{u}) - j(\boldsymbol{u}_\mu, \boldsymbol{u}_\mu) \leq 0,$$

and the previous equations can be rewritten as

$$\begin{aligned} &c(e_\mu; -\boldsymbol{u}_\mu, \boldsymbol{u} - \boldsymbol{u}_\mu) \leq -L(e_\mu; \boldsymbol{u} - \boldsymbol{u}_\mu), \\ &c(e; \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\mu) \leq L(e; \boldsymbol{u} - \boldsymbol{v}) + c(e; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_\mu) \quad \forall \boldsymbol{v} \in U. \end{aligned}$$

Keeping in mind that

$$\begin{aligned} -L(e_\mu; \boldsymbol{u} - \boldsymbol{u}_\mu) + L(e; \boldsymbol{u} - \boldsymbol{v}) &= L(e; \boldsymbol{u} - \boldsymbol{u}_\mu) - L(e_\mu; \boldsymbol{u} - \boldsymbol{u}_\mu) \\ &\quad + L(e; \boldsymbol{u} - \boldsymbol{v}) - L(e; \boldsymbol{u} - \boldsymbol{u}_\mu) \quad \forall \boldsymbol{v} \in U, \end{aligned}$$

and

$$\begin{aligned} c(e_\mu; -\boldsymbol{u}_\mu, \boldsymbol{u} - \boldsymbol{u}_\mu) + c(e; \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\mu) &= c(e_\mu; \boldsymbol{u} - \boldsymbol{u}_\mu, \boldsymbol{u} - \boldsymbol{u}_\mu) \\ &\quad + c(e; \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\mu) - c(e_\mu; \boldsymbol{u}, \boldsymbol{u} - \boldsymbol{u}_\mu), \end{aligned}$$

adding the previous inequalities and using properties (1), (4) and the inequality

$$ab \leq \epsilon a^2 + \frac{1}{4\epsilon} b^2, \quad a, b, \epsilon \in \mathbb{R}, \epsilon > 0,$$

we find that

$$\begin{aligned} \|\boldsymbol{u} - \boldsymbol{u}_\mu\|_V^2 \leq C \Big(&\|e - e_\mu\|_Y^2 + \|\boldsymbol{v} - \boldsymbol{u}_\mu\|_H + \|\boldsymbol{v} - \boldsymbol{u}_\mu\|_{[L^2(\Gamma_N)]^d} \\ &+ c(e; \boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}_\mu) \Big) \quad \forall \boldsymbol{v} \in U. \end{aligned}$$

Therefore, we obtain the following estimates for the displacement field

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_\mu\|_V \leq\ & C\Big(\|e-e_\mu\|_Y+\|\boldsymbol{v}-\boldsymbol{u}_\mu\|_H^{1/2}+\|\boldsymbol{v}-\boldsymbol{u}_\mu\|_{[L^2(\Gamma_N)]^d}^{1/2} \\ & +|c(e;\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}_\mu)|^{1/2}\Big) \quad \forall \boldsymbol{v}\in U.\end{aligned} \tag{12}$$

Combining now estimates (11) and (12) and using Gronwall's inequality we conclude that

$$\begin{aligned}\max_{0\le t\le T}\Big\{\|\boldsymbol{u}(t)-\boldsymbol{u}_\mu(t)\|_V+\|e(t)-e_\mu(t)\|_Y\Big\} \leq C\max_{0\le t\le T}\Big(\|\boldsymbol{v}(t)-\boldsymbol{u}_\mu(t)\|_H^{1/2} \\ +\|\boldsymbol{v}-\boldsymbol{u}_\mu\|_{[L^2(\Gamma_N)]^d}^{1/2}+|c(e(t);\boldsymbol{u}(t),\boldsymbol{v}(t)-\boldsymbol{u}_\mu(t))|^{1/2}\Big)\end{aligned} \tag{13}$$

for all $\boldsymbol{v}\in C([0,T];U)$. Taking into account that $j(\boldsymbol{u}_\mu(t),\boldsymbol{u}_\mu(t))\geq 0$, using property (1)(d) we find that

$$m_{\mathcal{C}}\|\boldsymbol{u}_\mu(t)\|_V^2 \leq c(e_\mu(t);\boldsymbol{u}_\mu(t),\boldsymbol{u}_\mu(t)) \leq L(e_\mu(t);\boldsymbol{u}_\mu(t)) \leq C\|\boldsymbol{u}_\mu(t)\|_V,$$

and therefore, there exists $M>0$, independent of μ, such that

$$\|\boldsymbol{u}_\mu(t)\|_V \leq M \quad \forall \mu>0.$$

Hence, there exists a subsequence of $\big(\boldsymbol{u}_\mu(t)\big)$ denoted by $\big(\boldsymbol{u}_{\mu_k}(t)\big)$ which is weakly convergent to an element $\tilde{\boldsymbol{u}}(t)$ belonging to V (since V is a closed space). Let us prove that $\tilde{\boldsymbol{u}}(t)\in U$, i.e. we shall verify that $\tilde{u}_\nu(t)=\tilde{\boldsymbol{u}}(t)\cdot\boldsymbol{\nu}\leq 0$ on Γ_C.

Using again properties (1) it follows that $c(e_{\mu_k}(t);\boldsymbol{u}_{\mu_k}(t),\boldsymbol{u}_{\mu_k}(t))\geq 0$, and thus we have,

$$j(\boldsymbol{u}_{\mu_k}(t),\boldsymbol{u}_{\mu_k}(t)) \leq L(e_{\mu_k}(t);\boldsymbol{u}_{\mu_k}(t)) \leq C\|\boldsymbol{u}_{\mu_k}(t)\|_V \leq CM.$$

where C is a positive constant independent of t, μ_k, $\boldsymbol{u}$ and e.

Taking limits it follows that

$$\lim_{\mu_k\to 0}\frac{1}{\mu_k}\int_{\Gamma_C}[(\boldsymbol{u}_{\mu_k}(t))_\nu]_+(\boldsymbol{u}_{\mu_k}(t))_\nu\,da \leq CM=\text{constant},$$

and, by Fatou's lemma, we have

$$0\leq\int_{\Gamma_C}\lim_{\mu_k\to 0}\Big\{[(\boldsymbol{u}_{\mu_k}(t))_\nu]_+(\boldsymbol{u}_{\mu_k}(t))_\nu\Big\}\,da \leq \lim_{\mu_k\to 0}\int_{\Gamma_C}[(\boldsymbol{u}_{\mu_k}(t))_\nu]_+(\boldsymbol{u}_{\mu_k}(t))_\nu\,da=0,$$

since $\boldsymbol{u}_{\mu_k}(t)$ converges strongly to $\tilde{\boldsymbol{u}}(t)$ on Γ_C (the trace operator from V into $[L^2(\Gamma_C)]^d$ is compact). Thus, we find that $\displaystyle\int_{\Gamma_C}[\tilde{u}_\nu(t)]_+\tilde{u}_\nu(t)\,da=0$, which implies that $\tilde{u}_\nu(t)\leq 0$ on Γ_C. Therefore we have proved that $\tilde{\boldsymbol{u}}(t)\in U$.

Keeping in mind that V is compactly embedded in H (Rellich-Kondrachov theorem) and that the trace operator is also compact from V into $[L^2(\Gamma_N)]^d$, the subsequence $\big(\boldsymbol{u}_{\mu_k}(t)\big)$ is also strongly convergent to $\tilde{\boldsymbol{u}}(t)$ in H and its trace is strongly convergent to the trace of $\tilde{\boldsymbol{u}}(t)$ in $[L^2(\Gamma_N)]^d$. Moreover, since $\boldsymbol{u}_{\mu_k}(t) \rightharpoonup \tilde{\boldsymbol{u}}(t)$ in H, taking $\boldsymbol{v}(t) = \tilde{\boldsymbol{u}}(t)$ in (13) we have

$$|c(e(t); \boldsymbol{u}(t), \tilde{\boldsymbol{u}}(t) - \boldsymbol{u}_{\mu_k}(t))|^{1/2} \to 0 \quad \text{as} \quad \mu_k \to 0,$$

and we conclude that

$$\max_{0\le t\le T} \Big\{ \|\boldsymbol{u}(t) - \boldsymbol{u}_{\mu_k}(t)\|_V + \|e(t) - e_{\mu_k}(t)\|_Y \Big\} \to 0 \quad \text{as} \quad \mu_k \to 0. \tag{14}$$

Finally, for any other subsequence $\big(\boldsymbol{u}_{\mu_i}(t)\big)$ of $\big(\boldsymbol{u}_{\mu}(t)\big)$ weakly convergent to another element $\hat{\boldsymbol{u}}(t) \in V$ we can repeat these arguments and we again obtain that $\hat{\boldsymbol{u}}(t) \in U$, so the limits are equal to zero, as in (14). Thus we can conclude that (14) is verified for all the sequence $\big(\boldsymbol{u}_{\mu}(t)\big)$.

$\square$

2. Numerical results with a fully discrete scheme

We now introduce a finite element algorithm for approximating solutions of both variational problems VP and VP_μ, and present some numerical results which demonstrate the convergence behaviour.

The discretization of the two problems is done in two steps. First, we consider the finite element spaces $V^h \subset V$ and $B^h \subset L^\infty(\Omega) \subset Y$ given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \; ; \; \boldsymbol{w}^h_{|Tr} \in [P_1(Tr)]^d, \, Tr \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma_D\},$$
$$B^h = \{\xi^h \in L^\infty(\Omega) \; ; \; \xi^h_{|Tr} \in P_0(Tr), \quad Tr \in \mathcal{T}^h\},$$

where Ω is a polyhedral domain, $\mathcal{T}^h$ denotes a triangulation of Ω compatible with the partition of the boundary $\Gamma = \partial\Omega$ into Γ_D, Γ_N and Γ_C, and $P_q(T_r)$, $q = 0, 1$, represents the space of polynomials of global degree less or equal to q in Tr. Here, $h > 0$ denotes the spatial discretization parameter. Moreover, we define the discrete admissible displacements convex set $U^h = U \cap V^h$; that is,

$$U^h = \{\boldsymbol{w}^h \in V^h \, ; \, w^h_\nu = \boldsymbol{w}^h \cdot \boldsymbol{\nu} \le s^h \quad \text{on} \quad \Gamma_C\},$$

where s^h is an appropriate approximation of the gap function s.

Secondly, the time derivatives are discretized by using a uniform partition of the time interval $[0, T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, and let k

be the time step size, $k = T/N$. Moreover, for a continuous function $f(t)$ we denote $f_n = f(t_n)$.

Using the forward Euler scheme, the fully discrete approximations of problems VP and VP_μ are as follows.

Problem VP^{hk}**.** *Find a discrete displacement field* $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset U^h$ *and a discrete bone remodeling function* $e^{hk} = \{e_n^{hk}\}_{n=0}^N \subset B^h$ *such that* $e_0^{hk} = e_0^h$ *and for* $n = 1, \dots, N$,

$$c(e_n^{hk}; \boldsymbol{u}_n^{hk}, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}) \geq L(e_n^{hk}; \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}) \quad \forall \boldsymbol{v}^h \in U^h,$$

$$\frac{e_n^{hk} - e_{n-1}^{hk}}{k} = a(e_{n-1}^{hk}) + \mathcal{A}(e_{n-1}^{hk}) : \boldsymbol{\varepsilon}(\boldsymbol{u}_{n-1}^{hk}),$$

where e_0^h is an appropriate approximation of the initial condition e_0 and $\boldsymbol{u}_0^{hk}$ is defined as the solution to the following problem,

$$\boldsymbol{u}_0^{hk} \in U^h, \quad c(e_0^h; \boldsymbol{u}_0^{hk}, \boldsymbol{v}^h - \boldsymbol{u}_0^{hk}) \geq L(e_0^h; \boldsymbol{v}^h - \boldsymbol{u}_0^{hk}) \quad \forall \boldsymbol{v}^h \in U^h.$$

Problem VP_μ^{hk}**.** *Find a discrete displacement field* $\boldsymbol{u}_\mu^{hk} = \{(\boldsymbol{u}_\mu)_n^{hk}\}_{n=0}^N \subset V^h$ *and a discrete function* $e_\mu^{hk} = \{(e_\mu)_n^{hk}\}_{n=0}^N \subset B^h$ *such that* $(e_\mu)_0^{hk} = e_0^h$ *and for* $n = 1, \dots, N$,

$$c((e_\mu)_n^{hk}; (\boldsymbol{u}_\mu)_n^{hk}, \boldsymbol{v}^h) + j((\boldsymbol{u}_\mu)_n^{hk}, \boldsymbol{v}^h) = L((e_\mu)_n^{hk}; \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in V^h,$$

$$\frac{(e_\mu)_n^{hk} - (e_\mu)_{n-1}^{hk}}{k} = a((e_\mu)_{n-1}^{hk}) + \mathcal{A}((e_\mu)_{n-1}^{hk}) : \boldsymbol{\varepsilon}((\boldsymbol{u}_\mu)_{n-1}^{hk}),$$

where e_0^h is an appropriate approximation of the initial condition e_0 (which we assumed, as in the previous section, equal to the initial condition for the Signorini's problem) and $(\boldsymbol{u}_\mu)_0^{hk}$ is the solution to the following problem,

$$(\boldsymbol{u}_\mu)_0^{hk} \in V^h, \quad c((e_\mu)_0^h; (\boldsymbol{u}_\mu)_0^{hk}, \boldsymbol{v}^h) + j((\boldsymbol{u}_\mu)_0^{hk}, \boldsymbol{v}^h) = L(e_0^h; \boldsymbol{v}^h) \quad \forall \boldsymbol{v}^h \in V^h.$$

From the properties (1), using classical results on nonlinear variational equations and nonlinear variational inequalities (see [9]), it is straightforward to obtain the existence and uniqueness of solution to both fully discrete problems (see also [5] for theoretical results for the approximation of an asymptotic bone remodeling rod model without contact).

We notice that the above discrete problems were solved by using a penalty-duality algorithm introduced in [1]. The numerical schemes were implemented on a 3.2Ghz PC using MATLAB.

2.1. Numerical results for a one-dimensional example. As a first example, we consider a one-dimensional setting. A bar which occupies the domain $\Omega = (0, 6)$ is assumed to be in contact with an obstacle (deformable or rigid) on its right corner, so $s = 0\,m$. A positive compression force is then applied on its left end, which enforces the body to maintain the contact with the obstacle (see Fig. 2).

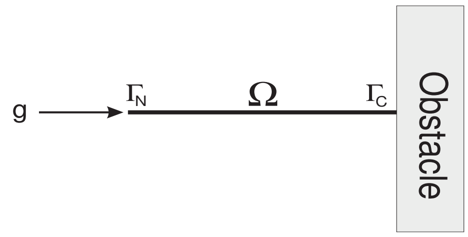

FIGURE 2. Example 1D: A bar in contact with an obstacle.

The following data have been employed in the numerical simulations (polynomial approximations as in [11] for $\mathcal{C}(e)$, $a(e)$ and $\mathcal{A}(e)$ are used):

$$
\begin{aligned}
&\Gamma_D = \emptyset, \quad T = 100\,\text{days}, \quad \mathcal{C}(e) = \frac{1}{\xi_0 + e}(C_0 + C_1 e), \quad a(e) = a_0 + a_1 e + a_2 e^2,\\
&\mathcal{A}(e) = A_0 + A_1 e, \quad A_0 = -216\,(100\text{ days})^{-1}, \quad A_1 = 216\,(100\text{ days})^{-1},\\
&\xi_0 = 0.892, \quad \gamma = 1740\,Kg/m^3, \quad f = 0\,N/m^3, \quad g = 8\times 10^{-2}\,GPa, \quad s = 0m,\\
&C_0 = 25.69\,GPa, \quad C_1 = 252.08\,GPa, \quad a_0 = -1296\times 10^{-4}\,(100\text{ days})^{-1},\\
&a_1 = -1296\times 10^{-2}\,(100\text{ days})^{-1}, \quad a_2 = 216\times 10^{-2}\,(100\text{ days})^{-1}.
\end{aligned}
$$

Moreover, we assume that the initial bone remodeling function is given by

$$
e_0(x) = 0 \quad \forall x \in (0, 6).
$$

Our aim here is to show the numerical convergence of the solution when the deformability coefficient μ tends to zero as stated in Theorem 3. Therefore, two fine uniform partitions of both the time interval and the domain have been considered with the discretization parameters $h = k = 0.001$.

In Fig. 3 we plot the evolution in time of the mechanical displacement of the contact point $x = 6$ for several values of μ (left-hand side) and the evolution, with respect to μ, of the mechanical displacement of this node at final time $T = 100$ days (right-hand side). As can be seen, the convergence to the Signorini case ($u(6, t) = 0$ for all $t \in [0, 100]$) is shown.

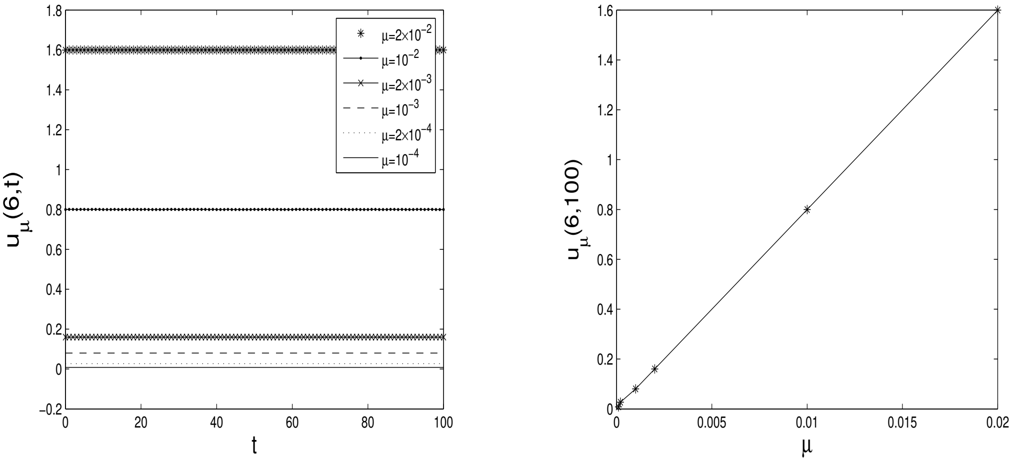

FIGURE 3. Convergence of the solution when $\mu \to 0$.

The errors, given by

$$E_\mu^{hk} = \max_{0 \le n \le N} \left\{ \|(\boldsymbol{u}_\mu)_n^{hk} - \boldsymbol{u}_n^{hk}\|_V + \|(e_\mu)_n - e_n^{hk}\|_Y \right\}$$

and obtained for different values of μ, are shown in Fig. 4. We notice that the convergence of the solution when $\mu \to 0$ is clearly observed (in fact, it seems that a linear behaviour is achieved).

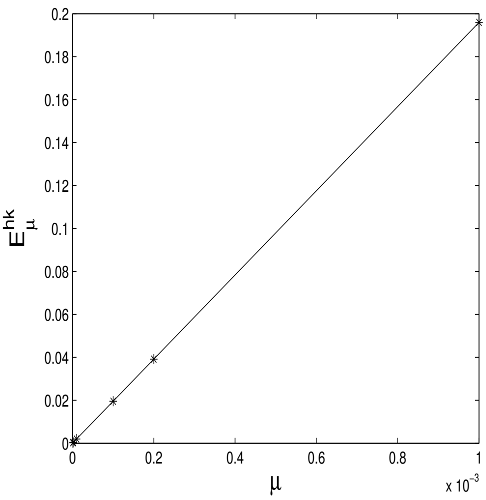

FIGURE 4. Asymptotic convergence depending on μ.

2.2. Numerical results for a two-dimensional problem ($d = 2$)**.** As a two-dimensional example, we consider the rectangular domain $\Omega = (0,6) \times (0,1.2)$ which is clamped on the boundary part $\Gamma_D = \{0\} \times [0,1.2]$. No volume forces are supposed to act in the body, a linearly increasing surface force acts on the boundary part $[0,6] \times \{1.2\}$ and, finally, the body is supposed to be in contact with a deformable obstacle on the contact boundary $\Gamma_C = [0,6] \times \{0\}$ (see Fig. 5).

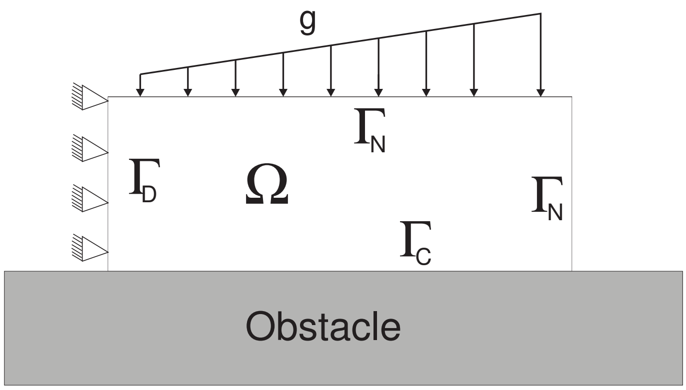

FIGURE 5. Example 2D: A rectangle in contact with an obstacle.

The following data were employed in this example:

$$\begin{aligned}
&T = 100\,\text{days}, \quad \boldsymbol{f} = \boldsymbol{0}\,N/m^3, \quad \boldsymbol{g}(x,y,t) = (0,-5x)\,MPa,\\
&\mathcal{C}(e) = \frac{1}{\xi_0 + e}(\mathcal{C}^0 + \mathcal{C}^1 e), \quad \mathcal{A}(e) = \mathcal{A}^0 + \mathcal{A}^1 e, \quad \xi_0 = 0.892, \quad \gamma = 1740\,Kg/m^3,\\
&s = 0\,m, \quad a(e) = a_0 + a_1 e + a_2 e^2, \quad a_0 = -1296 \times 10^{-4}\,(100\text{ days})^{-1},\\
&a_1 = -1296 \times 10^{-2}\,(100\text{ days})^{-1}, \quad a_2 = 216 \times 10^{-2}\,(100\text{ days})^{-1},
\end{aligned}$$

where the fourth-order tensors $\mathcal{C}^0 = (C^0_{ijkl})^2_{i,j,k,l=1}$ and $\mathcal{C}^1 = (C^1_{ijkl})^2_{i,j,k,l=1}$ and the second-order tensors $\mathcal{A}^0 = (A^0_{ij})^2_{i,j=1}$ and $\mathcal{A}^1 = (A^1_{ij})^2_{i,j=1}$ have the following components:

$$\begin{aligned}
&C^0_{1111} = 25.69\,GPa, \quad C^0_{2211} = 11.67\,GPa, \quad C^0_{2222} = 25.69\,GPa,\\
&C^0_{1211} = C^0_{1222} = 0\,GPa, \quad C^0_{1212} = 7\,GPa, \quad C^1_{1111} = 252.08\,GPa,\\
&C^1_{2211} = 114.58\,GPa, \quad C^1_{2222} = 252.08\,GPa,\\
&C^0_{1211} = C^0_{1222} = 0\,GPa, \quad C^1_{1212} = 68.75\,GPa,\\
&A^0_{11} = 216\,(100\text{days})^{-1}, \quad A^0_{22} = -216\,(100\text{days})^{-1}, \quad A^0_{12} = A^0_{21} = 0,\\
&A^1_{11} = 216\,(100\text{days})^{-1}, \quad A^1_{22} = 216\,(100\text{days})^{-1}, \quad A^0_{12} = A^0_{21} = 0.
\end{aligned}$$

Moreover, we assume that the initial bone remodeling function is given by

$$e_0(x,y) = 0 \quad \forall (x,y) \in (0,6) \times (0,1.2).$$

Taking $k = 0.01$, as the time discretization parameter, the mechanical displacement field (multiplied by 20) at final time together with the reference configuration are plotted in Fig. 6 for a deformable obstacle (left) and for a rigid one (right). We notice that, if we assume that the obstacle is deformable, a clear penetration is produced, but if the obstacle is rigid, then no penetration is obtained.

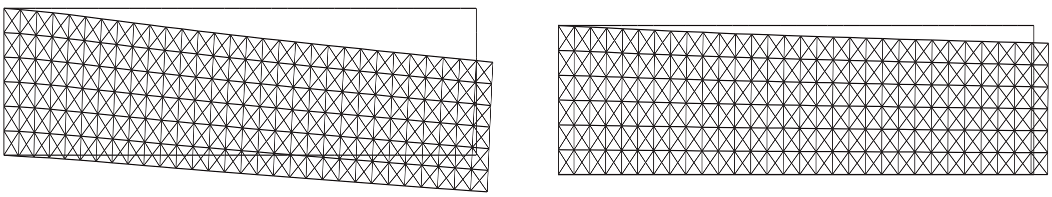

FIGURE 6. Example 2D: Deformed rectangle in contact with a deformable obstacle (left) and with a rigid one (right).

The transverse mechanical displacement of the contact boundary is plotted in Fig. 7 at final time T for different values of μ. As can be seen, the displacement converges to zero when $\mu \to 0$ since we assume that the body was in contact with the obstacle.

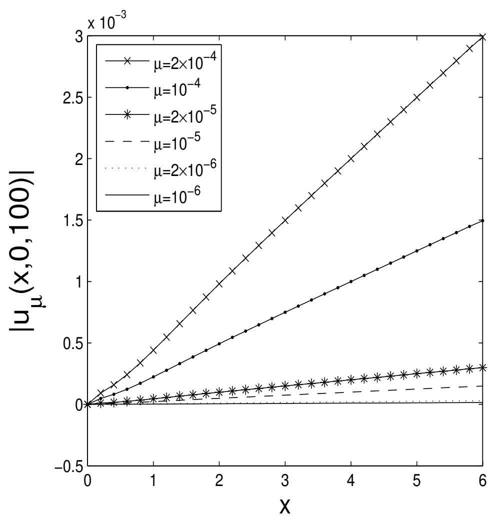

FIGURE 7. Example 2D: Contact boundary depending on μ.

Finally, the results concerning the bone remodeling function are presented in Figs. 8 and 9. This function is plotted, at final time, in Fig. 8 for the deformable obstacle (left) and for the rigid one (right). As can be seen, if the obstacle is assumed deformable, then the bone remodeling function has large negative values around the left lower corner while on the right part it has positive values. However, if the obstacle is rigid, the bone remodeling function is again negative on the left part (with smaller values) and positive on the right one, but it seems that it is constant through the transverse direction of the rectangle. This is shown clearly in Fig. 9, where we depict the convergence of the bone remodeling function, assuming contact with a deformable obstacle, at final time and on the contact boundary, to the bone remodeling function of the Signorini contact problem, at the same time and on the contact boundary.

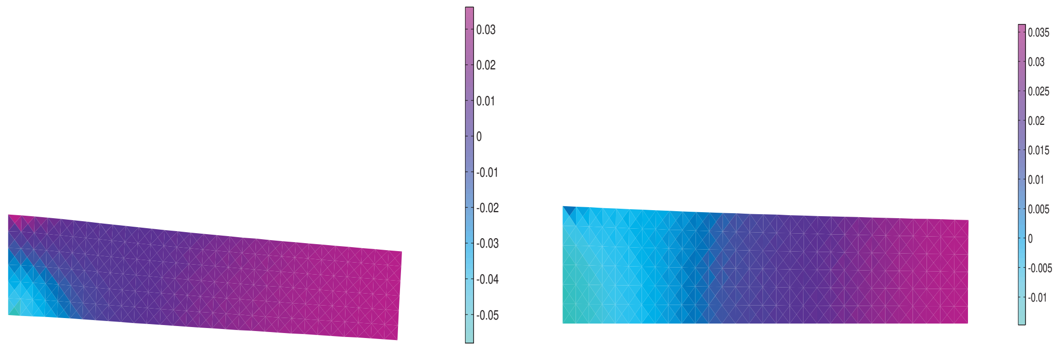

FIGURE 8. Example 2D: Bone remodeling function after contact with a deformable obstacle (left) and with a rigid one (right).

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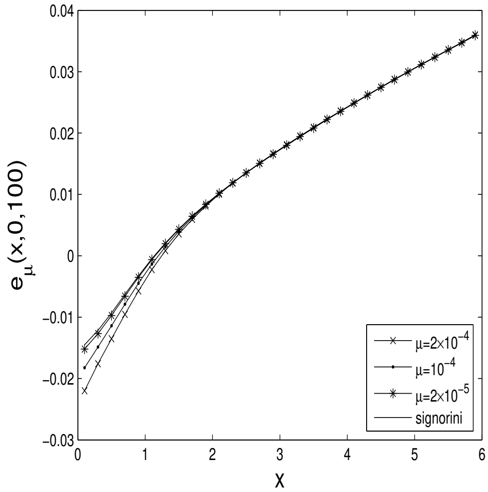

FIGURE 9. Example 2D: Bone remodeling function, on the contact boundary, depending on μ.

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