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A NOTE ON 3-QUASI-SASAKIAN GEOMETRY

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Abstract: 3-quasi-Sasakian manifolds were recently studied by the authors as a suitable setting unifying 3-Sasakian and 3-cosymplectic geometries. In this paper some geometric properties of this class of almost 3-contact metric manifolds are briefly reviewed, with an emphasis on those more related to physical applications.

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1. Introduction

The class of 3-quasi-Sasakian manifolds is the analogue in the setting of 3-structures of the class of quasi-Sasakian manifolds, introduced by Blair [3] and later studied among others by Tanno [13], Kanemaki [11], Olszak [12]. More recent are the examples of applications of quasi-Sasakian manifolds to string theory found by Friedrich and his collaborators [2, 9]. Just like quasi-Sasakian manifolds include Sasakian and cosymplectic manifolds, so 3-quasi-Sasakian manifolds unify 3-Sasakian and 3-cosymplectic geometry. A 3-quasi-Sasakian manifold can arise, for example, as the product of a 3-Sasakian manifold and a hyper-Kähler manifold (see Sect. 3 or [7]). The setting of 3-structures has been recently the object of a wider interest from both mathematicians and physicists due to the important role acquired by the 3-Sasakian and the related quaternionic structures in supergravity and superstring theory, where they appear in the so called hypermultiplet solutions (see e. g. [1, 2, 6, 15]). This note contains a concise review of the main properties of 3-quasi-Sasakian manifolds, recently studied by the authors in [7], together with some relevant properties of the two important subclasses of 3-Sasakian and 3-cosymplectic manifolds which were compared in [8].

2. 3-quasi-Sasakian geometry

An *almost contact metric manifold* is a $(2n+1)$-dimensional manifold M endowed with a field ϕ of endomorphisms of the tangent spaces, a vector field ξ,

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called *Reeb vector field*, a 1-form η satisfying $\phi^2 = -I + \eta \otimes \xi$, $\eta(\xi) = 1$ (where $I \colon TM \to TM$ is the identity mapping) and a *compatible* Riemannian metric g such that $g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y)$ for all $X, Y \in \Gamma(TM)$. The manifold is said to be *normal* if the tensor field $N^{(1)} = [\phi, \phi] + 2d\eta \otimes \xi$ vanishes identically. The 2-form Φ on M defined by $\Phi(X,Y) = g(X, \phi Y)$ is called the *fundamental 2-form* of the almost contact metric manifold (M, ϕ, ξ, η, g). Normal almost contact metric manifolds such that both η and Φ are closed are called *cosymplectic manifolds* and those such that $d\eta = \Phi$ are called *Sasakian manifolds*. The notion of quasi-Sasakian structure unifies those of Sasakian and cosymplectic structures. A *quasi-Sasakian manifold* is defined as a normal almost contact metric manifold whose fundamental 2-form is closed. A quasi-Sasakian manifold M is said to be of rank $2p$ (for some $p \leq n$) if $(d\eta)^p \neq 0$ and $\eta \wedge (d\eta)^p = 0$ on M, and to be of rank $2p+1$ if $\eta \wedge (d\eta)^p \neq 0$ and $(d\eta)^{p+1} = 0$ on M (cf. [3, 13]). Blair proved that there are no quasi-Sasakian manifolds of even rank. Just like Blair and Tanno did, we will only consider quasi-Sasakian manifolds of constant (odd) rank. If the rank of M is $2p+1$, then the module $\Gamma(TM)$ of vector fields over M splits into two submodules as follows: $\Gamma(TM) = \mathcal{E}^{2p+1} \oplus \mathcal{E}^{2q}$, $p + q = n$, where $\mathcal{E}^{2q} = \{X \in \Gamma(TM) \mid i_X d\eta = 0 \text{ and } i_X \eta = 0\}$ and $\mathcal{E}^{2p+1} = \mathcal{E}^{2p} \oplus \langle \xi \rangle$, $\mathcal{E}^{2p}$ being the orthogonal complement of $\mathcal{E}^{2q} \oplus \langle \xi \rangle$ in $\Gamma(TM)$. These modules satisfy $\phi \mathcal{E}^{2p} = \mathcal{E}^{2p}$ and $\phi \mathcal{E}^{2q} = \mathcal{E}^{2q}$ (cf. [13]).

We now come to the main topic of our paper, i.e. 3-quasi-Sasakian geometry, which is framed into the more general setting of almost 3-contact geometry. An *almost 3-contact metric manifold* is a $(4n+3)$-dimensional smooth manifold M endowed with three almost contact structures (ϕ_1, ξ_1, η_1), (ϕ_2, ξ_2, η_2), (ϕ_3, ξ_3, η_3) satisfying the following relations, for any even permutation (α, β, γ) of $\{1, 2, 3\}$,

$$\begin{aligned} \phi_\gamma &= \phi_\alpha \phi_\beta - \eta_\beta \otimes \xi_\alpha = -\phi_\beta \phi_\alpha + \eta_\alpha \otimes \xi_\beta, \\ \xi_\gamma &= \phi_\alpha \xi_\beta = -\phi_\beta \xi_\alpha, \quad \eta_\gamma = \eta_\alpha \circ \phi_\beta = -\eta_\beta \circ \phi_\alpha, \end{aligned} \tag{1}$$

and a Riemannian metric g compatible with each of them. It is well known that in any almost 3-contact metric manifold the Reeb vector fields ξ_1, ξ_2, ξ_3 are orthonormal with respect to the compatible metric g and that the structural group of the tangent bundle is reducible to $Sp(n) \times I_3$. Moreover, by putting $\mathcal{H} = \bigcap_{\alpha=1}^{3} \ker(\eta_\alpha)$ one obtains a $4n$-dimensional *horizontal* distribution on M and the tangent bundle splits as the orthogonal sum $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V} = \langle \xi_1, \xi_2, \xi_3 \rangle$ is the *vertical* distribution.

Definition 2.1. *A* 3-quasi-Sasakian manifold *is an almost* 3*-contact metric manifold* $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *such that each almost contact structure is quasi-Sasakian.*

The class of 3-quasi-Sasakian manifolds includes as special cases the well-known 3-Sasakian and 3-cosymplectic manifolds.

The following theorem combines the results obtained in Theorems 3.4 and 4.2 of [7].

Theorem 2.2. *Let* $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a* 3*-quasi-Sasakian manifold. Then the* 3*-dimensional distribution* $\mathcal{V}$ *generated by* ξ_1*,* ξ_2*,* ξ_3 *is integrable. Moreover,* $\mathcal{V}$ *defines a totally geodesic and Riemannian foliation of* M *and for any even permutation* (α, β, γ) *of* $\{1, 2, 3\}$ *and for some* $c \in \mathbb{R}$

$$[\xi_\alpha, \xi_\beta] = c\xi_\gamma.$$

Using Theorem 2.2 we may divide 3-quasi-Sasakian manifolds in two classes according to the behaviour of the leaves of the foliation $\mathcal{V}$: those 3-quasi-Sasakian manifolds for which each leaf of $\mathcal{V}$ is locally $SO(3)$ (or $SU(2)$) (which corresponds to take in Theorem 2.2 the constant $c \neq 0$), and those for which each leaf of $\mathcal{V}$ is locally an abelian group (this corresponds to the case $c = 0$).

The preceding theorem also allows to define a canonical metric connection on any 3-quasi-Sasakian manifold. Indeed, let ∇^B be the Bott connection associated to $\mathcal{V}$, that is the partial connection on the normal bundle $TM/\mathcal{V} \cong \mathcal{H}$ of $\mathcal{V}$ defined by $\nabla^B_V Z := [V, Z]_{\mathcal{H}}$ for all $V \in \Gamma(\mathcal{V})$ and $Z \in \Gamma(\mathcal{H})$. Following [14] we may construct an adapted connection on $\mathcal{H}$ putting

$$\tilde{\nabla}_X Y := \begin{cases} \nabla^B_X Y, & \text{if } X \in \Gamma(\mathcal{V}); \\ (\nabla_X Y)_{\mathcal{H}}, & \text{if } X \in \Gamma(\mathcal{H}). \end{cases}$$

This connection can be also extended to a connection on all TM by requiring that $\tilde{\nabla}\xi_\alpha = 0$ for each $\alpha \in \{1, 2, 3\}$. Some properties of this global connection have been considered in [8] for any almost 3-contact metric manifold. Now combining Theorem 2.2 with [8, Theorem 3.6] we have:

Theorem 2.3. *Let* $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ *be a* 3*-quasi-Sasakian manifold. Then there exists a unique metric connection* $\tilde{\nabla}$ *on* M *satisfying the following properties:*

(i) $\tilde{\nabla}\eta_\alpha = 0$, $\tilde{\nabla}\xi_\alpha = 0$, *for each* $\alpha \in \{1, 2, 3\}$,
(ii) $\tilde{T}(X, Y) = 2\sum_{\alpha=1}^{3} d\eta_\alpha(X, Y)\xi_\alpha$, *for all* $X, Y \in \Gamma(TM)$.

3. The rank of a 3-quasi-Sasakian manifold

For a 3-quasi-Sasakian manifold one can consider the ranks of the three structures $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$. The following theorem assures that these three ranks coincide.

Theorem 3.1 ([7]). *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of dimension $4n+3$. Then the 1-forms η_1, η_2 and η_3 have the same rank $4l+3$ or $4l+1$, for some $l \leq n$, according to $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$, or $[\xi_\alpha, \xi_\beta] = 0$, respectively.*

According to Theorem 3.1, we say that a 3-quasi-Sasakian manifold $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ has *rank* $4l+3$ or $4l+1$ if any quasi-Sasakian structure has such rank. We may thus classify 3-quasi-Sasakian manifolds of dimension $4n+3$, according to their rank. For any $l \in \{0, \ldots, n\}$ we have one class of manifolds such that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$, and one class of manifolds with $[\xi_\alpha, \xi_\beta] = 0$. The total number of classes amounts then to $2n+2$. In the following we will use the notation $\mathcal{E}^{4m} := \{X \in \Gamma(\mathcal{H}) \mid i_X d\eta_\alpha = 0\}$, while $\mathcal{E}^{4l}$ will be the orthogonal complement of $\mathcal{E}^{4m}$ in $\Gamma(\mathcal{H})$, $\mathcal{E}^{4l+3} := \mathcal{E}^{4l} \oplus \Gamma(\mathcal{V})$, and $\mathcal{E}^{4m+3} := \mathcal{E}^{4m} \oplus \Gamma(\mathcal{V})$.

We now consider the class of 3-quasi-Sasakian manifolds such that $[\xi_\alpha, \xi_\beta] = c\xi_\gamma$ with $c \neq 0$ and let $4l+3$ be the rank. In this case, according to [3], we define for each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ two $(1,1)$-tensor fields ψ_α and θ_α by putting

$$\psi_\alpha X = \begin{cases} \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4l+3}; \\ 0, & \text{if } X \in \mathcal{E}^{4m}; \end{cases} \qquad \theta_\alpha X = \begin{cases} 0, & \text{if } X \in \mathcal{E}^{4l+3}; \\ \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4m}. \end{cases}$$

Note that, for each $\alpha \in \{1,2,3\}$ we have $\phi_\alpha = \psi_\alpha + \theta_\alpha$. Next, we define a new (pseudo-Riemannian, in general) metric $\bar{g}$ on M setting

$$\bar{g}\left(X, Y\right) = \begin{cases} -d\eta_\alpha\left(X, \phi_\alpha Y\right), & \text{for } X, Y \in \mathcal{E}^{4l}; \\ g\left(X, Y\right), & \text{elsewhere.} \end{cases}$$

This definition is well posed by virtue of normality and of [7, Lemma 5.3]. $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, \bar{g})$ is in fact a hyper-normal almost 3-contact metric manifold, in general non-3-quasi-Sasakian. We are now able to formulate the following decomposition theorem, proven in [7].

Theorem 3.2. *Let $(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of rank $4l+3$ with $[\xi_\alpha, \xi_\beta] = 2\xi_\gamma$. Assume $[\theta_\alpha, \theta_\alpha] = 0$ for some $\alpha \in \{1,2,3\}$ and*

$\bar{g}$ positive definite on $\mathcal{E}^{4l}$. Then M^{4n+3} is locally the product of a 3-Sasakian manifold M^{4l+3} and a hyper-Kählerian manifold M^{4m} with $m = n - l$.

We now consider the class of 3-quasi-Sasakian manifolds such that $[\xi_\alpha, \xi_\beta] = 0$ and let $4l + 1$ be the rank. In this case we define for each structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ two $(1,1)$-tensor fields ψ_α and θ_α by putting

$$\psi_\alpha X = \begin{cases} \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4l}; \\ 0, & \text{if } X \in \mathcal{E}^{4m+3}; \end{cases} \qquad \theta_\alpha X = \begin{cases} 0, & \text{if } X \in \mathcal{E}^{4l}; \\ \phi_\alpha X, & \text{if } X \in \mathcal{E}^{4m+3}. \end{cases}$$

Note that for each α the maps $-\psi_\alpha^2$ and $-\theta_\alpha^2 + \eta_\alpha \otimes \xi_\alpha$ define an almost product structure which is integrable if and only if $[-\psi_\alpha^2, -\psi_\alpha^2] = 0$ or, equivalently, $[\psi_\alpha, \psi_\alpha] = 0$. Under this assumption the structure turns out to be 3-cosymplectic:

Theorem 3.3 ([7])**.** *Let $(M, \phi_\alpha, \xi_\alpha, \eta_\alpha, g)$ be a 3-quasi-Sasakian manifold of rank $4l+1$ such that $[\xi_\alpha, \xi_\beta] = 0$ for any $\alpha, \beta \in \{1,2,3\}$ and $[\psi_\alpha, \psi_\alpha] = 0$ for some $\alpha \in \{1,2,3\}$. Then M is a 3-cosymplectic manifold.*

As we have remarked before, 3-Sasakian and 3-cosymplectic manifolds belong to the class of 3-quasi-Sasakian manifolds, having respectively rank $4n + 3 = \dim(M)$ and rank 1. We now briefly collect some additional properties of these two important subclasses. We have seen that the vertical distribution $\mathcal{V}$ is integrable already in any 3-quasi-Sasakian manifold. Ishihara ([10]) has shown that if the foliation defined by $\mathcal{V}$ is regular then the space of leaves is a quaternionic-Kählerian manifold. Boyer, Galicki and Mann have proved the following more general result.

Theorem 3.4 ([5])**.** *Let $\left(M^{4n+3}, \phi_\alpha, \xi_\alpha, \eta_\alpha, g\right)$ be a 3-Sasakian manifold such that the Killing vector fields ξ_1, ξ_2, ξ_3 are complete. Then*

- **(i):** *M^{4n+3} is an Einstein manifold of positive scalar curvature equal to $2\,(2n+1)\,(4n+3)$.*
- **(ii):** *Each leaf of the foliation $\mathcal{V}$ is a 3-dimensional homogeneous spherical space form.*
- **(iii):** *The space of leaves $M^{4n+3}/\mathcal{V}$ is a quaternionic-Kählerian orbifold of dimension $4n$ with positive scalar curvature equal to $16n\,(n+2)$.*

We consider now the horizontal distribution: on the one hand, in the 3-Sasakian subclass $\mathcal{H}$ is never integrable. On the other hand, in any 3-cosymplectic manifold $\mathcal{H}$ is integrable since each η_α is closed. Furthermore, the projectability with respect to $\mathcal{V}$ is always granted, as the following theorem shows.

Theorem 3.5 ([8]). *Every regular* 3*-cosymplectic manifold projects onto a hyper-Kählerian manifold.*

As a corollary, it follows that every 3-cosymplectic manifold is Ricci-flat.

In [8] the horizontal flatness of such structures has been studied. In particular it has been proven to be equivalent to the existence of Darboux-like coordinates, that is local coordinates $\{x_1, \ldots, x_{4n}, z_1, z_2, z_3\}$ with respect to which, for each $\alpha \in \{1,2,3\}$, the fundamental 2-forms $\Phi_\alpha = d\eta_\alpha$ have constant components and $\xi_\alpha = a^1_\alpha \frac{\partial}{\partial z_1} + a^2_\alpha \frac{\partial}{\partial z_2} + a^3_\alpha \frac{\partial}{\partial z_3}$, a^β_α being functions depending only on the coordinates z_1, z_2, z_3. Consequently, in view of Theorem 3.4 and Theorem 3.5 we have the following result.

Theorem 3.6 ([8]). *A* 3*-Sasakian manifold does not admit any Darboux-like coordinate system. On the other hand, a* 3*-cosymplectic manifold admits a Darboux-like coordinate system around each of its points if and only if it is flat.*

4. Final Remarks

A number of natural questions arose during the development of our work on 3-quasi-Sasakian manifolds. We have seen that 3-Sasakian manifolds do not admit any Darboux coordinate system, while on 3-cosymplectic manifolds such coordinate exist if and only if the manifold is flat, so it is natural to wonder whether these coordinates do not exist on any 3-quasi-Sasakian manifold of rank greater than one. Another important topic would be to study the projectability of 3-quasi-Sasakian manifolds for understanding the general relation between this class and the quaternionic structures, since the 3-Sasakian manifolds project over quaternionic-Kähler structures while the structure of the leaf space turns out to be globally hyper-Kählerian in the 3-cosymplectic case. Finally, as both 3-Sasakian and 3-cosymplectic manifolds are Einstein manifolds a natural question would be to ask whether all 3-quasi-Sasakian manifolds are Einstein. However, since we have already found an example of an η-Einstein, non-Einstein 3-quasi-Sasakian manifold in [7], the natural problem now becomes to establish if there is any 3-quasi-Sasakian manifolds which is not η-Einstein. We will try to address some of these questions in the next future.

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