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POINTFREE FORMS OF DOWKER AND MICHAEL INSERTION THEOREMS

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Abstract: In this paper we prove two strict insertion theorems for frame homomorphisms. When applied to the frame of all open subsets of a topological space they are equivalent to the insertion statements of the classical theorems of Dowker and Michael regarding, respectively, normal countably paracompact spaces and perfectly normal spaces. In addition, a study of perfect normality for frames is made.

Keywords: Frame, sublocale, insertion theorem, extension theorem, Katětov relation, continuous real function on a frame, lower semicontinuous, upper semicontinuous, normal frame, perfectly normal frame, countably paracompact frame.
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1. Introduction

Theorems on the existence of continuous real functions on normal spaces rank among the fundamental results in point-set topology. They can, for instance, be divided into three groups: separation theorems (such as Urysohn's lemma), extension theorems (such as Tietze's theorem), and insertion theorems. The latter theorems are the strongest ones in the sense that they yield the former as very easy corollaries. It is therefore of importance to consider them in the more general setting of pointfree topology. This paper is a sequel to the authors' earlier papers regarding pointfree insertion (see [25], [14], [15] and [16]). For the reader's convenience we first record the three basic insertion theorems of Katětov-Tong [19, 29], Dowker [5] and Michael [24].

Theorem A. (Katětov-Tong) *A topological space X is normal if and only if, given $h, g : X \to \mathbb{R}$ such that $h \le g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h \le f \le g$.*

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Theorem B. (Dowker) *A topological space X is normal and countably paracompact if and only if, given $h, g : X \to \mathbb{R}$ such that $h < g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h < f < g$.*

Theorem C. (Michael) *A topological space X is perfectly normal if and only if, given $h, g : X \to \mathbb{R}$ such that $h \leq g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h \leq f \leq g$ and $h(x) < f(x) < g(x)$ whenever $h(x) < g(x)$.*

In pointfree setting, Theorem A has first been investigated by Li and Wang [23] with, however, some discrepancy between topological and frame semicontinuities. Right frame semicontinuities and right pointfree version of Theorem A have been fixed by Picado [25] and Gutiérrez García and Picado [14].

In this paper, we aim to provide some forms of Theorems B and C for, respectively, normal countably paracompact spaces and perfectly normal spaces. In the pointfree setting the situation becomes much more complex than in the topological case and we have not been able to provide pointfree assertions corresponding exactly to insertion statements of Theorems B and C. For instance, in both cases we assume $h = \mathbf{0}$. It should however be emphasized that both Theorems B and C easily follow from their pointfree versions established in this paper. These versions are corollaries of a rather general insertion lemma related to an arbitrary frame L with a certain extra order $\Subset$ which in turn is an abstract version of a result of Gutiérrez García and Kubiak [13] concerning a normal topology $\mathcal{O}X$ with $U \Subset V$ iff $\mathrm{int}(X \backslash U) \cup V = X$. We also establish some natural results regarding perfectly normal frames. These include separation and extension theorems for perfectly normal spaces. We have not been able to deduce them from our pointfree Michael theorem. These are deduced from our general insertion lemma.

2. Background in frames

I. Frames and locales. The category Frm of *frames* has as objects those complete lattices L in which

$$a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$$

for all $a \in L$ and $B \subseteq L$. Morphisms, called *frame homomorphisms*, are those maps between frames that preserve arbitrary joins (hence 1, the top) and finite meets (hence 0, the bottom). The set of all morphisms from L into M is denoted by $\mathsf{Frm}(L, M)$. The category of locales is the opposite category of Frm.

Motivating example: the lattice $\mathcal{O}X$ of all open subsets of a space X is a frame and if $f : X \to Y$ is a map, then $\mathcal{O}f : \mathcal{O}Y \to \mathcal{O}X$ defined by $\mathcal{O}f(U) = f^{-1}(U)$ is a frame homomorphism.

II. Heyting operator. With L a frame and $a \in L$, the map $a \wedge (\cdot) : L \to L$ preserves arbitrary joins and so has a right adjoint $a \to (\cdot) : L \to L$ determined by $c \leq a \to b$ iff $a \wedge c \leq b$. Thus, $a \to b = \bigvee\{c \in L : a \wedge c \leq b\}$. For all $a, b, c \in L$ and $B \subseteq L$ the following hold:

(H1) $a \to b = a \to (a \wedge b)$,
(H2) $a \wedge b = a \wedge c$ iff $a \to b = a \to c$,
(H3) $a \to \bigwedge B = \bigwedge_{b\in B}(a \to b)$.

The *pseudocomplement* of $a \in L$ is $a^* = a \to 0$. Clearly, $a \wedge a^* = 0$.

III. Sublocales. An $S \subseteq L$ is a *sublocale* of L if, given $A \subseteq S$ and $a \in L$, one has $\bigwedge A \in S$ and $a \to s \in S$ for all $s \in S$ (see [18, p. 50] and [26]). Each sublocale $S \subseteq L$ is a frame itself with $\wedge$ and $\to$ of L (the top of S is 1, while the bottom 0_S of S may differ from 0). It determines the surjection (frame quotient) $c_S : L \to S$ given by $c_S(x) = \bigwedge\{s \in S : x \leq s\}$. The sublocales of L form a complete lattice $(\mathcal{S}(L), \subseteq)$ with $\{1\}$ being the bottom $\mathsf{0}$, L being the top $\mathsf{1}$, and in which, given $\{S_j : j \in J\} \subseteq \mathcal{S}(L)$, one has

$$\bigwedge_{j\in J} S_j = \bigcap_{j\in J} S_j \quad \text{and} \quad \bigvee_{j\in J} S_j = \{\bigwedge A : A \subseteq \bigcup_{j\in J} S_j\}.$$

Then $\mathcal{S}(L)$ is a co-frame.

For any $a \in L$, the sets

$$\mathfrak{o}(a) = \{a \to b : b \in L\} \quad \text{and} \quad \mathfrak{c}(a) = {\uparrow}a$$

are sublocales of L called, respectively, *open* and *closed*. Clearly, the quotients $c_{\mathfrak{o}(a)}$ and $c_{\mathfrak{c}(a)}$ are given by

$$c_{\mathfrak{o}(a)}(x) = a \to x \quad \text{and} \quad c_{\mathfrak{c}(a)}(x) = a \vee x.$$

Properties 2.1. *We shall freely use the following properties:*

(1) $\mathfrak{o}(a) \subseteq \mathfrak{o}(b)$ *iff* $a \leq b$,

(2) $\mathfrak{o}(\bigvee A) = \bigvee_{a\in A} \mathfrak{o}(a)$,
(3) $\mathfrak{c}(a) \subseteq \mathfrak{o}(b)$ *iff* $a \vee b = 1$ *iff* $\mathfrak{c}(b) \subseteq \mathfrak{o}(a)$,
(4) $\mathfrak{o}(a)$ *and* $\mathfrak{c}(a)$ *are complemented to each other.*

IV. The frame of reals. A $G \subseteq L$ *generates* L if each element of L is a join of finite meets of G. Being algebraic, the category Frm allows definitions by generators and relations. Using this, one can constructively define the frame of reals in terms of $\mathbb{Q}$ [9]. Following the more recent detailed description in [2], the *frame of reals* $\mathfrak{L}(\mathbb{R})$ is one generated by $G \subseteq \mathbb{Q} \times \mathbb{Q}$ satisfying the following relations:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,
(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \le r < q \le s$,
(R3) $(p, q) = \bigvee\{(r, s) : p < r < s < q\}$,
(R4) $1 = \bigvee_{p,q\in\mathbb{Q}}(p, q)$.

We write: $(p, -) = \bigvee_{q>p}(p, q)$ and $(-, q) = \bigvee_{p<q}(p, q)$.

A morphism having $\mathfrak{L}(\mathbb{R})$ as a domain will be defined on the sets of their generators. Such a map uniquely determines a frame homomorphism if and only if it makes the relations holding for generators into identities (see [2] for details).

V. The lattice-ordered ring $\mathrm{C}(L)$. Members of

$$\mathrm{C}(L) = \mathsf{Frm}(\mathfrak{L}(\mathbb{R}), L)$$

are called *continuous real functions* [2] on L. The lattice-ordered ring structure of $\mathbb{Q}$ goes over to $\mathrm{C}(L)$ [7]. The following material comes from Banaschewski [2, Sect. 4].

Let $\langle p, q\rangle = \{r \in \mathbb{Q} : p < r < q\}$, let $\diamond \in \{+, \cdot, \max, \min\}$, and let $\langle r, s\rangle \diamond \langle t, u\rangle = \{x \diamond y : x \in \langle r, s\rangle$ and $y \in \langle t, u\rangle\}$. Given f_1, f_2, $f \in \mathrm{C}(L)$ and $r \in \mathbb{Q}$, we define

$$\begin{aligned}
(f_1 \diamond f_2)(p, q) &= \bigvee_{\langle r,s\rangle\diamond\langle t,u\rangle\subseteq\langle p,q\rangle} (f_1(r, s) \wedge f_2(t, u)),\\
(-f)(p, q) &= f(-q, -p),\\
\mathbf{r}(p, q) &= \begin{cases} 1 & \text{if } r \in \langle p, q\rangle,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

These operations satisfy all the lattice-ordered ring axioms in $\mathbb{Q}$ so that $(\mathrm{C}(L), +, \cdot, \leq)$ becomes a lattice-ordered ring with unit $\mathbf{1}$ where $f_1 \leq f_2$ iff $\max(f_1, f_2) = f_2$. It is well known that for all $p, q \in \mathbb{Q}$:

$$f_1 \leq f_2 \;\Leftrightarrow\; f_1(p, -) \leq f_2(p, -) \;\Leftrightarrow\; f_2(-, q) \leq f_1(-, q).$$

Let X be a topological space and let $\mathrm{C}(X)$ be the ring of all continuous real-valued functions on X. Then there is an isomorphism $\mathrm{C}(\mathcal{O}X) \to \mathrm{C}(X)$ determined by taking an f to $\widetilde{f}$ such that $p < \widetilde{f}(x) < q$ iff $x \in f(p, q)$ (cf. [2, p. 38]). This is the machinery which will convert our pointfree assertions into the topological ones when $L = \mathcal{O}X$.

Remark 2.2. For all $a \in L$ and $\diamond \in \{+, \max, \min\}$ one obviously has:

$$c_{\mathfrak{c}(a)} \circ (f_1 \diamond f_2) = (c_{\mathfrak{c}(a)} \circ f_1) \diamond (c_{\mathfrak{c}(a)} \circ f_2).$$

We shall only use products of the form $\mathbf{r} \cdot f$ denoted just by rf. Also, we do not distinguish in notation between the constants $\mathbf{r}$ having different range frames. In particular, $c_{\mathfrak{c}(a)} \circ (rf) = r(c_{\mathfrak{c}(a)} \circ f)$ and $c_{\mathfrak{c}(a)} \circ \mathbf{r} = \mathbf{r}$.

We may also use the *real unit interval frame* (cf. [2]):

$$\mathfrak{L}([0,1]) = \uparrow((-, 0) \vee (1, -)).$$

There is of course an obvious bijection

$$\begin{aligned} \mathsf{Frm}(\mathfrak{L}([0,1]), L) &\simeq \{f \in \mathrm{C}(L) : \mathbf{0} \leq f \leq \mathbf{1}\} \\ &= \{f \in \mathrm{C}(L) : f(-, 0) \vee f(1, -) = 0\}. \end{aligned}$$

VI. Generating continuous real functions on frames. A *scale* (*descending trail* in [2]) in L is a map $\varphi : \mathbb{Q} \to L$ such that $\varphi(r) \vee \varphi^*(s) = 1$ whenever $r < s$, and $\bigvee \varphi(\mathbb{Q}) = 1 = \bigvee \varphi^*(\mathbb{Q})$ where $\varphi^* = (\cdot)^* \circ \varphi$. In what follows we write φ_r rather than $\varphi(r)$.

Lemma 2.3. [2, Lemma 2] *Each scale φ in L generates a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ defined by*

$$f(p, q) = \bigvee \{\varphi_r \wedge \varphi_s^* : p < r < s < q\}.$$

Lemma 2.4. *If φ is a scale that generates a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$, then:*

(1) *$f(p, -) = \bigvee_{r>p} \varphi_r$ for all $p \in \mathbb{Q}$,*
(2) *$f(-, q) = \bigvee_{s<q} \varphi_s^*$ for all $q \in \mathbb{Q}$.*

Proof: To show (1), we calculate

$$\begin{aligned} f(p,-) &= \bigvee_{q>p} \bigvee_{p<r<s<q} \varphi_r \wedge \varphi_s^* \\ &= \bigvee_{q>p} \Big(\bigvee_{p<r} \varphi_r \wedge \bigvee_{s<q} \varphi_s^* \Big) \\ &= \bigvee_{p<r} \varphi_r \wedge \bigvee_{q>p} \bigvee_{s<q} \varphi_s^* \\ &= \bigvee_{p<r} \varphi_r \wedge \bigvee_{s\in\mathbb{Q}} \varphi_s^* \\ &= \bigvee_{p<r} \varphi_r, \end{aligned}$$

and similarly for (2). ∎

VII. The upper frame of reals and lower semicontinuity. Let $\mathfrak{L}_u(\mathbb{R})$ be the subframe of $\mathfrak{L}(\mathbb{R})$ generated by $\{(r,-) : r \in \mathbb{Q}\}$. A *lower semicontinuous* real function on L (see [14]) is a morphism $g : \mathfrak{L}_u(\mathbb{R}) \to L$ such that $\bigwedge_{r\in\mathbb{Q}} \mathfrak{o}(g(r,-)) = \mathsf{0}$. The collection of all lower semicontinuous real functions on L is denoted by $\mathrm{LSC}(L)$ and is partially ordered by:

$$g_1 \leq g_2 \quad \Leftrightarrow \quad g_1(r,-) \leq g_2(r,-) \text{ for all } r \in \mathbb{Q}.$$

Notation. Given $f \in \mathrm{C}(L)$ and $g \in \mathrm{LSC}(L)$, we write

$$f \leq g$$

if and only if

$$f_{|\mathfrak{L}_u(\mathbb{R})} \leq g.$$

We shall be concerned with members of $g \in \mathbf{Frm}(\mathfrak{L}_u(\mathbb{R}), L)$ which satisfy

$$g(1,-) = 0 \quad \text{and} \quad g(r,-) = 1 \quad \text{for all } r < 0,$$

i.e. $\mathbf{0} \leq f \leq \mathbf{1}$.

Remark. If $g : \mathfrak{L}_u(\mathbb{R}) \to L$ satisfies the above boundary conditions, then it is lower semicontinuous, since $\bigwedge_{r\in\mathbb{Q}} \mathfrak{o}(g(r,-)) \leq \mathfrak{o}(g(1,-)) = \mathfrak{o}(0) = \mathsf{0}$.

Following [25], for each $a \in L$ we define the *characteristic function* $l_a : \mathfrak{L}_u(\mathbb{R}) \to L$ by

$$l_a(p,-) = \begin{cases} 1 & \text{if } p < 0, \\ a & \text{if } 0 \leq p < 1, \\ 0 & \text{if } p \geq 1. \end{cases}$$

Clearly, $\mathbf{0} \leq l_a \leq \mathbf{1}$ is lower semicontinuous.

Other concepts will be defined when actually needed. For more information on frames and locales we refer to [18] and [27].

3. An insertion lemma for frames with an extra order

Sometimes, a complete lattice carries an extra order which is stronger than the lattice order. That extra order may have various names (modulo some conditions): proximity relation [11], strong inclusion [1], multiplicative auxiliary order [10], etc. Conditions (K0)–(K4) which follow are equivalent to the relation ρ investigated by Katětov [19] (cf. [17], [20] and [28]).

Definition 3.1. A binary relation $\Subset$ on a frame L is called a *Katětov relation* [21] if it satisfies the following conditions:

(K0) $a \Subset b$ implies $a \leq b$,
(K1) $a \leq b \Subset c \leq d$ implies $a \Subset d$,
(K2) $a, b \Subset c$ implies $a \vee b \Subset c$,
(K3) $a \Subset b, c$ implies $a \Subset b \wedge c$,
(K4) $a \Subset b$ implies $a \Subset c \Subset b$ for some $c \in L$. (Interpolation property)

We shall say that the Katětov relation $\Subset$ is *strong* [22] if

(K5) $a \Subset b$ implies $a \prec b$, where

$$a \prec b \quad \text{iff} \quad a^* \vee b = 1,$$

and is called the *well-inside order*.

Notation. Given $A \subseteq L$ and $b \in L$ we write $A \prec b$ whenever $a \prec b$ for all $a \in A$.

Recall that a frame L is called *normal* if, given $a, b \in L$ with $a \vee b = 1$, there exist $u, v \in L$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$. Equivalently, L is normal iff, whenever $a \vee b = 1$, there is an $u \in L$ such that $a \vee u = 1$ and $u \prec b$.

Examples 3.2. Among the frames with a strong Katětov relation are the following ones:

1. Each normal frame L with the well-inside order. It is normality of L which guarantees the interpolation property (K4) (see [27]).
2. In a frame L, a is *really inside* b, written $a \prec\!\!\prec b$, if there exists a family $\{\varphi_r : r \in \mathbb{Q} \cap [0,1]\}$ such that $a \leq \varphi_r \leq b$ and $\varphi_r \prec \varphi_s$ if $r < s$. If L is *completely regular*, i.e. $b = \bigvee_{a \prec\!\!\prec b} a$ for all $b \in L$, then $\prec\!\!\prec$ is a strong Katětov relation (see [18]).
3. Each continuous frame with a multiplicative *way below* relation (see [10]).

In [13], there is an insertion lemma holding for normal topologies $L = \mathcal{O}X$ with $U \Subset V$ iff $\overline{U} \subseteq V$. It continues to hold for arbitrary frames with a strong Katětov relation:

Lemma 3.3. (Insertion Lemma) *Let L be a frame endowed with a strong Katětov relation $\Subset$ and let $g : \mathfrak{L}_u(\mathbb{R}) \to L$ be lower semicontinuous with $\mathbf{0} \leq g \leq \mathbf{1}$. Let (a_n) be a non-decreasing sequence in L such that:*

(1) $a_n \Subset g(\frac{1}{n}, -)$ *for each* $n \in \mathbb{N}$,
(2) $\bigvee_n a_n = g(0, -)$.

Then there exists a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} \leq f \leq g$ and $g(0,-) = f(0,-)$.

Remark. We could write $g(0,-) \leq f(0,-)$, since the reverse inequality holds on account of $f \leq g$.

Proof: We need a family $\{\varphi_q : q \in (0,1) \cap \mathbb{Q}\}$ such that

$$\begin{cases} \varphi_q \Subset \varphi_r, & \text{if } r < q, \\ a_n \Subset \varphi_q \Subset g(\frac{1}{n}, -), & \text{if } q \in [\frac{1}{n+1}, \frac{1}{n}). \end{cases}$$

First, make $\mathbb{Q} \cap (0,1)$ into a sequence (q_n). The union $\bigcup_{m \in \mathbb{N}} [\frac{1}{m+1}, \frac{1}{m}) = (0,1)$, being disjoint, uniquely determines a sequence (μ_n) in $\mathbb{N}$ such that

$$\mu_n = m \quad \text{iff} \quad q_n \in [\tfrac{1}{m+1}, \tfrac{1}{m}).$$

One has: $q_l < q_n$ implies $\mu_l \geq \mu_n$. After these preparations, for each $n \geq 2$ we shall inductively construct a family $\Phi_n = \{\varphi_{q_i} : i < n\} \subseteq L$ such that:

$$\begin{cases} \varphi_{q_i} \Subset \varphi_{q_j}, & \text{if } q_j < q_i \text{ and } i, j < n, \\ a_{\mu_i} \Subset \varphi_{q_i} \Subset g(\frac{1}{\mu_i}, -), & \text{if } i < n. \end{cases} \tag{I_n}$$

The existence of φ_{q_1} satisfying (I_2) follows from the interpolation property of $\Subset$. Suppose now that Φ_n has already been defined and satisfies (I_n). Distinguish three cases:

Case 1. If $q_l = \max\{q_i : i < n\} < q_n$, then $\mu_l \geq \mu_n$ and

$$a_{\mu_n} = u \Subset w = \varphi_{q_l} \wedge g(\tfrac{1}{\mu_n}, -).$$

Case 2. If $q_n < q_r = \min\{q_i : i < n\}$, then $\mu_r \leq \mu_n$ and

$$a_{\mu_n} \vee \varphi_{q_r} = u \Subset w = g(\tfrac{1}{\mu_n}, -).$$

Case 3. If $q_l < q_n < q_r$, then $\mu_l \geq \mu_n \geq \mu_r$ and

$$a_{\mu_n} \vee \varphi_{q_r} = u \Subset w = \varphi_{q_l} \wedge g(\tfrac{1}{\mu_n}, -).$$

In all cases, select φ_{q_n} with $u \Subset \varphi_{q_n} \Subset w$. Then Φ_{n+1} satisfies (I_{n+1}).

Now, let $\varphi_q = 0$ for all $q \geq 1$ and $\varphi_q = 1$ for all $q \leq 0$. Then $\varphi : \mathbb{Q} \to L$ with $\varphi(q) = \varphi_q$ is a scale and, thus, determines a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ by

$$f(p,q) = \bigvee_{p<r<s<q} \varphi_r \wedge \varphi_s^*.$$

Actually, $\mathbf{0} \leq f \leq \mathbf{1}$ as

$$f(1,-) = \bigvee_{1<r} \varphi_r = 0 = \bigvee_{s<0} \varphi_s^* = f(-,0).$$

Also, we have $f_{|\mathfrak{L}_u(\mathbb{R})} \leq h$. Indeed, $f(p,-) = 1 = g(p,-)$ if $p < 0$, while $f(p,-) = 0 = g(p,-)$ if $p \geq 1$. If $0 \leq p < q < 1$, then $\varphi_q \leq g(\frac{1}{m}, -) \leq g(q,-)$ where $q \in [\frac{1}{m+1}, \frac{1}{m})$. So,

$$f(p,-) = \bigvee_{p<q} \varphi_q \leq \bigvee_{p<q} g(q,-) \leq g(p,-).$$

Finally, since $a_{s_n} \leq \varphi_{q_n}$ whenever $q_n \in [\frac{1}{s_n+1}, \frac{1}{s_n})$, we get

$$f(0,-) = \bigvee_{0<q} \varphi_q \geq \bigvee_n a_n = g(0,-). \qquad \blacksquare$$

This lemma will have many important consequences. To state our first corollary we recall that $a \in L$ is a *cozero element* [2] if there is an $f \in \mathrm{C}(L)$ such that $a = f((-,0) \vee (0,-))$. The set of all cozero elements of L will be denoted by $\operatorname{Coz} L$.

Corollary 3.4. [3, Prop. 1] *Let L be a frame and $a \in L$. If there exists a non-decreasing sequence (a_n) such that $a_n \prec\!\!\prec a$ and $\bigvee_n a_n = a$, then $a \in \operatorname{Coz} L$.*

Recall that a sublocale is G_δ, respectively, F_σ (or is a *G_δ-sublocale*, respectively, *F_σ-sublocale*) if it is a countable meet (resp., join) of open (resp., closed) sublocales.

Recall also that, given an $S \in \mathcal{S}(L)$, $\overline{f} : \mathfrak{L}(\mathbb{R}) \to L$ is called a *continuous extension* of $f : \mathfrak{L}(\mathbb{R}) \to S$ iff the following diagram commutes

$$\begin{tikzcd} & L \arrow[d, "c_S"] \\ \mathfrak{L}(\mathbb{R}) \arrow[ur, "\overline{f}"] \arrow[r, "f"'] & S \end{tikzcd}$$

i.e. $c_S \circ \overline{f} = f$.

Proposition 3.5. *Let L be a normal frame. For each $a \in L$ the following are equivalent:*

(1) *$\mathfrak{c}(a)$ is a G_δ-sublocale.*
(2) *$\mathfrak{o}(a)$ is a F_σ-sublocale.*
(3) *There is a countable $B \subseteq L$ such that $a = \bigvee B$ and $B \prec a$.*
(4) *For each continuous $f : \mathfrak{L}(\mathbb{R}) \to \mathfrak{c}(a)$ with $\mathbf{0} \le f \le \mathbf{1}$ there exists a continuous extension $\overline{f}$ to L such that $\overline{f}(0,1) \in \mathfrak{c}(a)$.*
(5) *$a \in \operatorname{Coz} L$.*

Proof: (1) $\Rightarrow$ (2): Since $\mathcal{S}(L)$ is a co-frame, the second De Morgan law $(\bigwedge_i S_i)^* = \bigvee_i S_i^*$ holds. Therefore,

$$\mathfrak{o}(a) = \mathfrak{c}(a)^* = (\bigwedge_n \mathfrak{o}(b_n))^* = \bigvee_n \mathfrak{c}(b_n).$$

(Note that normality is not used in the proof.)

(2) $\Rightarrow$ (3): Assume $\mathfrak{o}(a) = \bigvee_{d\in D} \mathfrak{c}(d)$ with a countable $D \subseteq L$. Then $\mathfrak{c}(d) \subseteq \mathfrak{o}(a)$, hence $a \vee d = 1$ for all d. By normality of L there exists $B = \{b_d : d \in D\}$ such that $d \vee b_d = 1$ (hence $\mathfrak{c}(d) \subseteq \mathfrak{o}(b_d)$) and $b_d \prec a$ (hence $\bigvee B \le a$). So,

$$\mathfrak{o}(a) = \bigvee_{d\in D} \mathfrak{c}(d) \subseteq \bigvee_{d\in D} \mathfrak{o}(b_d) = \mathfrak{o}(\bigvee B).$$

i.e., $a \le \bigvee B$. We have shown that $a = \bigvee B$ and $B \prec a$.

(3) $\Rightarrow$ (4): Write $B = \{b_n : n \in \mathbb{N}\}$ and let $f \in c(\mathfrak{c}(a))$ with $\mathbf{0} \le f \le \mathbf{1}$. By the localic Tietze's extension theorem (see, e.g., [4], [30] or [25]), there exists a continuous $f_1 : \mathfrak{L}(\mathbb{R}) \to L$ with $\mathbf{0} \le f_1 \le \mathbf{1}$ and $c_{\mathfrak{c}(a)} \circ f_1 = f$. Since $a = l_a(0,-) = l_a(\frac{1}{n},-)$ for all n, we have

$$\bigvee_{n\in\mathbb{N}} b_n = l_a(0,-) \quad \text{and} \quad b_n \prec l_a(\tfrac{1}{n},-)$$

Since l_a is lower semicontinuous, by Lemma 3.3 there is a continuous $f_2 : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} \le f_2 \le l_a$ and $a = l_a(0,-) = f_2(0,-)$. As in [13], we now define $\overline{f} : \mathfrak{L}(\mathbb{R}) \to L$ by

$$\overline{f} = \tfrac{1}{2}(\max(f_1 - f_2, \mathbf{0}) + \min(f_1 + f_2, \mathbf{1})).$$

Clearly, $\mathbf{0} \leq \overline{f} \leq \mathbf{1}$. We now show that $\overline{f}$ is the required extension of f. For this purpose observe first that $c_{\mathfrak{c}(a)} \circ f_2 = \mathbf{0}$. Indeed,

$$\begin{aligned} c_{\mathfrak{c}(a)} \circ f_2(p,q) &= a \vee f_2(p,q) \\ &= f_2((0,-) \vee (p,q)) \\ &= \begin{cases} 1, & \text{if } p < 0 < q, \\ a, & \text{otherwise} \end{cases} \\ &= \mathbf{0}(p,q) \end{aligned}$$

(cf. Remark 2.2). Since $c_{\mathfrak{c}(a)} \circ f_1 = f$ and $c_{\mathfrak{c}(a)} \circ f_2 = \mathbf{0}$ we have (using Remark 2.2 again):

$$c_{\mathfrak{c}(a)} \circ \overline{f} = \tfrac{1}{2}(\max(f - \mathbf{0}, \mathbf{0}) + \min(f + \mathbf{0}, \mathbf{1})) = f.$$

Finally, we prove that $a \leq \overline{f}(0,1)$. Since $\overline{f} \geq \frac{1}{2}\min(f_2, \mathbf{1})$, we have

$$\overline{f}(0,-) \geq (\tfrac{1}{2}\min(f_2, \mathbf{1}))(0,-) = \min(f_2, \mathbf{1})(0,-) = f_2(0,-) = a.$$

Since $\overline{f} \leq \max(\mathbf{1} - f_2, \mathbf{0})$, we get

$$\begin{aligned} \overline{f}(-,1) &\geq \max(\mathbf{1} - f_2)(-,1) \\ &= \textstyle\bigvee_{r \in \mathbb{Q}} \mathbf{1}(-,r) \wedge f_2(r-1,-) \\ &= \textstyle\bigvee_{r>1} f_2(r-1,-) \\ &= f_2(0,-) \\ &= a. \end{aligned}$$

Thus, $\overline{f}(0,1) \in \mathfrak{c}(a)$.

(4) $\Rightarrow$ (5): Consider $f = \mathbf{0}$ with values in $\mathfrak{c}(a)$. With $\overline{f} : \mathfrak{L}(\mathbb{R}) \to L$ satisfying $\mathbf{0} \leq f \leq \mathbf{1}$, $c_{\mathfrak{c}(a)} \circ \overline{f} = f$ and $\overline{f}(0,1) \geq a$ we get $\overline{f}(0,1) = a \vee \overline{f}(0,1) = f(0,1) = a$, hence $a \in \operatorname{Coz} L$.

(5) $\Rightarrow$ (1): Let $a = f((-,0) \vee (0,-))$ for some continuous $f : \mathfrak{L}(\mathbb{R}) \to L$. Take

$$a_n = f(\tfrac{1}{n}, \tfrac{1}{n}) \quad \text{and} \quad b_n = f((-,-\tfrac{1}{n}) \vee (\tfrac{1}{n},-))$$

for all $n \in \mathbb{N}$. Since $a \vee a_n = 1$, it follows that $\mathfrak{c}(a) \subseteq \mathfrak{o}(a_n)$ and, thus,

$$\mathfrak{c}(a) \subseteq \bigwedge_{n \in \mathbb{N}} \mathfrak{o}(a_n).$$

For the reverse inclusion, let $d \in \bigwedge_n \mathfrak{o}(a_n)$, i.e. $a_n \to d = d$ for all n. Since $a_n \wedge b_n = 0 \le d$, it follows that $a_n \le b_n \to d = d$ and so $a = \bigvee_n b_n \le d$. Hence $d \in \mathfrak{c}(a)$. ■

4. Perfectly normal frames

Perfect normality in pointfree topology was first considered by Charalambous [4] in the context of σ-frames. Recall that a lattice L with countable joins and finite meets is a *σ-frame* if finite meets distribute over countable joins. In [4], a σ-frame L is called *perfectly normal* if it is normal and for each $a \in L$ there exists a sequence (a_n) in L such that for all $b, c \in L$: $b \wedge a = c \wedge a$ iff $b \vee a_n = c \vee a_n$ for all n. Gilmour [12] observed that in the class of σ-frames perfect normality and regularity are equivalent concepts. Recall that a [σ-]frame is *regular* if for each $a \in L$ there exists a [countable] subset $B \subseteq L$ such that $a = \bigvee B$ and $B \prec a$.

As we shall see soon, when applied to arbitrary frames, the Charalambous' concept nevertheless yields the right notion of perfect normality (which of course is no longer equivalent to regularity). However, we adopt the following more natural definition of perfect normality for frames (cf. assertion (3) of Proposition 3.3 and [8, 1.5.K]).

Definition 4.1. A frame L is called *perfectly normal* if for each $a \in L$ there is a countable subset $B \subseteq L$ such that $a = \bigvee B$ and $B \prec a$.

Each perfectly normal frame is regular. Next, we gather together some characterizations of perfect normality. Part of them comes from Proposition 3.5, but we also add some new ones. We note that with L a σ-frame, the equivalence of (1) (with regularity instead of perfect normality) and (2) given below was proved by Gilmour [12, Prop. 1.1].

Proposition 4.2. *The following are equivalent for a frame L:*

(1) *L is perfectly normal.*
(2) [Charalambous' condition] *L is normal and for each $a \in L$ there is a countable $D \subseteq L$ such that for each $b, c \in L$ the following hold: $b \wedge a = c \wedge a \Leftrightarrow b \vee d = c \vee d$ for all $d \in D$.*
(3) *L is normal and each closed sublocale is G_δ.*
(4) *L is normal and each open sublocale is F_σ.*
(5) *L is normal and* $\operatorname{Coz} L = L$.

(6) [Tietze-type theorem] *Given a closed sublocale S, each continuous $f : \mathfrak{L}([0,1]) \to S$ extends continuously to an $\overline{f} : \mathfrak{L}([0,1]) \to L$ such that $\overline{f}(0,1) \in S$.*
(7) [Vedenisoff = Urysohn-type lemma] *Given $a, b \in L$ with $a \vee b = 1$, there exists a continuous $f : \mathfrak{L}([0,1]) \to L$ such that $f(0,-) = a$ and $f(-,1) = b$.*

Proof: (1) $\Rightarrow$ (2): For the normality let $a \vee b = 1$ in L. By perfect normality of L, there are non-decreasing sequences (a_n) and (b_n) such that $a = \bigvee_n a_n$, $b = \bigvee_n b_n$, $a_n \prec a$ and $b_n \prec b$ for all n. Define

$$u = \bigvee_{n \in \mathbb{N}} (a_n^* \wedge b_n) \quad \text{and} \quad v = \bigvee_{n \in \mathbb{N}} (a_n \wedge b_n^*).$$

We have

$$a \vee u = \bigvee_{n \in \mathbb{N}} ((a \vee a_n^*) \wedge (a \vee b_n)) = a \vee b = 1$$

and precisely in the same way, we have $b \vee v = 1$. In order to show that $u \wedge v = 0$ it suffices to check (using the frame distribution law) that for all n and m on has

$$c = (a_n^* \wedge b_n) \wedge (a_m \wedge b_m^*) = 0.$$

Indeed, if $n \leq m$, then $c \leq b_n \wedge b_n^* = 0$. Similarly, if $n > m$, then $c \leq a_m^* \wedge a_m = 0$.

The additional condition is rather obvious: Let $a = \bigvee A$ with $A \prec a$. It suffices to observe that $D = \{x^* : x \in A\}$ does the job. Let us check the implication ($\Leftarrow$). Assume $b \vee x^* = c \vee x^*$ for all $x \in A$. Then

$$b \wedge a = \bigvee_{x \in A} b \wedge x = \bigvee_{x \in A} ((b \vee x^*) \wedge x) = \bigvee_{x \in A} ((c \vee x^*) \wedge x) = \bigvee_{x \in A} c \wedge x = c \wedge a.$$

(2) $\Rightarrow$ (1): Let $a \in L$. Put $b = 1$ and $c = a$ (cf. [12]). Let (d_n) be the sequence given by hypothesis. Then $d_n \vee a = 1$ for all n. By normality there exists an $A = \{a_n : n \in \mathbb{N}\} \subseteq L$ such that $d_n \vee a_n = 1 = a \vee a_n^*$ for all n. Thus, $a_n \prec a$ and $\bigvee A \leq a$. Moreover, $d_n \vee \bigvee A = 1 = d_n \vee a$. Thus, by hypothesis, $a \wedge \bigvee A = a$, i.e., $a \leq \bigvee A$.

(1) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6): See Proposition 3.5.

(6) $\Rightarrow$ (7): Let $a \vee b = 1$. Define $f : \mathfrak{L}(\mathbb{R}) \to \mathfrak{c}(a \wedge b)$ by

$$f(p,q) = \begin{cases} a, & \text{if } 0 \leq p < 1 < q, \\ b, & \text{if } p < 0 < q \leq 1, \\ 1, & \text{if } p < 0 < 1 < p, \\ a \wedge b, & \text{otherwise.} \end{cases}$$

Note that f defines a frame homomorphism precisely because $a \vee b = 1$. Then $\mathbf{0} \leq f \leq \mathbf{1}$ and f extends continuously to an $\overline{f}$ with $\mathbf{0} \leq \overline{f} \leq \mathbf{1}$ and $a \wedge b \leq \overline{f}(0,1)$. Thus,

$$a = f(0,-) = (c_{\mathfrak{c}(a \wedge b)} \circ \overline{f})(0,-) = (a \wedge b) \vee \overline{f}(0,-) = \overline{f}(0,-)$$

and, similarly, $b = f(-,1) = \overline{f}(0,-)$

(7) $\Rightarrow$ (1): Let $b = 1$ and $a \in L$ be arbitrary. With the hypothesized f, put $a_n = f(\frac{1}{n},-)$ for all $n \in \mathbb{N}$. Then $a = \bigvee_n a_n$ and

$$a \vee a_n^* = f(0,-) \vee (f(\tfrac{1}{n},-))^* \geq f(0,-) \vee f(-,\tfrac{1}{n}) = 1 \quad \text{for all } n \in \mathbb{N}. \quad \blacksquare$$

Now it is obvious that perfect normality implies complete regularity. Indeed, we have shown that in a perfectly normal L one has $\operatorname{Coz} L = L$, while L is completely regular iff it is generated by $\operatorname{Coz} L$ (see [2, Corollary 2]).

Regarding hereditariness, we have: perfect normality $\Rightarrow$ hereditary normality $\Rightarrow$ normality. In fact, the following holds (see also [15, Proposition 3.1]):

Proposition 4.3. *Any sublocale of a perfectly normal frame is perfectly normal.*

Proof: Let S be a sublocale of a perfectly normal frame L. For each $a \in S$ there is a countable $B \subseteq L$ such that $a = \bigvee B$ and $B \prec a$. It follows that $a = c_S(\bigvee B) = \bigvee_S c_S(B)$ and

$$1 = c_S(1) = c_S(b^* \vee a) = c_S(b^*) \vee_S c_S(a) = c_S(b^*) \vee_S a.$$

Since $c_S(b^*) \wedge c_S(b) = c_S(b^* \wedge b) = c_S(0) = 0_S$, it follows that $c_S(b^*) \leq c_S(b)^{*_S}$ and so $c_S(b)^{*_S} \vee_S a = 1$ (where $(-)^{*_S}$ means the pseudocomplement in S). Consequently, the countable subset $c_S(B)$ is the desired set in S. $\blacksquare$

5. Two insertion theorems for frames

In this section we prove the two pointfree forms of Dowker's and Michael's insertion theorems.

Before formulating our pointfree version of Dowker's insertion theorem, we introduce some notation. Given $g \in \mathrm{LSC}(L)$ and $f \in \mathrm{C}(L)$, we put

$$\iota(f, g) = \bigvee_{r \in \mathbb{Q}} f(-, r) \wedge g(r, -).$$

We write $f < g$ iff $\iota(f, g) = 1$. Clearly $\mathbf{0} < g$ iff $g(0, -) = 1$.

Remark 5.1. When applied to $L = \mathcal{O}X$, one gets $\iota(f, g) = X$ iff $\widetilde{f}(x) < \widetilde{g}(x)$ for all $x \in X$ where $\widetilde{f}$ and $\widetilde{g}$ are the real-valued functions on X associated to f and g.

According to [6], a frame L is *countably paracompact* if each countable cover of L has a locally finite refinement. Rather than recalling the definition of the latter concept, we just quote the following characterization which is enough for our purpose. Before doing this, we recall that $A \subseteq L$ is a *cover* if $\bigvee A = 1$. A cover $(a_j)_{j \in J}$ is *shrinkable* [6] if there is a cover $(b_j)_{j \in J}$ such that $b_j \prec a_j$ for all $j \in J$.

Proposition 5.2. [6, Prop. 7] *A frame L is countably paracompact if and only if each countable non-decreasing cover is shrinkable.*

Proposition 5.3. *Each perfectly normal frame is countably paracompact.*

Proof: Let L be perfectly normal and let (c_n) be a non-decreasing cover. By perfect normality, for each n there exists a family $\{b_{n,m} : m \in \mathbb{N}\}$ such that $c_n = \bigvee_{m \in \mathbb{N}} b_{n,m}$ and $b_{n,m} \prec c_n$. Let $a_n = \bigvee_{i,j \leq n} b_{i,j}$ for each n. The sequence (a_n) is a non-decreasing cover which shrinks (c_n):

$$a_n = \bigvee_{i,j \leq n} b_{i,j} \prec \bigvee_{i \leq n} c_i = c_n \quad \text{and} \quad \bigvee_{n \in \mathbb{N}} a_n = \bigvee_{n \in \mathbb{N}} \bigvee_{i,j \leq n} b_{i,j} = \bigvee_{n \in \mathbb{N}} c_n = 1. \qquad \blacksquare$$

We can prove now the following result with the help of Lemma 3.3.

Proposition 5.4. *If L is normal and countably paracompact frame, then for each lower semicontinuous $g : \mathfrak{L}_u(\mathbb{R}) \to L$ with $\mathbf{0} < g \leq \mathbf{1}$, there exists a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} < f \leq g$.*

Proof: We have $1 = g(0,-) = \bigvee_n g(\frac{1}{n},-)$, a countable non-decreasing open cover. By Proposition 5.2 there is a cover (c_n) such that $c_n^* \vee g(\frac{1}{n},-) = 1$ for each $n \in \mathbb{N}$. Put $a_n = \bigvee_{i\le n} c_i$ for each n. The sequence (a_n) is non-decreasing and is a cover, for one has

$$\bigvee_n a_n = \bigvee_n \bigvee_{i\le n} c_i = \bigvee_n c_n = 1.$$

In particular, $\bigvee_n a_n = g(0,-)$ and

$$a_n = \bigvee_{i\le n} c_i \prec \bigvee_{i\le n} g(\tfrac{1}{i},-) = g(\tfrac{1}{n},-).$$

The required $f \le g$ with $f(0,-) = g(0,-) = 1$ is given by Lemma 3.3. ■

Even if Proposition 5.4 looks quite modest in comparison with its classical counterpart, when applied to $\mathcal{O}X$ for a normal and countably paracompact space X it nevertheless yields the harder part of Dowker's theorem. We shall refer to Theorem A (Katětov-Tong) as is, for instance, the case in [19].

Corollary 5.5. (Dowker [5]) *A space X is normal and countably paracompact if and only if, whenever $h, g : X \to \mathbb{R}$ are such that $h < g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h < f < g$.*

Proof: We prove the *only if* part. Consider the normal and countably paracompact frame $\mathcal{O}X$. We may assume $h, g : X \to [0,1]$. Thus $\mathbf{0} < g - h$ and by (the spatial version of) Theorem 5.4 there is a continuous $k : X \to [0,1]$ such that $\mathbf{0} < k \le g - h$. Since $h + \frac{k}{2} \le g - \frac{k}{2}$, by Theorem A there is a continuous $f : X \to [0,1]$ such that $h + \frac{k}{2} \le f \le g - \frac{k}{2}$. Since $k > \mathbf{0}$, we have $h < f < g$. ■

In the class of normal frames we can formulate an iff criterion for the strict insertion which resembles classical Dowker's result in a better way:

Theorem 5.6. (Insertion theorem) *For L a normal frame, the following are equivalent:*

(1) *L is countably paracompact.*
(2) *For each lower semicontinuous $g : \mathfrak{L}_u(\mathbb{R}) \to L$ with $\mathbf{0} < g \le \mathbf{1}$, there exists a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} < f < g$.*

Proof: (1) ⇒ (2): Let $g : \mathfrak{L}_u(\mathbb{R}) \to L$ lower semicontinuous with $\mathbf{0} < g \leq \mathbf{1}$. Then, by Proposition 5.4 there exists a continuous $f_1 : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} < f_1 \leq g$. Then $f = \frac{1}{2} f_1 : \mathfrak{L}(\mathbb{R}) \to L$ is such that $\mathbf{0} < f < g$. Indeed $f(0, -) = \frac{1}{2} f_1(0, -) = f_1(0, -) = 1$ and

$$\begin{aligned} \iota(f, g) &= \bigvee_{r \in \mathbb{Q}} f(-, r) \wedge g(r, -) \\ &= \textstyle\bigvee_{r \in \mathbb{Q}} f_1(-, 2r) \wedge g(r, -) \\ &= \bigvee_{0<r} f_1(-, 2r) \wedge g(r, -) \\ &\geq \textstyle\bigvee_{0<r} g(r, 2r) \\ &= g(0, -) = 1. \end{aligned}$$

(2) ⇒ (1): Let (c_n) be a non-decreasing cover. Define $g : \mathfrak{L}_u(\mathbb{R}) \to L$ by

$$g(p, -) = \begin{cases} 1 & \text{if } p \leq 0, \\ c_n & \text{if } \frac{1}{n+1} \leq p < \frac{1}{n}, \\ 0 & \text{if } p \geq 1. \end{cases}$$

Clearly, $\mathbf{0} < g \leq \mathbf{1}$ is lower semicontinuous and there exists a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} < f < g$. Let $a_n = f(\frac{1}{n}, -)$ for each n. The sequence (a_n) is a cover which shrinks (c_n):

$$a_n = f(\tfrac{1}{n}, -) \prec f(\tfrac{1}{n+1}, -) \leq g(\tfrac{1}{n+1}, -) = c_n$$

and

$$\bigvee_n a_n = f(0, -) = 1. \qquad \blacksquare$$

Now we move to the case of Michael insertion theorem.

Theorem 5.7. (Insertion theorem) *For L a frame, the following are equivalent:*

(1) *L is perfectly normal.*
(2) *L is normal and for each $g : \mathfrak{L}_u(\mathbb{R}) \to L$ lower semicontinuous with $\mathbf{0} \leq g \leq \mathbf{1}$ there exists a continuous $f : \mathfrak{L}(\mathbb{R}) \to L$ such that $\mathbf{0} \leq f \leq g$ and $f(0, -) = g(0, -)$.*

Proof: (1) ⇒ (2): For each $n \in \mathbb{N}$ there exists a family $\{b_{n,m} : m \in \mathbb{N}\}$ such that

$$g(\tfrac{1}{n}, -) = \bigvee_{m \in \mathbb{N}} b_{n,m} \quad \text{and} \quad b_{n,m} \prec g(\tfrac{1}{n}, -).$$

Put $a_n = \bigvee_{i,j\leq n} b_{i,j}$ for each n. The sequence (a_n) is non-decreasing and

$$\bigvee_{n\in\mathbb{N}} a_n = \bigvee_{n\in\mathbb{N}} \bigvee_{i,j\leq n} b_{i,j} = g(0,-)$$

as well as

$$a_n = \bigvee_{i,j\leq n} b_{i,j} \prec \bigvee_{i\leq n} g(\tfrac{1}{i},-) = g(\tfrac{1}{n},-).$$

The required assertion follows by Lemma 3.3 applied to the well-inside relation $\prec$.

(2) $\Rightarrow$ (1): Let $a \in L$. Then $\mathbf{0} \leq l_a$ and there exists a continuous real function f on L such that $\mathbf{0} \leq f \leq l_a$ and $f(0,-) = l_a(0,-) = a$. Thus, $a \in \operatorname{Coz} L$. By Proposition 3.5, L is perfectly normal. ■

Applied to $\mathcal{O}X$ for a perfectly normal space X, Theorem 5.7, also with the help of Theorem A, yields Theorem C.

Corollary 5.8. (Michael [24]) *A space X is perfectly normal if and only if, whenever $h, g : X \to \mathbb{R}$ are such that $h \leq g$, h is upper semicontinuous and g is lower semicontinuous, there is a continuous $f : X \to \mathbb{R}$ such that $h \leq f \leq g$ and $h(x) < f(x) < g(x)$ whenever $h(x) < g(x)$.*

Proof: We prove the *only if* part. Without loss of generality we assume $h, g : X \to [0,1]$ and $h \leq g$. Then $g - h$ is lower semicontinuous and by (the spatial version of) Theorem 5.7 there is a continuous $k : X \to [0,1]$ such that $\mathbf{0} \leq k \leq g - h$ and $(g-h)^{-1}(0,+\infty) = k^{-1}(0,+\infty)$. Since $h + \frac{k}{2} \leq g - \frac{k}{2}$, by Theorem A there is a continuous $f : X \to [0,1]$ such that $h \leq h + \frac{k}{2} \leq f \leq g - \frac{k}{2} \leq g$. Finally, if $h(x) < g(x)$, then $k(x) > 0$ and thus $h(x) < f(x) < g(x)$. ■

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