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BOOLEAN ALGEBRAS AND SPECTRUM

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ABSTRACT: For any Boolean algebra we compute the spectrum of its associated undirected Hasse graph.

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1. Overview

Given a finite graph, one of the most celebrated problem, see e.g. [7], is to study its spectrum, i.e. the eigenvalues of its adjacency matrix.
We remark that in general this can turn out to be a very tough and daunting problem because even the computation of the determinant can be extremely hard, see e.g. [10, 11, 12, 13, 14] and references therein.

We follow [2, 9, 20] for poset notations and terminology and we refer to [5, 8] for comprehensive references about graph theory.
Given a finite poset $(P, \preceq)$ we consider its undirected *Hasse graph*, called also "cover graph", see e.g. [6, 15, 16, 17, 19], viz. the graph having P as vertex–set and such that there is an edge between $x, y \in P$ if and only if one element covers the other one, i.e. $x \lessdot y$ or $y \lessdot x$. We define the *spectrum* of $(P, \preceq)$ as the spectrum of its associated graph, and we denote it by $\mathrm{Spec}\,(P)$.

For any $n \in N$, let $\mathfrak{B}_n$ be the *Boolean algebra* of rank n, i.e. $\mathfrak{B}_n$ is the poset of all subset of $[n] = \{t \in \mathbb{N} : 1 \leq t \leq n\}$ (thus $[0] = \emptyset$) under the inclusion ordering.
In this paper we compute the spectrum of $\mathfrak{B}_n$, for all $n \in N$. Quoting [4, §2.1] "Boolean lattices are certainly the most popular example of Macaulay posets", therefore these objects are very natural to study: information about spectrum of Macaulay posets are very valuable in order to gain a deeper insight of their structure, see [4].

We point out there is another way to associate an undirected graph to any given finite poset $(P, \preceq)$, which is to write to consider its *comparability graph*.

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For any Boolean algebra in [3] the determinant, but not the spectrum, of the adjacency matrix of such graph is computed.

2. Preparation

In this section we establish some results about the undirected Hasse graph $\mathcal{G}_n$ of $\mathfrak{B}_n$ and its adjacency matrix $\mathcal{A}_n$, $n \in \mathbb{N}$.

We collect some definitions and notation. The cardinality of a set $\mathcal{X}$ will be denoted by $\#\mathcal{X}$. For two sets $\mathcal{X}, \mathcal{Y}$ we denote with $\mathcal{X} \biguplus \mathcal{Y}$ the *disjoint union* of $\mathcal{X}$ and $\mathcal{Y}$, with $\mathcal{X} \setminus \mathcal{Y} = \{x : x \in \mathcal{X},\ x \notin \mathcal{Y}\}$ the difference set, and with $\mathcal{X} \Delta \mathcal{Y} = (\mathcal{X} \bigcup \mathcal{Y}) \setminus (\mathcal{X} \bigcap \mathcal{Y}) = (\mathcal{X} \setminus \mathcal{Y}) \biguplus (\mathcal{Y} \setminus \mathcal{X})$ the symmetric difference. For any $n, m \in \mathbb{N}$, $n \leq m$, we let $[n, m] = \{t \in \mathbb{N} : n \leq t \leq m\}$, thus $[n] = [1, n]$.
We denote with I_n the identity $n \times n$ square matrix, $n \in \mathbb{N} \setminus \{0\}$.

We choose the following linear extension φ of $\mathfrak{B}_n$ in order to represent $\mathcal{A}_n$:

$$\begin{aligned} \varphi : \ \mathfrak{B}_n & \stackrel{\sim}{\longrightarrow} \ [2^n] \\ \mathcal{X} & \longmapsto \ 1 + \sum_{j=0}^{n-1} \gamma_j 2^j \end{aligned} \tag{1}$$

where

$$\gamma_j = \begin{cases} 1 & \text{if } j+1 \in \mathcal{X}, \\ 0 & \text{if } j+1 \notin \mathcal{X}. \end{cases}$$

Proposition 2.1. *For all $n \in \mathbb{N}$,*

- **:** $\circledast$ $\mathcal{G}_n$ *is a n–regular graph and* $\operatorname{diam}(\mathcal{G}_n) = n$,
- **:** $\circledast$ *for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{B}_n$, the number of different paths between $\mathcal{X}$ and $\mathcal{Y}$ is*

$$(\#(\mathcal{X} \Delta \mathcal{Y}))!$$

- **:** $\circledast$ $\mathcal{A}_0 = 0$ *and*

$$\mathcal{A}_{n+1} = \begin{pmatrix} \mathcal{A}_n & I_{2^n} \\ I_{2^n} & \mathcal{A}_n \end{pmatrix}.$$

Proof: For any $\mathcal{X} \in \mathfrak{B}_n$

$$\begin{aligned} \{\mathcal{Z} \in \mathfrak{B}_n \mid \mathcal{X} \lhd \mathcal{Z}\} &= \biguplus_{j \in [n] \setminus \mathcal{X}} \left(\mathcal{X} \biguplus \{j\} \right) \\ \{\mathcal{T} \in \mathfrak{B}_n \mid \mathcal{T} \lhd \mathcal{X}\} &= \biguplus_{j \in \mathcal{X}} (\mathcal{X} \setminus \{j\}), \end{aligned} \tag{2}$$

therefore $\mathcal{G}_n$ is a n–regular graph.

From (2) we get that for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{B}_n$ there is an edge between them in $\mathcal{G}_n$ if and only if $\#(\mathcal{X}\Delta\mathcal{Y}) = 1$, hence any path from $\mathcal{X}$ to $\mathcal{Y}$ in $\mathcal{G}_n$ is given adding all elements in $\mathcal{Y} \setminus \mathcal{X}$ and removing all elements in $\mathcal{X} \setminus \mathcal{Y}$ in whatever order, thus $\operatorname{dist}(\mathcal{X}, \mathcal{Y}) = \#(\mathcal{X}\Delta\mathcal{Y})$ and the number of all possible paths is $(\#(\mathcal{X}\Delta\mathcal{Y}))!$
Furthermore $\operatorname{diam}(\mathcal{G}_n) = n$ and it is achieved when $\{\mathcal{X}, \mathcal{Y}\}$ is a partition of $\mathfrak{B}_n$.

Now consider the adjacency matrix $\mathcal{A}_{n+1}$; for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{B}_{n+1}$ there are four cases

(1) $n+1 \notin \mathcal{X} \bigcup \mathcal{Y}$, therefore $\mathcal{X}, \mathcal{Y} \in \mathfrak{B}_n \subsetneqq \mathfrak{B}_{n+1}$,
(2) $n+1 \in \mathcal{X} \bigcap \mathcal{Y}$,
(3) $n+1 \in \mathcal{Y} \setminus \mathcal{X}$,
(4) $n+1 \in \mathcal{X} \setminus \mathcal{Y}$.

In the first case, from (1) we have $\varphi(\mathcal{X}), \varphi(\mathcal{Y}) \in [2^n]$, and the corresponding adjacency submatrix is given from the first 2^n rows and columns of $\mathcal{A}_{n+1}$ and obviously it equals $\mathcal{A}_n$.

In the second case from (1) we have $\varphi(\mathcal{X}), \varphi(\mathcal{Y}) \in \left[2^n + 1, 2^{n+1}\right]$ so the corresponding adjacency submatrix is given by $\left[2^n + 1, 2^{n+1}\right] \times \left[2^n + 1, 2^{n+1}\right]$ rows and columns. Clearly there is an edge in $\mathcal{G}_{n+1}$ between $\mathcal{X}$ and $\mathcal{Y}$ if and only if there is an edge between $\mathcal{X} \setminus \{n+1\} \in \mathfrak{B}_n$ and $\mathcal{Y} \setminus \{n+1\} \in \mathfrak{B}_n$, hence again the corresponding adjacency submatrix equals $\mathcal{A}_n$.

In the third case we have $\mathcal{X} \in \mathfrak{B}_n$ and $\mathcal{Y} \in \mathfrak{B}_{n+1} \setminus \mathfrak{B}_n$, and from (2) we get that there is an edge between $\mathcal{X}$ and $\mathcal{Y}$ if and only if $\mathcal{Y} = \mathcal{X} \biguplus \{n+1\}$, thus $\varphi(\mathcal{Y}) = 2^n + \varphi(\mathcal{X})$, and hence the corresponding adjacency submatrix is given by $[2^n] \times \left[2^n + 1, 2^{n+1}\right]$ rows and columns and it equals the identity I_{2^n}.

Patently the forth case is equivalent to the third one, and the desired result follows. ■

We need the following result, whose proof can be found in [18, Chap. 1, §3].

Theorem 2.2 ([18, Chap. 1, Thm 3.1.1]). *Let $p, q \in \mathbb{N} \setminus \{0\}$ and*

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a $(p+q)\times(p+q)$ *square matrix, such that* A *is a* $p\times p$ *square matrix,* B *is a* $p\times q$ *matrix,* C *is a* $q\times p$ *matrix, and* D *is a* $q\times q$ *square matrix.*

If $\det(A)\neq 0$ *then* $D-CA^{-1}B$ *is called the* Schur complement *of* A *in* M *and*

$$\det(M)=\det(A)\det\left(D-CA^{-1}B\right).$$

3. Main Results

In this section we compute the spectrum of $\mathcal{A}_n$.
Let $t\in\mathbb{R}$; we set

$$\mathcal{A}_n(t)=\mathcal{A}_n+t\cdot I_{2^n},$$

hence $\mathcal{A}_n=\mathcal{A}_n(0)$.

Theorem 3.1. *For any* $n\in\mathbb{N}$ *and* $t\in\mathbb{R}$

$$\det(\mathcal{A}_{n+2}(t))=(\det(\mathcal{A}_n(t)))^2\det(\mathcal{A}_n(t-2))\det(\mathcal{A}_n(t+2))$$

holds.

Proof: From Proposition 2.1 we have

$$\mathcal{A}_{n+2}(t)=\begin{pmatrix}\mathcal{A}_n(t) & I_{2^n} & I_{2^n} & 0\\ I_{2^n} & \mathcal{A}_n(t) & 0 & I_{2^n}\\ I_{2^n} & 0 & \mathcal{A}_n(t) & I_{2^n}\\ 0 & I_{2^n} & I_{2^n} & \mathcal{A}_n(t)\end{pmatrix},$$

and now we perform block–wise row and column operation. We replace the first row with the first row minus the forth one and the second one with the second one minus the third one and we obtain

$$\begin{pmatrix}\mathcal{A}_n(t) & 0 & 0 & -\mathcal{A}_n(t)\\ 0 & \mathcal{A}_n(t) & -\mathcal{A}_n(t) & 0\\ I_{2^n} & 0 & \mathcal{A}_n(t) & I_{2^n}\\ 0 & I_{2^n} & I_{2^n} & \mathcal{A}_n(t)\end{pmatrix}.$$

Now we replace the forth column with the first one plus the forth one and we get

$$\begin{pmatrix}\mathcal{A}_n(t) & 0 & 0 & 0\\ 0 & \mathcal{A}_n(t) & -\mathcal{A}_n(t) & 0\\ I_{2^n} & 0 & \mathcal{A}_n(t) & 2I_{2^n}\\ 0 & I_{2^n} & I_{2^n} & \mathcal{A}_n(t)\end{pmatrix},$$

and expanding along the first row we have

$$\det\left(\mathcal{A}_{n+2}(t)\right)=\det\left(\mathcal{A}_n(t)\right)\cdot\det\begin{pmatrix}\mathcal{A}_n(t) & -\mathcal{A}_n(t) & 0\\ 0 & \mathcal{A}_n(t) & 2I_{2^n}\\ I_{2^n} & I_{2^n} & \mathcal{A}_n(t)\end{pmatrix}.$$

Now consider the matrix

$$\begin{pmatrix}\mathcal{A}_n(t) & -\mathcal{A}_n(t) & 0\\ 0 & \mathcal{A}_n(t) & 2I_{2^n}\\ I_{2^n} & I_{2^n} & \mathcal{A}_n(t)\end{pmatrix}$$

and replace the second column with the first one plus the second one to obtain

$$\begin{aligned}\det\left(\mathcal{A}_{n+2}(t)\right) &= \det\left(\mathcal{A}_n(t)\right)\cdot\det\begin{pmatrix}\mathcal{A}_n(t) & 0 & 0\\ 0 & \mathcal{A}_n(t) & 2I_{2^n}\\ I_{2^n} & 2I_{2^n} & \mathcal{A}_n(t)\end{pmatrix}\\ &= \left(\det\left(\mathcal{A}_n(t)\right)\right)^2\det\begin{pmatrix}\mathcal{A}_n(t) & 2I_{2^n}\\ 2I_{2^n} & \mathcal{A}_n(t)\end{pmatrix}.\end{aligned}$$

It is very easy to see that

$$\det\begin{pmatrix}\mathcal{A}_n(t) & 2I_{2^n}\\ 2I_{2^n} & \mathcal{A}_n(t)\end{pmatrix}=(-1)^{2^n}\det\begin{pmatrix}2I_{2^n} & \mathcal{A}_n(t)\\ \mathcal{A}_n(t) & 2I_{2^n}\end{pmatrix}$$

(note that we can have $n=0$), and applying Theorem 2.2 we get

$$\begin{aligned}(-1)^{2^n}\det\begin{pmatrix}2I_{2^n} & \mathcal{A}_n(t)\\ \mathcal{A}_n(t) & 2I_{2^n}\end{pmatrix} &= (-1)^{2^n}\det\left(2I_{2^n}\left(2I_{2^n}-\frac{1}{2}\mathcal{A}_n(t)^2\right)\right)\\ =\det\left(\mathcal{A}_n(t)^2-4I_{2^n}\right) &= \det\left(\mathcal{A}_n(t-2)\right)\det\left(\mathcal{A}_n(t+2)\right).\end{aligned}$$

The desired result follows. ■

Corollary 3.2. $\det\left(\mathcal{A}_n(t)\right)\in\mathbb{Z}\left[t^2\right]$ *for any* $n\in\mathbb{N}\setminus\{0\}$.

Proof: Obviously $\det\left(\mathcal{A}_n(t)\right)\in\mathbb{Z}[t]$ therefore it is enough to prove that

$$\det\left(\mathcal{A}_n(t)\right)=\det\left(\mathcal{A}_n(-t)\right)$$

for any $n\in\mathbb{N}\setminus\{0\}$ and all $t\in\mathbb{R}$. It is easily seen that this is true if $n\in[2]$, and using induction and Theorem 3.1 the desired result follows. ■

Therefore we have $Spec\left(\mathfrak{B}_n\right)=\{t\in\mathbb{R}:\det\left(\mathcal{A}_n(t)\right)=0\}$ for all $n\in\mathbb{N}$. From Corollary 3.2 we get that if λ is an eigenvalue of $\mathcal{A}_n$ with multiplicity m_λ, then $-\lambda$ is an eigenvalue too with the same multiplicity.

Theorem 3.3. *For all* $n \in \mathbb{N}$

$$Spec\,(\mathfrak{B}_n) = \{-n + 2k \text{ with multiplicity } \binom{n}{k} \mid k = 0, \dots, n\}.$$

Proof: Evidently the statement holds for $n \in \{0, 1\}$. Using induction and Theorem 3.1 the desired result follows noticing that

$$\binom{n+2}{k} = \binom{n+1}{k-1} + \binom{n+1}{k} = \binom{n}{k-2} + 2\binom{n}{k-1} + \binom{n}{k}.$$

■

The following consequence is immediate.

Corollary 3.4. *Let* $k \in \mathbb{N}$;

- **:** ⊛ $\det(\mathcal{A}_{2k}) = 0$ *and* $\operatorname{rank}(\mathcal{A}_{2k}) = 4^k - \binom{2k}{k}$,
- **:** ⊛ $\det(\mathcal{A}_1) = -1$,
- **:** ⊛ *for all* $k \geq 1$

$$\det(\mathcal{A}_{2k+1}) = \left(\prod_{j=1}^{k} (2j+1)^{\binom{2k+1}{k-j}}\right)^2.$$

■

4. Remarks

Studying the spectrum of other types of posets is a very interesting and challenging question. In particular, a very tempting and appealing choice would be to consider the poset of S_n, the symmetric group of n elements equipped with the Bruhat ordering. However, this should probably be a very hard and difficult task to tackle, because so far not so much is known about the structure of the Bruhat graph of S_n, see [1].

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