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# DISTRIBUTIONAL EQUATION FOR LAGUERRE-HAHN FUNCTIONALS ON THE UNIT CIRCLE

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Abstract: Let $u$ be a hermitian linear functional defined in the linear space of Laurent polynomials and $F$ its corresponding Carathéodory function. We establish the equivalence between a Riccati differential equation with polynomial coefficients for $F$, $zAF' = BF^2 + CF + D$ and a distributional equation for $u$, $\mathcal{D}(Au) = B_1u^2 + C_1u + H_1\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, and the polynomials $B_1, C_1, D_1$ are defined in terms of the polynomials $A, B, C, D$.

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## 1. Introduction

In this paper we introduce the concept of Laguerre-Hahn class of hermitian functionals on the unit circle. Let $u$ be a hermitian regular linear functional defined in the linear space of Laurent polynomials and $F$ the corresponding Carathéodory function. The functional $u$ (respectively, the corresponding Carathéodory function, $F$) is said to be *Laguerre-Hahn* if $F$ satisfies a Riccati differential equation with polynomial coefficients,

$$zAF' = BF^2 + CF + D, \ A \not\equiv 0 \,. \tag{1}$$

We shall call the set of all such functionals (respectively, Carathéodory function, $F$) the *Laguerre-Hahn class on the unit circle*. We remark that the Laguerre-Hahn class on the unit circle can be regarded as an extension of the Laguerre-Hahn class on the real line, studied in [5, 7, 8].

If $B \equiv 0$ in (1) we obtain the Laguerre-Hahn affine class on the unit circle (see [2, 3]). If $B \equiv 0$ and $C, D$ specific polynomials, we obtain the semi-classical class on the unit circle (see [2, 6, 11]). Moreover, the Laguerre-Hahn class on the unit circle includes the class of second degree functionals on

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the unit circle (see [4]) and includes, as we will see on section 3, the linear-fractional transformations of Carathéodory functions which are Laguerre-Hahn.

Analogously to what has been done on the real case (cf. [7, 8]), we establish the equivalence between (1) and a distributional equation for the corresponding $u$,

$$\mathcal{D}(Au) = B_1 u^2 + C_1 u + H_1 \mathcal{L}\,, \tag{2}$$

where $\mathcal{L}$ is the Lebesgue functional, and $B_1, C_1, D_1$ polynomials defined in terms of $A, B, C, D$.

This paper is organized as follows. In section 2 we give the definitions and introduce some notations which will be used in the forthcoming sections. In section 3 we study the stability in the Laguerre-Hahn class, and give some examples of sequences of polynomial orthogonal with respect to a functional of Laguerre-Hahn type. In section 4 we state some auxiliary results which enable us to establish, in section 5, the equivalence between (1) and (2).

## 2. Preliminaries and notations

Let $\Lambda = \operatorname{span}\{z^k : k \in \mathbb{Z}\}$ be the space of Laurent polynomials with complex coefficients, $\Lambda'$ its algebraic dual space, $\mathbb{P} = \operatorname{span}\{z^k : k \in \mathbb{N}\}$ the space of complex polynomials, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ (or, using the parametrization $z = e^{i\theta}$, $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\,\})$ the unit circle.

Let $u \in \Lambda'$ be a linear functional. We denote by $\langle u, f \rangle$ the action of $u$ over $f \in \Lambda$.

Given the sequence of moments $(c_n)$ of $u$, $c_n = \langle u, \xi^{-n} \rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, the minors of the Toeplitz matrix are defined by

$$\Delta_{-1} = 1\,,\ \Delta_0 = c_0\,,\ \Delta_k = \begin{vmatrix} c_0 & \cdots & c_k \\ \vdots & & \vdots \\ c_{-k} & \cdots & c_0 \end{vmatrix}\,,\ k \in \mathbb{N}\,.$$

**Definition 1** (cf. [10])**.** The linear functional $u$ is:

a) *hermitian* if $c_{-n} = \overline{c_n}, \forall n \geq 0$;

b) *regular* or *quasi-definite* if $\Delta_n \neq 0, \forall n \geq 0$;

c) *positive definite* if $\Delta_n > 0, \forall n \geq 0$.

As an example of hermitian functional we have the *Lebesgue functional*, $\mathcal{L}$, defined in terms of its moments by $\langle \mathcal{L}, \xi^{-n} \rangle = \delta_{0,n}$, $n \in \mathbb{Z}$.

If $u$ is a positive definite hermitian functional, then there exists a non-trivial probability measure $\mu$ supported on the unit circle such that

$$\langle u, \xi^{-n} \rangle = \int_0^{2\pi} \xi^{-n} d\mu(\theta) \,, \quad \xi = e^{i\theta} \,, \quad n \in \mathbb{Z} \,.$$

**Definition 2.** Let $\{\phi_n\}$ be a sequence of complex polynomials with deg$(\phi_n)$ $= n$ and $u$ a hermitian linear functional. We say that $\{\phi_n\}$ is a *sequence of orthogonal polynomials* with respect to $u$ (or $\{\phi_n\}$ is a sequence of orthogonal polynomials on the unit circle) if

$$\langle u, \phi_n(\xi)\overline{\phi_m}(1/\xi) \rangle = K_n \delta_{n,m} \,, \ \xi = e^{i\theta} \,, \ K_n \neq 0 \,, \ n, m \in \mathbb{N} \,.$$

If for each $n \in \mathbb{N}$ the leading coefficient $\phi_n$ is 1, then $\{\phi_n\}$ is said to be a *sequence of monic orthogonal polynomials* and will be denoted by MOPS.

*Remark* . In the positive definite case, as the hermitian functional $u$ has an integral representation in terms of a measure $\mu$, we will also say that $\{\phi_n\}$ is orthogonal with respect to $\mu$.

For a polynomial $P$ with degree $n$, the reciprocal polynomial $P^*$ is defined by $P^*(z) = z^n \overline{P}(1/z)$ . It is well known (see [10]) that a given sequence of complex polynomials $\{\phi_n\}$ is orthogonal on the unit circle if, and only if, $\{\phi_n\}$ satisfy a recurrence relation of the following type,

$$\phi_n(z) = z\phi_{n-1}(z) + a_n \phi^*_{n-1}(z), \ \ n \geq 1 \,, \tag{3}$$

with $|a_n| \neq 1$ , and initial conditions $\phi_0(z) = 1, \ \phi_{-1}(z) = 0$.

Given a sequence of monic orthogonal polynomials $\{\phi_n\}$ with respect to $u$, the *sequence of associated polynomials of the second kind* $\{\Omega_n\}$ is defined by

$$\Omega_0(z) = 1 \,, \quad \Omega_n(z) = \langle u_\theta, \frac{e^{i\theta} + z}{e^{i\theta} - z} \left( \phi_n(e^{i\theta}) - \phi_n(z) \right) \rangle, \ n = 1, 2, \ldots \,,$$

and verify $\phi^*_n(z)\Omega_n(z) + \phi_n(z)\Omega^*_n(z) = 2K_n z^n$ , for all $n = 1, 2, \ldots$ (see for instance [10]).

We consider the formal series associated with the hermitian linear functional $u$,

$$F_u(z) = c_0 + 2\sum_{k=1}^{+\infty} c_k z^k \,, \ |z| < 1 \,, \tag{4}$$

$$F_u(z) = -c_0 - 2\sum_{k=1}^{+\infty} c_{-k} z^{-k} \,, \ |z| > 1 \,. \tag{5}$$

Since, for each $\theta \in [0, 2\pi[$, the following expansions take place,

$$\begin{aligned} \frac{e^{i\theta}+z}{e^{i\theta}-z} &= 1+2\sum_{k=1}^{+\infty}(e^{i\theta})^{-k}z^k, \ |z|<1\,, \\ \frac{e^{i\theta}+z}{e^{i\theta}-z} &= -1-2\sum_{k=1}^{+\infty}(e^{i\theta})^{k}z^{-k}, \ |z|>1 \end{aligned}$$

then, taking into account the definition of the moments of $u$, formally we can write

$$\langle u_\theta, \frac{e^{i\theta}+z}{e^{i\theta}-z}\rangle = F_u(z)\,, \quad z \in \mathbb{C}\setminus\mathbb{T} \tag{6}$$

where $\langle u_\theta,\ .\ \rangle$ denotes the action of $u$ over the variable $\theta$, $\theta \in [0, 2\pi[$. Thus, we will also say that the series in (4), (5) formally correspond to the function $F_u$ defined by (6).

In the positive definite case, $F_u$ is the *Carathéodory function* corresponding to $u$, and is represented by

$$F_u(z) = \int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}d\mu(\theta)\,, \quad z \in \mathbb{C}\setminus\mathbb{T}\,,$$

where $\mu$ is the probability measure associated with $u$.

Next we see some operations in the linear space $\Lambda'$. Given $u \in \Lambda'$, $f, g \in \Lambda$, the functionals $fu$ and $\mathcal{D}u$, both elements of $\Lambda'$, are defined by

$$\langle fu, p\rangle = \langle u, fp\rangle\,, \quad \langle \mathcal{D}u, p\rangle = -i\langle u, \xi p'\rangle\,, \quad p \in \Lambda$$

thus, $\langle \mathcal{D}(gu), p\rangle = -i\langle u, \xi g p'\rangle,\ p \in \Lambda\,.$ We remark that if $u$ is hermitian, then $\mathcal{D}u$ is also hermitian.

We consider the *generating function of the moments for a hermitian functional,* $u$, defined in $\mathbb{C}\setminus\mathbb{T}$ by

$$\mathcal{F}_u(z) = \sum_{n=0}^{+\infty} c_n z^n\,,\ |z|<1\,, \quad \mathcal{F}_u(z) = -\sum_{n=1}^{+\infty} c_{-n} z^{-n}\,,\ |z|>1\,.$$

Then the following relation between the Carathéodory function and the generating function of the moments holds,

$$\mathcal{F}_u(z) = \frac{F_u(z)+1}{2}\,, \ \ z \in \mathbb{C}\setminus\mathbb{T}\,. \tag{7}$$

The next definition is the analogue of the definition given in [8] for the real case.

**Definition 3.** Let $u, v \in \Lambda'$ be hermitian linear functionals and $\mathcal{F}_u, \mathcal{F}_v$ the corresponding generating functions of the moments. The *product of* $u$ *and* $v$, $uv$, is the functional defined in terms of its moments by

$$(uv)_n = \sum_{\substack{\nu+k=n \\ \mathrm{sgn}(\nu)=\mathrm{sgn}(k)=\mathrm{sgn}(n)}} u_\nu v_k \,, \qquad n \in \mathbb{Z}\,, \tag{8}$$

that satisfies

$$\mathcal{F}_u(z)\, \mathcal{F}_v(z) = \mathcal{F}_{uv}(z) \tag{9}$$

*Remark* . The functional $uv$ defined by (8) is hermitian.

As a consequence, we get the following result.

**Lemma 1.** *Let* $u$ *be a hermitian linear functional and* $F_u$ *the corresponding formal Carathéodory function. Then,*

$$(F_u)^2 = 2F_{u^2} - 2F_u + 1\,. \tag{10}$$

*Proof*: Put $u = v$ in (9) and get, after using (7) that $F_{u^2} + 1 = ((F_u)^2 + 2F_u + 1)/2$, and the required follows. ∎

## 3. Stability in the Laguerre-Hahn class on the unit circle. Examples.

We begin this section studying the stability of the Laguerre-Hahn Carathéodory functions under a special kind of linear-fractional transformations.

**Theorem 1.** *Let* $F_1$ *be a Carathéodory function satisfying a Riccati differential equations with polynomials coefficients,*

$$A_1 F_1' = B_1 F_1^2 + C_1 F_1 + D_1,\ A_1 \not\equiv 0, \tag{11}$$

*and* $F_2$ *a linear-fractional transformation of* $F_1$ *of the following type*

$$F_2 = \frac{\alpha_1 - \beta_1 F_1}{-\alpha_2 + \beta_2 F_1},\ \ \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{P},\ \alpha_1\beta_2 - \alpha_2\beta_1 \not\equiv 0\,.$$

*Then,* $F_2$ *satisfies*

$$A_2 F_2' = B_2 F_2^2 + C_2 F_2 + D_2\,, \tag{12}$$

*with*

$$A_2 = -(\alpha_1\beta_2 - \alpha_2\beta_1)A_1 \not\equiv 0\,, \tag{13}$$
$$B_2 = (\alpha_2\beta_2' - \alpha_2'\beta_2)A_1 + \alpha_2^2 B_1 + \alpha_2\beta_2 C_1 + \beta_2^2 D_1\,, \tag{14}$$
$$C_2 = (\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)A_1 + 2\alpha_1\alpha_2 B_1 + (\alpha_1\beta_2 + \alpha_2\beta_1)C_1 + 2\beta_1\beta_2 D_1\,, \tag{15}$$
$$D_2 = (\alpha_1\beta_1' - \alpha_1'\beta_1)A_1 + \alpha_1^2 B + \alpha_1\beta_1 C_1 + \beta_1^2 D_1\,. \tag{16}$$

*Thus, if $F_2$ is a Carathéodory function, then $F_2$ is Laguerre-Hahn.*

*Proof*: We have $F_2 = \dfrac{\alpha_1 - \beta_1 F_1}{-\alpha_2 + \beta_2 F_1}$ or equivalently $F_1 = \dfrac{\alpha_1 + \alpha_2 F_2}{\beta_1 + \beta_2 F_2}$. By substituting $F_1 = \dfrac{\alpha_1 + \alpha_2 F_2}{\beta_1 + \beta_2 F_2}$ in (11) we obtain (12) with coefficients given by (13)-(16). Thus, the assertion follows. ■

Next we see some examples of Laguerre-Hahn sequences.

**Corollary 1.** *Let $\{\phi_n\}$ be a MOPS on $\mathbb{T}$, $\{\Omega_n\}$ the sequence of associated polynomials of the second kind and $F, F_1$ the corresponding Carathéodory functions. If $F$ is Laguerre-Hahn and satisfies $zAF' = BF^2 + CF + D$, with $A \not\equiv 0$, then $F_1$ is Laguerre-Hahn and satisfies $zAF_1' = -DF_1^2 - CF_1 - B$.*

*Proof*: It is well known that $\{\Omega_n\}$ is orthogonal with respect to $F_1 = 1/F$, with $F$ the Carathéodory function associated to $\{\phi_n\}$ (cf. for example [9]). Thus, the assertion follows from Theorem 1. ■

**Definition 4** (Peherstorfer [9])**.** Let $\{\phi_n\}$ be a MOPS on the unit circle, $(a_n)$ the corresponding sequence of reflection coefficients and $N \in \mathbb{N}$. The sequence $\{\phi_n^N\}$ defined by (3) with $\phi_n^N(0) = a_{n+N}$, $n = 0, 1, \ldots$ is said to be the *sequence of associated polynomials of $\{\phi_n\}$ of order $N$.*

Let $\{\phi_n\}$ be a MOPS with respect to a Carathéodory function $F$ and $\{\Omega_n\}$ be the sequence of orthogonal polynomials of the first kind. In [9] it is established that $\{\phi_n^N\}$ is orthogonal with respect to the Carathéodory function $F^N$ given by

$$F^N = \frac{(\Omega_N - \Omega_N^*) + (\phi_N + \phi_N^*)F}{(\Omega_N + \Omega_N^*) + (\phi_N - \phi_N^*)F}\,, \tag{17}$$

and $(\Omega_N - \Omega_N^*)(\phi_N - \phi_N^*) - (\phi_N + \phi_N^*)(\Omega_N + \Omega_N^*) = -2K_N z^N \not\equiv 0\,.$

**Corollary 2.** *Let $F$ be a Carathéodory function, $\{\phi_n\}$ the corresponding MOPS, $\{\Omega_n\}$ the sequence of associated polynomials of the second kind and $F^N$ the Carathéodory function corresponding to the sequence of associated polynomials of $\{\phi_n\}$ of order $N$, $\{\phi_n^N\}$. If $F$ is Laguerre-Hahn and satisfies*

$$zAF' = BF^2 + CF + D, A \not\equiv 0\,,$$

*then $F^N$ is Laguerre-Hahn and satisfies*

$$A_N\left(F^N\right)' = B_N\left(F^N\right)^2 + C_N F^N + D_N\,,$$

*with $A_N = 4K_N z^{N+1} A \not\equiv 0$ and*

$$\begin{aligned}
B_N &= zA\left\{(\Omega_N+\Omega_N^*)'(\phi_N-\phi_N^*) - (\phi_N-\phi_N^*)'(\Omega_N+\Omega_N)^*\right\}\\
&\quad + (\Omega_N+\Omega_N^*)^2 B - (\Omega_N+\Omega_N^*)(\phi_N-\phi_N^*)C + (\phi_N-\phi_N^*)^2 D\,,\\
C_N &= zA\left\{(\Omega_N-\Omega_N^*)(\phi_N-\phi_N^*)' - (\Omega_N+\Omega_N^*)'(\phi_N+\phi_N^*)\right\}\\
&\quad - 2(\Omega_N^2-(\Omega_N^*)^2)B + 2(\phi_N\Omega_N+\phi_N^*\Omega_N^*)C - 2(\phi_N^2-(\phi_N^*)^2)D\,,\\
D_N &= (\Omega_N-\Omega_N^*)^2 B - (\Omega_N-\Omega_N^*)(\phi_N+\phi_N^*)C + (\phi_N+\phi_N^*)^2 D\,.
\end{aligned}$$

*Proof*: From (17) it follows that $F = \dfrac{(\Omega_N+\Omega_N^*)F^N - (\Omega_N-\Omega_N^*)}{(\phi_N+\phi_N^*) - (\phi_N-\phi_N^*)F^N}$, and using Theorem 1 the asssertion follows. ■

## 4. Auxiliary results

Next we establish results that will be used in next section. We consider hermitian linear functionals $u \in \Lambda$ and $F_u$ the corresponding formal Carathéodory function defined by (6).

**Lemma 2** (cf. [2])**.** *Let $u$ be a hermitian linear functional and $A, B$ polynomials. Then, for all $z \in \mathbb{C}\setminus\mathbb{T}$,*

$$\langle B(\xi)u, \frac{\xi+z}{\xi-z}\rangle = P_{u,B}(z) + B(z)F_u(z)\,, \tag{18}$$

$$A(z)F_u'(z) = -A'(z)F_u(z) + Q_{u,A}(z) + \frac{1}{iz}\langle \mathcal{D}(Au), \frac{\xi+z}{\xi-z}\rangle \tag{19}$$

*where $P_{u,B}$, $Q_{u,A}$ are polynomials with* $\deg(P_{u,B}) = \deg(B)$, $\deg(Q_{u,A}) = \deg(A)-1$, *defined by*

$$P_{u,B}(z) = \langle u, \frac{\xi+z}{\xi-z}\,(B(\xi)-B(z))\rangle\,, \tag{20}$$

$$Q_{u,A}(z) = -A'(z) - \langle u, 2\xi\sum_{k=2}^{\deg(A)}\frac{A^{(k)}(z)}{k!}(\xi-z)^{k-2}\rangle \tag{21}$$

*Remark* . From (20) and (21) follows $P_{u,B+\tilde{B}} = P_{u,B}+P_{u,\tilde{B}}\,, \quad P_{u,\alpha B} = \alpha P_{u,B}\,,$ with $\alpha\in\mathbb{C}$ and $B,\tilde{B}\in\mathbb{P}$.

**Lemma 3** (cf. [1]). *Let $u\in\Lambda'$ be a hermitian linear functional and $A$ a polynomial. Then, $(A(z)+\overline{A}(1/z))u$ is hermitian.*

Next lemma is a generalization of Theorem 3.4 of [1].

**Lemma 4.** *Let $u$ be a hermitian linear functional. If there exist polynomials $A,B,C,H$ such that $\mathcal{D}(Au) = Bu^2+Cu+H\mathcal{L}$, with $\mathcal{L}$ the Lebesgue functional, then*

$$\mathcal{D}(A+\overline{A})u = (B+\overline{B})u^2+(C+\overline{C})u+(H+\overline{H})\mathcal{L} \tag{22}$$

*Conversely, if (22) holds, then $u$ satisfies the distributional equation with polynomial coefficients*

$$\mathcal{D}(A_1u) = B_1u^2+(C_1+isA_1)u+H_1\mathcal{L} \tag{23}$$

*where* $s = \max\{\deg(A),\deg(B),\deg(C),\deg(H)\}$, *and the polynomials $A_1$, $B_1,C_1,H_1$ are given by* $A_1(z) = z^s(A(z)+\overline{A}(1/z))$, $B_1(z) = z^s(B(z)+\overline{B}(1/z))$, $C_1(z) = z^s(C(z)+\overline{C}(1/z))$, $H_1(z) = z^s(H(z)+\overline{H}(1/z))$.

*Proof*: If $\mathcal{D}(Au) = Bu^2+Cu+H\mathcal{L}$, then

$$\langle\mathcal{D}(Au),\xi^k\rangle = \langle Bu^2+Cu+H\mathcal{L},\xi^k\rangle,\ \forall k\in\mathbb{Z}\,.$$

Applying conjugates follows

$$\langle\mathcal{D}(\overline{A}u),\xi^{-k}\rangle = \langle\overline{B}u^2+\overline{C}u+\overline{H}\mathcal{L},\xi^{-k}\rangle,\ \forall k\in\mathbb{Z}\,.$$

Therefore, we obtain

$$\langle\mathcal{D}((A+\overline{A})u),\xi^n\rangle = \langle(B+\overline{B})u^2+(C+\overline{C})u+(H+\overline{H})\mathcal{L},\xi^n\rangle,\ \forall n\in\mathbb{Z}\,,$$

and (22) follows.

Conversely, if $u$ satisfies (22), then we successively get for all $k \in \mathbb{Z}$

$$\begin{aligned}\langle \mathcal{D}((A+\overline{A})u), \xi^k\rangle &= \langle (B+\overline{B})u^2, \xi^k\rangle + \langle (C+\overline{C})u, \xi^k\rangle + \langle (H+\overline{H})\mathcal{L}, \xi^k\rangle\,, \\ -ik\langle u, (A(\xi)+\overline{A}(1/\xi))\xi^k\rangle &= \langle u^2, (B(\xi)+\overline{B}(1/\xi))\xi^k\rangle \\ &\quad + \langle (C(\xi)+\overline{C}(1/\xi))\xi^k\rangle + \langle \mathcal{L}, (H(\xi)+\overline{H}(1/\xi))\xi^k\rangle\,.\end{aligned}$$

Let $s = \max\{\deg(A), \deg(B), \deg(C), \deg(H)\}$, then the last equation is given by

$$\begin{aligned}&-ik\langle u, \xi^s(A(\xi)+\overline{A}(1/\xi))\xi^{k-s}\rangle = \langle u^2, \xi^s(B(\xi)+\overline{B}(1/\xi))\xi^{k-s}\rangle \\ &+ \langle u, \xi^s(C(\xi)+\overline{C}(1/\xi))\xi^{k-s}\rangle + \langle \mathcal{L}, \xi^s(H(\xi)+\overline{H}(1/\xi))\xi^{k-s}\rangle,\ \forall k \in \mathbb{Z} \quad (24)\end{aligned}$$

If we write $m = k - s$ and define $A_1, B_1, C_1, H_1$ as in the statement of the lemma, the equation (24) is given by

$$\begin{aligned}&-i(s+m)\langle u, A_1(\xi)\xi^m\rangle \\ &\qquad = \langle u^2, B_1(\xi)\xi^m\rangle + \langle u, C_1(\xi)\xi^m\rangle + \langle \mathcal{L}, H_1(\xi)\xi^m\rangle,\ \forall m \in \mathbb{Z}\,,\end{aligned}$$

and so,

$$\begin{aligned}&-im\langle u, A_1(\xi)\xi^m\rangle \\ &\quad = \langle u^2, B_1(\xi)\xi^m\rangle + \langle u, (C_1(\xi)+isA_1(\xi))\xi^m\rangle + \langle \mathcal{L}, H_1(\xi)\xi^m\rangle,\ \forall m \in \mathbb{Z}\,.\end{aligned}$$

From the definition of $\mathcal{D}$, the previous equation is equivalent to

$$\langle \mathcal{D}(A_1 u), \xi^m\rangle = \langle B_1 u^2, \xi^m\rangle + \langle (C_1 + isA_1)u, \xi^m\rangle + \langle H_1\mathcal{L}, \xi^m\rangle,\ \forall m \in \mathbb{Z}\,,$$

and we get (23). ■

**Lemma 5.** *Let $u$ be a hermitian linear functional and $F_u$ the corresponding formal Carathéodory function. If $F_u$ satisfies $zAF_u' = BF_u^2 + CF_u + D$ in $\mathbb{C} \setminus \mathbb{T}$, then $u$ satisfies the distributional equation*

$$\langle \mathcal{D}(Au) + L(\xi)u - 2iB(\xi)u^2, \frac{\xi+z}{\xi-z}\rangle = iH(z) \quad (25)$$

*with*

$$L = i\,(-zA' + 2B - C)\ ,\ H = B - 2P_{u^2,B} + P_{u,-zA'+2B-C} - zQ_{u,A} + D. \quad (26)$$

*Proof*: If we use (19) and (10) in $zAF_u' = BF_u^2 + CF_u + D$ we get

$$(-zA' + 2B - C)F_u - i\langle \mathcal{D}(Au), \frac{\xi + z}{\xi - z}\rangle = 2BF_{u^2} + B + D - zQ_{u,A} \quad (27)$$

On the other hand, from (18) and (19) we know that,

$$\begin{aligned}(-zA' + 2B - C)F_u & \\ &= \langle(-\xi A'(\xi) + 2B(\xi) - C(\xi))u, \frac{\xi + z}{\xi - z}\rangle - P_{u,-zA'+2B-C}(z) \\ BF_{u^2} &= \langle B(\xi)u^2, \frac{\xi + z}{\xi - z}\rangle - P_{u^2,B}\,,\end{aligned}$$

and substituting these two equations in (27) we get

$$\begin{aligned}&\langle \mathcal{D}(Au) + i\,(-\xi A' + 2B - C)\,u - 2iBu^2, \frac{\xi + z}{\xi - z}\rangle \\ &\qquad = i\,(B - 2P_{u^2,B} + P_{u,-zA'+2B-C} - zQ_{u,A} + D)\ .\end{aligned}$$

Thus, we get (25) with $L, H$ given in (26). ■

**Theorem 2.** *Let $u$ be a hermitian linear functional and $F_u$ the corresponding formal Carathéodory function. If $F_u$ satisfies*

$$zAF_u' = BF_u^2 + CF_u + D,\ |z| < 1$$

*then $u$ satisfies the distributional equation with polynomial coefficients*

$$\mathcal{D}(A_1 u) = B_1 u^2 + (isA_1 - L_1)u + H_1\mathcal{L}\,,$$

*where* $s = \max\{\deg(A), \deg(B), \deg(L), \deg(H)\}$ *, and the polynomials* $A_1$*,* $B_1, L_1, H_1$ *are given by* $A_1(z) = z^s(A(z) + \overline{A}(1/z))$ *,* $B_1(z) = z^s(2iB(z) + \overline{2iB}(1/z))$ *,* $L_1(z) = z^s(L(z) + \overline{L}(1/z))$ *,* $H_1(z) = z^s(iH(z) + \overline{iH}(1/z))$ *, and* $L, H$ *are defined in* (26)*.*

*Proof*: If $F_u$ satisfies $zAF_u' = BF_u^2 + CF_u + D,\ |z| < 1$, then, from Lemma 5, we obtain (25), $\langle \mathcal{D}(Au) + L(\xi)u - 2iB(\xi)u^2, \frac{\xi + z}{\xi - z}\rangle = iH(z)$. Applying conjugates and the transformation $Z = 1/z$ to previous equation we get $\langle \mathcal{D}(\overline{A}u) + \overline{L}u - \overline{2iB}u^2, \frac{\overline{\xi} + 1/z}{\overline{\xi} - 1/z}\rangle = -i\overline{H}(1/z)$. Since $-\frac{1/\xi + 1/z}{1/\xi - 1/z} = \frac{\xi + z}{\xi - z}$, we obtain

$$\langle \mathcal{D}(\overline{A}u) + \overline{L}u - \overline{2iB}u^2, \frac{\xi + z}{\xi - z}\rangle = i\overline{H}(1/z)\,. \quad (28)$$

Summing (25) with (28) we get

$$\langle \mathcal{D}\left((A+\overline{A})u\right) + (L+\overline{L})u - (2iB+\overline{2iB})u^2, \frac{\xi+z}{\xi-z}\rangle = i(H(z)+\overline{H}(1/z)) .$$

Therefore, if we compute the moments of the hermitian functional

$$\mathcal{D}\left((A+\overline{A})u\right) + (L+\overline{L})u - (2iB+\overline{2iB})u^2$$

(using the asymptotic expansion of $\dfrac{\xi+z}{\xi-z}$ in $|z|<1$ and in $|z|>1$) we get

$$\mathcal{D}\left((A+\overline{A})u\right) + (L+\overline{L})u - (2iB+\overline{2iB})u^2 = (iH+\overline{iH})\mathcal{L} .$$

From Lemma 4 we obtain the required functional equation. ■

If we take $B=0$ in previous Theorem we obtain the result for the Laguerre-Hahn affine class.

**Corollary 3.** *Let $u$ be a hermitian linear functional. If $F_u$ satisfies*

$$zAF_u' = CF_u + D \quad \textit{for} \quad |z|<1 ,$$

*then $u$ satisfies*

$$\mathcal{D}(A_1u) = (isA_1 - L_1)u + H_1\mathcal{L} ,$$

*where $s=\max\{\deg(A),\deg(L),\deg(H)\}$, and $A_1, B_1, L_1, H_1, L, H$ are given by $A_1(z) = z^s(A(z)+\overline{A}(1/z))$, $L_1(z) = z^s(L(z)+\overline{L}(1/z))$, $H_1(z) = z^s(iH+\overline{iH}(1/z))$, $L(z) = i\left(-zA'(z) - C(z)\right)$, $H(z) = P_{u,-zA'-C}(z) - zQ_{u,A}(z) + D(z)$.*

## 5. The characterization theorem

**Theorem 3.** *Let $u$ be a hermitian linear functional and $F_u$ the corresponding formal Carathéodory function. If $u$ satisfies $\mathcal{D}(Au) = Bu^2 + Cu + \xi H\mathcal{L}$, where $\mathcal{L}$ is the Lebesgue functional, then $F_u$ satisfies the differential equations, in $\mathbb{C}\setminus\mathbb{T}$,*

$$zAF_u' = -\frac{iB}{2}F_u^2 + (-zA'-iB-iC)F_u + \frac{iB}{2} - iP_{u^2,B} + zQ_{u,A} - iP_{u,C} + E , \quad (29)$$

*with $E(z) = -2izH(z)\,\mathbb{I}(z)$, where* $\mathbb{I}(z) = \begin{cases} 1 & , \quad |z|<1 \\ 0 & , \quad |z|>1 \end{cases} .$

*Conversely, if $F_u$ satisfies the differential equations* (29)*, then $u$ satisfies the distributional equation*

$$\mathcal{D}(Au) = Bu^2 + Cu + \xi H\mathcal{L} , \quad (30)$$

*where $\mathcal{L}$ is the Lebesgue functional.*

*Proof*: If we substitute $\mathcal{D}(Au) = Bu^2 + Cu + \xi H\mathcal{L}$ in (19) we get

$$\begin{aligned} & zA(z)F_u'(z) \\ & = -zA'(z)F_u + zQ_{u,A}(z) - i\langle Bu^2, \frac{\xi+z}{\xi-z}\rangle - i\langle Cu, \frac{\xi+z}{\xi-z}\rangle - i\langle \xi H\mathcal{L}, \frac{\xi+z}{\xi-z}\rangle \,. \end{aligned}$$

From (18) follows

$$\begin{aligned} zA(z)F_u' = -zA'(z)F_u + zQ_{u,A}(z) - i\left(P_{u^2,B} + B(z)F_{u^2}\right) \\ - i\left(P_{u,C} + C(z)F_u\right) - i\langle \xi H\mathcal{L}, \frac{\xi+z}{\xi-z}\rangle \,. \end{aligned} \quad (31)$$

Since $\langle \xi H\mathcal{L}, \frac{\xi+z}{\xi-z}\rangle = 2zH(z)\,\mathbb{I}(z)$ then, in $|z|<1$, (31) is equivalent to

$$\begin{aligned} zA(z)F_u' = -iB(z)F_{u^2} + (-zA'(z) - iC(z))F_u \\ + zQ_{u,A}(z) - iP_{u^2,B}(z) - iP_{u,C}(z) - 2izH(z) \end{aligned}$$

and, in $|z|>1$, (31) is equivalent to

$$\begin{aligned} zA(z)F_u' = -iB(z)F_{u^2} + (-zA'(z) - iC(z))F_u \\ + zQ_{u,A}(z) - iP_{u^2,B}(z) - iP_{u,C}(z) \,. \end{aligned}$$

Using (10) we get (29).

We now prove the converse. If $F$ satisfies

$$\begin{aligned} zAF_u' = -\frac{iB}{2}F_u^2 + (-zA' - iB - iC)F_u + \frac{iB}{2} - iP_{u^2,B} \\ + zQ_{u,A}(z) - iP_{u,C}(z) + E(z),\ |z|<1 \,, \end{aligned}$$

then, from (25), we get

$$\langle V, \frac{\xi+z}{\xi-z}\rangle = 2zH\,\mathbb{I}(z) \quad \text{with} \quad V = \mathcal{D}(Au) - Bu^2 - Cu \,.$$

Using the asymptotic expansion of $\dfrac{\xi+z}{\xi-z}$, in $|z|<1$ and in $|z|>1$, the two previous equations are, respectively, equivalent to

$$\begin{aligned} \langle V, 1+2\sum_{k=1}^{+\infty}\xi^{-k}z^{k}\rangle &= 2zH(z)\,, \quad |z|<1\,, && (32)\\ \langle V, -1-2\sum_{k=1}^{+\infty}\xi^{k}z^{-k}\rangle &= 0\,, \quad |z|>1\,. && (33)\end{aligned}$$

Writing $zH(z)=h_1z+\cdots+h_lz^l$ then, from (32) we get

$$\langle V,1\rangle = 0\,, \quad \langle V,\xi^{-k}\rangle = h_k\,,\ k=1,\ldots,l\,, \quad \langle V,\xi^{-k}\rangle = 0\,,\ k\geq l+1 \qquad (34)$$

and, from (33) we get

$$\langle V,1\rangle = 0\,, \quad \langle V,\xi^{k}\rangle = 0\,,\ k\geq 1\,. \qquad (35)$$

Finally, from (34) and (35), we conclude that $V=zH\mathcal{L}$ and (30) holds. ■

If we do $B=0$ in previous Theorem we get the following result to the Laguerre-Hahn affine class (see [2, 3]).

**Corollary 4.** *Let $u$ be a hermitian linear functional and $F_u$ the corresponding formal Carathéodory function. If $u$ satisfies $\mathcal{D}(Au)=Cu+\xi H(\xi)\mathcal{L}$, then $F_u$ is such that*

$$zA(z)F_u'(z) = (-zA'(z)-iC(z))\,F_u(z) + zQ_{u,A}(z) - iP_{u,C}(z) + E(z)\,, \quad (36)$$

*for $z\in\mathbb{C}\setminus\mathbb{T}$, with $E(z)=-2izH(z)\,\mathbb{I}(z)$.*
*Conversely, if $F_u$ satisfies (36), then $u$ satisfies $\mathcal{D}(Au)=Cu+\xi H(\xi)\mathcal{L}$.*

As a consequence of previous Theorem, we get a characterization for semi-classical functionals (see [2]).

**Corollary 5.** *Let $u$ be a hermitian linear functional and $F$ the corresponding formal Carathéodory function, satisfying a differential equation with polynomial coefficients*

$$zA(z)F'(z) = (-zA'(z)-iC(z))F(z) + P(z)\,, \quad z\in\mathbb{C}\setminus\mathbb{T}\,. \qquad (37)$$

*A necessary and sufficient condition for $u$ to be semi-classical and satisfy $\mathcal{D}(Au)=Cu$ is that the polynomial $P$ of (37) satisfy $P=zQ_{u,A}-iP_{u,C}$, where $Q_{u,A}$ and $P_{u,C}$ are given by (20) and (21).*

# References

[1] M. Alfaro and L. Moral, *Quasi-orthogonality on the unit circle and semi-classical forms*, Portugaliæ Mathematica, **51** (1) (1994), 47-62.

[2] A. Branquinho and M.N. Rebocho, *Characterizations of Laguerre-Hahn affine orthogonal polynomials on the unit circle*, to appear in J. Comput. Anal. Appl.

[3] A. Cachafeiro and C. Pérez, *A study of the Laguerre-Hahn affine functionals on the unit circle*, J. Comput. Anal. Appl., **6** (2) (2004), 107-123.

[4] A. Cachafeiro and C. Pérez, *Second degree functionals on the unit circle*, Integral Transforms and Special Functions, **15** (4) (2004), 281-294.

[5] A. Magnus, *Riccati acceleration of the Jacobi continued fractions and Laguerre-Hahn orthogonal polynomials*, in "Padé Approximation and its Applications", Proc. Bad Honnef 1983, Lect. Notes in Math. **1071** (H. Werner and H.T. Bunger, eds.), Springer-Verlag, Berlin, 1984, 213-230.

[6] F. Marcellán, *Orthogonal polynomials and Toeplitz matrices: Some applications*, in "Seminario Matemático García Galdeano" (M. Alfaro, ed.), Univ. Zaragoza, 1989, 31-57.

[7] P. Maroni, *Le calcul des formes linéaires et les polynômes orthogonaux semi-classiques*, in Lect. Notes in Math., vol. 1329, Springer Verlag, Berlin, 1988, 279-290.

[8] P. Maroni, *Une théorie algébrique des polynômes orthogonaux. Application aux polynômes orthogonaux semi-classiques*. In "Orthogonal polynomials and their application" (C. Brezinski, L. Gori and A. Ronveux, eds.) J.C.Baltzer AG. Basel IMACS Annals on Computing and Applied Mathematics, **9** (1991), 95-130.

[9] F. Peherstorfer, *A special class of polynomials orthogonal on the unit circle including the associated polynomials*, Constr. Approx., **12** (2) (1996), 161-185.

[10] G. Szegő, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., vol. 23, Providence Rhode Island, 1975 (fourth edition).

[11] C. Tasis, *Propriedades diferenciales de los polinomios ortogonales relativos a la circunferencia unidad*, PhD Thesis, Universidad de Cantabria, 1989 (in spanish).

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