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ON $(0,1)$-MATRICES WITH PRESCRIBED ROW AND COLUMN SUM VECTORS

C.M. DA FONSECA AND RICARDO MAMEDE

Abstract: Given partitions R and S with the same weight, the Robinson-Schensted-Knuth correspondence establishes a bijection between the class $\mathcal{A}(R,S)$ of $(0,1)$-matrices with row sum R and column sum S and pairs (P,Q) of Young tableaux of conjugate shapes λ and λ^*, with $S \preccurlyeq \lambda \preccurlyeq R^*$. An algorithm for constructing a matrix in $\mathcal{A}(R,S)$ whose insertion tableaux has a prescribed shape λ, with $S \preccurlyeq \lambda \preccurlyeq R^*$, is provided. We generalize some recent constructions due to R. Brualdi for the extremal cases $\lambda = S$ and $\lambda = R^*$.

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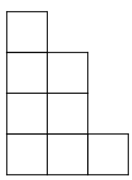
1. Preliminaries

A *partition* λ of weight $t \geq 0$ is a decreasing sequence of integers $\lambda = (\lambda_1, \lambda_2, \ldots)$ whose sum is t. The *weight* is denoted by $|\lambda|$. The number of nonzero elements in λ is called the *length* of λ and is denoted by $\ell(\lambda)$. When $\ell(\lambda) = n$ we usually write $\lambda = (\lambda_1, \ldots, \lambda_n)$. The set of all partitions with weight t is denoted by $P(t)$, with $P(0)$ consisting of the empty partition $\emptyset$. A partition λ is identified with its Ferrer diagram, which is the left-justified arrangement of boxes with λ_i boxes in the i-th row, $i = 1, 2, \ldots, n$ (we use "French" notation for Ferrer diagrams). The *conjugate* of λ is the partition λ^*, the transpose of the Young diagram of λ. For instance, the partition $\lambda = (3,2,2,1)$ has length 4 and weight 8, and corresponds to the Young diagram

 .

The conjugate partition $\lambda^* = (4,3,1)$ is obtained by reading the sequence of column lengths of the Ferrer diagram of λ from left to right.

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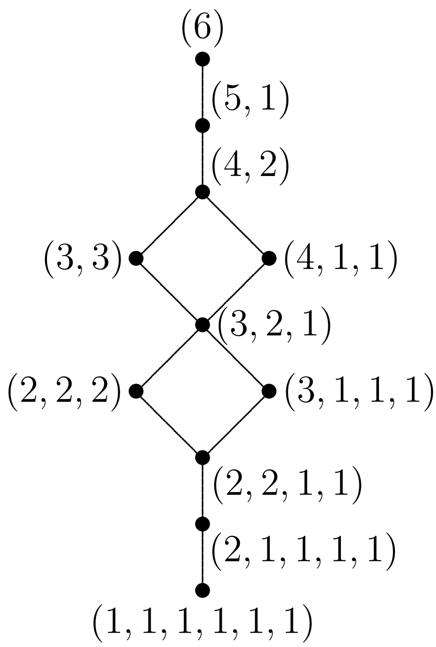
In $P(t)$, a partition λ is dominated by the partition μ, denoted $\lambda \preccurlyeq \mu$, if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i,$$

for $i = 1, \ldots, \ell(\lambda)$. The set of all partitions of a given integer is partially ordered by the relation $\preccurlyeq$ and forms a lattice (Cf. [9]). The poset $(P(t), \preccurlyeq)$ is a chain if and only if $t \leq 5$. For $t \geq 6$, the relation $\preccurlyeq$ is a partial order, as there are pairs of partitions which are not comparable. For instance, $(3,3), (4,1,1)$ and $(3,1,1,1), (2,2,2)$ are the only pairs in $P(6)$ which are not comparable, as we may check in the Hasse diagram of $P(6)$:



From the identification of a partition with its Ferrer diagram, it is clear that the map $\lambda \mapsto \lambda^*$ is an anti-automorphism in $P(t)$, with respect to the order $\preccurlyeq$, i.e., for $\lambda, \mu \in P(t)$, $\lambda \preccurlyeq \mu$ if and only if $\mu^* \preccurlyeq \lambda^*$.

Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be two partitions in $P(t)$. A Young tableau $\mathcal{P}$ of shape λ and content μ is a filling of the Ferrer diagram of λ with μ_i integers i, $i = 1, \ldots, n$, weakly increasing across each row and strictly increasing down each column. An example of a Young tableau of shape $(3,2,2,1)$ and content $(2,2,2,1,1)$ is

$$\mathcal{P} = \begin{array}{lll} 5 & & \\ 3 & 4 & \\ 2 & 3 & \\ 1 & 1 & 2 \end{array} . \tag{1.1}$$

We point out that the boxes in a Young tableau are often eliminated.

The number of Young tableaux with the shape λ and content μ, denoted by $K_{\lambda,\mu}$, is called a Kostka number. It is easy to see that the number $K_{\lambda,\mu}$ is 1 when $\lambda = \mu$, and $K_{\lambda,\mu} \neq 0$ if and only if $\mu \preccurlyeq \lambda$.

In this paper we describe a direct algorithm for constructing a matrix with prescribed row sum partition R and column sum partition S, whose insertion tableau has any prescribed shape λ, with $S \preccurlyeq \lambda \preccurlyeq R^*$.

2. The class $\mathcal{A}(R, S)$

Given two partitions $R = (r_1, \dots, r_m)$ and $S = (s_1, \dots, s_n)$ in $P(t)$, we denote by $\mathcal{A}(R, S)$ the classe of all $m \times n$ $(0,1)$-matrices with row sum vector R and column sum vector S. By Gale-Riser Theorem (cf. [2, 4, 7]), the class $\mathcal{A}(R, S)$ is nonempty if and only if $S \preccurlyeq R^*$. Notice that this condition implies the obvious relation $r_1 \leq n$, i.e., the number of ones in any row is less than or equal to the number of columns on any matrix in $\mathcal{A}(R, S)$.

There is a one-to-one correspondence between $m \times n$ $(0,1)$-matrices and biwords with no repeated billeters over the alphabet $\{1, \dots, m\} \times \{1, \dots, n\}$, in lexicographic order, established by the map

$$A = (a_{ij}) \mapsto \Theta_A = \begin{pmatrix} i_k \\ j_k \end{pmatrix},$$

where the ordered pairs (i_k, j_k) are the positions of A occupied by the ones of A. For example, the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \in \mathcal{A}(R, S),$$

with $R = (2, 2, 2)$ and $S = (2, 2, 1, 1) \preccurlyeq R^* = (3, 3)$, corresponds to the biword

$$\Theta_A = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 \\ 1 & 2 & 1 & 3 & 2 & 4 \end{pmatrix}. \tag{2.1}$$

The *Robinson-Schensted-Knuth* (RSK) *correspondence* (cf. [3, 6]) establishes a one-to-one correspondence between biwords with no repeated billeters and pairs of tableaux with conjugate shapes. It is based on the column-bumping or column-insertion algorithm, described in [8], which takes a tableau $\mathcal{P}$ and a positive integer x, and constructs a new tableau $\mathcal{P}'$ as follows: if x is strictly larger than all entries of first column, we put x at the end of this column; otherwise, it bumps the smallest entry in that column which is larger or equal to x. This bumped entry moves to the next column, going to the end of it if possible, and bumping an element to the next column otherwise. The process continues until the bumped entry can go to the end of the next column, or until it becomes the only entry of a new column.

For instance, the column-insertion of the integer 2 into the tableau (1.1), gives:

$$2 \to \begin{array}{lll} 5 & & \\ 3 & 4 & \\ \mathbf{2} & 3 & \\ 1 & 1 & 2 \end{array}, \qquad 2 \to \begin{array}{lll} 5 & & \\ 3 & 4 & \\ 2 & \mathbf{3} & \\ 1 & 1 & 2 \end{array}, \qquad 3 \to \begin{array}{lll} 5 & & \\ 3 & 4 & \\ 2 & 2 & \\ 1 & 1 & 2 \end{array}, \qquad \begin{array}{lll} 5 & & \\ 3 & 4 & \\ 2 & 2 & 3 \\ 1 & 1 & 2 \end{array}.$$

The image of a biword $\Theta = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$ in lexicographic order, with no repeat billeters, by the RSK correspondence, is obtained as follows: start with the pair $(\mathcal{P}, \mathcal{Q})$ of empty tableaux, and recursively construct $\mathcal{P}$ and $\mathcal{Q}$ by column-inserting in $\mathcal{P}$ the integer j_k, and then inserting in $\mathcal{Q}$ the integer i_k in the conjugate position of the new element in $\mathcal{P}$, for $k = 1, 2, \ldots, n$. $\mathcal{P}$ is called the *insertion tableau*, and $\mathcal{Q}$ is the *recording tableau*. For example, this construction applied to the biword (2.1) gives:

$$1 \to \begin{array}{l} 2 \\ 1 \end{array} \to \begin{array}{ll} 2 & \\ 1 & 1 \end{array} \to \begin{array}{ll} 3 & \\ 2 & \\ 1 & 1 \end{array} \to \begin{array}{ll} 3 & \\ 2 & 2 \\ 1 & 1 \end{array} \to \begin{array}{ll} 4 & \\ 3 & \\ 2 & 2 \\ 1 & 1 \end{array}$$

and

$$1 \to 1 \;\; 1 \to \begin{array}{ll} 2 & \\ 1 & 1 \end{array} \to \begin{array}{lll} 2 & & \\ 1 & 1 & 2 \end{array} \to \begin{array}{lll} 2 & 3 & \\ 1 & 1 & 2 \end{array} \to \begin{array}{llll} 2 & 3 & & \\ 1 & 1 & 2 & 3 \end{array}.$$

Thus, Θ_A is in RSK correspondence with the pair $(\mathcal{P}, \mathcal{Q})$ of tableaux of conjugate shape and content $S = (2, 2, 1, 1)$ and $R = (2, 2, 2)$, respectively, where

$$\mathcal{P} = \begin{array}{ll} 4 & \\ 3 & \\ 2 & 2 \\ 1 & 1 \end{array} \quad \text{and} \quad \mathcal{Q} = \begin{array}{llll} 2 & 3 & & \\ 1 & 1 & 2 & 3 \end{array}. \tag{2.2}$$

Reciprocally, the inverse image of a pair $(\mathcal{P}, \mathcal{Q})$ of tableaux with conjugate shapes, by the RSK correspondence, is obtained by constructing a sequence

$$(\mathcal{P}, \mathcal{Q}) = (\mathcal{P}_t, \mathcal{Q}_t), (\mathcal{P}_{t-1}, \mathcal{Q}_{t-1}), \ldots, (\mathcal{P}_1, \mathcal{Q}_1),$$

of pairs of tableaux with conjugate shapes, each having one fewer box than the preceding one. To construct $(\mathcal{P}_{k-1}, \mathcal{Q}_{k-1})$ from $(\mathcal{P}_k, \mathcal{Q}_k)$, identify the box in $\mathcal{Q}_k$ which has the largest entry; if there are several equal entries, choose the rightmost box. Then, $\mathcal{P}_{k-1}$ is obtained performing the reverse column-insertion to $\mathcal{P}_k$ using the conjugate of this box to start, and $\mathcal{Q}_{k-1}$ is obtained from $\mathcal{Q}_k$ by removing this box. If j_k is the entry removed from $\mathcal{P}_k$, and i_k is the entry removed from $\mathcal{Q}_k$, $k = 1, \ldots, t$, we get the biword

$$\Theta = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$$

in RSK correspondence with the pair $(\mathcal{P}, \mathcal{Q})$.

In [5], C. Greene gave a combinatorial interpretation for the lengths of the columns of a Young tableau, generalizing Schensted result in [8] which states that the length of the longest strictly increasing subword of a given word is given by the length of the first column of its insertion tableau. Given a sequence w of nonnegative integers with length $m > 1$, denote by $\ell(w, k)$ the maximum of the sum of the lengths of k strictly increasing subsequences of w, for $k = 1, \ldots, m$.

Proposition 2.1. [5] *Let w be a sequence of nonnegative integers with length ≥ 1, and let us assume that its insertion tableau $\mathcal{P}$ has the shape λ, with $\lambda^* = (\lambda_1^*, \ldots, \lambda_n^*)$. Then, for each $k = 1, \ldots, n$, $\lambda_k^* = \ell(w, k) - \ell(w, k-1)$, with $\ell(w, 0) = 0$.*

It is easy to check that the longest strictly increasing subsequence of the bottom line $w = \underline{12}1\underline{3}2\underline{4}$ of the biword (2.1) is given by the underlined letters $\underline{1234}$. So $\ell(1) = 4$ and thus $\ell(2) = 6$. This means that the first and second columns of the insertion tableau of w have lengths 4 and $2 = 6 - 4$, respectively, as shown in (2.2).

When restricted to $\mathcal{A}(R, S)$, the RSK correspondence establishes a bijection between matrices in $\mathcal{A}(R, S)$ and pairs of tableaux $(\mathcal{P}, \mathcal{Q})$ of conjugate shape and content S and R, respectively. From this bijection, it follows that the cardinality of $\mathcal{A}(R, S)$ is given by

$$|\mathcal{A}(R, S)| = \sum_{\lambda} K_{\lambda^*, R} K_{\lambda, S},$$

where the sum is over all partitions in $P(t)$. Since the Kostka number $K_{\lambda,\mu}$ is nonzero if and only if $\mu \preccurlyeq \lambda^*$, it follows that

$$\begin{aligned} K_{\lambda^*\!,R}K_{\lambda,S} \neq 0 &\text{ if and only if } R \preccurlyeq \lambda^* \text{ and } S \preccurlyeq \lambda, \\ &\text{ if and only if } \lambda \preccurlyeq R^* \text{ and } S \preccurlyeq \lambda, \\ &\text{ if and only if } S \preccurlyeq \lambda \preccurlyeq R^*, \end{aligned}$$

and therefore,

$$|\mathcal{A}(R,S)| = \sum_{S \preccurlyeq \lambda \preccurlyeq R^*} K_{\lambda^*\!,R}K_{\lambda,S}.$$

Thus, for each pair $(\mathcal{P},\mathcal{Q})$ of tableaux of conjugate shape λ and λ^*, respectively, where $\mathcal{P}$ has content S and $\mathcal{Q}$ has content R, there is one and only one matrix in $\mathcal{A}(R,S)$ in RSK correspondence with $(\mathcal{P},\mathcal{Q})$. Therefore, given partitions $S \preccurlyeq \lambda \preccurlyeq R^*$, we may use the RSK correspondence to construct a $(0,1)$-matrix in $\mathcal{A}(R,S)$ whose insertion tableau has shape λ.

Example 2.1. Let $R = (3,3,2,2,2)$ and $S = (4,3,3,2) \preccurlyeq R^* = (5,5,2)$. To construct a $(0,1)$-matrix in $\mathcal{A}(R,S)$, whose insertion tableau has shape λ with $S \preccurlyeq \lambda = (5,4,3) \preccurlyeq R^*$, we start by considering a pair $(\mathcal{P},\mathcal{Q})$ of Young tableaux with shape λ and λ^*, and contents S and R, respectively. This may be achieved by inserting, from top to bottom, and from right to left, x_k integers k, for $k = m,\ldots,1$, in the boxes of the Ferrer diagram λ and λ^*, where $(x_1,\ldots,x_m)$ is S and R, respectively:

$$\mathcal{P} = \begin{array}{|c|c|c|c|c|} \hline 3 & 4 & 4 \\ \cline{1-3} 2 & 2 & 3 & 3 \\ \cline{1-4} 1 & 1 & 1 & 1 & 2 \\ \hline \end{array} \qquad \text{and} \qquad \mathcal{Q} = \begin{array}{|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 5 \\ \cline{1-2} 3 & 3 & 4 \\ \hline 2 & 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array}\ .$$

Using now the RSK correspondence, we obtain the biword

$$\begin{pmatrix} 1\,1\,1\,2\,2\,2\,3\,3\,4\,4\,5\,5 \\ 2\,3\,4\,1\,3\,4\,1\,2\,1\,3\,1\,2 \end{pmatrix},$$

which corresponds to the $(0,1)$-matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \in \mathcal{A}(R,S).$$

In [1] R. Brualdi established two direct algorithms for the construction of matrices A_S and A_{R^*} in $\mathcal{A}(R,S)$, whose insertion tableaux have shapes S and R^*, respectively. In next section we describe a direct algorithm for constructing a matrix A_λ in $\mathcal{A}(R,S)$ whose insertion tableaux has any prescribed shape λ, with $S \preccurlyeq \lambda \preccurlyeq R^*$.

3. Fillings of Ferrer diagrams and $(0,1)$-matrices

A *filling* of a Ferrer diagram is an assignment of positive integers to all boxes of the diagram. Here, we restrict ourselves to those fillings with no repetitions in the same column, and identify those fillings differing from one another by the order in which the integers are placed inside the same column. The *content* of a filling of a Ferrer diagram is the sequence formed by the number of integers i assigned to the boxes of the diagram, for all $i \geq 0$. For example,

$$\begin{array}{cccc} 1 & & & \\ 2 & & & \\ 5 & 1 & 3 & \\ 4 & 3 & 2 & 2 \end{array} \qquad \text{and} \qquad \begin{array}{cccc} 5 & & & \\ 4 & & & \\ 2 & 3 & 3 & \\ 1 & 1 & 2 & 2 \end{array} \tag{3.1}$$

are equal fillings of the Ferrer diagram $(4,3,1,1)$ with content $(3,3,2,1,1)$.

In particular, if the filling of a Ferrer diagram is weakly increasing across each row and strictly increasing down each column, we get a Young tableau.

Definition 3.1. An east chain, or for short e-chain, of length k, in a filling of a Ferrer diagram is a sequence of k non-zero entries, such that each entry is greater than or equal to and strictly to the right of the preceding entry in the sequence.

For instance, the sequence $(1,1,2,2)$ is the only e-chain of length 4 in (3.1), but this filling has four e-chains of length 3, namely $(1,1,2), (1,2,2)$, $(1,3,3)$ and $(2,3,3)$.

Given two partitions $S, R \in P(t)$, with $S \preccurlyeq R^*$, there is a one-to-one correspondence between the fillings of the Ferrer diagram S^* with content R, and the matrices in $\mathcal{A}(R,S)$ defined as follows: if $(a_{ij}) \in \mathcal{A}(R,S)$, for $i = 1, \ldots, \ell(R)$, and for all $j \geq 1$, put the integer i in a box on column j of the Ferrer diagram with columns lengths S whenever $a_{ij} = 1$. Notice that this corresponds to a filling of the diagram with content R.

Example 3.1. Consider the partitions $R = (3, 2, 2, 2, 2)$ and $S = (3, 3, 2, 2, 1) \preccurlyeq R^*$ in $P(11)$, and the $(0, 1)$-matrix

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} \in \mathcal{A}(R, S).$$

By the bijection defined above, this matrix corresponds to the following filling of the Ferrer diagram with columns lengths S:

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 \\ \cline{1-4} 2 & 3 & 1 & 3 \\ \hline 5 & 4 & 4 & 5 & 2 \\ \hline \end{array}\ .$$

Thus, presenting a matrix in $\mathcal{A}(R, S)$ is equivalent to the presentation of a filling of the Ferrer diagram with column lengths S and content R.

Definition 3.2. Given partitions $R = (r_1, \ldots, r_m) \preccurlyeq \lambda = (\lambda_1, \ldots, \lambda_p)$ in $P(t)$, let $R^1_\lambda = R$ and define for each $k = 2, \ldots, p$,

$$R^k_\lambda = (0, \ldots, 0, r^k_b, r^k_{b+1}, \ldots, r^k_m),$$

where $r^k_l = r_l$, for $l = b+1, \ldots, m$, $0 < r^k_b \leq r_b$, and $r_1+\cdots+r_{b-1}+(r_b-r^k_b) = \lambda_1 + \cdots + \lambda_{k-1}$, for some $b \leq m$. Define also $r^k_\lambda := r^k_b$, the leftmost nonzero entry in R^k_λ, for $k = 1, \ldots, p$.

For example, if $R = (3, 2, 2, 2, 2)$ and $\lambda = (3, 3, 2, 2, 1)$, we have $R^2_\lambda = (0, 2, 2, 2, 2)$, $R^3_\lambda = (0, 0, 1, 2, 2)$, $R^4_\lambda = (0, 0, 0, 1, 2)$ and $R^5_\lambda = (0, 0, 0, 0, 1)$, and thus $r^1_\lambda = 3, r^2_\lambda = 2$ and $r^3_\lambda = r^4_\lambda = r^5_\lambda = 1$.

Taking the equivalence between $(0, 1)$-matrices in $\mathcal{A}(R, S)$ and fillings of the Ferrer diagram, with column lengths S and content R, into account, we present a first algorithm for constructing a matrix in $\mathcal{A}(R, S)$ whose insertion tableau has a prescribed shape $\lambda = (\lambda_1, \ldots, \lambda_q)$, where the partition λ satisfies $S \preccurlyeq \lambda \preccurlyeq R^*$, and is such that its conjugate partition $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_p)$ satisfies $r^k_{\lambda^*} \leq \lambda^*_k$, for all $k = 1, \ldots, p$.

Algorithm 1. *Let* $R = (r_1, \ldots, r_m), S = (s_1, \ldots, s_n)$ *and* $\lambda = (\lambda_1, \ldots, \lambda_q)$ *be partitions of a fixed weight with* $S \preccurlyeq \lambda \preccurlyeq R^*$*, and such that the conjugate partition* $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_p)$ *of* λ *satisfy* $r^k_{\lambda^*} \leq \lambda^*_k$*, for all* $k = 1, \ldots, p$*. Consider the Ferrer diagram with column lengths* S*. Set* $R' := R$*,* $S' := S$*,*

$\lambda' := \lambda$, and denote by r'_b the leftmost nonzero entry in R'. For $k = 1, \ldots, p$, repeat the following steps:

(P_1): *If $r'_b = \lambda^*_k$, search for the rightmost configuration of λ^*_k top boxes in the Ferrer diagram with column lengths S', including necessarily the rightmost top box, with the property that the Ferrer diagram H^* obtained by removing those boxes satisfies $H \preccurlyeq \mu$, where $\mu^* = (\lambda^*_{k+1}, \ldots, \lambda^*_p)$. Then, put integers b in those boxes and set $R' := R^{k+1}_{\lambda^*}$, $S' := H$, and $\lambda' := \mu$.*

(P_2): *If $r'_b < \lambda^*_k$, search for the rightmost configuration of λ^*_k top boxes in the Ferrer diagram $(S')^*$, including necessarily the rightmost top box, with the property that the Ferrer diagram H^*, obtained by removing those boxes, satisfies $H \preccurlyeq \mu$, where $\mu^* = (\lambda^*_{k+1}, \ldots, \lambda^*_p)$. Then, construct an e-chain with length λ^*_k, filling that λ^*_k boxes with r'_b integers b, and the maximum number of integers integers j, for $j = b+1, \ldots, m$. Set $R' := R^{k+1}_{\lambda^*}$, $S' := H$ and $\lambda' := \mu$.*

(P_3): *While applying (P_1) or (P_2) in step k, if at least one of the last set of integers, say b with cardinality z, is placed in a column having already the letter b, then let $z = \alpha + \beta$, where α is the number of b's placed in step k in columns having already a letter b.*

If the number of letters $b-1$ placed in step $k-1$ is greater or equal to z, replace the z letters b placed in step k with the z rightmost letters $b-1$ placed in the previous step.

Otherwise, there are at least α letters $b-1$ placed at the rightmost positions of the e-chain obtained in step $k-2$. In this case replace the α rightmost letters b placed in step k with the α rightmost letters $b-1$ placed in step $k-2$.

Repeat (P_3) with the letter $b-1$ if necessary, that is, if in the process some letter $b-1$ is placed in a column having already a letter $b-1$.

In our main result, Theorem 3.1 we shall prove that the filling of the Ferrer diagram with column lengths S obtained from the application of our algorithm corresponds to a $(0,1)$-matrix in $\mathcal{A}(R,S)$ whose insertion tableau has shape λ, when the conjugate partition $\lambda^* = (\lambda^*_1, \ldots, \lambda^*_p)$ of λ satisfies the condition $r^k_{\lambda^*} \leq \lambda^*_k$, for all $k = 1, \ldots, p$. Before, however, we present some examples.

Example 3.2. Consider the partitions $R = (4,4,2,1), S = (2,2,2,2,2,1)$ and $\lambda^* = (6,2,2,1)$, which satisfy $S \preccurlyeq \lambda = (4,3,1,1,1,1) \preccurlyeq R^* = (4,3,2,2)$. Following the algorithm above, we have in the first step

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & \multicolumn{1}{c}{} \\ \hline & & & & & 2 \\ \hline \end{array}\ ,$$

since $\lambda_1^* = 6$ and the Ferrer diagram $H^* = (5)$ obtained from S^* by removing the labeled boxes satisfies $H \preccurlyeq \mu = (3,2)$, where μ is the conjugate of $(2,2,1)$. So, we set $R' = (0,2,2,1)$, $S' = (5)$, and $\lambda' = (3,2)$.

In the next step, we search for the rightmost configuration of 2 boxes in $(S')^*$ in the required condition of (P_1), and so we get

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & \multicolumn{1}{c}{} \\ \hline & & & \mathbf{2} & \mathbf{2} & 2 \\ \hline \end{array}\ .$$

But since one of two letters 2 (in bold) is placed in a column having already a letter 2, we replace this two letters with the two rightmost letters 1 placed in the previous step. Thus, we get

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \hline & & & \mathbf{1} & \mathbf{1} & 2 \\ \hline \end{array}\ .$$

Proceeding with the application of the algorithm we get

$$\begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \hline & 3 & 3 & 1 & 1 & 2 \\ \hline \end{array} \quad \text{and finally} \quad \begin{array}{|c|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 2 & \multicolumn{1}{c}{} \\ \hline 4 & 3 & 3 & 1 & 1 & 2 \\ \hline \end{array}\ .$$

The filling of the Ferrer diagram of S^* obtained corresponds to the $(0,1)$-matrix

$$\begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{A}(R,S).$$

The associated biword is $\begin{pmatrix} 1111 & 2222 & 33 & 4 \\ 1245 & 3456 & 23 & 1 \end{pmatrix}$ and its insertion tableau

$$\begin{array}{cccc} 6 \\ 5 \\ 4 \\ 3 \\ 2 & 3 & 5 \\ 1 & 1 & 2 & 4 \end{array}$$

has the shape $\lambda = (4, 3, 1, 1, 1, 1)$.

Example 3.3. Consider now $R = (3, 2, 2, 2, 2)$ and $S = (3, 3, 2, 2, 1) \preccurlyeq R^*$ in $P(11)$.

(*i*) If $\lambda = S$, we have $S^* = (5, 4, 2)$. Following the proposed algorithm, we have successively,

$$\begin{ytableau} 1 & 1 \\ {} & {} & 1 & 2 \\ {} & {} & {} & {} & 2 \end{ytableau} \longrightarrow \begin{ytableau} 3 & 3 \\ {} & {} & 4 & 4 \end{ytableau} \longrightarrow \begin{ytableau} 5 & 5 \end{ytableau}\ ,$$

since $(2, 2, 1, 1) \preccurlyeq (2, 2, 1, 1)$ and $(1, 1) \preccurlyeq (1, 1)$ (when there is no possibility of repetition of integers in the same column, in each step we eliminate the boxes filled in the previous step). Therefore, we get the following filling of the Ferrer diagram with column lengths S, and the corresponding $(0, 1)$-matrix in $\mathcal{A}(R, S)$:

$$\begin{ytableau} 1 & 1 \\ 3 & 3 & 1 & 2 \\ 5 & 5 & 4 & 4 & 2 \end{ytableau} \leftrightarrow A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The biword associated with A is $\Theta_A = \begin{pmatrix} 1\,1\,1\,2\,2\,3\,3\,4\,4\,5\,5 \\ 1\,2\,3\,4\,5\,1\,2\,3\,4\,1\,2 \end{pmatrix}$, and its insertion tableau

$$\begin{array}{lll} 5 & & \\ 4 & 4 & \\ 3 & 3 & \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{array}$$

has the shape S.

(*ii*) If $\lambda = R^* = (5, 5, 1)$, then the successive steps of Algorithm 1 gives:

$$\begin{ytableau} {} & {} \\ {} & {} & 1 & 1 \\ {} & {} & {} & {} & 1 \end{ytableau} \longrightarrow \begin{ytableau} {} & {} \\ {} & {} \\ {} & {} & 2 & 2 \end{ytableau} \longrightarrow \begin{ytableau} 3 & 3 \\ {} & {} \\ {} & {} \end{ytableau} \longrightarrow \begin{ytableau} 4 & 4 \\ {} & {} \end{ytableau} \longrightarrow \begin{ytableau} 5 & 5 \end{ytableau},$$

since $(3, 3, 1, 1) \preccurlyeq (4, 4)$, $(3, 3) \preccurlyeq (3, 3)$, $(2, 2) \preccurlyeq (2, 2)$ and $(1, 1) \preccurlyeq (1, 1)$. Therefore, we get the following filling of the Ferrer diagram with column lengths S, and the corresponding $(0,1)$-matrix in $\mathcal{A}(R,S)$:

$$\begin{array}{ccccc} 3 & 3 & & & \\ 4 & 4 & 1 & 1 & \\ 5 & 5 & 2 & 2 & 1 \end{array} \quad \leftrightarrow \quad A' = \begin{pmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The corresponding biword is $\Theta_{A'} = \begin{pmatrix} 1\,1\,1\,2\,2\,3\,3\,4\,4\,5\,5 \\ 3\,4\,5\,3\,4\,1\,2\,1\,2\,1\,2 \end{pmatrix}$ and it has insertion tableau

$$\begin{array}{ccccc} 5 & & & & \\ 2 & 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 3 & 3 \end{array}$$

with shape R^*.

(iii) Setting now $\lambda = (3,3,3,2)$, with $\lambda^* = (4,4,3)$, and noticing that the rightmost configuration of 4 top boxes in the Ferrer diagram S^*, whose removal gives a Ferrer diagram H^* with $H \leq (2,2,2,1)$ is

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & & & \\ \hline & & & 1 & \\ \hline & & & & 2 \\ \hline \end{array}\,.$$

Following the steps of Algorithm 1, since $(2,2,2,1) \preccurlyeq (2,2,2,1)$ and $(1,1,1) \preccurlyeq (1,1,1)$, we get

$$\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & & & \\ \hline & & & 1 & \\ \hline & & & & 2 \\ \hline \end{array} \;\rightarrow\; \begin{array}{|c|c|c|c|} \hline 2 & 3 & 3 & \\ \hline & & & 4 \\ \hline \end{array} \;\rightarrow\; \begin{array}{|c|c|c|} \hline 4 & 5 & 5 \\ \hline \end{array}\,.$$

Hence, we obtain the filling of the Ferrer diagram S^* and the corresponding $(0,1)$-matrix in $\mathcal{A}(R,S)$:

$$\begin{array}{ccccc} 1 & 1 & & & \\ 2 & 3 & 3 & 1 & \\ 4 & 5 & 5 & 4 & 2 \end{array} \quad \leftrightarrow \quad A'' = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

The insertion tableau associated with the biword

$$\Theta_{A''} = \begin{pmatrix} 1\,1\,1\,2\,2\,3\,3\,4\,4\,5\,5 \\ 1\,2\,4\,1\,5\,2\,3\,1\,4\,2\,3 \end{pmatrix}$$

is

$$\begin{array}{ccc} 4 & 5 & \\ 3 & 3 & 4 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{array}$$

and has the shape λ.

Theorem 3.1. *Let R and S be partitions in $P(t)$ such that $S \preccurlyeq R^*$. Then, for each partition $\lambda \in P(t)$ with $S \preccurlyeq \lambda \preccurlyeq R^*$, such that the conjugate partition $\lambda^* = (\lambda_1^*, \dots, \lambda_p^*)$ of λ satisfies $r_{\lambda^*}^k \leq \lambda_k^*$, for all $k = 1, \dots, p$, the filling of the Ferrer diagram S^* obtained by Algorithm 1 corresponds to a $(0,1)$-matrix in $\mathcal{A}(R,S)$, whose insertion tableau has the shape λ.*

Proof: It is clear that the boxes filled at each step of Algorithm 1 are such that the remaining diagram is still a Ferrer diagram, and that the filling of the Ferrer diagram of S^* has no repetitions in the same column, that is, is an admissible filling. Also, notice that the biword Θ corresponding to the filling of the Ferrer diagram of S^* achieved by Algorithm 1 is obtained reading, from left to right, the columns of this filling having the integer k, for $k = 1, 2, \dots, \ell(R)$.

In each step k of the algorithm, the relation $H \preccurlyeq \mu$ means that the Kostka number K_{H^*,μ^*} is nonzero. This means that it is possible to fill the boxes of the Ferrer diagram with column lengths H with integers forming e-chains of length $\mu_{k+1}^*, \dots, \mu_p^*$. Since $r_{\lambda^*}^{k+1} \leq \lambda_{k+1}^*$, we find that step $k+1$ is admissible.

Finally, from the procedure of placing the integers in the boxes of the diagram, we find that the e-chain of length λ_k^* obtained in each step $k = 1, \dots, m$, of the algorithm, is the longest e-chain in the filling of the diagram. Notice that since $r_{\lambda^*}^k \leq \lambda_k^*$, for each $k = 1, \dots, p$, we cannot have a e-chain with leftmost letter b achieved in step k followed by another e-chain with only letters b. Notice also that the replacements described in (P_3) are admissible since R is a partition, and that in these replacements, the length of the e-chains obtained in each step is kept. Going back to the biword Θ, this means that the maximum of the sum of the lengths of k strictly increasing subsequences of the bottom word of Θ is $\lambda_1^* + \cdots + \lambda_k^*$, for $k = 1, \dots, m$. By Proposition 2.1, the insertion tableau associated with Θ must have shape λ. ■

In next example we show that Algorithm 1 may fail when the partition $\lambda^* = (\lambda_1^*, \ldots, \lambda_p^*)$ does not satisfy $r_{\lambda^*}^k \leq \lambda_k^*$, for all $k = 1, \ldots, p$. Therefore, for the general, case we need an extra step in our algorithm.

Example 3.4. With $R = (3,2,2,2,2), S = (3,3,2,2,1)$ and $\lambda = (5,2,2,2)$, we have $S \preccurlyeq \lambda \preccurlyeq R^* = (5,5,1)$, with $\lambda^* = (4,4,1,1,1)$. Following the algorithm we have, respectively:

$$\begin{array}{|c|c|ccc}\hline 1 & 1 & \multicolumn{3}{|c}{} \\ \hline \; & \; & \multicolumn{1}{c|}{\;} & \multicolumn{1}{c|}{1} & \\ \hline \; & \; & \multicolumn{1}{c|}{\;} & \multicolumn{1}{c|}{\;} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \;\longrightarrow\; \begin{array}{|c|c|c|c}\hline 2 & 3 & 3 & \\ \hline \; & \; & \; & \multicolumn{1}{c|}{4} \\ \hline \end{array} \;\longrightarrow\; \begin{array}{|c|c|c|}\hline \; & \; & 4 \\ \hline \end{array} \;\longrightarrow\; \begin{array}{|c|c|}\hline \; & 5 \\ \hline \end{array} \;\longrightarrow\; \begin{array}{|c|}\hline 5 \\ \hline \end{array}\,,$$

since $(2,2,2,1) \preccurlyeq (4,1,1,1), (1,1,1) \preccurlyeq (3)$ and $(1,1) \preccurlyeq (2)$. Observe that in step 4, we have $R' = (0,0,0,0,2)$ and $(\lambda')^* = (1,1)$, and thus $r_{\lambda^*}^4 > \lambda_4^*$. Therefore, we get the filling of the diagram of S^* and the associated $(0,1)$-matrix in $\mathcal{A}(R,S)$:

$$\begin{array}{|c|c|c|c|c|}\hline 1 & 1 & \multicolumn{3}{|c}{} \\ \hline 2 & 3 & 3 & 1 & \multicolumn{1}{c}{} \\ \hline 5 & 5 & 4 & 4 & 2 \\ \hline \end{array} \quad \longleftrightarrow \quad \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

whose biword is $\begin{pmatrix} 111 & 22 & 33 & 44 & 55 \\ 124 & 15 & 23 & 34 & 12 \end{pmatrix}$. We remark that the two e-chains of length 1 given in the last two steps of the algorithm are in fact one single e-chain of length 2 in the filling of the diagram of S^*. The associated insertion tableau

$$\begin{array}{cccc} 4 & 5 & & \\ 3 & 3 & & \\ 2 & 2 & 4 & \\ 1 & 1 & 2 & 2 \end{array}$$

has shape $(4,3,2,2)$, which is different from λ.

4. The general procedure

Consider now the general case, that is, let $R = (r_1, \ldots, r_m), S = (s_1, \ldots, s_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_q)$ be partitions with weight t and such that $S \preccurlyeq \lambda \preccurlyeq R^*$, with $\lambda^* = (\lambda_1^*, \ldots, \lambda_p^*)$. The general algorithm follows from the procedure established in the previous section with an extra step.

Algorithm 2. *Consider the Ferrer diagram with column lengths S, and set $R' := R$, $S' := S$ and $\lambda' := \lambda$. Let $\varepsilon := \zeta := 0$. For each $k = \varepsilon + 1, \ldots, p - \zeta$, repeat the following steps:*

- **(P_1'):** *equal to (P_1).*
- **(P_2'):** *equal to (P_2).*
- **(P_3'):** *equal to (P_3).*
- **(P_4'):** *If in step k this procedure provides an e-chain with a letter b placed to the left of a previously constructed e-chain with leftmost letter $b' < b$, and the letters of these e-chains form an e-chain with greater length, then remove all entries placed until step 2. Apply $(P_1'), (P_2')$ and (P_3') to construct an e-chain with length $\lambda^*_{p-\zeta}$. Let R' and S' be obtained by removing the integers, respectively, the boxes used to construct the e-chain of length $\lambda^*_{p-\zeta}$, and let λ'^* be obtained eliminating the last entry $\lambda^*_{p-\zeta}$. Let $\zeta := \zeta + 1$, $\varepsilon := 1$ and re-initiate the algorithm.*

If it is necessary to apply (P_4') more than once, then on each time, place the respective e-chain to the left of the last e-chain constructed by (P_4'), and using distinct integers each time.

Theorem 4.1. *Let R and S be partitions in $P(t)$ such that $S \preccurlyeq R^*$. Then, for each partition $\lambda \in P(t)$ with $S \preccurlyeq \lambda \preccurlyeq R^*$, the filling of the Ferrer diagram of S^* obtained by Algorithm 2 corresponds to a $(0,1)$-matrix in $\mathcal{A}(R, S)$, whose insertion tableau has shape λ.*

Proof: It follows the proof of Theorem 3.1, noticing that by (P_4'), each e-chain of length λ^*_k constructed by Algorithm 2 is the longest e-chain in the filling of the Ferrer diagram of S^* after removing the boxes corresponding to the e-chains of lengths λ^*_l, for $\ell = 1, \ldots, k-1$. Henceforth, the result is immediately from Proposition 2.1. ∎

Example 4.1. Consider the partitions $R = (3,2,2,2,2)$ and $S = (3,3,2,2,1) \preccurlyeq \lambda = (5,2,2,2) \preccurlyeq R^*$ in $P(11)$, with $\lambda^* = (4,4,1,1,1)$. Notice that $R^4_{\lambda^*} = (0,0,0,0,2)$, and so $r^4_{\lambda^*} > \lambda^*_4 = 1$. Following the algorithm above for $k = 1, \ldots, 5$, we get

$$\begin{array}{|c|c|c|c|c|}\cline{1-2} 1&1\\ \cline{1-4} &&&1\\ \hline &&&&2\\ \hline\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\cline{1-2} 1&1\\ \cline{1-4} 2&3&3&1\\ \hline &&&4&2\\ \hline\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\cline{1-2} 1&1\\ \cline{1-4} 2&3&3&1\\ \hline &&4&4&2\\ \hline\end{array} \rightarrow \begin{array}{|c|c|c|c|c|}\cline{1-2} 1&1\\ \cline{1-4} 2&3&3&1\\ \hline &5&4&4&2\\ \hline\end{array} \rightarrow$$

$$\rightarrow \begin{ytableau} 1 & 1 \\ 2 & 3 & 3 & 1 \\ \mathbf{5} & 5 & 4 & 4 & 2 \end{ytableau} .$$

Since the e-chain of length 1 placed in step 5 is formed by the letter 5 and its on the left of a previous constructed e-chain with leftmost letter 5, we must follow (P_4') and remove all entries placed until step 2. Next we build an e-chain with length $\lambda_5^* = 1$ following $(P_1'), (P_2')$ and (P_3'):

$$\begin{ytableau} 1 & 1 \\ \ & \ & \ & 1 \\ \ & \ & \ & \ & 2 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 \\ \ & \ & 2 & 1 \\ \ & \ & \ & \ & 2 \end{ytableau}$$

Set $\zeta := 1$, and re-initiate the algorithm for $k = 2, \ldots, 4$. We get

$$\begin{ytableau} 1 & 1 \\ \ & \ & 2 & 1 \\ \ & \ & \ & \ & 2 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 \\ 3 & 3 & 2 & 1 \\ \ & \ & 4 & 4 & 2 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 \\ 3 & 3 & 2 & 1 \\ \ & 5 & 4 & 4 & 2 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 \\ 3 & 3 & 2 & 1 \\ \mathbf{5} & 5 & 4 & 4 & 2 \end{ytableau} .$$

Again the same situation occurs with the e-chains of length 1. Thus, we remove all entries placed until step 2, and, following the algorithm, obtain:

$$\begin{ytableau} 1 & 1 \\ \ & \ & 2 & 1 \\ \ & \ & \ & \ & 2 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 \\ \ & 3 & 2 & 1 \\ \ & \ & \ & \ & 2 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 \\ 3 & 3 & 2 & 1 \\ \ & 4 & 4 & 5 & 2 \end{ytableau} \rightarrow \begin{ytableau} 1 & 1 \\ 3 & 3 & 2 & 1 \\ 5 & 4 & 4 & 5 & 2 \end{ytableau} .$$

The corresponding filling of S^* corresponds to the following $(0,1)$-matrix in $\mathcal{A}(R,S)$:

$$\begin{ytableau} 1 & 1 \\ 3 & 3 & 2 & 1 \\ 5 & 4 & 4 & 5 & 2 \end{ytableau} \leftrightarrow \widetilde{A} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix} .$$

The insertion tableau associated with the biword $\Theta_{\widetilde{A}} = \begin{pmatrix} 1\,1\,1\,2\,2\,3\,3\,4\,4\,5\,5 \\ 1\,2\,4\,3\,5\,1\,2\,2\,3\,1\,4 \end{pmatrix}$ is

$$\begin{array}{ccccc} 4 & 5 \\ 3 & 3 \\ 2 & 2 \\ 1 & 1 & 1 & 2 & 4 \end{array}$$

and has the shape λ.

A final example is now presented, showing that there are partitions $R \preccurlyeq \lambda^* \preccurlyeq S^*$, where although λ does not satisfies $r_{\lambda^*}^k \leq \lambda_k^*$, for all k, the resulting $(0,1)$-matrix is obtained using algorithm 1, that is, step (P_4') of algorithm 2 is not used.

Example 4.2. Consider now partitions $R = (4,3,3,3,3) \preccurlyeq \lambda^* = (5,5,2,2,2) = S^*$ in $P(16)$, noticing that $r^3_{\lambda^*} = 3 > \lambda^*_3 = 2$. Following the proposed algorithm we get

$$\begin{array}{ccccc} 1 & 1 & & & \\ \square & \square & & & \\ \square & \square & & & \\ \square & \square & 1 & 1 & 2 \\ \square & \square & \square & \square & \square \end{array} \rightarrow \begin{array}{ccccc} 1 & 1 & & & \\ 2 & 2 & & & \\ \square & \square & & & \\ \square & \square & 1 & 1 & 2 \\ \square & \square & 3 & 3 & 3 \end{array} \rightarrow \begin{array}{ccccc} 1 & 1 & & & \\ 2 & 2 & & & \\ 4 & 4 & & & \\ \square & \square & 1 & 1 & 2 \\ \square & \square & 3 & 3 & 3 \end{array} \rightarrow \begin{array}{ccccc} 1 & 1 & & & \\ 2 & 2 & & & \\ 4 & 4 & & & \\ \mathbf{4} & 5 & 1 & 1 & 2 \\ \square & \square & 3 & 3 & 3 \end{array}.$$

Since the last letter **4** was placed in a column having already a letter 4, we use (P_3') to obtain

$$\begin{array}{ccccc} 1 & 1 & & & \\ 2 & 2 & & & \\ \mathbf{3} & 4 & & & \\ 4 & 5 & 1 & 1 & 2 \\ \square & \square & 3 & 3 & \mathbf{4} \end{array} \quad \text{and then} \quad \begin{array}{ccccc} 1 & 1 & & & \\ 2 & 2 & & & \\ 3 & 4 & & & \\ 4 & 5 & 1 & 1 & 2 \\ 5 & \mathbf{5} & 3 & 3 & 4 \end{array}.$$

But now we have a letter **5** in a column having already a letter 5. Thus, again by (P_3'), we get, successively:

$$\begin{array}{ccccc} 1 & 1 & & & \\ 2 & 2 & & & \\ 3 & 4 & & & \\ 4 & \mathbf{4} & 1 & 1 & 2 \\ 5 & 5 & 3 & 3 & \mathbf{5} \end{array} \quad \text{and} \quad \begin{array}{ccccc} 1 & 1 & & & \\ 2 & 2 & & & \\ 3 & \mathbf{3} & & & \\ 4 & 4 & 1 & 1 & 2 \\ 5 & 5 & 3 & \mathbf{4} & 5 \end{array}.$$

The corresponding filling of S^* corresponds to the following $(0,1)$-matrix in $\mathcal{A}(R,S)$:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

The insertion tableau associated with the biword

$$\Theta_A = \begin{pmatrix} 1111 & 222 & 333 & 444 & 555 \\ 1234 & 125 & 123 & 124 & 125 \end{pmatrix}$$

is

$$\begin{array}{ccccc} 5 & 5 & & & \\ 4 & 4 & & & \\ 3 & 3 & & & \\ 2 & 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{array}$$

and has the shape $\lambda = (5, 5, 2, 2, 2)$. Notice that although the partitions R and λ^* satisfy $r^3_{\lambda^*} = 3 > \lambda^*_3 = 2$, we did not use (P'_4) of Algorithm 2. Thus, in this case, the required $(0, 1)$-matrix was achieved by Algorithm 1.

5. Relation with other algorithms

We finish this paper showing that the output of our algorithm differs from the output of the algorithm given by Brualdi in [1] for the case where the partition λ is equal to R^*. When $\lambda = S$ the two algorithms seem to provide the same output, as we may check applying Brualdi's algorithm to our examples 4.2 and 3.3 (i). It would be interesting to check if the two algorithms give, in fact, the same output when $\lambda = S$.

In fact, let us consider Example 2 given in [1], setting $\lambda^* = R = (4, 4, 3, 3, 2)$ and $S = (4, 3, 3, 3, 3)$. The construction described by Brualdi yields the following $(0, 1)$-matrix in $\mathcal{A}(R, S)$:

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix},$$

while the application of Algorithm 1 originates the following filling of the Ferrer diagram of S^*, and the corresponding $(0, 1)$-matrix in $\mathcal{A}(R, S)$:

$$\begin{array}{|c|cccc} \cline{1-1} 2 & & & & \\ \hline 3 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} & \multicolumn{1}{c|}{1} \\ \hline 4 & \multicolumn{1}{|c|}{4} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} & \multicolumn{1}{c|}{2} \\ \hline 5 & \multicolumn{1}{|c|}{5} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \leftrightarrow \quad A' = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The insertion tableaux associated with the matrices A and A' are, respectively,

$$\begin{array}{ccccc} 5 & 5 & & & \\ 4 & 4 & 4 & 5 & \\ 2 & 2 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 2 \end{array} \quad \text{and} \quad \begin{array}{ccccc} 5 & 5 & & & \\ 3 & 4 & 4 & 5 & \\ 2 & 2 & 3 & 3 & 4 \\ 1 & 1 & 1 & 1 & 2 \end{array},$$

having both the shape $\lambda = (5, 5, 4, 2)$.

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C.M. da Fonseca
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal

E-mail address: `cmf@mat.uc.pt`

Ricardo Mamede
CMUC, Department of Mathematics, University of Coimbra, 3001-454 Coimbra, Portugal

E-mail address: `mamede@mat.uc.pt`