Pré-Publicações do Departamento de Matemática
Universidade de Coimbra
Preprint Number 07–25

PRIESTLEY SPACES: THE THREEFOLD WAY

MARGARIDA DIAS

ABSTRACT: Three different ways of describing Priestley spaces are presented: as the objects of a category which arises in the equivalence induced by an adjunction $F \dashv U : \mathcal{O}rd\mathcal{T}op^{op} \to \mathcal{L}at$, as limits of (suitable) finite topologically-discrete preordered spaces (i.e. as profinite preorders) and as the *2-compact ordered* spaces, in the sense of Engelking and Mrówka [5], three situations where, for discrete-ordered topological spaces, one obtains Stone spaces instead of Priestley spaces.

KEYWORDS: ordered (preordered) topological spaces, Priestley spaces.
AMS SUBJECT CLASSIFICATION (2000): 18D30, 18A20, 18C20.

0. Introduction

A *Stone space* is a compact, Hausdorff and totally disconnected space. The full subcategory of $\mathcal{T}op$ whose objects are the Stone spaces will be denoted by $\mathcal{S}tone$. An *ordered topological space* is a triple $(X, \tau, \leq)$ where X is a set, τ is a topology and $\leq$ is an partial order on X. They are the objects of the category $\mathcal{O}rd\mathcal{T}op$ whose morphisms are the continuous maps which preserve the order and the same for $\mathcal{P}reord\mathcal{T}op$.

An ordered topological space $(X, \tau, \leq)$ is called *totally order-disconnected* if for $x, x' \in X$ such that $x' \not\leq x$ there exists a closed and open (clopen, for short) decreasing subset U of X (i.e. if $y \leq x \in U$ then $y \in U$) containing x but not x'. The compact topological spaces with a partial order which are totally order-disconnected are called the *Priestley spaces*. The full subcategory of $\mathcal{O}rd\mathcal{T}op$ whose objects are the Priestley spaces will be denoted by $\mathcal{PS}p$. The full subcategory of $\mathcal{P}reord\mathcal{T}op$ with objects the Stone spaces with a preorder which are totally preordered-disconnected, in the obvious sense, will be denoted by $\mathcal{P}reord\mathcal{P}$.

A lattice is a partially ordered set with meets and joins of finite subsets, including $\wedge\emptyset = 1, \vee\emptyset = 0$. Homomorphisms of lattices are functions that preserve finite meets and joins and so, in particular, preserve 1 and 0. Let $\mathcal{L}at$ denote the category of lattices and lattice homomorphisms. The full

Received September 20, 2007.
The author gratefully acknowledges financial support from FCT/Centro de Matemática da Universidade de Coimbra .

subcategory of $\mathcal{L}at$ with objects the distributive lattices will be denoted by $\mathcal{DL}at$. By $\mathcal{B}ool$ we denote the full subcategory of $\mathcal{DL}at$ with objects the Boolean algebras.

We deviate from important sources like L. Nachbin [8] and others. These define a preordered (ordered) topological spaces X as being a topological space with an order whose graph is closed in $X \times X$. However, for Priestley spaces $(X, \tau, \leq)$ this holds, i.e. the order relation is always closed in $X \times X$.

1. The duality induced by $F \dashv U : \mathcal{O}rd\mathcal{T}op^{op} \to \mathcal{L}at$

In this section, using well-known results (see e.g. [4]), we show that Priestley duality, as well as Stone duality, arise as the largest equivalences induced by dual adjunctions between $\mathcal{O}rd\mathcal{T}op$ and $\mathcal{L}at$, in the first case, and $\mathcal{T}op$ and $\mathcal{L}at$, in the second one.

We recall that an adjunction $F \dashv U : \mathcal{A} \to \mathcal{B}(\eta, \varepsilon)$, between categories $\mathcal{A}$ and $\mathcal{B}$ induces an equivalence between the full subcategories $\mathcal{A}_0$ of $\mathcal{A}$ and $\mathcal{B}_0$ of $\mathcal{B}$ where

$$\mathcal{A}_0 = Fix\varepsilon \equiv \{A \in \mathcal{A} | \varepsilon_A \text{is an isomorphism}\}$$

$$\mathcal{B}_0 = Fix\eta \equiv \{B \in \mathcal{B} | \eta_B \text{ is an isomorphism}\}$$

Let 2_l be the two chain $0 < 1$ and 2_{do} be 2_l with the discrete topology. The contravariant hom functors $Hom(-, 2_l) : \mathcal{L}at \to \mathcal{S}et$ and $Hom(-, 2_{do}) : \mathcal{O}rd\mathcal{T}op \to \mathcal{S}et$ can be lifted to $\mathcal{O}rd\mathcal{T}op$ and to $\mathcal{L}at$, respectively. Indeed, for a lattice L, we take

$$F(L) = (\mathcal{F}_p(L), \tau, \subseteq),$$

the set of all prime filters of L, identifying each $f \in Hom(L, 2_l)$ with the prime filter $f^{-1}(1)$, with the topology whose subbasis of open subsets is the set

$$S = \{U_b | b \in L\} \cup \{\mathcal{F}_p(L) - U_b | b \in L\} \text{ with } U_b = \{F \in \mathcal{F}_p(L) | b \in F\}. \quad (1)$$

For an ordered topological space X, we take

$$U(X) = (DClopen(X), \cap, \cup),$$

the (distributive) lattice of all clopen decreasing subsets of X, identifying each $g \in Hom(X, 2_{do})$ with the decreasing clopen set $g^{-1}(0)$.

The functor F is left adjoint to $U : \mathcal{O}rd\mathcal{T}op^{op} \to \mathcal{L}at$, where the components of the unit $\eta_L : L \to UF(L)$ and co-unit $\varepsilon_X : X \to FU(X)$ are defined by

$\eta_L(a) = \Gamma_a = \{F \in \mathcal{F}_p(L) | a \in F\}$ and $\varepsilon_X(x) = \Sigma_x = \{A \in DClopen(X) | x \in A\}$, respectively.

Proposition 1. *An ordered topological space X is a Priestley space if and only if ε_X is an isomorphism.*

Proof: The implications follow from the well-know facts:
(i) $(\mathcal{F}_p(L), \tau, \subseteq)$ is a Priestley space for every distributive lattice L.
(ii) If X is a Priestley space then ε_X is an isomorphism. ∎

Proposition 2. *A lattice L is distributive if and only if η_L is an isomorphism.*

Proof: . The "only if" and the "if" part are immediate consequences of the following:
(i) $U(X) = (DClopen(X), \cap, \cup)$ is a distributive lattice for every space X.
(ii) If L is a distributive lattice then η_L is an isomorphism. ∎

Consequently, the dual adjunction between $\mathcal{O}rd\mathcal{T}op$ and $\mathcal{L}at$ induces a dual equivalence between the categories $Fix\varepsilon = \mathcal{PS}p$ and $Fix\eta = \mathcal{DL}at$ as displayed in the diagram

$$\begin{tikzcd}
\mathcal{O}rd\mathcal{T}op^{op} \arrow[r, shift left, "U"] & \mathcal{L}at \arrow[l, shift left, "F"] \\
\mathcal{PS}p^{op} \arrow[u, hook] \arrow[r, phantom, "\sim"] & \mathcal{DL}at \arrow[u, hook]
\end{tikzcd}$$

which is the well-known Priestley duality.

Also the Stone duality arises from a the dual adjunction between $\mathcal{T}op$, that can be considered as the category of discretely ordered topological spaces, and $\mathcal{L}at$. The adjunction is defined by lifting to $\mathcal{T}op$ and to $\mathcal{L}at$ the functors $Hom(-, 2_l) : \mathcal{L}at \to \mathcal{S}et$ and $Hom(-, 2_d) : \mathcal{O}rd\mathcal{T}op \to \mathcal{S}et$, where 2_d is the two point discrete space. In this case, for a topological space X, $U(X) = (Clopen(X), \cap, \cup)$ is the set of its clopen subsets which is a Boolean algebra, and, for each Boolean algebra B, the set $F(B) = \mathcal{F}_p(L)$ of its prime filters with the topology defined by (1), is a Stone space.

In a completely analogous way, the dual adjunction $F \dashv U : \mathcal{T}op \to \mathcal{L}at$ induces a duality between $Fix\varepsilon = \mathcal{S}tone$ that $Fix\eta = \mathcal{B}ool$.

2. Priestley Spaces are Profinite Preorders

Priestley spaces are the profinite orders: they are exactly the limits of finite topologically-discrete ordered spaces, the later being essentially the objects of $\mathcal{P}os_f$. This follows from the fact that the embedding of $\mathcal{P}os_f \to \mathcal{PS}p$ is (equivalent) to the procompletion of $\mathcal{P}os_f$ ([7], Corollary 3.3 (ii)). It also follows from 4.6 and 4.7 in [6].

Here we are going to show that the Priestley spaces are limits of finite preordered spaces that we specify next.

It is well-known that the profinite spaces(= limits of finite topologically-discrete sets) are exactly the Stone spaces (See e.g. Theorem 3.4.7 of [2]). There Borceux and Janelidze consider, for each $X \in \mathcal{S}tone$, the set $\mathcal{R}$ of all equivalence relations R on X such that the topological quotient space is finite and has the discrete topology.

Considering the set $\mathcal{R}$, ordered by inclusion, as a category, the functor $D : \mathcal{R} \to Stone$, defined on objects by $D(R) = X/R$, and its limit $(\lambda_R : L \to X/R)_{R\in\mathcal{R}}$, it is proved there that the unique morphism $\varphi : X \to L = LimD$ such that $\lambda_R \circ \varphi = p_R$, for every $R \in \mathcal{R}$, is an homeomorphism.

To find an ordered version of this result we have to consider a more general setting than $\mathcal{PS}p$. We are going to show that the profinite preorders are exactly the objects of the full subcategory $\mathcal{P}reord\mathcal{P}$ of $\mathcal{P}reord\mathcal{T}op$ with objects the Stone spaces equipped with a preorder with respect to which they are totally preordered-disconnected.

We first give an example to show that, even for $X \in \mathcal{PS}p$, the relation induced in X/R by transitive closure is not, in general, an order relation.

Example 3. Let $\mathbb{N}_\infty$ be the Alexandroff compactification of $\mathbb{N}$ (the discrete space of natural numbers) equipped with the order $\{(1, a)|a \in \mathbb{N}_\infty\} \cup \{(2, 3)\} \cup \Delta_{\mathbb{N}_\infty}$ and $f : \mathbb{N}_\infty \to \{0, 1\}$ defined by

$$f(n) = \begin{cases} 1 & \text{if } n = 1, 3 \\ 0 & \text{otherwise} \end{cases}$$

Considering in X/R, where R is the equivalence relation induced in X by f, the induced relation by transitive closure, X/R is not ordered. In fact, $0 < 1$ in X/R, because $2 < 3$ in $\mathbb{N}_\infty$ and $f(2) = 0, f(3) = 1$, and $1 < 0$ in X/R because $1 < 2$ in $\mathbb{N}_\infty$ and $f(1) = 1, f(2) = 0$.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "s"'] & {2_{do}}^S \arrow[d, "p_s"] \\ & 2_{do} \end{tikzcd}$$

Let $x, x' \in X$ such that $x \neq x'$, then $x \not\leq x'$ or $x' \not\leq x$.

If $x \not\leq x'$ then there exists a clopen decreasing subset U of X such that $x' \in U$ and $x \notin U$. Then $x' \in U$ with U clopen decreasing subset of X and $x \in (X - U)$ with $X - U$ a clopen increasing subset of X. Hence, there exists a morphism $s_0 : X \to 2_{do}$ in S such that $s_0(U) = 0$ e $s_0(X - U) = 1$. Therefore $e(x) = (s(x))_{s\in S} \neq e(x') = (s(x'))_{s\in S}$ and so φ is injective. Indeed we proved more than that: we really proved that if $x \not\leq x'$ then $e(x) \not\leq e(x')$, that is that $\varphi : X \to e(X)$ is an order isomorphism.

Thus, the Priestley space X is isomorphic to the space $e(X)$ which is a compact subspace of the Hausdorff space ${2_{do}}^S$, and so $e(X)$ is closed of ${2_{do}}^S$. Therefore, X being a closed subspace of a power of 2_{do} belongs to $\mathcal{O}rd\mathcal{C}omp_{2_{do}}$. ■

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Margarida Dias
Departamento de Matemática, Universidade dos Açores, 9501-801 Ponta Delgada, Portugal
CMUC, Centre of Mathematics of the University of Coimbra, 3001-454 Coimbra, Portugal
E-mail address: mdias@notes.uac.pt

(1) (i) Taking $E = I = [0,1]$ with the trivial order, we have

$$\mathcal{C}Reg_I \underset{\perp}{\overset{\beta}{\rightleftarrows}} \mathcal{C}omp_I$$

where $\mathcal{C}Reg_I$ is the category of Tychonoff spaces, $\mathcal{C}omp_I$ is the category of compact Hausdorff spaces and $R = \beta$ is the Stone - Čech compactification.

(ii) Let $E = 2_d$ the discrete topological space with two points. Then

$$\mathcal{CR}eg_{2_d} \underset{\perp}{\overset{\zeta}{\rightleftarrows}} \mathcal{C}omp_{2_d}$$

where $\mathcal{CR}eg_{2_d}$ is the category of Hausdorff zero-dimensional spaces, $\mathcal{C}omp_{2_d}$ is the category of Stone spaces and $R = \zeta$ is the reflection, as proved by Banaschewski in [1].

(2) (i) Taking $E = I_o = [0,1]$ we have

$$\mathcal{O}rd\mathcal{CR}eg_{I_o} \underset{\perp}{\overset{\beta_o}{\rightleftarrows}} \mathcal{O}rd\mathcal{C}omp_{I_o}$$

where $\mathcal{O}rd\mathcal{CR}eg_{I_o}$ is the category of completely regular ordered spaces, $\mathcal{O}rd\mathcal{C}omp_{I_o}$ is the category of compact ordered spaces and $R = \beta_o$ is the Nachbin - Stone - Čech ordered compactification [8].

(ii) Let $E = 2_{do}$ be the two chain with discrete topology. There is a reflection

$$\mathcal{O}rd\mathcal{CR}eg_{2_{do}} \underset{\perp}{\overset{\zeta_o}{\rightleftarrows}} \mathcal{O}rd\mathcal{C}omp_{2_{do}}$$

that gives us a new way to describe the Priestley spaces.

Theorem 7. *$\mathcal{PS}p$ is the category $\mathcal{O}rd\mathcal{C}omp_{2_{do}}$.*

Proof: For each set S, 2_{do}^S is a Priestley space and so is each closed subset of powers of 2_{do}, because closed subspaces of compact spaces are compact and subspaces of totally order disconnected spaces are totally order disconnected.

Conversely, each Priestley space is a closed subspace of some power of 2_{do}. Indeed, let X be a Priestley space. For $S = \mathcal{O}rd\mathcal{T}op(X, 2_{do})$ we consider the commutative diagram

It is an easy exercise to show that the category $\mathcal{P}reord\mathcal{P}$ is complete. We show now that a preordered topological space is an object of $\mathcal{P}reord\mathcal{P}$ if and only if it is the limit of finite topologically-discrete preordered spaces.

Proposition 4. *$\mathcal{P}reord\mathcal{P}$ is the category of profinite preorders.*

Proof: Discrete finite preordered spaces are totally disconnected with respect to every preorder. Since $\mathcal{P}reord\mathcal{P}$ is complete, one of the implications is trivial.

Conversely, let $X \in \mathcal{P}reord\mathcal{P}$ and $\mathcal{R}$ the set of all equivalence relations R on X such that the topological quotient space with respect to the canonical projection $p_R : X \to X/R$ is finite, discrete and preordered by the transitive closure of $p_R \times p_R(\preceq_X)$.

For $D_o : \mathcal{R} \to \mathcal{P}reord\mathcal{P}$ defined by $D_o(R) = X/R$, let $(\lambda_R : L \to X/R)_{R\in\mathcal{R}}$ be the limit of D_o in $\mathcal{P}reord\mathcal{P}$ and φ the unique morphism such that $\lambda_R \circ \varphi = p_R$ for every $R \in \mathcal{R}$.

We know that φ is an homeomorphism. So, we have just to prove that φ is an order isomorphism, that is that

$$\varphi(x) \preceq \varphi(x') \Rightarrow x \preceq x'.$$

Let us assume that $x \npreceq x'$. Then, as X is totally preordered disconnected, there exists a clopen decreasing subset U of X such that $x' \in U$ and $x \notin U$.

Let us take R_U the equivalence relation on X corresponding to the partition

$$X = U \cup (X - U),$$

therefore, $R_U \in \mathcal{R}$, $[x]_{R_U} = X - U$, $[x']_{R_U} = U$ and $[x]_{R_U} \npreceq [x']_{R_U}$. Indeed, if $[x]_{R_U} \preceq [x']_{R_U}$ then there would exists a finite sequence

$$x_1 \preceq x'_1,\ x_2 \preceq x'_2,\ ...\ x_n \preceq x'_n$$

such that $[x]_{R_U} = [x_1]_{R_U}$, $[x'_i]_{R_U} = [x_{i+1}]_{R_U}$ for $i = 1, 2, ..., n-1$ and $[x']_{R_U} = [x'_n]_{R_U}$.

But, since $X/R_U = \{X - U, U\}$, we would have, for same $1 \leq k < n$, $x_k \in X - U$ and $x_{k+1} \in U$ with $x_k \preceq x_{k+1}$ what it would imply that $x_k \in U$, because U is a decreasing subset, so we would have a contradiction.

Thus, we showed that there exists a equivalence relation R_U on X such that $[x]_{R_U} \npreceq [x']_{R_U}$, therefore $\varphi(x) \npreceq \varphi(x')$ and this concludes the proof. ∎

The way $\mathcal{P}reord\mathcal{P}$ sits between $\mathcal{PS}p$ and $\mathcal{P}reord\mathcal{T}op$ is studied in [3]: $\mathcal{PS}p$ is a regular-epireflective subcategory of $\mathcal{P}reord\mathcal{P}$ (2.6) and $\mathcal{P}reord\mathcal{P}$is an

bireflective subcategory of $\mathcal{P}reord\mathcal{S}tone$ (2.4), the later being reflective in $\mathcal{P}reord\mathcal{T}op$ as it follows from the reflectiveness of the topological side.

From the above we obtain our claim. More precisely, we conclude the following:

Corollary 5. *A preordered topological space X is a Priestley space if and only if the limit object of $D_o : \mathcal{R} \to \mathcal{P}reord\mathcal{P}$ is an ordered space.*

3. Priestley spaces are the 2-*compact ordered* spaces

For a topological space E and a set S we denote by E^S the product of S copies of E.

Let E be an Hausdorff space.

Using the terminology introduced by Engelking and Mrówka in [5], the *E-completely regular* spaces and the *E-compact* spaces are the subspaces and the closed subspaces of some power of E, respectively.

We are going to consider the full subcategories of $\mathcal{T}op$, $\mathcal{CR}eg_E$, with objects the E-completely regular and $\mathcal{C}omp_E$ with objects the E-compact spaces.

If, furthermore, E is equipped with an order, that is, E is an ordered Hausdorff space, then $\mathcal{O}rd\mathcal{CR}eg_E$ and $\mathcal{O}rd\mathcal{C}omp_E$ denote the full subcategories of $\mathcal{O}rd\mathcal{T}op$ which objects are subspaces and closed subspaces of some power of E, respectively, with the induced order.

If $E = I$, the unit interval with usual topology and trivial order, then $\mathcal{CR}eg_E$ is the category of Tychonoff spaces and $\mathcal{C}omp_E$ is the category of compact spaces. If $E = I_o$, the space I with the usual order, then $\mathcal{O}rd\mathcal{CR}eg_E$ and $\mathcal{O}rd\mathcal{C}omp_E$ are, respectively, the categories of completely regular ordered spaces and compact ordered spaces, as defined in [8]. In both cases we have well-known reflections.

We are going to describe the reflection R of $\mathcal{O}rd\mathcal{CR}eg_E$ in $\mathcal{O}rd\mathcal{C}omp_E$, for an arbitrary ordered Hausdorff space E.

Proposition 6. *For every Hausdorff ordered space E, $\mathcal{O}rd\mathcal{C}omp_E$ is a reflective subcategory of $\mathcal{O}rd\mathcal{CR}eg_E$.*

Proof: Let X be an E-completely regular ordered space and S be the set of the continuous and order-preserving maps for X in E, that is $S = \mathcal{O}rd\mathcal{T}op(X, E)$. We have the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\varphi"] \arrow[dr, "s"'] & E^S \arrow[d, "p_s"] \\
 & E
\end{tikzcd}$$

where s are continuous and an order-preserving maps, p_s are the corresponding projections and $\varphi = ev_{X,s}$ is the induced map, the evaluation map, which is defined by $\varphi(x) = (s(x))_{s\in S}$. Then, being a subspace of some power of E, X is also a subspace of E^S and so it is isomorphic to $\varphi(X)$.

We define $R(X)$ as the closure $\overline{\varphi(X)}$ of $\varphi(X)$ in E^S. Consequently, $R(X)$ is an object of the category $\mathcal{O}rd\mathcal{C}omp_E$.

For a morphism $f : X \to Y$ in $\mathcal{O}rd\mathcal{CR}eg_E$, we have the following diagram

$$\begin{tikzcd}
X \arrow[r, "f"] & Y \arrow[r, "\eta_Y"] \arrow[drr, "q"'] & R(Y) \arrow[r, hook] & \prod_{S'} E \arrow[d, "p_q"] \\
 & & & E
\end{tikzcd}$$

where $S' = \mathcal{O}rd\mathcal{T}op(Y, E)$ and $q \in S'$. So $q \circ f \in S$, then there exists a unique morphism $h : \prod_S E \to \prod_{S'} E$, defined by $h((e_s)_{s\in S}) = (\hat{e}_{s'})_{s'\in S'}$ where $\hat{e}_{s'} = e_{s'\circ f}$, such that $p_q \circ h = p_{q\circ f}$. Therefore, $h(\overline{\varphi(X)}) \subseteq \overline{h(\varphi(X))} \subseteq \overline{\varphi(Y)}$, that is $h(R(X)) \subseteq R(Y)$, and so the restriction h' of h to $R(X)$ is a continuous order-preserving map from $R(X)$ to $R(Y)$ such that $h' \circ \eta_X = \eta_Y \circ f$,

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[rr, bend left, "\eta_X"] \arrow[d, "f"] & \varphi(X) \arrow[r, hook] & R(X) \arrow[r, hook] \arrow[d, "h'"] & \prod_S E \arrow[d, "h"] \\
Y \arrow[r, "\cong"] \arrow[rr, bend left, "\eta_Y"] & \varphi(Y) \arrow[r, hook] & R(Y) \arrow[r, hook] & \prod_{S'} E
\end{tikzcd}$$

If Y is a closed subspace of $E^{S'}$, then $Y \cong \varphi(Y) \cong \overline{\varphi(Y)} = R(Y)$ and η_Y is an isomorphism. For $\overline{f} = \eta_Y^{-1} \circ h'$ we have that $\overline{f} \circ \eta_X = f$ and the morphism $\overline{f}$ is unique because η_X, being a dense map in $\mathcal{O}rd\mathcal{H}aus$, is an epimorphism. Therefore $\mathcal{O}rd\mathcal{C}omp_E$ is a reflective subcategory of $\mathcal{O}rd\mathcal{CR}eg_E$. ■

Examples