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THE ORTHOGONAL SUBCATEGORY PROBLEM AND THE SMALL OBJECT ARGUMENT

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ABSTRACT: A classical result of P. Freyd and M. Kelly states that in "good" categories, the Orthogonal Subcategory Problem has a positive solution for all classes $\mathcal{H}$ of morphisms whose members are, except possibly for a subset, epimorphisms. We prove that under the same assumptions on the base category and on $\mathcal{H}$, the generalization of the Small Object Argument of D. Quillen holds - that is, every object of the category has a cellular $\mathcal{H}$-injective weak reflection. In locally presentable categories, we prove a sharper result: a class of morphisms is called quasi-presentable if for some cardinal λ every member of the class is either λ-presentable or an epimorphism. Both the Orthogonal Subcategory Problem and the Small Object Argument are valid for quasi-presentable classes. Surprisingly, in locally ranked categories (used previously to generalize Quillen's result), this is no longer true: we present a class $\mathcal{H}$ of morphisms, all but one being epimorphisms, such that the orthogonality subcategory $\mathcal{H}^\perp$ is not reflective and the injectivity subcategory $\mathrm{Inj}\,\mathcal{H}$ is not weakly reflective. We also prove that in locally presentable categories, the Injectivity Logic and the Orthogonality Logic are complete for all quasi-presentable classes.

1. Introduction

Our paper is devoted to two "classical" Galois connections between objects A and morphisms h of a given category $\mathcal{A}$: *orthogonality*

$$A \perp h,$$

which means that the hom functor $\mathcal{A}[-, A] : \mathcal{A}^{op} \to \mathbf{Set}$ takes h to an isomorphism, and *injectivity*

$$A \in \mathrm{Inj}\,\{h\}$$

which means that $\mathcal{A}[-, A]$ takes h to an epimorphism.

(I) ORTHOGONALITY. Given a class $\mathcal{H}$ of morphisms in $\mathcal{A}$, the full subcategory

$$\mathcal{H}^\perp$$

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of all objects orthogonal to all the members of $\mathcal{H}$ is clearly closed under limits. In [9], Peter Freyd and Max Kelly posed the *Orthogonal Subcategory Problem*:

$$\text{is } \mathcal{H}^\perp \text{ a reflective subcategory of } \mathcal{A}\,?$$

They proved that in "suitable" categories, the answer is affirmative whenever $\mathcal{H}$ is the union of a set of morphisms and of a class of epimorphisms:

$$\mathcal{H}^\perp \text{ reflective if } \mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_e \text{ with } \mathcal{H}_0 \text{ small and } \mathcal{H}_e \subseteq \mathrm{Epi}.$$

We call such classes $\mathcal{H}$ *quasi-small.* The "suitability" conditions considered in [9] included the existence of limits; later Kelly [15] showed that this condition is not needed. We call categories satisfying the weakened conditions *locally bounded.* For example, the category **Top** of topological spaces is locally bounded, but also, as we will show below, every locally presentable category of Peter Gabriel and Friedrich Ulmer [10].

It has been known that the result of Freyd and Kelly extends to more general classes of morphisms, assuming that the base category is locally presentable:

(i) Under the set-theoretical Vopěnka's Principle, $\mathcal{H}^\perp$ is reflective for all (!) classes $\mathcal{H}$, as proved by Jiří Rosický and the first author [5].

Without set-theoretical restrictions, we can also say something more general by using the concept of *λ-presentable morphism* as in [13]: it is a morphism $f : A \to B$ which, in the slice category $A \downarrow \mathcal{A}$, is a λ-presentable object. A class of morphisms is called *presentable* if there exists a cardinal number λ such that all its members are λ-presentable. Every set of morphisms (in locally presentable categories) is presentable, but one easily finds presentable proper classes.

(ii) For all presentable classes $\mathcal{H}$, the subcategory $\mathcal{H}^\perp$ is reflective, see [2].

We are going to prove a joint generalization of the Freyd-Kelly result and of (ii), in the context of locally presentable categories:

$$\mathcal{H}^\perp \text{ is reflective for all classes } \mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_e \text{ with } \mathcal{H}_0 \text{ presentable and } \mathcal{H}_e \subseteq \mathrm{Epi}.$$

We call such classes $\mathcal{H}$ *quasi-presentable.* The results above concern the "object side" of the Galois correspondence $\perp$ between objects and morphisms. The "morphism side" is the field of *Orthogonality Logic*: given a class $\mathcal{H}$ of morphisms, characterize all its *orthogonality consequences*, that is, all morphisms k such that all objects orthogonal to $\mathcal{H}$ are also orthogonal to k.

Notation:

$$\mathcal{H} \models k \ \text{ iff } \ \mathcal{H}^{\perp} \subseteq \{k\}^{\perp}$$

Inspired by the work of Grigori Roşu [21], we have recently formulated a simple deduction system for the Orthogonality Logic, and proved that it is sound and complete in every locally presentable category, for all classes $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_{se}$, where $\mathcal{H}_0$ is small and the members of $\mathcal{H}_{se}$ are strong epimorphisms. Here we improve this result: the Orthogonality Logic is sound and complete for all quasi-presentable classes of morphisms. Not too surprisingly: assuming Vopěnka's Principle, soundness and completeness hold for *all* classes of morphisms. By applying the technique of Max Kelly, we also prove that in every locally bounded category, the soundness and completeness hold for all quasi-small classes of morphisms.

(II) INJECTIVITY. Here, the corresponding full subcategory

$$\operatorname{Inj} \mathcal{H}$$

of all objects injective with respect to all members of $\mathcal{H}$ is closed under products and split subobjects, but in general not under limits. Thus, we do not expect it to be reflective. However, in homotopy theory, *weak reflections* in subcategories of the form Inj $\mathcal{H}$ are often used. Recall that a weak reflection of an object A in Inj $\mathcal{H}$ is a morphism $r_A : A \to A'$ which is weakly initial in the slice category $A \downarrow \operatorname{Inj} \mathcal{H}$. Since weak reflections are not essentially unique, we may wish the morphism r_A to have additional qualities. In homotopy theory, the requirement is that r_A be *$\mathcal{H}$-cellular*, which means that it is a transfinite composite of morphisms, each of which is a pushout of a member of $\mathcal{H}$, see 3.2 below. A classical result of Daniel Quillen [20], called the *Small Object Argument*, states that if the object A is "sufficiently small", then it has a cellular weak reflection in Inj $\mathcal{H}$. A generalization is proved in [4]: for "sufficiently nice" categories, given any set $\mathcal{H}$ of morphisms, all objects have a cellular weak reflection in Inj $\mathcal{H}$. In the present paper, we prove that the generalization of Quillen's Small Object Argument works:

(i) for quasi-small classes of morphisms in locally bounded categories, as well as

(ii) for quasi-presentable classes of morphisms in locally presentable categories.

Our proof of (i) is based on ideas of Max Kelly [15]. The categories used in [4] were quite similar to locally bounded ones, but did not allow the above generalization (from "small" to "quasi-small"): we demonstrate this in the last section by finding a category $\mathcal{A}$ satisfying the assumptions of [4], a class $\mathcal{H}$ whose members except one are epimorphisms and such that $\mathcal{H}^{\perp}$ is not reflective, and another such class with $\mathrm{Inj}\,\mathcal{H}$ not weakly reflective. In order to show this, we use substantially results from [18] and [19] on continuous algebras in CPO.

There is, again, a "morphism side" to our Galois connection: the *Injectivity Logic*. Given a class $\mathcal{H}$ of morphisms, we want to characterize all its *injectivity consequences*, that is, all morphisms k such that every object injective with respect to $\mathcal{H}$ is also injective with respect to k. Notation:

$$\mathcal{H} \models k \ \text{ iff } \ \mathrm{Inj}\,\mathcal{H} \subseteq \mathrm{Inj}\,\{k\}.$$

The Injectivity Logic has the same origin than the Orthogonality Logic, since Roşu ([21]) was dealing with classes of epimorphisms, in which case orthogonality and injectivity coincide. His logic was finitary (i.e., the deduction rules involved just a finite number of morphisms), and he proved the completeness for sets of finitely presentable epimorphisms with projective domains.

In the paper [8], the completeness was proved for *finitary* epimorphisms (i.e., with finitely presentable domains and codomains), considering a slight modification of Roçu's deduction system; moreover, it was shown that the completeness of the finitary logic may fail for sets of finitely presentable epimorphisms. In [1], this logic (and the corresponding completeness theorem) was extended to all finitary morphisms. Infinitary logic was also considered, and the completeness of the corresponding rules was proved for all classes of finitely presentable morphisms. Here we prove that the Injectivity Logic is sound and complete:

(i) for quasi-small classes of morphisms in locally bounded categories, as well as

(ii) for quasi-presentable classes of morphisms in locally presentable categories.

It is worth noting that, as opposed to the orthogonality case, no set-theoretical condition is needed to find a class $\mathcal{H}$ in a locally presentable category for which the Injectivity Logic is not complete: such an example is provided in [2] for the category of graphs.

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2. Locally Presentable and Locally Bounded Categories

Throughout the paper we assume that the category $\mathcal{A}$ is cocomplete and $\mathcal{E}$-cowellpowered, where $(\mathcal{E}, \mathcal{M})$ is some fixed *proper* factorization system (i.e., $\mathcal{E} \subseteq$ Epi and $\mathcal{M} \subseteq$ Mono). Moreover, we will often work in categories in which each object is "λ-small" for some infinite regular cardinal λ. In the case of locally presentable categories, "λ-small" objects are just the λ-presentable ones. Recall that an object A is λ-presentable if $\mathcal{A}[A, -] : \mathcal{A} \to \mathbf{Set}$ preserves λ-filtered colimits. However Peter Freyd and Max Kelly used a different type of smallness, called bound:

2.1. Definition An object A is said to have *bound* λ (an infinite cardinal), if $\mathcal{A}[A, -]$ preserves λ-directed unions of $\mathcal{M}$-subobjects.

Explicitly: given a λ-directed collection of subobjects $m_i : M_i \to M \ \ (i \in I)$ in $\mathcal{M}$ whose union is M, in the sense that the morphism $[m_i] : \coprod_{i \in I} M_i \to M$ lies in $\mathcal{E}$, then every morphism from A to M factors through m_i for some $i \in I$.

2.2. Definition We call $\mathcal{A}$ *locally* $\mathcal{E}$*-bounded* (or *locally bounded w.r.t.* $(\mathcal{E}, \mathcal{M})$), provided that every object in $\mathcal{A}$ has a bound. *Locally bounded* means locally Epi-bounded.

2.3. Remark Locally $\mathcal{E}$-bounded categories with limits were considered by P. Freyd and M. Kelly in [9] for the study of the Orthogonal Subcategory Problem, although they did not give them a name. Later, M. Kelly [15] showed that the results of [9] remain valid without the assumption of the existence of limits.

2.4. Examples (a) Recall from [10] that $\mathcal{A}$ is called *locally* λ*-presentable* if it is cocomplete and has a set of λ-presentable objects whose closure under λ-filtered colimits is all of $\mathcal{A}$.

As noted in [9], all locally presentable categories are locally bounded w.r.t. the factorization system $(StrongEpi, Mono)$. This is an easy consequence of the fact that in a locally λ-presentable category each λ-directed diagram with monomorphisms as connecting morphisms fulfils the following: (i) its colimit consists of monomorphisms, and (ii) for every compatible cocone of monomorphisms the induced factorizing morphism is monic; see 1.62 and 1.63 of [5]. We will prove below that although none of (i) and (ii) is true for strong monomorphisms in general, locally presentable categories are nevertheless locally bounded (i.e., w.r.t. the factorization system $(Epi, StrongMono)$).

(b) The category **Top** of topological spaces and continuous functions is locally bounded (cf. [9]).

(c) Let $\mathcal{A}$ be locally bounded w.r.t. $(\mathcal{E}, \mathcal{M})$. Then every full $\mathcal{E}$-reflective subcategory $\mathcal{L}$ (i.e., one whose reflection maps lie in $\mathcal{E}$) is locally bounded with respect to the factorization system $(\mathcal{E}', \mathcal{M}')$, where $\mathcal{E}' = \mathcal{E} \cap Mor\mathcal{L}$ and $\mathcal{M}' = \mathcal{M} \cap Mor\mathcal{L}$. Since $\mathcal{L}$ is closed under $\mathcal{M}$-subobjects in $\mathcal{A}$, we obtain a factorization system $(\mathcal{E}', \mathcal{M}')$ in $\mathcal{L}$. Every object of $\mathcal{L}$ of bound λ in $\mathcal{A}$ has bound λ in $\mathcal{L}$ w.r.t. $(\mathcal{E}', \mathcal{M}')$. In fact, let $m_i : M_i \to M$ $(i \in I)$ be morphisms in $\mathcal{M}'$ having union M in $\mathcal{L}$, then it is easy to see that they have union M in $\mathcal{A}$ too. Thus, every morphism $L \to M$ in $\mathcal{L}$ factorizes through some m_i.

(d) The category **Haus** of Hausdorff topological spaces is locally bounded w.r.t. the factorization system $(Surjections, Embeddings)$, see (c). However it is not locally bounded w.r.t. $(StrongEpi, Mono)$ nor $(Epi, StrongMono)$ ([9], 3.3.5). For the results presented below the natural factorization system is $(Epi, StrongMono)$. That is the reason why we use simply the term "locally bounded" in this case. However, the category **Haus** demonstrates that we should not restrict ourselves to that factorization system.

2.5. Lemma *For every locally presentable category $\mathcal{A}$, there exist (arbitrary large) cardinals λ such that the category $\mathcal{A}^{\twoheadrightarrow}$ of all epimorphisms (as a full subcategory of the morphism category $\mathcal{A}^{\to}$) is locally λ-presentable, and the inclusion functor $\mathcal{A}^{\twoheadrightarrow} \hookrightarrow \mathcal{A}^{\to}$ preserves λ-filtered colimits and λ-presentable objects.*

Proof (1) We can consider $\mathcal{A}^{\twoheadrightarrow}$ as the category of all models of the sketch consisting of a single pushout

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "f"'] & b \arrow[d, "id_b"] \\ b \arrow[r, "id_b"'] & b \end{tikzcd}$$

in the category $\mathcal{A}$. Consequently, $\mathcal{A}^{\twoheadrightarrow}$ is accessible (see Theorem 2.60 in [5]), and being closed under colimits in $\mathcal{A}^{\to}$, it is cocomplete. The local presentability of $\mathcal{A}^{\twoheadrightarrow}$ follows from 2.47 in [5].

(2) Since $\mathcal{A}^{\twoheadrightarrow}$ is closed under colimits in $\mathcal{A}^{\to}$, by Theorem 2.19 of [5], there exists a (arbitrary large) regular cardinal λ such that $\mathcal{A}^{\twoheadrightarrow}$ is locally λ-presentable and its embedding into $\mathcal{A}^{\to}$ preserves λ-filtered colimits and λ-presentable objects. $\square$

2.6. Theorem *Every locally presentable category is locally bounded.*

Proof (1) Let λ be a cardinal such that the statement of 2.5 holds. We prove that for $k \geq \lambda$, every k-directed union of strong monomorphisms is a colimit.

Let

$$f_i : A_i \to B \ (i \in I)$$

be a k-directed diagram of strong monomorphisms in $\mathcal{A}$. Denote by

$$x_i : A_i \to A \ (i \in I)$$

the colimit cocone of the diagram of all connecting strong monomorphisms, then the factorizing morphism

$$f : A \to B \ \text{ with } f_i = f \cdot x_i \ \ (i \in I)$$

is an epimorphism. To conclude the proof we verify that f is a split monomorphism. Then $B = \operatorname{colim} A_i$.

Express f as a k-filtered colimit of k-presentable objects g_u of $\mathcal{A}^{\to}$ with the colimit cocone

$$(t_u, s_u) : (\ T_u \xrightarrow{g_u} S_u\ ) \longrightarrow (\ A \xrightarrow{f} B\ ) \ \ (u \in U).$$

Since $(x_i, \mathrm{id}_B) : (\ A_i \xrightarrow{f_i} B\ ) \longrightarrow (\ A \xrightarrow{f} B\ )$ is a k-filtered colimit in $\mathcal{A}^{\to}$, for every $u \in U$ there exists $i(u) \in I$ such that (t_u, s_u) factors through $(x_{i(u), \mathrm{id}_B})$ in $\mathcal{A}^{\to}$:

$$\begin{tikzcd}
& T_u \arrow[r, "g_u"] \arrow[d, "t'_u"'] \arrow[ddl, "t_u"'] & S_u \arrow[dl, dashed, "p_u"'] \arrow[d, "s_u"] \arrow[ddr, "s_u"] & \\
& A_{i(u)} \arrow[r, "f_{i(u)}"'] \arrow[dl, "x_{i(u)}"] & B \arrow[dr, equal] & \\
A \arrow[rrr, "f"'] & & & B
\end{tikzcd}$$

The diagonal fill-in property gives a morphism $p_u : S_u \to A_{i(u)}$ such that the above diagram commutes. It is easy to verify that the morphisms $x_{i(u)} \cdot p_u$ form a compatible cocone of the diagram of $S_u \,(u \in U)$, whose colimit is B (use the fact that the morphisms g_u are epimorphic); thus, we have a unique

$$p : B \to A \ \text{ with } p \cdot s_u = x_{i(u)} \cdot p_u \ (u \in U).$$

We conclude that

$$p \cdot f \cdot t_u = t_u \text{ for all } u \in U$$

and since (t_u) is a colimit cocone, we deduce $p \cdot f = \mathrm{id}$, as required.

(2) Every object A is k-presentable for some regular cardinal k. Assuming (without loss of generality) that $k \geq \lambda$, it follows from (1) that A has bound k w.r.t. $(Epi, StrongMono)$. □

2.7. Definition ([13]) A morphism $f : A \to B$ of a category $\mathcal{A}$ is said to be λ-*presentable*, for an infinite regular cardinal λ, if it is a λ-presentable object in the slice category $A \downarrow \mathcal{A}$.

2.8. Examples (i) A morphism $f : A \to B$ in **Set** is finitely presentable iff it is

a. "almost epic", i.e., the set $B - f[A]$ is finite

and

b. "almost monic", i.e., the set $ker f - \Delta_A$ is finite.

(ii) Analogously, in the category of R-modules, an embedding $A \hookrightarrow B$ is finitely presentable iff the quotient B/A is a finitely presentable module ([13]).

2.9. Facts The following facts will be used throughout. Where no reference is given, they can be easily verified.

(i) Every composition of two λ-presentable morphisms is λ-presentable. (Observe that if $f : A \to B$ and $g : B \to C$ are two λ-presentable morphisms, $x'_i : x_i \to (\, A \xrightarrow{x} X \,)$ are the colimit morphisms of a λ-filtered diagram $x_{ij} : (\, A \xrightarrow{x_i} X_i \,) \longrightarrow (\, A \xrightarrow{x_j} X_j \,)$, and $k : g \cdot f \to x$ is a morphism in $A \downarrow \mathcal{A}$, then there exists i and a morphism t such that $k \cdot g = x'_i \cdot t$, and $k \cdot g$ is the colimit of a λ-filtered diagram $x_{jj'} : x_{ij} \cdot t \to x_{ij'} \cdot t$ in $B \downarrow \mathcal{A}$.)

(ii) Every morphism $f : A \to B$ with A and B λ-presentable (in $\mathcal{A}$) is a λ-presentable morphism. (These are precisely the λ-presentable objects of $\mathcal{A}^{\to}$.) Conversely, if $f : A \to B$ and A are λ-presentable, then B is also λ-presentable. This implies that all λ-presentable morphisms with a common domain form essentially a set (since each object A is γ-presentable for some γ). This is also a consequence of the fact that $A \downarrow \mathcal{A}$ is locally λ-presentable for every A, see [5].

(iii) In every pushout

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d] & {} \arrow[d] \\ {} \arrow[r, "g"'] & {} \end{tikzcd}$$

if f is λ-presentable, then so is g. Theorem 2.3 of [13] shows that in locally λ-presentable categories, λ-presentable morphisms are actually characterized as the pushouts of the morphisms with λ-presentable domains and codomains.

2.10. Theorem *Let $\mathcal{A}$ be a locally λ-presentable category. Then*

(i) every strong epimorphism $f : A \to B$ is a λ-filtered colimit (in $A \downarrow \mathcal{A}$) of λ-presentable strong epimorphisms; and

(ii) there exists a regular cardinal $\bar{\lambda}$ such that every epimorphism $f : A \to B$ is a $\bar{\lambda}$-filtered colimit (in $A \downarrow \mathcal{A}$) of $\bar{\lambda}$-presentable epimorphisms.

Observe the difference between this theorem and Lemma 2.5: here we are working in the slice categories.

Proof (i) Since $A \downarrow \mathcal{A}$ is locally λ-presentable (see [5]), f is the colimit of the canonical diagram made of all the λ-presentable morphisms with domain A through which f factors. Let $(l_j : (e_j, A_j) \to (e, C))$ be the colimit diagram of the full subdiagram (also λ-filtered) of all λ-presentable strong epimorphisms. We show that the induced $g : C \to B$ is an isomorphism. Since it is certainly a strong epimorphism, we need only to show it is monomorphic. For that it suffices to check that it has the diagonal fill-in property with respect to strong epimorphisms between λ-presentable objects, see [12] or 5.7 of [7]. Let $q : X \to Y$ be a strong epimorphism with X and Y λ-presentable, and let u and v be morphisms such that $v \cdot q = g \cdot u$. The λ-presentability of X assures the existence of some morphism $\bar{u} : X \to A_j$ with $j \in J$ such that $u = l_j \cdot \bar{u}$. Let $(\hat{q}, \hat{u})$ be the pushout of $(q, \bar{u})$.

$$\begin{tikzcd} X \arrow[rrr, "q"] \arrow[dr, "\bar{u}"] \arrow[dd, "u"'] & & & Y \arrow[dl, "\hat{u}"'] \arrow[dd, "v"] \\ & A_j \arrow[r, "\hat{q}"] \arrow[dl, "l_j"] & \bullet \arrow[dll, "l_k"] & \\ C \arrow[rrr, "g"'] & & & B \end{tikzcd}$$

Then $\hat{q}$ is a λ-presentable strong epimorphism, and so is its composition with e_j. That is, $\hat{q} \cdot e_j = e_k$ for some $k \in J$. The equality $l_k \cdot \hat{q} = l_j$ follows because

e_j is an epimorphism. Thus, we easily conclude that $l_k \cdot \hat{u}$ is the desired diagonal morphism.

(ii) Choose a cardinal $\bar{\lambda} \geq \lambda$ such that Lemma 2.5 holds for it. We use the same argument of the proof of Proposition 1.2 of [12]. Given an epimorphism $f : A \to B$, let $(a_i, b_i) : f_i \to f$ be the colimit morphisms of a $\bar{\lambda}$-filtered diagram $(f_i \to f_j)$, with all f_i $\bar{\lambda}$-presentable objects of $\mathcal{A}^{\twoheadrightarrow}$. If f_i^* is the pushout of f_i by a_i, then f is a $\bar{\lambda}$-filtered colimit in $A \downarrow \mathcal{A}$ of the obvious diagram with objects the f_i^*. Since, by 2.5, each f_i is a $\bar{\lambda}$-presentable object in $\mathcal{A}^{\to}$, then, by 2.9, the morphisms f_i^* are $\bar{\lambda}$-presentable epimorphisms. □

2.11. Remark In the statement (ii) of the preceding theorem, it is not always possible to choose $\bar{\lambda} = \lambda$. This is demonstrated by Example 1.5(b) of [12].

3. The Small Object Argument

We explained in the introduction that our aim is to extend the generalized Small Object Argument as formulated in [4], from sets to (certain) classes of morphisms. The statement of the Small Object Argument is that in every "reasonable" category $\mathcal{A}$, and given a set (or a "reasonable" class) $\mathcal{A}$ of morphisms, then

(1) Inj$\mathcal{H}$ is a weakly reflective subcategory,

and

(2) the weak reflection of every object can be chosen to be *cellular* (see below).

We are going to prove that if $\mathcal{A}$ is a locally bounded category, then (1) and (2) hold for classes $\mathcal{H}$ of morphisms such that all members but a subset are epimorphisms. This nicely corresponds to the result of Max Kelly [15] that under precisely the same assumptions,

$$\mathcal{H}^{\perp} \text{ is a reflective subcategory of } \mathcal{A}.$$

Indeed, we are going to simply modify the technique of using *pointed endofunctors* which Kelly employed brilliantly in his paper. He used well-pointed functors, that is, the given natural transformation $\sigma : Id \to S$ satisfied $S\sigma = \sigma S$. Our pointed functors are unfortunately not well-pointed, therefore, some modifications are necessary.

We also give a surprising negative result: although the "locally ranked" categories used in [4] are but a small variation of locally bounded ones, they cannot be used for the generalization above: in the last section, a counterexample is presented.

3.1. Definition Given a class $\mathcal{H}$ of morphisms in $\mathcal{A}$, *the Small Object Argument is valid*, provided that every object of $\mathcal{A}$ has a weak reflection in $\operatorname{Inj}\mathcal{H}$ which lies in the class

$$\operatorname{cell}\mathcal{H}$$

of ($\mathcal{H}$-)*cellular morphisms*, that is, the closure of $\mathcal{H}$ under composition, pushout and wide pushout. Here, closing $\mathcal{H}$ under pushouts means adding every pushout of a member of $\mathcal{H}$ (by any morphism), and closing under wide pushouts means adding any wide pushout of a (small) family of members of $\mathcal{H}$ (that is, for any set of morphisms $h_i : A \to A_i$, $i \in I$, in $\mathcal{H}$, the coproduct in $A \downarrow \mathcal{A}$ of the objects h_i is a morphism of $\mathcal{H}$).

The concept of cellular morphism plays an important role in Homotopy Theory. There, a different but equivalent formulation is used:

3.2. Definition Let α be an ordinal.

(1) A chain $C : \alpha \to \mathcal{A}$ is called *smooth* if it preserves directed colimits (that is, for every limit ordinal $i \in \alpha$, we have $C_i = \operatorname{colim}_{j<i} C_j$).

(2) A morphism h is an α-*composite* of morphisms h_i ($i < \alpha$) provided that there exists a smooth chain $C : \alpha + 1 \to \mathcal{A}$ such that $h_i = C_{i,i+1}$ for every $i < \alpha$ and $h = C_{0,\alpha}$. The case $\alpha = 0$ expresses precisely that h is an identity morphism, while the case $\alpha = 2$ is the usual concept of composition.

(3) A *transfinite composite* is an α-composite for some ordinal α.

Given a class $\mathcal{H}$ of morphisms, it is easy to see that $\operatorname{cell}\mathcal{H}$ is the closure of $\bar{\mathcal{H}}$ under transfinite composition, where $\bar{\mathcal{H}}$ is the closure of $\mathcal{H}$ under pushouts.

3.3. Terminology Let $\mathcal{A}$ be a category with a proper factorization system $(\mathcal{E}, \mathcal{M})$. A class of morphisms in $\mathcal{A}$ is called *quasi-small* if all members but a set lie in $\mathcal{E}$.

3.4. Construction Given a quasi-small class $\mathcal{H}$ in a (cocomplete and $\mathcal{E}$-cowellpowered) category $\mathcal{A}$, we define a pointed endofunctor

$$\sigma : Id_{\mathcal{A}} \to S$$

as follows: given an object A, consider all spans (f, h), where $h \in \mathcal{H}$ and f has codomain A. For each such (f, h), form their pushout (h_f, f')

$$\begin{tikzcd} & P \arrow[dl, "f"'] \arrow[dr, "h"] & \\ A \arrow[dr, "h_f"'] & & Q \arrow[dl, "f'"] \\ & A_{f,h} & \end{tikzcd}$$

To define S and σ on the object A, form the wide pushout of all the h_f : $A \to A_{f,h}$ as above:

$$\begin{tikzcd} & A \arrow[dl, "h_f"'] \arrow[dd, "\sigma_A"] \\ A_{f,h} \arrow[dr, "\hat{h}_f"'] & \\ & SA \end{tikzcd}$$

This pushout exists because $\mathcal{A}$ is $\mathcal{E}$-cowellpowered and all but a set of morphisms h_f lie in $\mathcal{E}$. The definition of S on morphisms $u : A \to A'$ is obvious: Su is the unique morphism such that $Su \cdot \hat{h}_f = \hat{h}_{u \cdot f} \cdot \bar{u}$, with $\bar{u}$ the pushout of u along h_f.

3.5. Remark (i) Every object injective w.r.t. $\mathcal{H}$ is injective w.r.t. σ_A. This follows immediately from the definition of σ_A.

(ii) A is injective w.r.t. $\mathcal{H}$ iff σ_A is a split monomorphism.

3.6. Theorem *In every locally $\mathcal{E}$-bounded category, the Small Object Argument is valid for all quasi-small classes.*

Proof Let $\mathcal{H} = \mathcal{H}_o \cup \mathcal{H}_e$ be a class of morphisms with $\mathcal{H}_o$ small and $\mathcal{H}_e \subseteq \mathcal{E}$.

For every object A we present a weak reflection in $\mathrm{Inj}\, \mathcal{H}$.

(1) Let

$$X : \mathrm{Ord} \to \mathcal{A}$$

be the chain of objects X_i and morphisms x_{ij} $(i \leq j)$ defined by transfinite induction as follows:

$X_0 = A$	
$X_{i+1} = SX_i$ and $x_{i,i+1} = \sigma_{X_i}$	for all ordinals i
$X_i = \mathrm{colim}_{j<i} X_j$ with the colimit cocone x_{ji}	for all limit ordinals i.

Since we work in a cocomplete $\mathcal{E}$-cowellpowered $(\mathcal{E}, \mathcal{M})$-category, it follows from [17] 8.5 that there exist an embedding $\varphi : \mathrm{Ord} \to \mathrm{Ord}$ preserving joins, a chain $Y : \mathrm{Ord} \to \mathcal{A}$, and a natural transformation

$$\beta_i : Y_i \to X_{\varphi(i+1)} \qquad (i \in \mathrm{Ord}),$$

with the following properties (where we will use the notation $\hat{i} = \varphi(i+1)$):

(i) For every ordinal i, the composites

$$Y_i \xrightarrow{\beta_i} X_{\hat{i}} \xrightarrow{x_{\hat{i}j}} X_j$$

lie in $\mathcal{M}$ for all $j \geq \hat{i}$.

(ii) For every limit ordinal i, the cocone

$$Y_j \xrightarrow{\beta_j} X_{\hat{j}} \xrightarrow{x_{\hat{j}\varphi(i)}} X_{\varphi(i)} \quad (j < i)$$

is a colimit of the chain of all Y_j for $j < i$.

(2) Since $\mathcal{H}_o$ is small, there exists a regular cardinal k such that all domains and codomains of morphisms of $\mathcal{H}_o$ are bounded by k. We prove that if

$$e : X_{\varphi(k)} \to Z \quad \text{ is a morphism of } \mathcal{E}$$

such that

$$n_i \equiv (\, Y_i \xrightarrow{x_{\hat{i}\varphi(k)} \cdot \beta_i} X_{\varphi(k)} \xrightarrow{e} Z \,) \text{ lies in } \mathcal{M} \ \ (i < k), \tag{3.1}$$

then Z is injective w.r.t. $\mathcal{H}_o$.

From (ii) of (1), one concludes that the morphisms

$$Y_i \xrightarrow{\beta_i} X_{\hat{i}} \xrightarrow{x_{\hat{i}\varphi(k)}} X_{\varphi(k)}\ , \ \ i < k$$

are the colimit morphisms of a k-directed colimit whose connecting morphisms lie in $\mathcal{M}$. Since e belongs to $\mathcal{E}$, it follows that Z is the union of the $\mathcal{M}$-subobjects $n_i : Y_i \to Z$ $(i < k)$. Now, given $h : A \to B$ in $\mathcal{H}_o$ and a morphism $f : A \to Z$, since A is bounded by k there is some ordinal i and some morphism $\bar{f} : A \to Y_i$ such that

$$n_i \cdot \bar{f} = f.$$

From the definition of (S, σ) we see that there exists a commutative diagram of the form

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "\bar{f}"'] & B \arrow[dd, "g"] \\
Yi \arrow[d, "\beta_i"'] & \\
X_{\hat{\imath}} \arrow[r, "\sigma_{X_{\hat{\imath}}}"] & X_{\hat{\imath}+1} = SX_{\hat{\imath}}
\end{tikzcd}$$

Consequently,

$$(e \cdot x_{\hat{\imath}+1,\varphi(k)} \cdot g) \cdot h = e \cdot x_{\hat{\imath}+1,\varphi(k)} \cdot \sigma_{X_{\hat{\imath}}} \cdot \beta_i \cdot \bar{f} = e \cdot x_{\hat{\imath},\varphi(k)} \cdot \beta_i \cdot \bar{f} = n_i \cdot \bar{f} = f.$$

(3) By using $\mathcal{H}_e$ in place of $\mathcal{H}$, we obtain a new pointed endofunctor $(\bar{S}, \bar{\sigma})$ and the components of $\bar{\sigma}$ lie in $\mathcal{E}$. Let us define a chain $Z : \mathrm{Ord} \to \mathcal{A}$ analogous to the chain X in (1) by putting $Z_0 = X_{\varphi(k)}$, $Z_{i+1} = \bar{S}Z_i$ and $Z_i = \mathrm{colim}_{j<i} Z_j$ for limit ordinals i. Every morphism z_{ij} (for $i \leq j$) of this chain belongs to $\mathcal{E}$. Since $\mathcal{E}$ is cowellpowered, the chain converges, i.e., there is some $l \in \mathrm{Ord}$ such that z_{lj} is an isomorphism for every $j \geq l$. We are going to show that Z_l is $\mathcal{H}$-injective, which will finish the proof.

The object Z_l is clearly $\mathcal{H}_e$-injective because $\bar{\sigma}_{Z_l} = z_{l,l+1}$ is an isomorphism. In order to show that Z_l is $\mathcal{H}_o$-injective, we apply (2): it suffices to show that, for each $i < \varphi(k)$, the morphism

$$Y_i \xrightarrow{\quad x_{\hat{\imath},\varphi(k)} \cdot \beta_i \quad} X_{\varphi(k)} \xrightarrow{z_{0l}} Z_l$$

belongs to $\mathcal{M}$.

Observe that we have an obvious natural transformation α:

$$\begin{tikzcd}
& \mathrm{Id} \arrow[dl, "\bar{\sigma}"'] \arrow[dr, "\sigma"] & \\
\bar{S} \arrow[rr, "\alpha"] & & S
\end{tikzcd}$$

whose components are given by the unique factorization of a smaller wide pushout through a bigger one. This yields a natural transformation

$$\gamma_i : Z_i \to X_{\varphi(k)+i} \ \ (i \in \mathrm{Ord})$$

given by $\gamma_0 = \mathrm{id}_{Z_0}$, $\gamma_i = \mathrm{colim}_{j<i}\,\gamma_j$ for limit ordinals i, and

$$\gamma_{i+1} \equiv (\ \bar{S}Z_i \xrightarrow{\alpha_{Z_i}} SZ_i \xrightarrow{S\gamma_i} SX_{\varphi(k)+i} = X_{\varphi(k)+i+1}\ ).$$

This transformation satisfies $\gamma_i \cdot z_{0i} = x_{\varphi(k),\varphi(k)+i}$ for all i (as it is easily seen by induction). We obtain a commutative diagram

$$\begin{tikzcd}
Y_i \arrow[r, "\beta_i"] \arrow[ddrr, "x_{\hat{i},\varphi(k)+l}\cdot\beta_i"'] & X_{\hat{i}} \arrow[r, "x_{\hat{i}\varphi(k)}"] & X_{\varphi(k)} = Z_0 \arrow[d, "z_{0l}"] \\
& & Z_l \arrow[d, "\gamma_l"] \\
& & X_{\varphi(k)+l}
\end{tikzcd}$$

whose lower path lies in $\mathcal{M}$, thus, the morphism $z_{0l} \cdot x_{\hat{i}\varphi(k)} \cdot \beta_i$ lies in $\mathcal{M}$. Consequently, (2) implies

$$Z_l \in \mathrm{Inj}\mathcal{H}.$$

(4) An easy transfinite induction shows that every object injective w.r.t. $\mathcal{H}$ is injective w.r.t. the $\mathcal{H}$-cellular morphisms $x_{0\varphi(k)} : A \to X_{\varphi(k)}$ and $z_{0l} : X_{\varphi(k)} \to Z_l$, then injective w.r.t. the $\mathcal{H}$-cellular morphism

$$r_A \equiv (\ A \xrightarrow{x_{0\varphi(k)}} X_{\varphi(k)} \xrightarrow{z_{0l}} Z_l\ ).$$

Consequently, since Z_l belongs to $\mathrm{Inj}\,\mathcal{H}$, r_A is the desired weak reflection. $\square$

3.7. Remark Theorem 3.6 is completely analogous to the result of Peter Freyd and Max Kelly on the Orthogonal Subcategory Problem. There are, however, differences between that problem and the Small Object Argument. For example, if the set-theoretical Vopěnka's Principle is assumed, then for all (!) classes $\mathcal{H}$ of morphisms in locally presentable categories, the subcategory $\mathcal{H}^{\perp}$ is reflective, see [5]. In contrast, independently of set theory, there are situations where in an "everyday" locally finitely presentable category, the Small Object Argument fails:

3.8. Example of a class $\mathcal{H}$ of graph homomorphisms such that the Small Object Argument is not valid.

Here $\mathcal{A}$ is the (locally finitely presentable) category of graphs. Let $\mathcal{H}$ be the class of all

$$0 \to C_n \quad (n \in \mathrm{Card})$$

where 0 is the initial (empty) graph and C_n is a *clique* on n nodes (i.e., where two nodes form an edge iff they are distinct). It is rather obvious that

$$\operatorname{Inj}\mathcal{H} = \text{ the class of all graphs with a loop.}$$

Then, no nonempty graph without loops has a cellular weak reflection in $\operatorname{Inj}\mathcal{H}$. In fact, this is proved by an analogous argument to that of Example 6.1 in [2].

3.9. Definition A class $\mathcal{H}$ of morphisms is called

(i) *presentable* if there exists an infinite cardinal λ such that all members of $\mathcal{H}$ are λ-presentable morphisms (see 2.7);

(ii) *quasi-presentable* if

$$\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_e \text{ with } \mathcal{H}_0 \text{ presentable and } \mathcal{H}_e \subseteq \text{Epi}.$$

3.10. Remark A presentable class need not have, in any reasonable sense, a set of representatives: consider for example the classes $\mathcal{H}$ referred to in Example 2.8: the finitely presentable morphisms in **Set**, and the finitely presentable monomorphisms in R-modules. Note that, in the case of R-modules, $\operatorname{Inj}(\mathcal{H})$ turns out to be the class of all FP-injective (or absolutely pure) R-modules (see [14]).

3.11. Theorem *In every locally presentable category the Small Object Argument is valid for all quasi-presentable classes of morphisms.*

Proof Let $\mathcal{H}$ be a quasi-presentable class of morphisms in a locally presentable category $\mathcal{A}$. Then by Theorem 2.10, there exists a cardinal λ such that

(i) $\mathcal{A}$ is locally λ-presentable,

(ii) every epimorphism with domain X is a λ-filtered colimit of λ-presentable epimor-

phisms with domain X,

and

(iii) all members of $\mathcal{H}$ are λ-presentable or epimorphisms.

We use a modification of the Construction 3.4:

For every object A, consider all λ-presentable morphisms $p : A \to P$ which have a "prolongation" $A \xrightarrow{p} P \xrightarrow{q} Q$ such that $q \cdot p$ is $\mathcal{H}$-cellular. Let

$$\{p_t : A \to P_t \ \ ; \ \ t \in T_A\}$$

be a set of representatives of all λ-presentable morphisms prolongable to $\mathcal{H}$-cellular morphisms. For each t, choose one

$$q_t : P_t \to Q_t \text{ with } q_t \cdot p_t \in \operatorname{cell} \mathcal{H}.$$

We form the wide pushout

$$\begin{tikzcd} & A \arrow[dl, "q_t \cdot p_t"'] \arrow[dd, "\sigma_A"] \\ Q_t \arrow[dr, "s_t"'] & \\ & SA \end{tikzcd}$$

Since σ_A is a wide pushout of cellular morphisms, it is cellular. We will iterate λ times by defining a chain of objects A_i $(i < \lambda)$ and cellular morphisms

$$r_{ij} : A_i \to A_j \quad \text{ for } i \le j \le \lambda.$$

We will then prove that

$$r_{0,\lambda} : A \to A_\lambda$$

is a weak reflection of A in $\operatorname{Inj} \mathcal{H}$.

The chain is defined by transfinite induction:

First step: $A_0 = A$.

Iterated step: $A_{i+1} = SA_i$ and $r_{i,i+1} = \sigma_{A_i}$.

Limit step: $A_j = \operatorname{colim}_{i<j} A_i$ for all limit ordinals $j \le \lambda$ and $r_{i,j}(i < j)$ is the colimit cocone.

It is obvious that

$$r_{i,j} \in \operatorname{cell} \mathcal{H} \text{ for all } i, j,$$

and it remains to show that $A_\lambda \in \operatorname{Inj} \mathcal{H}$ and $r_{0,\lambda}$ is a weak reflection of A.

(a) Injectivity of A_λ w.r.t. $\mathcal{H}$:

Given a span $(f : X \to A_\lambda, h : X \to Y)$ with $h \in \mathcal{H}$, we prove that f factors through h. Form a pushout h' of h along f:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "f'"] \\ A_\lambda \arrow[r, "h'"'] & Q' \end{tikzcd}$$

We know that h is either λ-presentable or an epimorphism. In the latter case, express h as a λ-filtered colimit of λ-presentable epimorphisms

$$h_d : X \to Y_d \quad (d \in \mathrm{D})$$

with the colimit cocone

$$\begin{tikzcd} & Y_d \arrow[d, "y_d"] \\ X \arrow[ur, "h_d"] \arrow[r, "h"'] & Y \end{tikzcd}$$

in $X \downarrow \mathcal{A}$. In the former case, just put $\mathrm{D} = \{d\}$ and $h = h_d$.

By 2.9(iii), the λ-presentable morphism h_d is a pushout of some $h_d^* : X_d^* \to Y_d^*$, where X_d^* and Y_d^* are λ-presentable objects:

$$\begin{tikzcd}
 & X_d^* \arrow[r, "h_d^*"] \arrow[dl, "v_d"'] \arrow[dd, "u_d"] & Y_d^* \arrow[dr, "v'_d"] \arrow[dd, "u'_d"] & \\
A_i \arrow[rrr, "\hat{h}_d"] \arrow[d, "\sigma_{A_i}"'] & & & \hat{Q}_d \arrow[dddl, "w_d"] \\
SA_i = A_{i+1} \arrow[ddr, "r_{i+1,\lambda}"'] & X \arrow[r, "h_d"] \arrow[dr, "h"] \arrow[dd, "f"'] & Y_d \arrow[d, "y_d"] & \\
 & & Y \arrow[d, "f'"'] & \\
 & A_\lambda \arrow[r, "h'"'] & Q' &
\end{tikzcd}$$

Since X_d^* is λ-presentable, the morphism $f \cdot u_d : X_d^* \to A_\lambda = \operatorname{colim}_{i<\lambda} A_i$ factorizes through some of the colimit morphisms. That is, there exists an ordinal $i < \lambda$ (depending on d) and a morphism $v_d : X_d^* \to A_i$ such that

$$f \cdot u_d = r_{i,\lambda} \cdot v_d = r_{i+1,\lambda} \cdot \sigma_{A_i} \cdot v_d.$$

Next we form the pushout

$$\hat{h}_d : A_i \to \hat{Q}_d$$

of h_d^* along v_d and observe that, due to 2.9,

$$\hat{h}_d \text{ is a } \lambda\text{-presentable morphism.}$$

The obvious factorization morphism $w_d : \hat{Q}_d \to Q'$ is a prolongation of $\hat{h}_d$ to a cellular morphism:

$$w_d \cdot \hat{h}_d = h' \cdot r_{i,\lambda} \in \operatorname{cell} \mathcal{H}.$$

Consequently, in the above definition of SA_i we can assume that $\hat{h}_d$ is one of the morphisms p_t:

$$\hat{h}_d = p_t \text{ for some } t \in T_{A_i}.$$

This implies $\sigma_{A_i} = s_t \cdot q_t \cdot \hat{h}_d$, and by composing this equality with $r_{i+1,\lambda}$, we get

$$r_{i,\lambda} = r_{i+1,\lambda} \cdot s_t \cdot q_t \cdot \hat{h}_d = f_d \cdot \hat{h}_d,$$

where $f_d = r_{i+1,\lambda} \cdot s_t \cdot q_t$. Thus,

$$f \cdot u_d = r_{i,\lambda} \cdot v_d = f_d \cdot \hat{h}_d \cdot v_d = f_d \cdot v'_d \cdot h^*_d.$$

The universal property of the pushout h_d (of h^*_d) thus yields a unique factorization $z_d : Y_d \to A_\lambda$:

$$\begin{tikzcd}
X^*_d \arrow[r, "h^*_d"] \arrow[d, "u_d"'] & Y^*_d \arrow[d, "u'_d"'] \arrow[dr, "v'_d"] & \\
X \arrow[r, "h_d"] \arrow[dr, "f"'] & Y_d \arrow[d, dotted, "z_d"] & \hat{Q}_d \arrow[dl, "f_d"] \\
& A_\lambda &
\end{tikzcd}$$

These morphisms are compatible, that is, for every connecting morphism $y_{d,d'} : Y_d \to Y_{d'}$ $(d, d' \in \mathrm{D})$ of the diagram at the beginning, we have $z_d = z_{d'} \cdot y_{d,d'}$. This is true by default if $\mathrm{D} = \{d\}$. If h is an epimorphism, then h_d is one, and we have

$$z_d \cdot h_d = f = z_{d'} \cdot h_{d'} = (z_{d'} \cdot y_{d,d'}) \cdot h_d.$$

This yields a unique

$$y : Y \to A_\lambda \text{ with } y \cdot y_d = z_d \text{ for } d \in \mathrm{D}.$$

This is the desired factorization of f through h:

$$f = z_d \cdot h_d = y \cdot y_d \cdot h_d = y \cdot h.$$

(b) Every morphism $b : A \to B$ with $B \in \operatorname{Inj} \mathcal{H}$ factorizes through $r_{0,\lambda}$. In fact, we find a compatible cocone $b_i : A_i \to B$ of the diagram defining A_λ (with $b_o = b$), and then $b = b_\lambda \cdot r_{0,\lambda}$. The limit steps $b_j : \operatorname{colim}_{i<j} A_i \to B$ are clear from $A_j = \operatorname{colim}_{i<j} A_i$. For the isolated steps, just observe that since B is injective w.r.t. $\mathcal{H}$, it is injective w.r.t. σ_{A_i} for every i. □

4. Injectivity Logic

The logic of injectivity concerns the relation

$$\mathcal{H} \models k \qquad \text{(injectivity consequence)}$$

which states that every object injective w.r.t. $\mathcal{H}$ is also injective w.r.t. k. We will use the deduction system introduced in [8] as a slight modification of the deduction system of Grigore Roçu [21]. In our previous paper [1] we proved the completeness of the injectivity logic for all quasi-small classes of morphisms. However, the assumptions on the base category were far less pleasant than local boundedness. And also the proof was unnecessarily long. Here we are going to derive the completeness of the logic rather easily from the Small Object Argument.

We also present a new completeness result for locally presentable categories: the logic is complete for all quasi-presentable classes.

4.1. Definition The Injectivity Deduction System for morphisms s, t, ... of a cocomplete category consists of the three deduction rules:

PUSHOUT $\dfrac{s}{t}$ for every pushout $\begin{tikzcd} \cdot \arrow[r, "s"] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, "t"'] & \cdot \end{tikzcd}$

TRANSFINITE COMPOSITION $\dfrac{s_i\,(i < \alpha)}{t}$ if t is an α-composite of the s_i $(i < \alpha)$

CANCELLATION $\dfrac{s \cdot t}{t}$

Given a class $\mathcal{H}$ of morphisms we say that a morphism k is *provable* from it, notation

$$\mathcal{H} \vdash k$$

if there exists a formal proof of k using as assumptions elements of $\mathcal{H}$ and using the Injectivity Deduction System.

4.2. Remark The Injectivity Logic is infinitary, thus, proofs have in general an ordinal length. A proof of k from $\mathcal{H}$ of length β is a sequence of morphisms

$$k_i \ (i \leq \beta)$$

such that $k = k_\beta$ and for each $i \leq \beta$, either

$k_i \in \mathcal{H}$ (k_i is an assumption)

or

k_i is the conclusion of one of the three deduction rules whose assumption(s) lie(s) in $\{k_j\}_{j<i}$.

4.3. Corollary *In locally $\mathcal{E}$-bounded categories, the Injectivity Logic is sound and complete for quasi-small classes $\mathcal{H}$ of morphisms. That is, given a morphism k then*

$$\mathcal{H} \models k \qquad \textit{iff} \qquad \mathcal{H} \vdash k.$$

Proof Let $\mathcal{H} \models k : A \to B$, and let $r_A : A \to \bar{A}$ be an $\mathcal{H}$-cellular weak reflection in $\operatorname{Inj}\mathcal{H}$, which exists by 3.6. Then there exists $t : B \to \bar{A}$ such that $t \cdot k = r_A$. Being $\mathcal{H}$-cellular, r_A is provable from $\mathcal{H}$, hence via CANCELLATION, we derive k. □

It is clear that, arguing as in 4.3, we obtain, as a consequence of Theorem 3.11, the completeness of the Injectivity Logic for quasi-presentable classes in locally presentable categories. In 4.4 we give a different proof of this fact, showing that, moreover, there is a *functorial* weak reflection for Inj $\mathcal{H}$.

4.4. Theorem *For quasi-presentable classes $\mathcal{H}$ of morphisms in locally presentable categories:*

(i) Inj $\mathcal{H}$ has a functorial weak reflection;

(ii) the Injectivity Logic is sound and complete.

Proof (i) Let $\mathcal{A}$ be a locally λ-presentable category and $\mathcal{H}$ a class of λ-presentable morphisms in $\mathcal{A}$. For each $A \in \mathcal{A}$, we construct a weak reflection r_A in $\operatorname{Inj}\mathcal{H}$ such that $\mathcal{H} \vdash r_A$ as follows. We define $r_A = r_{0,\lambda}$ for the chain of morphisms $r_{i,j} : A_i \to A_j$, all provable from $\mathcal{H}$, defined by the following transfinite induction:

First step: $A_0 = A$;
Isolated step: $r_{i,i+1} : A_i \to A_{i+1}$ is the wide pushout of all λ-presentable morphisms with domain A_i provable from $\mathcal{H}$, and $r_{j,i+1} = r_{i,i+1} \cdot r_{j,i}$; it exists due to 2.7.
Limit step: Given a limit ordinal i, then $r_{j,i}$ is the colimit of the chain $(r_{j,k})_{k<i}$.

We put $r_A = r_{0,\lambda}$. It is clear that $\mathcal{H} \vdash r_A$, and it remains to show that A_λ belongs to $\operatorname{Inj}\mathcal{H}$.

Let $h : X \to Y$ belong to $\mathcal{H}$. Since h is λ-presentable, there exist a morphism $h^\star : X^\star \to Y^\star$ with λ-presentable domain and codomain such that h is the pushout of $h^\star$ along some morphism u (see 2.9(iii)), as illustrated by the left-hand square below. Given $f : X \to A_\lambda$, it is our task to prove that f factorizes through h. In fact let h' be the pushout of h along f as in the right-hand square:

$$\begin{tikzcd} X^\star \arrow[r, "u"] \arrow[d, "h^\star"'] & X \arrow[r, "f"] \arrow[d, "h"] & A_\lambda \arrow[d, "h'"] \\ Y^\star \arrow[r, "v"'] & Y \arrow[r, "f'"'] & Q \end{tikzcd}$$

Since $X^\star$ is λ-presentable, there exists a morphism $g : X^\star \to A_i$, for some $i < \lambda$, such that $r_{i,\lambda} \cdot g = f \cdot u$. Let p be the pushout of $h^\star$ along g, and consider the pushout h' of p along $r_{i,\lambda}$; since $r_{i,\lambda} \cdot g = f \cdot u$, we conclude that h' is also the pushout of $h^\star$ along $f \cdot u$:

$$\begin{tikzcd} X^\star \arrow[r, "g"] \arrow[d, "h^\star"'] & A_i \arrow[r, "r_{i,\lambda}"] \arrow[d, "p"] & A_\lambda \arrow[d, "h'"] \\ Y^\star \arrow[r, "g'"'] & P \arrow[r, "r'"'] & Q \end{tikzcd}$$

Taking into account that $r_{i,\lambda}$ and h' are both provable from $\mathcal{H}$, we have that $\mathcal{H} \vdash h' \cdot r_{i,\lambda} = r' \cdot p$. Being the pushout of $r_{i,\lambda}$, the morphism r' is also provable from $\mathcal{H}$, therefore, by applying CANCELLATION, we get $\mathcal{H} \vdash p$. Furthermore, the morphism p is λ-presentable, because it is the puhout of $h^\star$. Hence p is one of the branches forming the wide pushout $r_{i,i+1} : A_i \to A_{i+1}$. Consequently, there is a factorization $r_{i,i+1} = s \cdot p$. This yields

$$1_{A_\lambda} \cdot (f \cdot u) = r_{i,\lambda} \cdot g = r_{i+1,\lambda} \cdot r_{i,i+1} \cdot g = r_{i+1,\lambda} \cdot s \cdot p \cdot g = (r_{i+1,\lambda} \cdot s \cdot g') \cdot h^\star.$$

By the universality of the pushout Q, there exists a morphism $t : Q \to A_\lambda$ such that $t \cdot h' = 1_{A_\lambda}$. Thus, we get the desired factorization: $f = t \cdot h' \cdot f = (t \cdot f') \cdot h$.

Now, given $\mathcal{H} \models k$, we see that r_A factorizes through k, then $\mathcal{H} \vdash r_A$ implies $\mathcal{H} \vdash k$ by CANCELLATION.

Finally, the functoriality of r_A is obvious from its construction.

(ii) Let $\mathcal{H}$ be a quasi-presentable class, and choose a regular cardinal λ such that (a) $\mathcal{A}$ is locally λ-presentable , (b) $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_e$ with $\mathcal{H}_0$ λ-presentable and $\mathcal{H}_e \subseteq \mathit{Epi}$ and (c) every epimorphism of $\mathcal{A}$ is expressible as a λ-filtered colimit of λ-presentable epimorphisms, see Theorem 2.6. For each $f \in \mathcal{H}_e$ select a family of λ-presentable epimorphisms of which f is a colimit. Let $\hat{\mathcal{H}}_e$ be the union of all of these families. Then it is easy to see that $\mathrm{Inj}\mathcal{H}_e = \mathrm{Inj}\hat{\mathcal{H}}_e$. Consequently the result follows from (i) applied to the λ-presentable class $\mathcal{H}_o \cup \hat{\mathcal{H}}_e$. □

4.5. Example of a class of morphisms in a locally presentable category for which the Injectivity Logic is incomplete. Recall the example of cliques in graphs (3.8). Since every graph in $\mathrm{Inj}\,\mathcal{H}$ has a loop we have

$$\mathcal{H} \models k \qquad \text{for the unique } k : 0 \to 1.$$

We showed in [1] that, however, k cannot be proved from $\mathcal{H}$.

5. The Orthogonal Subcategory Problem and its Logic

The Orthogonal Subcategory Problem asks whether, given a collection $\mathcal{H}$ of morphisms, the orthogonality class $\mathcal{H}^\perp$ is reflective, where $\mathcal{H}^\perp$ is the full subcategory of all objects orthogonal to every member of $\mathcal{H}$. From Max Kelly, we know that the answer is affirmative for all quasi-small classes $\mathcal{H}$:

5.1. Theorem (see [15], 10.1 and 10.2) *In every locally $\mathcal{E}$-bounded category the Orthogonal Subcategory Problem has an affirmative answer for all quasi-small classes of morphisms.*

5.2. Remark (i) In locally presentable categories, under the set-theoretic Weak Vopěnka's Principle, the Orthogonal Subcategory Problem has an affirmative answer for every class $\mathcal{H}$ (see Section 7 for more on that). We now prove that even without any large-cardinal set-theoretic assumption, the Orthogonal Subcategory Problem is affirmative if $\mathcal{H}$ is quasi-presentable.

(ii) In the locally bounded category **Top** there exist classes of morphisms for which the Orthogonal Subcategory Problem has a negative answer. In fact, by [6] there are full reflective subcategories $\mathcal{A}$, $\mathcal{B}$ of **Top** such that $\mathcal{A} \cap \mathcal{B}$ is not reflective. Let then $\mathcal{H}_a$ be the class of all reflection morphisms in $\mathcal{A}$, analogously for $\mathcal{H}_b$. Then $(\mathcal{H}_a \cup \mathcal{H}_b)^{\perp}$ is not reflective.

5.3. Theorem *In a locally presentable category, $\mathcal{H}^{\perp}$ is reflective for every quasi-presentable class $\mathcal{H}$.*

Proof (1) If $\mathcal{H}$ is presentable, then $\mathcal{H}^{\perp}$ is reflective by Proposition 3.16 of [2].

(2) Let $\mathcal{H}$ be a quasi-presentable class, and choose a regular cardinal λ such that (i) $\mathcal{A}$ is locally λ-presentable , (ii) $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_e$ with $\mathcal{H}_0$ formed by λ-presentable morphisms and $\mathcal{H}_e \subseteq Epi$, and (iii) every epimorphism of $\mathcal{A}$ is expressible as a λ-filtered colimit of λ-presentable epimorphisms, see Theorem 2.10. For each $f \in \mathcal{H}_e$ select a family of λ-presentable epimorphisms of which f is a λ-filtered colimit. Let $\hat{\mathcal{H}}_e$ be the union of all of these families. Then it is easy to see that $\mathcal{H}_e^{\perp} = \hat{\mathcal{H}}_e^{\perp}$. Consequently the result follows from (1) applied to the presentable class $\mathcal{H}_0 \cup \hat{\mathcal{H}}_e$. □

5.4. Remark In [2] we formulated an Orthogonality Logic which studies *orthogonality consequences* of a class $\mathcal{H}$ of morphisms. This means those morphisms k such that every object orthogonal to $\mathcal{H}$ is also orthogonal to k. We improve the results of [2] by proving that the Orthogonality Logic is complete for

(i) all quasi-small classes in locally bounded categories

and

(ii) all quasi-presentable classes in locally presentable categories.

Throughout this section, $\models$ denotes the orthogonality consequence and $\vdash$ denotes provability in the following deduction system, presented in [2]:

5.5. Orthogonality Deduction System consists of four deduction rules:

PUSHOUT	$\dfrac{s}{t}$	for every pushout $\begin{array}{ccc} \cdot & \xrightarrow{s} & \cdot \\ \downarrow & & \downarrow \\ \cdot & \xrightarrow[t]{} & \cdot \end{array}$
COEQUALIZER	$\dfrac{s}{t}$	if $\overset{f}{\underset{g}{\rightrightarrows}} \xrightarrow{t}$ is a coequalizer and $f \cdot s = g \cdot s$
TRANSFINITE COMPOSITION	$\dfrac{s_i\,(i < \alpha)}{t}$	if t is an α-composite of the s_i's (see 3.2)
WEAK CANCELLATION	$\dfrac{u \cdot t \quad v \cdot u}{t}$	

5.6. Remark The Orthogonality Logic is sound in every category with colimits (for all classes $\mathcal{H}$). That is,

$$\mathcal{H} \vdash k \qquad \text{implies} \qquad \mathcal{H} \models k$$

see [2].

5.7. Theorem *In a locally bounded category the Orthogonality Logic is complete for all quasi-small classes.*

Proof (1) For every morphism $k : M \to N$, recall from [15] the pointed endofunctor $\sigma : Id \to S$ of $\mathcal{A}$ defined on objects X by using the following pushout

$$\begin{array}{ccc} \coprod_{f:M\to X} M + \coprod_{g:N\to X} N & \xrightarrow{[\bar{k},\bar{l}]} & \coprod_{f:M\to X} N \\ {\scriptstyle [\eta_M,\eta_N]}\downarrow & & \downarrow \\ X & \xrightarrow[\sigma_X]{} & SX \end{array}$$

Here η_M and $\bar{k}$ have their f-component equal to f and to $k \cdot i_f$, respectively, where i_f is the injection indexed by f, and η_N and $\bar{l}$ have their g-component g and the injection $i_{g \cdot k}$, respectively. We will prove that σ_X is an orthogonality consequence of k:

$$k \vdash \sigma_X$$

This is done by describing the above pushout as a certain wide pushout and observing that wide pushouts are transfinite composites of (ordinary) pushouts. For every morphism $f : M \to X$ form the pushout

$$\begin{tikzcd} M \arrow[r, "k"] \arrow[d, "f"'] & N \arrow[d, "\hat{f}"] \\ X \arrow[r, "s_f"'] & S_f \end{tikzcd}$$

We have of course $k \vdash s_f$. Given a morphism $g : N \to X$ with $f = g \cdot k$, form a coequalizer

$$N \underset{s_f \cdot g}{\overset{\hat{f}}{\rightrightarrows}} S_f \xrightarrow{c_{f,g}} C_{f,g} \ .$$

Since

$$\hat{f} \cdot k = s_f \cdot g \cdot k$$

we have, due to COEQUALIZER,

$$k \vdash c_{f,g}$$

Thus, if $c_f = a_g \cdot c_{f,g} : S_f \to C_f$ is the wide pushout of all $c_{f,g}$, where g ranges over all the morphisms with $f = g \cdot k$, then

$$k \vdash c_f,$$

and, consequently,

$$k \vdash c_f \cdot s_f.$$

Let us form a wide pushout of these morphisms

$$\begin{tikzcd} X \arrow[r, "\sigma_X"] \arrow[d, "c_f \cdot s_f"'] & SX \\ C_f \arrow[ur, "t_f"'] & \end{tikzcd}$$

Then, once again,

$$k \vdash \sigma_X.$$

We claim that the square

$$\begin{tikzcd} \coprod_{f:M\to X} M + \coprod_{g:N\to X} N \arrow[r, "{[\bar{k},\bar{l}]}"] \arrow[d, "{[\eta_M,\eta_N]}"'] & \coprod_{f:M\to X} N \arrow[d, "{[t_f\cdot c_f\cdot \hat{f}]}"] \\ X \arrow[r, "\sigma_X"'] & SX \end{tikzcd}$$

is a pushout:

(a) The square above commutes. In fact, the f-component commutes since

$$\begin{aligned} \sigma_X \cdot f &= t_f \cdot c_f \cdot s_f \cdot f \\ &= t_f \cdot c_f \cdot \hat{f} \cdot k. \end{aligned}$$

The g-component commutes because for $f = g \cdot k$ we have that

$$\begin{aligned} \sigma_X \cdot g &= t_f \cdot c_f \cdot s_f \\ &= t_f \cdot a_g \cdot c_{f,g} \cdot s_f \cdot g \\ &= t_f \cdot a_g \cdot c_{f,g} \cdot \hat{f} \\ &= t_f \cdot c_f \cdot \hat{f}. \end{aligned}$$

(b) Suppose that the square

$$\begin{tikzcd} \coprod_{f:M\to X} M + \coprod_{g:N\to X} N \arrow[r, "{[\bar{k},\bar{l}]}"] \arrow[d, "{[\eta_M,\eta_N]}"'] & \coprod_{f:M\to X} N \arrow[d, "{[\tilde{f}]}"] \\ X \arrow[r, "p"'] & P \end{tikzcd}$$

commutes. That is, to every $f : M \to X$ a morphism $\tilde{f} : N \to P$ is assigned with

$$p \cdot f = \tilde{f} \cdot k$$

and

$$p \cdot g = \tilde{f} = \widetilde{gk} \qquad \text{for } f = g \cdot k.$$

For each f we have a unique $u_f : S_f \to P$ with

$$u_f \cdot s_f = p \qquad \text{and} \qquad u_f \cdot \hat{f} = \tilde{f}.$$

Given g with $f = g \cdot k$, then

$$u_f \cdot \hat{f} = u_f \cdot s_f \cdot g$$

because the left-hand side is $\tilde{f}$, and the right-hand one $p \cdot g = \widetilde{g \cdot k} = \tilde{f}$.

Thus we get a unique $b_g : C_{f,g} \to P$ with $u_f = b_g \cdot c_{f,g}$ and this yields a unique

$$v_f : C_f \to P \quad \text{ with } \ v_f \cdot a_g = b_g.$$

Since

$$\begin{aligned} v_f \cdot c_f \cdot s_f &= v_f \cdot a_g \cdot c_{f,g} \cdot s_f \\ &= b_g \cdot c_{f,g} \cdot s_f \\ &= u_f \cdot s_f \\ &= p \end{aligned}$$

we obtain a unique $v : SX \to P$ with

$$v_f = v \cdot t_f \qquad \text{and} \qquad p = v \cdot \sigma_X.$$

This morphism has then the desired properties:

$$p = v \cdot \sigma_X \quad \text{ and } \quad \tilde{f} = v \cdot (t_f \cdot c_f \cdot \hat{f}) \quad \text{ for all } f : M \to X;$$

the latter equality follows because

$$\tilde{f} = u_f \cdot \hat{f} = b_g \cdot c_{f,g} \cdot \hat{f} = v_f \cdot a_g \cdot c_{f,g} \cdot \hat{f} = v_f \cdot c_f \cdot \hat{f} = v \cdot t_f \cdot c_f \cdot \hat{f}.$$

Conversely, these two properties determine v uniquely.

(2) From the proof of Theorem 10.2 in [15] it follows that for every object A of $\mathcal{A}$ the reflection $r_A : A \to \bar{A}$ exists "constructively", where the construction uses transfinite composition and wide pushouts of morphisms of the form σ_X for various members $k \in \mathcal{H}$ and various objects X of $\mathcal{A}$. Thus, (1) implies that $\mathcal{H} \vdash r_A$.

(3) We are ready to prove the completeness of the Orthogonality Logic for $\mathcal{H}$. Suppose that $k : A \to B$ is an orthogonality consequence of $\mathcal{H}$, and let $r_A : A \to \bar{A}$ be a reflection in $\mathcal{H}^{\perp}$. Then $\bar{A}$ is orthogonal to r_A (since $\mathcal{H} \models r_A$) which yields $u : B \to A$ with $r_A = u \cdot k$. Due to (2), we have

$$\mathcal{H} \vdash u \cdot k.$$

Since $\mathcal{H} \models u \cdot k$ and $\mathcal{H} \models k$ we easily deduce $\mathcal{H} \models u$. Thus arguing as above with u in the role of k, we conclude that there exists v with

$$\mathcal{H} \vdash v \cdot u.$$

Then we obtain $\mathcal{H} \vdash k$ by WEAK CANCELLATION. □

5.8. Example For general classes of morphisms in locally bounded categories the Orthogonality Logic is not complete in general. In fact, let $\mathcal{K}$ be the category of all triples (A, α, a) where (A, α) is a topological space and $a \subseteq A \times A$ is a binary relation (a directed graph). Morphisms are all continuous graph homomorphisms. It is easy to see, arguing as in **Top**, that $\mathcal{K}$ is locally bounded. For the unique morphism

$$v : 0 \to 1$$

from the initial (empty) object 0 to the terminal (singleton loop) object 1 we present a class $\mathcal{H}$ such that

$$\mathcal{H} \models v \qquad \text{but} \qquad \mathcal{H} \not\vdash v.$$

We use the result of Václav Koubek [16] that **Top** is almost universal. In particular there exists a proper class of topological spaces (A_i, α_i), $i \in I$, such that there are no continuous maps between them except the constant maps and the identity self-maps. For each i let a_i be the clique on A_i:

$$a_i = A_i \times A_i - \Delta_{A_i}.$$

This defines a morphism

$$h_i : 0 \to (A_i, \alpha_i, a_i)$$

in $\mathcal{K}$ and we put

$$\mathcal{H} = \{h_i\}_{i \in I}.$$

(1) We prove $\mathcal{H} \models v$, that is, whenever an object (B, β, b) is orthogonal to $\mathcal{H}$, then the graph contains a unique loop. In fact, choose i such that A_i has cardinality bigger than B. Since $h_i \perp (B, \beta, b)$, we have a unique morphism $f : (A_i, \alpha_i, a_i) \to (B, \beta, b)$. There exists $x \neq x'$ in A_i with $f(x) = f(x')$. Since $x a_i x'$, we conclude that $f(x) = f(x')$ is a loop. The loop is unique because every loop defines a constant morphism $(A_i, \alpha_i, a_i) \to (B, \beta, b)$.

(2) $\mathcal{H} \not\vdash v$. Assuming the contrary we derive a contradiction: it is sufficient to find a category $\bar{\mathcal{K}}$ in which $\mathcal{K}$ is a full subcategory closed under colimits and such that

$$\mathcal{H} \not\models v \qquad \text{in } \bar{\mathcal{K}}.$$

In fact, then a proof of v from $\mathcal{H}$ in $\mathcal{K}$ is also a proof of v from $\mathcal{H}$ in $\bar{\mathcal{K}}$, in contradiction to Remark 5.6.

We extend $\mathcal{K}$ by a single new object $\bar{K}$: the intention is that $\bar{K}$ is the coproduct of all (A_i, α_i, a_i). Therefore, the morphisms of $\bar{\mathcal{K}}$ are

(a) all morphisms of $\mathcal{K}$;

(b) endomorphisms of $\bar{K}$ are precisely all families $(f_i)_{i \in I}$ where $f_i : (A_i, \alpha_i, a_i) \to (A_{i'}, \alpha_{i'}, a_{i'})$ is a morphism of $\mathcal{K}$ for some $i' \in I$ (recall that, then, f_i is constant, or $i = i'$ and $f_i = \mathrm{id}$);

(c) analogously, morphisms from $\bar{K}$ to $(B, \beta, b) \in \mathcal{K}$ are precisely all families $(f_i)_{i \in I}$ of morphisms $f_i : (A_i, \alpha_i, a_i) \to (B, \beta, b)$ in $\mathcal{K}$
and
(d) morphisms from (B, β, b) in $\mathcal{K}$ to $\bar{K}$ are precisely the morphisms

$$f : (B, \beta, b) \to \coprod_{i \in J} (A_i, \alpha_i, a_i)$$

where $J \subseteq I$ is a set; to make these well defined we assume that J is the least such set, i.e., every $j \in J$ fulfils

$$A_j \cap f[B] \neq \emptyset.$$

The composition in $\bar{\mathcal{K}}$ is obvious: it is easy to see that $\mathcal{K}$ is a full subcategory of $\bar{\mathcal{K}}$ closed under colimits. We have

$$\bar{K} \in \mathcal{H}^{\perp}, \text{ but } \bar{K} \notin \{v\}^{\perp}.$$

In fact $\bar{K}$ is not orthogonal to v because $\hom(1, \bar{K}) = \emptyset$. And $\bar{K}$ is orthogonal to h_i because the unique morphism from (A_i, α_i, a_i) is that obtained from id_{A_i} in $\mathcal{K}$. Indeed, let $f : (A_i, \alpha_i, a_i) \to \coprod_{j \in J} (A_j, \alpha_j, a_j)$ be a morphism of type (d). Since a_i is a clique on A_i, it easily follows that there exists $j \in J$ with $f[A_i] \subseteq A_j$. This implies $J = \{j\}$ by our minimality requirement above. Since (A_i, a_i) is a clique and (A_j, a_j) has no loops, f must be one-to-one. This implies $i = j$ and $f = \mathrm{id}_{A_i}$ by our choice at the beginning.

5.9. Theorem *In locally presentable categories the Orthogonality Logic is complete for all quasi-presentable classes of morphisms.*

Proof In [2] we proved the completeness for presentable classes. To generalize this to quasi-presentable classes, we use the idea (and notation) of the proof of Theorem 5.3, part (2). Given

$$\mathcal{H} \models k,$$

since $\mathcal{H}_e^{\perp} = \hat{\mathcal{H}}_e^{\perp}$, we have that $\mathcal{H}_0 \cup \hat{\mathcal{H}}_e \models k$. Hence, by the completeness for presentable classes, $\mathcal{H}_0 \cup \hat{\mathcal{H}}_e \vdash k$. To conclude that $\mathcal{H} \vdash k$, it suffices to show that $\mathcal{H}_e \vdash g$ for each member g of $\hat{\mathcal{H}}_e$. Now $g \in \hat{\mathcal{H}}_e$ implies there exists a colimit cocone $c_h : (\, A \xrightarrow{h} C_h \,) \longrightarrow (\, A \xrightarrow{f} B \,)$ in $A \downarrow \mathcal{A}$ with $f \in \mathcal{H}_e$ such that g is one of the morphisms h. Consequently, $f = c_g \cdot g$. Since g is an epimorphism, from $\mathcal{H} \vdash f$ we deduce $\mathcal{H} \vdash g$, see [2], 3.6(vii). $\square$

6. A Counter-example

A generalization of the Small Object Argument was presented in [4] for sets of morphisms in categories "almost like" locally bounded ones: they are called *locally ranked* and the definition differs from 2.2 above only in modifying (ii) as follows:

for every object A there is a regular cardinal λ (called *rank*) such that $\hom(A, -)$ preserves λ–directed colimits of monomorphisms in $\mathcal{M}$.

We are going to demonstrate that, in the spite of the formal similarity, Theorems 3.6 and 5.1 fail in general for locally ranked categories. We are using unary algebras defined on *cpo*'s: the corresponding category **UCPO** will be proved to have the form Inj $\mathcal{H}$ for a quasi-small class $\mathcal{H}$ in a locally ranked category, and yet, **UCPO** is not weakly reflective there.

In the following definition we assume the standing hypotheses formulated in Section 2:

6.1. Definition ([4]). We call $\mathcal{A}$ *locally* $\mathcal{E}$*-ranked* provided that every object in $\mathcal{A}$ has a rank. *Locally ranked* means locally Epi-ranked.

6.2. Example We give an example of a locally ranked category

$$\mathbf{UCPO}^*$$

and its quasi-small class $\mathcal{H}$ of morphisms with

$$\mathbf{UCPO} = \mathcal{H}^\perp$$

such that **UCPO** is not reflective in $\mathbf{UCPO}^*$.

(a) We recall the well-known category **CPO** whose objects, the (strict) *cpo*'s, are posets with a least element $\perp$ and with joins of directed subsets. Morphisms are functions which are

(i) *continuous*, i.e., preserve directed joins,

and

(ii) *strict*, i.e., preserve $\perp$.

We denote by

$$\mathbf{UCPO}$$

the category of unary algebras on *cpo*'s: objects are pairs (A, α) where A is a *cpo* and α is an order-preserving unary operation on A. Morphisms are the continuous and strict algebra homomorphisms.

(b) Our next category is the category

$$\mathbf{CPO}^*$$

of *partial cpo's*. These are pairs (A, Z_A) consisting of a poset A with a least element $\perp$, and a set $Z_A \subseteq \exp A$ of directed subsets which have a join in A. Every *cpo* is considered as a partial *cpo* with $Z_A =$ all directed subsets of A.

Morphisms $f : A \to B$ of $\mathbf{CPO}^*$ are the strict, order-preserving functions which are *continuous* in the expected sense: if $M \in Z_A$, then $f[M] \in Z_B$ and f preserves the join of M.

(c) Our main category is the category

$$\mathbf{UCPO}^*$$

of all pairs (A, α) where A is a partial *cpo* and α is a binary relation on its underlying set. Every unary *cpo*-algebra is considered as an object with α equal to the graph of the unary operation.

Morphisms $f : (A, \alpha) \to (B, \beta)$ are the strict, continuous functions $f : A \to B$ which are graph homomorphisms, that is, from $a\alpha a'$ it follows that $f(a)\beta f(a')$.

6.3. Notation (1) For every partial *cpo*, A, we denote by

$$\varrho_A : A \to \bar{A}$$

its reflection in $\mathbf{CPO}$. That is:

(i) $\bar{A}$ is a *cpo*,

(ii) A is a subposet closed under Z_A-joins, i.e., the embedding $\varrho : A \hookrightarrow \bar{A}$ is strict and continuous,

and

(iii) every strict continuous function $f : A \to B$ where B is a *cpo* has a unique strict continuous extension to $\bar{f} : \bar{A} \to B$.

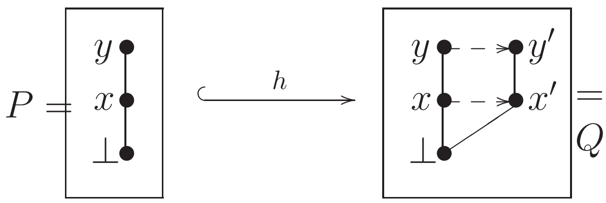
This poset $\bar{A}$ always exists, see [18]. In the articles [18] and [19] Ana Pasztor works with a subset system Z on the category of posets, but all her results remain true if one works with one poset A and one collection Z_A.

(2) Let $\mathcal{H}_e$ be the class of all morphisms

$$\varrho_A : (A, \emptyset) \to (\bar{A}, \emptyset)$$

where A is a partial *cpo* considered as an object of $\mathbf{UCPO}^*$ via the empty relation.

(3) Let $h : P \to Q$ denote the following morphism in $\mathbf{UCPO}^*$ carried by the inclusion map



where the edges $|$ and $\dashrightarrow$ correspond to the ordering $\leq$ and the relation α, respectively.

6.4. Lemma *We have* $\mathbf{UCPO} = \mathcal{H}^\perp$ *for* $\mathcal{H} = \mathcal{H}_e \cup \{h\}$ *and every member of* $\mathcal{H}_e$ *is an epimorphism in* $\mathbf{UCPO}^*$*. However* $\mathbf{UCPO}$ *is not reflective in* $\mathbf{UCPO}^*$*.*

Remark In fact, we will see that $\mathbf{UCPO}$ is not weakly reflective in $\mathbf{UCPO}^*$.

Proof An object (A, α) is orthogonal to $\mathcal{H}_e$ iff A is a *cpo*: this follows from the fact that $\mathcal{H}_e$ is the class of reflections of all objects of $\mathbf{CPO}^*$ in the subcategory $\mathbf{CPO}$. Let A be a *cpo* and let

$$(A, \alpha) \perp h.$$

For every $a \in A$ the morphism $P \to A$ with $x \mapsto a$ and $y \mapsto a$ factors uniquely through h, thus, there exists a unique a' with $a\alpha a'$. This tells us that α is (the graph of) a unary operation. And α is obviously order preserving: for $a \leq b$ in A consider the morphism $P \to A$ with $x \mapsto a$, $y \mapsto b$. Conversely, whenever α is an order preserving unary operation, then clearly $(A, \alpha) \perp h$. This proves

$$\mathcal{H}^\perp = \mathbf{UCPO}.$$

To see that this is not a reflective subcategory of the category $\mathbf{UCPO}^*$, observe that the latter category has a (one-element) initial object, and the category $\mathbf{UCPO}$ does not have any weakly initial object. In fact, suppose (A, α) is initial in $\mathbf{UCPO}$ and define a chain of elements $a_i \in A$ for all ordinals i by transfinite induction as follows:

$$a_0 = \perp,$$
$$a_{i+1} = \alpha a_i,$$

and for limit ordinals j

$$a_j = \bigvee_{i<j} a_i.$$

It follows by easy transfinite induction that $i \le i'$ implies $a_i \le a_{i'}$. Thus, the last join exists. We prove that $i < j$ implies $a_i \ne a_j$ – a contradiction. In fact, let (B, β) be the object of **UCPO** where B is the *cpo* of all ordinals $k \le j$ and β is defined by $\beta(k) = k+1$ for $k < j$ and $\beta(j) = j$. The unique morphism $f : (A, \alpha) \to (B, \beta)$ fulfils $f(a_k) = k$ for all $k \le j$, as easily proved by transfinite induction on k. Since $f(a_i) < f(a_j)$, we conclude $a_i \ne a_j$.

Finally, each $\varrho_A \in \mathcal{H}_e$ is an epimorphism because given morphisms $u, v : (\bar{A}, \emptyset) \to (B, \beta)$ with $u \cdot \varrho_A = v \cdot \varrho_A$, the morphisms $\varrho_B \cdot u$, $\varrho_B \cdot v : \bar{A} \to \bar{B}$ are continuous, therefore $(\varrho_B \cdot u) \cdot \varrho_A = (\varrho_B \cdot v) \cdot \varrho_A$ implies $\varrho_B \cdot u = \varrho_B \cdot v$ by the universal property of ϱ_A. Since ϱ_B is a monomorphism, we conclude $u = v$. $\square$

6.5. Theorem *The category* $\mathbf{UCPO}^*$ *is locally ranked, and* $\mathcal{H} = \mathcal{H}_e \cup \{h\}$ *is a class of morphisms whose members are all but one epimorphisms and such that* $\mathcal{H}^\perp$ *is not reflective.*

Proof In view of the above lemma, it remains to show that $\mathbf{UCPO}^*$ is locally ranked.

(1) Proof of the cocompleteness of $\mathbf{CPO}^*$: coproducts in $\mathbf{CPO}^*$ are the obvious disjoint unions with all bottom elements merged to one. To describe coequalizers, let $u, v : A \to B$ be strict and continuous functions and let $\bar{u}, \bar{v} : \bar{A} \to \bar{B}$ be their unique strict and continuous extensions, see 6.3. Form a coequalizer, $\bar{c} : \bar{B} \to \bar{C}$, of $\bar{u}$ and $\bar{v}$ in the (cocomplete) category **CPO**:

$$\begin{tikzcd}
\bar{A} \arrow[r, shift left, "\bar{u}"] \arrow[r, shift right, "\bar{v}"'] & \bar{B} \arrow[r, "\bar{c}"] & \bar{C} \\
A \arrow[u, "\varrho_A"] \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[u, "\varrho_B"'] \arrow[r, "c"'] & C \arrow[u, "i"']
\end{tikzcd}$$

Denote by $i : C \hookrightarrow \bar{C}$ the subposet of $\bar{C}$ on the image of $\bar{c} \cdot \varrho_B$, considered as a partial *cpo* via

$$Z_C = \{c[M];\ M \in Z_B\}.$$

Then $c : B \to C$ is a morphism of $\mathbf{CPO}^*$: it is strict and order-preserving (being equal to $\bar{c} \cdot \varrho_B$). Also, the definition of Z_C makes c obviously continuous (since $\bar{c}$ is). We claim that c is a coequalizer of u and v.

Given a morphism $d : B \to D$ of $\mathbf{CPO}^*$ with $d \cdot u = d \cdot v$, the corresponding morphism $\bar{d} : \bar{B} \to \bar{D}$ fulfils $\bar{d} \cdot \bar{u} = \bar{d} \cdot \bar{v}$:

$$\begin{tikzcd}
\bar{A} \arrow[r, shift left, bend left, "\bar{u}"] \arrow[r, shift right, bend right, "\bar{v}"'] & \bar{B} \arrow[rr, "\bar{c}"] \arrow[dr, "\bar{d}"'] & & \bar{C} \arrow[dl, "\bar{h}"] \\
& & \bar{D} & \\
& & D \arrow[u, "\varrho_D"'] & \\
A \arrow[uuu, "\varrho_A"] \arrow[r, shift left, bend left, "u"] \arrow[r, shift right, bend right, "v"'] & B \arrow[uuu, "\varrho_B"] \arrow[rr, "c"'] \arrow[ur, "d"] & & C \arrow[uuu, "i"'] \arrow[ul, dashed, "h"']
\end{tikzcd}$$

Therefore, there is a strict, continuous $\bar{h} : \bar{C} \to \bar{D}$ with $\bar{h} \cdot \bar{c} = \bar{d}$. Since c is surjective and r_D monic, there exists a diagonal fill-in morphism h in **Set**, and we only need to show that h is strict (which is obvious from the strictness of d) and continuous. In fact, for every member of Z_C of the form $c[M]$, $M \in Z_B$, we know that $d[M] \in Z_D$ and $d(\vee M) = \vee d[M]$. From $d = h \cdot c$ we conclude

$$(h \cdot c)[M] \in Z_D$$

and

$$\vee (h \cdot c)[M] = \vee d[M] = d(\vee M) = h \cdot c(\vee M).$$

Finally, $\bar{c} \cdot \varrho_B$ is strict and continuous, therefore $\bar{c} \cdot \varrho_B$ preserves the join of M. Consequently, $\vee M$ lies in the image of B under $\bar{c} \cdot \varrho_B$, which is C. We conclude that c, the codomain restriction of $\bar{c} \cdot \varrho_B$, also preserves the join of M, finishing the proof of

$$h \cdot c(\vee M) = h(\vee c[M]).$$

(2) $\mathbf{UCPO}^*$ is cocomplete. In fact, the forgetful functor to $\mathbf{CPO}^*$ makes $\mathbf{UCPO}^*$ topological over $\mathbf{CPO}^*$, therefore cocomplete, see [3], 21.1 and 21.16.

(3) $\mathbf{UCPO}^*$ is cowellpowered. It is sufficient to prove that $\mathbf{CPO}^*$ is cowellpowered, see [3], 21.16.

We will use the description of epimorphisms of $\mathbf{CPO}^*$ presented by Ana Pasztor [19]. Let

$$f : A \to B \qquad \text{be a morphism in } \mathbf{CPO}^*$$

and for every set $Y \subseteq B$ let

$$cl(Y)$$

be the least subset W of B containing Y such that $\vee M \in W$ for every set $M \in Z_B$ contained in Y. The morphism f is an epimorphism iff

for every $b \in B$ there exists an ordinal i with $b \xrightarrow{i} b$,

where $\xrightarrow{i}$ are relations on the underlying set of B defined by transfinite induction as follows:

(i) Intial case: $b \xrightarrow{0} b'$ iff $b \geq f(a) \geq b'$ for some $a \in A$.

(ii) Successor ordinals: $b \xrightarrow{i+1} b'$ iff $b' \in cl(Y)$ where $Y = \{y \in B;\ b \xrightarrow{i} y\ \}$

(iii) Limit ordinals j: $\xrightarrow{j}$ is the union of all $\xrightarrow{i}$ with $i < j$.

Let $\hat{A}$ be a free completion of A under joins and let $\hat{f} : \hat{A} \to \hat{B}$ be the extension of f preserving joins. We prove that every element of B lies in the image of $\hat{f}$ by transfinite induction on i with $b \xrightarrow{i} b$. This follows from the following more general fact:

$$b \xrightarrow{i} b' \qquad \text{implies} \qquad b \geq \hat{f}(\hat{a}) \geq b' \qquad \text{for some} \qquad \hat{a} \in \hat{A}.$$

Case $i = 0$ is obvious.

Case $i + 1$: Given $Y \subseteq B$ with $b \xrightarrow{i} y$ for $y \in Y$, there exist $\hat{a}_y \in \hat{A}$ with $b \geq \hat{f}(\hat{a}_y) \geq y$, by induction hypothesis. Let $\hat{a} = \bigvee_{y \in Y} \hat{a}_y$. We have $b \geq \hat{f}(\hat{a})$, since $\hat{f}$ preserves joins, and we prove $\hat{f}(\hat{a}) \geq b'$. In fact, the set C of all $c \in B$ with $\hat{f}(\hat{a}) \geq c$ contains Y and is closed under joins, therefore, it contains $cl(Y)$. Thus, we just use $b' \in cl(Y)$.

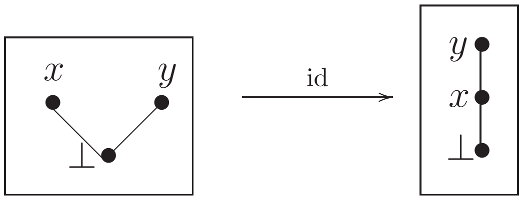
The limit case is obvious.

Since $\mathrm{card}\hat{A} \leq 2^{\mathrm{card}A}$, we have just established that for every epimorphism $f : A \to B$ we have $\mathrm{card}B \leq 2^{\mathrm{card}A}$. This proves the cowellpoweredness of $\mathbf{CPO}^*$.

(4) Strong subobjects of $\mathbf{UCPO}^*$: Suppose that

$$m : (C, \gamma) \to (D, \delta)$$

is a strong monomorphism. The diagonal fill-in property w.r.t. the epimorphism



tells us that for every $x, y \in C$ we have

$$x \leq y \text{ in } C \quad \text{iff} \quad m(x) \leq m(y) \text{ in } D.$$

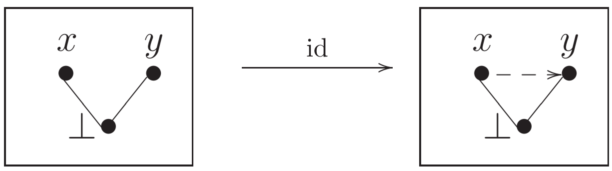
Consequently, we can assume without loss of generality that C is a subposet of D and m is the inclusion map. Moreover, C is closed under the chosen joins in the following sense: put

$$Z_C = Z_D \cap (\exp C)$$

then the join of a set $M \in Z_C$ is the same in C and D. In fact, this follows from the diagonal fill-in property applied to the epimorphism

$$M \cup \{\perp\} \hookrightarrow M \cup \{\perp, m\}$$

where the domain, A, is the subposet of D with $Z_A = \emptyset$, and the codomain, B, is the subposet obtained from A by adding $m = \vee M$ and putting $Z_B = \{M\}$. Finally, the relation γ is simply $\delta \cap (C \times C)$: use the epimorphism



(5) Colimits of directed diagrams D of strong subobjects are unions. We can suppose that D has objects (D_i, δ_i) where i ranges through a λ-directed poset I (with λ an infinite cardinal), and that for $i \leq j$ the connecting map $(D_i, \delta_i) \to (D_j, \delta_j)$ is an embedding of a subposet D_i of D_j closed under Z_{D_j}-joins and with the induced set Z_{D_i}, see (4) above. Let

$$C = \cup_{i \in I} D_i$$

be the poset with the ordering and Z_C induced by the D_i's:

$$x \leq y \text{ in } C \qquad \text{iff} \qquad x \leq y \text{ in } D_i \text{ for some } i$$

and

$$Z_C = \bigcup_{i \in I} Z_{D_i}.$$

For every $M \in Z_C$ we have a join m of M in some D_i, and the same m is the join of M in D_j for all $j \geq i$ – therefore $m = \vee M$ in C. We now define the binary relation γ on C as that induced by the given binary relations: $\gamma = \cup_{i \in I} \delta_i$. It is easy to verify that the embbeddings

$$(D_i, \delta_i) \hookrightarrow (C, \gamma)$$

form a colimit in $\mathbf{UCPO}^*$.

(6) Every object (A, α) of cardinality

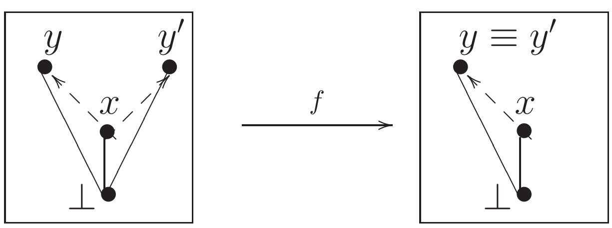
$$\mathrm{card} A < \lambda \qquad (\lambda \text{ an infinite cardinal})$$

has rank λ. This follows immediately from (5): given a morphism $f : (A, \alpha) \to (C, \gamma)$, the image of f lies in D_i for some $i \in I$. Then the fact that the subposet D_i is closed in C as well as in each D_j, $j \geq i$, under the directed sets of Z_{D_i}, their joins, and the relation (as proved in (4)) makes it clear that

$$f' : (A, \alpha) \to (D_i, \delta_i)$$

is a morphism. Therefore, $\hom(A, -)$ preserves the colimit as described in (5).

6.6. Remark We now want to have an analogous example for injectivity: a quasi-small class $\mathcal{H}$ with $\mathrm{Inj}\, \mathcal{H} = \mathbf{UCPO}$. In fact, all we need is to add to the above class $\mathcal{H}_e$ an epimorphism f such that injectivity w.r.t. f is equivalent to the statement that the binary relation is (the graph of a) unary operation. Here is such an epimorphism:



6.7. Corollary *In the locally ranked category* $\mathbf{UCPO}^*$ *the class* $\mathcal{H} = \mathcal{H}_e \cup \{f, h\}$ *has all members except* h *epimorphisms; however* $\mathbf{UCPO} =$ *Inj*$\mathcal{H}$ *is not weakly reflective.*

In fact, it follows by using the argument in the proof of Lemma 6.4.

7. Summary and Open Problems

An excellent solution of the orthogonal subcategory problem was presented by Max Kelly and Peter Freyd [9], [15]: for every quasi-small class $\mathcal{H}$ of morphisms in a locally bounded category the orthogonal subcategory of $\mathcal{H}$ is reflective. Based on the constructive approach to reflections due to Max Kelly we have proved that, under the same assumptions, our Orthogonality Logic [2] is complete. It appears that local boundedness is a more suitable concept than that of a locally ranked category, introduced in [4]: we have presented an example of a quasi-small class $\mathcal{H}$ in the locally ranked category $\mathbf{UCPO}^*$ whose orthogonal subcategory is not reflective.

In locally presentable categories of P. Gabriel and F. Ulmer a stronger result holds: whenever $\mathcal{H}$ is a quasi-presentable class, then its orthogonal subcategory is reflective and the Orthogonality Logic is complete. Moreover, under the set-theoretical Vopěnka's Principle no restrictions on the class $\mathcal{H}$ of morphisms are needed: (i) the orthogonal subcategory is always reflective, and (ii) the Orthogonality Logic is always complete. Actually, the statement (i) is equivalent to Weak Vopěnka's Principle, see 6.22-23 in [5], and the statement (ii) is equivalent to Vopěnka's Principle, see 4.6 in [2].

Open Problem Under which conditions on the base category does the reflectivity of $\mathcal{H}^\perp$ always imply the completeness of the Orthogonality Logic for $\mathcal{H}$? Under which conditions does the converse implication always hold?

The main result of our paper is that Quillen's Small Object Argument can be generalized to precisely the same situations studied by Freyd and Kelly: in every locally bounded category, given a quasi-small class $\mathcal{H}$ of morphisms, then cellular weak $\mathcal{H}$-injective reflections exist, and the Injectivity Logic is complete. Again, we have proved a stronger result for locally presentable categories: here quasi-presentability of the class of morphisms is sufficient.

As mentioned above, in [4] the authors gave, for every set $\mathcal{H}$ of morphisms in a locally ranked category, a construction for a cellular weak reflection in the subcategory Inj $\mathcal{H}$. But they gave as well a construction for a functorial (they called it "natural") weak reflection, and stated as an open problem whether there exists a weak reflection which would be simultaneously cellular and functorial. It appears that we have a similar situation here: given a quasi-presentable class $\mathcal{H}$ of morphisms in a locally presentable category, Theorem 3.11 gives a construction for a cellular weak reflection into Inj $\mathcal{H}$, and Theorem 4.4 a functorial one. However, we do not know if there exists a cellular and functorial weak reflection.

The situations for injectivity and for orthogonality are no longer symmetric when arbitrary classes of morphisms are considered. We now have that: (i) Vopěnka's Principle implies that the injectivity subcategory is always weakly reflective, which itself implies Weak Vopěnka's Principle (see 6.26-27 in [5]); however, (ii) the Injectivity Logic is not always complete (independently of set-theory): Examples 3.8 and 4.5 describe a (weakly reflective) subcategory of the locally finitely presentable category of graphs of the form Inj $\mathcal{H}$ for which the Injectivity Logic is not complete. Thus here we are left with the following:

Open Problem Under which conditions on the base category does the completeness of the Injectivity Logic always imply the weak reflectivity of the injective subcategory?

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