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THE J-NUMERICAL RANGE OF A J-HERMITIAN MATRIX AND RELATED INEQUALITIES

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Abstract: Recently, indefinite versions of classical inequalities of Schur, Ky Fan and Rayleigh-Ritz on Hermitian matrices have been obtained for J-Hermitian matrices that are J-unitarily diagonalizable, $J = I_r \oplus (-I_s)$, $r, s > 0$. The inequalities were obtained in the context of the theory of numerical ranges of operators on indefinite inner product spaces. In this paper, the subject is revisited, relaxing the constraint of the matrices being J-unitarily diagonalizable.

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1. Introduction

For $J = I_r \oplus (-I_s) = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$, $r, s > 0$, consider $\mathbb{C}^{r+s}$ endowed with an indefinite inner product $[\cdot, \cdot]$ defined by $[\xi, \eta] = \langle J\xi, \eta\rangle$, where $\langle \xi, \eta\rangle = \xi^*\eta$. Let $M_n(\mathbb{C})$ denote the algebra of $n \times n$ matrices over the field $\mathbb{C}$ of complex numbers. A matrix $A \in M_n(\mathbb{C})$ is said to be J-*Hermitian* if $A = A^\sharp$, where $A^\sharp = JA^*J$ denotes the J-*adjoint* of A. A matrix $A \in M_n(\mathbb{C})$ which commutes with its J-adjoint is called J-*normal*. A J-normal matrix $U \in M_n(\mathbb{C})$ is said to be J-*unitary* if $UU^\sharp = U^\sharp U = I_n$. The J- unitary matrices form a group denoted by $U(r, s)$ [2, 7]. The study of operators on indefinite inner product spaces has various motivations. There are many articles on this subject in mathematical physics, operator theory and operator algebras. (We cite some recent ones [13, 1, 12].)

For $A, C \in M_n(\mathbb{C})$, consider the J-*tracial numerical range*

$$W_C^J(A) = \{\mathrm{tr}(CUAU^{-1}) : U \in U(r, s)\}, \tag{1}$$

which was investigated for J-Hermitian matrices A, C under certain conditions [5, 18]. For $J = I_n$, $W_C^J(A)$ is called the C-*numerical range of* A, and is simply denoted by $W_C(A)$. Recently applications of the C-numerical range to NMR spectroscopy and to quantum control and quantum information theory have been discussed ([6, 23]). We also consider the well-known *classical*

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numerical range defined by

$$W(A) = \{\langle A\xi, \xi\rangle/\langle \xi, \xi\rangle : \xi \in \mathbb{C}^{r+s}, \langle \xi, \xi\rangle \neq 0\},$$

and the following sets

$$W^J_+(A) = \{[A\xi, \xi]/[\xi, \xi] : \xi \in \mathbb{C}^{r+s}, [\xi, \xi] > 0\},$$
$$W^J_-(A) = \{[A\xi, \xi]/[\xi, \xi] : \xi \in \mathbb{C}^{r+s}, [\xi, \xi] < 0\},$$
$$W^J(A) = \{[A\xi, \xi]/[\xi, \xi] : \xi \in \mathbb{C}^{r+s}, [\xi, \xi] \neq 0\}.$$

While the sets $W^J_+(A)$, $W^J_-(A)$ are convex, $W^J(A) = W^J_+(A) \cup W^J_-(A)$ is *pseudo-convex*, that is, for any pair of distinct points $x, y \in W_J(A)$, either $W_J(A)$ contains the closed line segment $tx + (1-t)y$, $0 \leq t \leq 1$ or $W_J(A)$ contains the half-lines $tx + (1-t)y$ where $0 \geq t$ or $t \geq 1$ [15] (see also [3] and [4]). The boundary of $W^J(A)$ has been discussed in [22]. For diagonal matrices $C_1 = I_1 \oplus 0_{r+s-1}, C_2 = 0_r \oplus (-I_1) \oplus 0_{s-1}$, we get $W^J_+(A) = W_{C_1}(A)$, $W^J_-(A) = W_{C_2}(A)$. If A, C are J-Hermitian, then it is not difficult to show that $\overline{\mathrm{tr}(CUAU^{-1})} = \mathrm{tr}(CUAU^{-1})$, for every $U \in U(r,s)$, and so $W^J_C(A)$ is a subset of $\mathbb{R}$.

For $X \in M_n(\mathbb{C})$, we denote by $\sigma_+(X)$ ($\sigma_-(X)$) the set of the eigenvalues of X with associated eigenvectors ξ with $[\xi, \xi] > 0$ ($[\xi, \xi] < 0$). Throughout, we assume that the eigenvalues of C are $c_1, c_2, \cdots, c_r \in \sigma_+(C)$ and $c_{r+1}, c_{r+2}, \cdots, c_{r+s} \in \sigma_-(C)$, and are non-increasingly ordered

$$c_1 \geq c_2 \geq \cdots \geq c_r; \quad c_{r+1} \geq c_{r+2} \geq \cdots \geq c_{r+s}; \quad c_r \geq c_{r+1}. \tag{2}$$

In Theorem 2.1 of [5], it has been shown that if the eigenvalues of A are not all real, and assuming that $c_r > c_{r+1}$, then $W^J_C(A)$ is the whole real line. Notice that having in mind the inequality $c_1 > c_{r+s}$, this constraint may be removed. Thus, in the sequel we assume that the spectrum of A, denoted by $\sigma(A)$, is real. We recall a tracial spectral inequality on Hermitian matrices ([17, 20]). Suppose that $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ and $A = \mathrm{diag}(a_1, a_2, \ldots, a_n)$ are real diagonal matrices, whose eigenvalues are non-increasingly ordered: $c_1 \geq c_2 \geq \cdots \geq c_n$ and $a_1 \geq a_2 \geq \cdots \geq a_n$. Then the inequalities

$$\sum_{i=1}^{n} c_i a_{n+1-i} \leq \mathrm{tr}(CUAU^{-1}) \leq \sum_{i=1}^{n} c_i a_i \tag{3}$$

hold for any unitary matrix U. These inequalities can be deduced from the inequality between the diagonal entries of UAU^{-1} and the eigenvalues of

UAU^{-1} discovered by Schur in 1923 ([16], p. 218). If A is positive semi-definite, Schur's result can be obtained from Ky Fan's result of 1951 ([16], p. 228, [10]). The inequalities (3) can be interpreted in the context of the C-numerical range of A ([8]), and are related to the numerical computation of the eigenvalues of a Hermitian matrix, especially, Rayleigh-Ritz procedure. It is also remarkable that Schur's result is crucial for a proof given by Poon, based on the theory of majorization [9], of an important theorem on numerical ranges due to Westwick, by using Morse theory ([21, 25]). A treatment of majorization based on the theory of numerical ranges was given in [8]. In [5], spectral inequalities for the trace of the product of two J-Hermitian matrices that are J-unitarily diagonalizable have been stated and indefinite versions of the above mentioned classical inequalities have been obtained. The inequalities were obtained in the context of the theory of numerical ranges of operators on indefinite inner product spaces. In this note, the subject is revisited, with the aim of removing certain imposed restrictions, like relaxing the constraint of both matrices being J-unitarily diagonalizable.

This paper is organized as follows. In Section 2, the characterization of $W^J(A)$ is revisited and the so obtained results are crucial for the development of the subsequent Sections. In Section 3, we study conditions under which $W^J_C(A)$ equals $\mathbb{R}$. In Section 4, we investigate the case of $W^J_C(A)$ being a half-line. In Section 5, $W^J_C(A)$ is characterized for A being a J-Hermitian nilpotent matrix and C being a J-unitarily diagonalizable matrix. The main results are Theorem 5.4 and the Corollaries 5.2 and 5.3 , which generalize the principal results of [5], namely the mentioned indefinite versions of Ky Fan's, and Schur's results for J-Hermitian matrices.

2. $W^J(A)$ for J-Hermitian A revisited

For $\lambda \in \sigma(A) \subset \mathbb{R}$, we set

$$X_\lambda = \{\xi \in \mathbb{C}^{r+s} : (\lambda I - A)^{r+s}\xi = 0\}. \tag{4}$$

For easier reference we state the well-known result [7].

Proposition 2.1. *Let A be a J-Hermitian matrix with real spectrum, such that $\lambda_1 > \lambda_2 > \cdots > \lambda_m$ are all the distinct eigenvalues of A. Then the subspaces $X_{\lambda_1}, X_{\lambda_2}, \ldots, X_{\lambda_m}$ are mutually orthogonal with respect to $[\cdot,\cdot]$. Moreover, X_{λ_j} is a non-degenerate indefinite inner product space. If X_{λ_j} is 1-dimensional, $j = 1, 2, \ldots, m$, then A is J-unitarily diagonalizable.*

For the sake of completeness, and inspired by the characterization of $W_+^J(A)$ in terms of $W(JA+i\,J)$ (Theorem 4.1 of [14] and Lemma 1.1 of [15]), we give a short proof of the following result.

Proposition 2.2. *Let A be a J-Hermitian matrix. Then:*

$$W_+^J(A) = \{x \in \mathbb{R} : t(x+i) \in W(JA+i\,J) \text{ for some } 0 < t \leq 1\}, \tag{5}$$

$$W_-^J(A) = \{x \in \mathbb{R} : t(-x-i) \in W(JA+i\,J) \text{ for some } 0 < t \leq 1\}. \tag{6}$$

Proof. We prove (5). Let $x \in W_+^J(A)$. Then, there exists a non-zero vector $\xi \in \mathbb{C}^n$ such that $[A\xi, \xi] = x$ and $[\xi, \xi] = 1$. Thus,

$$\frac{\langle (JA+i\,J)\xi, \xi\rangle}{\langle \xi, \xi\rangle} = \frac{x+i}{\langle \xi, \xi\rangle} \in W(JA+i\,J),$$

being obviously $0 < \langle \xi, \xi\rangle^{-1} \leq 1$. The proof of the reversed inclusion is easy and (6) follows similarly to (5). ■

We briefly survey some basic facts on the *boundary generating curve*, or *Kippenhahn curve*, of $W(JA+i\,J)$, where JA and J are Hermitian matrices ([11, 26]). For Hermitian matrices $H, K \in M_n(\mathbb{C})$, we consider the complex projective curve Γ defined by

$$\Gamma = \{[(t,x,y)] \in CP^2 : \det(tI_n + x\,H + y\,K) = 0\},$$

and its dual curve

$$\Gamma^\wedge = \{[(T,X,Y)] \in CP^2 : T\,t + X\,x + Y\,y = 0 \text{ is a tangent of } \Gamma\}.$$

The real affine part of $\Gamma^\wedge$, $\Gamma^* = \{(X,Y) \in \mathbb{R}^2 : (1,X,Y) \in \Gamma^\wedge\}$, is called the boundary generating curve, or Kippenhahn curve, of $W(H+i\,K)$. (For details on plane algebraic curves, see e.g. [24].) Identifying the real affine plane $\mathbb{R}^2$ with the Gaussian plane $\mathbb{C}$, $W(H+i\,K)$ is the convex hull of the curve Γ^*. Every boundary point of $W(H+i\,K)$ lies on the curve Γ^* or on its *bitangent*, that is, a tangent (at least) at 2 distinct points. By duality, a real line $a\,\Re(z)+b\,\Im(z)+1 = 0$, $(0,0) \neq (a,b) \in \mathbb{R}^2$, is tangent to Γ^* if and only if $\det(I_n + a\,H + b\,K) = 0$. A real line $a\,\Re(z)+b\,\Im(z) = 0$, $(0,0) \neq (a,b) \in \mathbb{R}^2$, is tangent to Γ^* if and only $\det(a\,H+b\,K) = 0$. The boundary generating curve Γ^* passes through the origin $z = 0$ if and only if the line $t = 0$ is tangent to the curve Γ at some point $[(t,x,y)] = [(0,x,y)] \in CP^2$.

Proposition 2.3. *Let A be a non-scalar J-Hermitian matrix. Then λ is a real eigenvalue of A if and only if $x = \lambda y$ is a tangent line to Γ^* passing through the origin.*

Proof. Consider the family of lines $x = \lambda y$, whose homogeneous line coordinates are $(1 : -\lambda : 0)$. A line $x = \lambda y$ is tangent to the Kippenhahn curve of $W(JA + i\,J)$, if and only if

$$\det(JA - \lambda J) = 0,$$

which is clearly equivalent to λ being a (real) eigenvalue of A. ■

Proposition 2.4. [15, 22] *Let A be a J-Hermitian matrix such that*

$$W_+^J(A) = [M_1, +\infty) \text{ or } W_+^J(A) = (M_1, +\infty)$$

and

$$W_-^J(A) = (-\infty, M_2] \text{ or } W_-^J(A) = (-\infty, M_2)$$

for $M_1, M_2 \in \mathbb{R}$. Then M_1, M_2 are eigenvalues of A.

Proof. By (5) and (6), the lines $\Re(z) = M_1 \Im(z)$, $\Re(z) = M_2 \Im(z)$ are real tangents to the Kippenhahn curve Γ^* of $W(JA + i\,J)$. In fact, if $W_+^J(A) = [M_1, +\infty)$, then $x = M_1 y$ is tangent to the Kippenhahn curve of $W(JA + i\,J)$, at least at one point with positive imaginary coordinate. If $W_+^J(A) = (M_1, +\infty)$, then $x = M_1 y$ is tangent to the Kippenhahn curve at the origin of the affine plane. An analogous analysis is valid for $W_-^J(A)$. Thus, by Proposition 2.3 M_1, M_2 are eigenvalues of A. ■

The boundary of a compact subset K of the complex plane is denoted by ∂K. A *support line* of K is a line containing at least one point of K and such that K is contained in one of the two half-planes defined by it.

Proposition 2.5. *Let A be a non-scalar J-Hermitian matrix. Then, $W_+^J(A) = W_-^J(A) = \mathbb{R}$ if and only if there exists a non-degenerate closed interval $[p, q]$ simultaneously contained in $W_+^J(A)$ and $W_-^J(A)$.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Assume that $[p, q] \subset W_+^J(A)$ and $[p, q] \subset W_-^J(A)$. Consider $p \in W_+^J(A)$ and $q \in W_-^J(A)$. We show that $(-\infty, p] \subset W_+^J(A)$. Having in mind (5) and (6), there exist complex numbers $z, w \in W(JA + i\,J)$ such that

$$z = t(p + i),\ 0 < t \leq 1,\ w = -t'(q + i),\ 0 < t' \leq 1.$$

Since $W(JA + i\,J)$ is convex, then $[z, w] \subset W(JA + i\,J)$. The endpoint z lies in the upper half-plane, while w lies in the lower half-plane. The line segment $[z, w]$ intersects the real line to the left at the origin. This implies that $(-\infty, p] \subset W_+^J(A)$. Similarly, we can show that $[q, +\infty) \subset W_+^J(A)$, and it follows from the connectedness of $W_+^J(A)$ that $W_+^J(A) = \mathbb{R}$. By analogous arguments we can prove that $W_-^J(A) = \mathbb{R}$. ■

Theorem 2.1. *Let A be a non-scalar J-Hermitian matrix. Then $W_+^J(A)$ is a right half-line $W_+^J(A) = [M_1, +\infty)$, or $W_+^J(A) = (M_1, +\infty)$, for some $M_1 \in \mathbb{R}$ if and only if $W_-^J(A)$ is a left half-line $W_-^J(A) = (-\infty, M_2]$, or $W_-^J(A) = (-\infty, M_2)$, for some $M_2 \in \mathbb{R}$ and $M_1 \geq M_2$.*

Proof. $(\Rightarrow)$ $W_+^J(A)$ and $W_-^J(A)$ are unbounded connected subsets of $\mathbb{R}$ ([15], Theorem 2.3 (c),(iv)). Suppose that $W_+^J(A) = [M_1, +\infty)$ or $W_+^J(A) = (M_1, +\infty)$. Then, for $M_2 \in \mathbb{R}$, we consider the following possibilities:

$$\begin{aligned}
&(a) \quad W_-^J(A) = [M_2, +\infty) \text{ or } W_-^J(A) = (M_2, +\infty)\\
&(b) \quad W_-^J(A) = (-\infty, M_2], \;\; M_1 \geq M_2;\\
&(c) \quad W_-^J(A) = (-\infty, M_2], M_1 < M_2;\\
&(d) \quad W_-^J(A) = \mathbb{R}.
\end{aligned}$$

If (a), (c) or (d) occurs, then there exists a real non-degenerate closed interval contained in $W_+^J(A)$ and in $W_-^J(A)$. By Proposition 2.5, $W_+^J(A) = W_-^J(A) = \mathbb{R}$, contradicting the hypothesis. $(\Leftarrow)$ This assertion is proved by similar arguments. ■

The following theorem gives necessary and sufficient conditions for $W_+^J(A)$ to be a half-line or the whole real line.

Theorem 2.2. *Let A be a non-scalar J-Hermitian matrix. Then:*
(1) $W_+^J(A) = W_-^J(A) = W^J(A) = \mathbb{R}$ if and only if 0 is an interior point of $W(JA + i\,J)$.
(2) $W_+^J(A)$ is a half-line if and only if either 0 is a regular boundary point of $W(JA + i\,J)$ or $0 \notin W(JA + i\,J)$.

Proof. (1) $(\Rightarrow)$ If $W_+^J(A) = W_-^J(A) = W^J(A) = \mathbb{R}$, then $\{1, -1\} \subset W_+^J(A)$, $\{1, -1\} \subset W_-^J(A)$, and there exist $0 < t_1, t_2, t_3, t_4 \leq 1$ such that $t_1(1 + i), t_2(-1 + i), -t_3(1 + i), -t_4(-1 + i) \in W(JA + i\,J)$. The origin is an interior point of the convex quadrilateral with vertices $t_1(1 + i), t_2(-1 + i), -t_3(1 + i), -t_4(-1 + i)$, and which is contained in $W(JA + i\,J)$.

(1) ($\Leftarrow$) If 0 is an interior point of $W(JA + i\,J)$, then there exists a ball centered at the origin contained in $W(JA + i\,J)$ and (5) and (6) imply that $W_+^J(A) = W_-^J(A) = \mathbb{R}$.

(2) ($\Rightarrow$) Let A be a non-scalar J-Hermitian matrix, $J \neq \pm I$, and suppose that $W_+^J(A)$ does not coincide with the real axis. If 0 is an interior point of $W(JA+i\,J)$, then $W_+^J(A) = \mathbb{R}$, a contradiction. Thus, either $0 \notin W(JA+i\,J)$ or $0 \in \partial W(JA + i\,J)$. The hypothesis 0 is a boundary sharp point leads to a contradiction, since it implies that there exists $\xi \in \mathbb{C}^{r+s}$ such that $(JA+i\,J)\xi = 0$ ([10], p. 50, Theorem 1.6.3) so that $A\xi = -i\xi$, which implies $W_+^J(A) = \mathbb{R}$ ([5], Theorem 2.1), a contradiction.

(2) ($\Leftarrow$) Firstly, suppose that $0 \notin W(JA + i\,J)$. Then there exist two support lines of $W(JA+i\,J)$ passing through 0, which are $a_1\Re(z)+b_1\Im(z) = 0$ and $a_2\Re(z) + b_2\Im(z) = 0$ satisfying $(0, 0) \neq (a_1, b_1)$, $(a_2, b_2) \in \mathbb{R}^2$, $a_1a_2 \neq 0$.

Let the points $x_j + iy_j$, $x_j, y_j \in \mathbb{R}$, $j = 1, 2$, with $y_1 > 0, y_2 < 0$, belong to the mentioned support lines and to $W(JA + i\,J)$. Hence, the support lines are of the form $\Re(z) = M_1\Im(z)$, $\Re(z) = M_2\Im(z)$ and they are tangents to $\partial W(JA + i\,J)$ at the tangency points $z_1 = t_1(M_1 + i)$ and $z_2 = -t_2(M_2 + i)$, for some $0 < t_1, t_2 \leq 1$. By Proposition 2.3 and Proposition 2.4, $W_+^J(A)$ is a half-line with endpoint M_1.

Now, suppose that $0 \in \partial W(JA + i\,J)$ is a regular point. Then there exists a unique support line $\Re(z) = M\,\Im(z)$ of $W(JA + i\,J)$ passing through it. Then, by Proposition 2.2 $W_+^J(A)$ is a half-line with endpoint M. ■

Now, we fix some notation:

$$\sigma_0(A) = \{\lambda \in \mathbb{R} : A\xi = \lambda\,\xi \,\text{for some}\, 0 \neq \xi \in \mathbb{C}^{r+s} \,\text{with}\, [\xi, \xi] = 0\}, \quad (7)$$

$$X_{\lambda,+} = \{\xi \in \mathbb{C}^{r+s} : A\xi = \lambda\xi, [\xi, \xi] > 0\} \cup \{0\}, \quad (8)$$

$$X_{\lambda,-} = \{\xi \in \mathbb{C}^{r+s} : A\xi = \lambda\xi, [\xi, \xi] < 0\} \cup \{0\}, \quad (9)$$

$$Y_\lambda = \{\xi \in X_\lambda : [\xi, \eta] = [\xi, \zeta] = 0,\ \forall\eta \in X_{\lambda,+}, \forall\zeta \in X_{\lambda,-}\}. \quad (10)$$

By Proposition 2.1, there exists a basis $\{\tilde{\xi}_1, \ldots, \tilde{\xi}_{r'}, \tilde{\xi}_{r'+1}, \ldots, \tilde{\xi}_{r'+s'}\}$ of X_{λ_j} satisfying $[\tilde{\xi}_j, \tilde{\xi}_k] = 0$ for $j \neq k$ and $[\tilde{\xi}_j, \tilde{\xi}_j] = 1, j = 1, \cdots, r', [\tilde{\xi}_j, \tilde{\xi}_j] = -1, j =$

$r'+1, \cdots, r'+s'$. The space X_{λ_j} is not invariant under J, but a new J-operator for which the invariance holds may be defined on X_{λ_j} as follows:

$$J'(\sum_{j=1}^{r'+s'} x_j \tilde{\xi}_j) = \sum_{j=1}^{r'} x_j \tilde{\xi}_j - \sum_{j=r'+1}^{r'+s'} x_j \tilde{\xi}_j, \ \ x_k \in \mathbb{C}. \tag{11}$$

Theorem 2.3. *Let A be a J-Hermitian matrix with real spectrum. If $\sigma_0(A)$ in (7) contains two distinct eigenvalues λ_1, λ_2, then $W_+^J(A) = \mathbb{R}$.*

Proof. The spaces Y_{λ_1}, Y_{λ_2} in (10) are invariant under A. Denote by A_{λ_j}, $j = 1, 2$, the restriction of A to Y_{λ_j} and let J' be defined in (11). If $W_+^{J'}(A_{\lambda_j}) = \mathbb{R}$, for $j = 1$ or for $j = 2$, then the relation $W_+^J(A) = \mathbb{R}$ is clear. If $W_+^{J'}(A_{\lambda_1}) = (\lambda_1, +\infty)$ and $W_+^{J'}(A_{\lambda_2}) = (-\infty, \lambda_2)$, then by the convexity $W_+^J(A) = \mathbb{R}$. So, we may assume that $W_+^{J'}(A_{\lambda_1}) = (\lambda_1, +\infty)$ and $W_+^{J'}(A_{\lambda_2}) = (\lambda_2, +\infty)$. Since the operator A_{λ_j} has a unique eigenvalue, then $\Re(z) = \lambda_j \Im(z)$ is the unique support line of $W(J'A_j + i\, J')$, $j = 1, 2$, from the origin. Hence, the origin is an interior point of the convex hull of the curves $\partial W(J'A_{\lambda_j} + i\, J')$, $j = 1, 2$. Thus, 0 is an interior point of $W(JA + iJ)$, and by Theorem 2.2 (1), $W_+^J(A) = \mathbb{R}$. ■

3. When $W_C^J(A)$ is the whole real line

We denote by $X[p, \ldots, q]$ the principal submatrix of $X \in M_n(\mathbb{C})$ in rows and columns $p, \ldots, q$.

Proposition 3.1. *Let A be a J-Hermitian matrix on a Krein space of type (r, s) satisfying $W_+^J(A) = \mathbb{R}$. Then there exists a J-unitary matrix U such that for $B = UAU^{-1}$ and at least one of the pairs*

$$B_1 = B[1, r+1], \ \ J_1 = J[1, r+1], \tag{12}$$

$$B_2 = B[1, 2, r+1], \ \ J_2 = J[1, 2, r+1], \tag{13}$$

$$B_3 = B[1, r+1, r+2], \ \ J_3 = J[1, r+1, r+2], \tag{14}$$

satisfies $W_+^{J_k}(B_k) = \mathbb{R}$, $k = 1, 2, 3$.

Proof. Let A be a J-Hermitian matrix such that $W_+^J(A) = \mathbb{R}$, and so A is non-scalar. By Theorem 2.2 (1), 0 is an interior point of $W(JA+iJ)$, and so there exist vectors ξ_1, ξ_2, ξ_3 in the Krein space $\mathbb{C}^{r+s}$ satisfying $[A\xi_1, \xi_1] > 0$, $[\xi_1, \xi_1] > 0$, $[A\xi_2, \xi_2] < 0$, $[\xi_2, \xi_2] > 0$, and $[A\xi_3, \xi_3] = 0$, $[\xi_3, \xi_3] < 0$, ensuring that the triangle with vertices $[A\xi_j, \xi_j] + i\,[\xi_j, \xi_j]$, $j = 1, 2, 3$, contains 0 as an interior point.

The linear space X spanned by ξ_1, ξ_2, ξ_3 is 2- or 3-dimensional. If X is 2-dimensional, it has a basis $\{\eta_1,\ \eta_2\}$ satisfying $[\eta_1, \eta_1] = 1$, $[\eta_2, \eta_2] = -1$, and we may assume that $a = [\eta_1, \eta_2] \geq 0$. Suppose that there exists a vector $\eta = c_1\eta_1 + c_2\eta_2 \in X$, $(c_1, c_2) \neq (0, 0)$, which is orthogonal to X. Then $[\eta, \eta_1] = c_1 + c_2\,a = 0$, $[\eta, \eta_2] = -c_2 + c_1\,a = 0$, and so $c_1 = c_2 = 0$, which is a contradiction. Hence, the space X is non-degenerate and the pair (12) provides the desired result.

If X is 3-dimensional, the space $X + J(X)$ is non-degenerate and is isometrically imbedded in a Krein space of type $(3, 3)$. So we may assume that X is a linear subspace of a Krein space $\mathbb{C}^6$ of type $(3, 3)$, $J = I_3 \oplus -I_3$, with a basis $\{\xi_1, \xi_2, \xi_3\}$, whose vectors satisfy $[\xi_1, \xi_1] > 0$, $[\xi_2, \xi_2] > 0$, $[\xi_3, \xi_3] < 0$. Let us assume that there exists a basis $\{\eta_1, \eta_2, \eta_3\}$ of X such that the vectors η_1, η_2 are orthogonal to X. We get $[y_1\eta_1 + y_2\eta_2 + y_3\eta_3, y_1\eta_1 + y_2\eta_2 + y_3\eta_3] = |y_3|^2\,[\eta_3, \eta_3]$. This contradicts the existence of vectors ξ_1, ξ_3 satisfying $[\xi_1, \xi_1] > 0$, $[\xi_3, \xi_3] < 0$. Thus, the space

$$X \cap X^{\perp} = \{\xi \in X : [\xi, \eta] = 0, \quad \forall \eta \in X\}$$

is at most 1-dimensional.

If $X \cap X^{\perp}$ is 1-dimensional, we can replace X by a non-degenerate space $X_\epsilon \subset X + J(X)$ as follows. Let $\{\eta_1, \eta_2, \eta_3\}$ be a basis of $X \subset \mathbb{C}^6$, where $\eta_1 = (1, 0, 0, 1, 0, 0)$, $\eta_2 = (0, 1, 0, 0, 0, 0)$, $\eta_3 = (0, 0, 0, 0, 1, 0)$. In this basis, the vectors ξ_1, ξ_2, ξ_3 are expressed as $\xi_1 = u_1\eta_1 + u_2\eta_2 + u_3\eta_3$, $\xi_2 = v_1\eta_1 + v_2\eta_2 + v_3\eta_3$, $\xi_3 = w_1\eta_1 + w_2\eta_2 + w_3\eta_3$. For $\eta_4 = (1, 0, 0, -1, 0, 0) \in J(X)$ and a sufficiently small $\epsilon > 0$, let $\xi_1(\epsilon) = u_1(\eta_1 + \epsilon\eta_4) + u_2\eta_2 + u_3\eta_3$, $\xi_2(\epsilon) = v_1(\eta_1 + \epsilon\eta_4) + v_2\eta_2 + v_3\eta_3$, $\xi_3(\epsilon) = w_1(\eta_1 + \epsilon\eta_4) + w_2\eta_2 + w_3\eta_3$. For $\epsilon \to 0$, we get $[A\xi_j(\epsilon), \xi_j(\epsilon)] \to [A\xi_j, \xi_j]$, and $[\xi_j(\epsilon), \xi_j(\epsilon)] \to [\xi_j, \xi_j]$, $j = 1, 2, 3$. The origin is an interior point of the triangle with vertices $[A\xi_j(\epsilon), \xi_j(\epsilon)] + i[\xi_j(\epsilon), \xi_j(\epsilon)]$, $j = 1, 2, 3$, for sufficiently small ϵ. The space $X(\epsilon)$ spanned by $\eta_1 + \epsilon\eta_4, \eta_2, \eta_3$, is non-degenerate for $0 < \epsilon < 1$ and the pair (13) provides the required result, having in mind Theorem 2.2 (1).

For the pair (14) a similar treatment can be used. ■

If a J-Hermitian matrix A on the 3-dimensional Krein space $\mathbb{C}^3$ of type $(2, 1)$ has an eigenvector ξ with $[\xi, \xi] \neq 0$, then the J-orthogonal space Y of $\mathbb{C}\xi$ is also invariant under A. The restriction B of A to the 2-dimensional non-degenerate space Y satisfies $W_+^J(B) = \mathbb{R}$ if $W_+^J(A) = \mathbb{R}$. So we may assume that A is irreducible and there does not exist an eigenvector ξ of A with $[\xi, \xi] \neq 0$.

Lemma 3.1. *Let A be an irreducible J-Hermitian matrix on a Krein space of type $(2,1)$ with real spectrum and such that $W_+^J(A) = \mathbb{R}$. Then there exists a J-unitary matrix U for which the principal submatrix of $B = UAU^{-1}$ in the second and third rows and columns has imaginary eigenvalues.*

Proof. By the irreducibility of A it has a unique eigenvalue λ. By replacing A by $A - \lambda I$, we may assume that A is nilpotent. Moreover, by a rotation and a dilation, we may suppose that for some $a, b, c \in \mathbb{R}$ and a certain $U \in U(r,s)$,

$$B = UAU^{-1} = \begin{pmatrix} -a-b & c & 0 \\ c & b & 1 \\ 0 & -1 & a \end{pmatrix}. \tag{15}$$

Considering the $(3,3)$-entry of $B^3 = 0$, we obtain $b = a^3 - 2a$ and taking the $(3,2)$-entry of $B^3 = 0$, we find $a^2 + b^2 + c^2 + ab - 1 = 0$. This equation has real solutions b if and only if $3a^2 \leq 4(1 - c^2)$. Since $a^2 \leq 4/3$ and $b = a^3 - 2a$, it follows that $|b - a| \leq 2$, being $b - a = 2$ for $a = -1$ and $b - a = -2$ for $a = 1$. In any case, we get $c = 0$, which is impossible by the irreducibility. Therefore, $|b - a| < 2$, and the result follows. ∎

Theorem 3.1. *Let C be a non-scalar J-Hermitian and J-unitarily diagonalizable matrix, whose eigenvalues satisfy (2). Let A be a J-Hermitian matrix acting on $\mathbb{C}^{r+s}$ and let J' be defined in (11). If there exists a non-degenerate subspace X such that the restriction A' of A to X satisfies $W_+^{J'}(A') = \mathbb{R}$, then $W_C^J(A) = \mathbb{R}$.*

Proof. We apply Proposition 3.1. Firstly, suppose that there exists a 2-dimensional restriction B_1 of $B = UAU^{-1}$ for which $W_+^{J_1}(B_1) = \mathbb{R}$, being J_1 defined in (12). By a translation and a dilation, we may assume that $c_1 = 1$, $c_{r+1} = 0$. The operator B_1 acts on the non-degenerate space X, being its orthogonal complement $X^\perp$ also non-degenerate. We have $\xi_1, \xi_{r+1} \in X, [\xi_1, \xi_1] = 1, [\xi_{r+1}, \xi_{r+1}] = -1, [\xi_1, \xi_{r+1}] = 0$. Let $\{\xi_2, \ldots, \xi_r, \xi_{r+2}, \ldots, \xi_{r+s}\}$ be an orthogonal basis of $X^\perp$ satisfying $[\xi_j, \xi_j] = 1, j = 2, \cdots, r, [\xi_j, \xi_j] = -1, j = r+2, \cdots, r+s$. For

$$\begin{aligned} W^{J_1}(B_1) = \{&c_1[A\eta_1, \eta_1] - c_{r+1}[A\eta_{r+1}, \eta_{r+1}] : \eta_1, \eta_{r+1} \in X, [\eta_1, \eta_1] = 1, \\ &[\eta_{r+1}, \eta_{r+1}] = -1, [\eta_1, \eta_{r+1}] = 0\}, \end{aligned}$$

the following inclusion holds:

$$W_C^J(A) \supset W^{J_1}(B_1) + \sum_{j=2}^{r} c_j[A\xi_j, \xi_j] - \sum_{j=2}^{s} c_{r+j}[A\xi_{r+j}, \xi_{r+j}],$$

and so $W_C^J(A) = \mathbb{R}$.

Now, suppose that there exists a 3-dimensional restriction B_2 of $B = UAU^{-1}$ for which $W_+^{J_2}(B_2) = \mathbb{R}$, where J_2 is defined in (13). Suppose that B_2 acts on X. If B_2 is irreducible, by Lemma 3.1 there exists a 2-dimensional restriction B' of A satisfying $W_+^{J'}(B') = \mathbb{R}$.

Assume now that B_2 is reducible and so there exists a vector ξ_1 satisfying $[\xi_1, \xi_1] = 1$ and $B_2\xi_1 = \lambda\xi_1$ for some $\lambda \in \sigma(B_2)$. We may suppose that the space X is invariant under J, and there exists an orthogonal basis $\{\xi_1, \xi_2, \xi_{r+1}\}$ of X satisfying $[\xi_1, \xi_1] = [\xi_2, \xi_2] = 1, [\xi_{r+1}, \xi_{r+1}] = -1$, and such that

$$B_2\xi_1 = \lambda\xi_1, \; X'' = \mathbb{C}\xi_2 + \mathbb{C}\xi_{r+1}, \; B_2(X'') \subset X'',$$

where B'' is the restriction of B_2 to X''. In the sequel, J'' and J''' are defined analogously to J' in (11). If $W_+^{J''}(B'') = \mathbb{R}$, it is easy to prove that $W_C^J(A) = \mathbb{R}$. So, we only have to consider the case $(M, +\infty) \subset W_+^{J''}(B'') \subset [M, +\infty)$. We remark that $W(J_2B_2 + iJ_2)$ is the convex hull of the point $\lambda + i$ and the (possibly degenerate) elliptical disc $\mathcal{E} = W(J''B'' + iJ'')$, and so

$$W(J_2B_2 + iJ_2) \supset \mathcal{E} \supset \{\frac{[B''\xi_2, \xi_2] + i}{\langle \xi_2, \xi_2\rangle}, \frac{[B''\xi_{r+1}, \xi_{r+1}] - i}{\langle \xi_{r+1}, \xi_{r+1}\rangle}\}.$$

Let $p\,\xi_2 + q\,\xi_{r+1} \in \mathbb{C}^{r+s}$ be a vector satisfying $[p\,\xi_2 + q\,\xi_{r+1}, p\,\xi_2 + q\,\xi_{r+1}] = |p|^2 - |q|^2 < 0$ and $\langle p\xi_2 + q\,\xi_{r+1}, p\,\xi_2 + q\xi_{r+1}\rangle = 1$, and such that the line through the points $\lambda + i$ and $[(B'' + iI_2)(p\,\xi_2 + q\xi_{r+1}), p\xi_2 + q\xi_{r+1}]$ is a support line to the ellipse $\partial W(J''B'' + iJ'')$ (possibly degenerate). We set

$$\tilde{\xi}_{r+1} = \frac{p\xi_2 + q\xi_{r+1}}{\sqrt{|q|^2 - |p|^2}},$$

and we take $p = 0$ and $q = 1$ if the ellipse degenerates. The restriction B''' of B_2 to the non-degenerate space spanned by ξ_1 and $\tilde{\xi}_{r+1}$ satisfies $W_+^{J'''}(B''') \supset (-\infty, \lambda)$ for the above mentioned real number λ. For a vector $\tilde{\xi}_2$ with $[\tilde{\xi}_2, \tilde{\xi}_2] = 1$, $\{\xi_1, \tilde{\xi}_2, \tilde{\xi}_{r+1}\}$ is an orthogonal basis of X. Let $\{\xi_3, \ldots, \xi_r, \xi_{r+2}, \ldots, \xi_{r+s}\}$ be an orthogonal basis of $X^\perp$ satisfying $[\xi_j, \xi_j] = 1, j = 3, \cdots, r, [\xi_j, \xi_j] = -1, j = r + 2, \cdots, r + s$.

By a translation and a dilation, we may assume that $c_1 = 1$, $c_{r+1} = 0$. Then, for

$$\begin{aligned} W_+^{J'''}(B''') &= \{c_1[A\eta_1, \eta_1] - c_{r+1}[A\eta_{r+1}, \eta_{r+1}] : \eta_1, \eta_{r+1} \in \mathbb{C}\xi_1 + \mathbb{C}\tilde{\xi}_{r+1}, \\ &\quad [\eta_1, \eta_1] = 1, [\eta_1, \eta_{r+1}] = 0, [\eta_{r+1}, \eta_{r+1}] = -1\}, \end{aligned}$$

the following inclusion holds:

$$W_C^J(A) \supset W_+^{J'''}(B''') + c_2[A\tilde{\xi}_2, \tilde{\xi}_2] + \sum_{s=3}^{r} c_s[A\xi_s, \xi_s] - \sum_{t=2}^{s} c_{r+t}[A\xi_{r+t}, \xi_{r+t}],$$

and so $W_C^J(A)$ contains a left half-line. Considering B'', we can show that $W_C^J(A)$ contains a right half-line and by the connectedness of $W_C^J(A)$, we conclude that $W_C^J(A) = \mathbb{R}$. ■

4. When $W_C^J(A)$ is a half-line

Theorem 4.1. *Let C be a non-scalar J-Hermitian and J-unitarily diagonalizable matrix satisfying (2). Let A be a J-Hermitian matrix with real spectrum acting on $\mathbb{C}^{r+s}$, whose eigenvalues are non-increasingly ordered $\alpha_1, \alpha_2, \cdots, \alpha_{r-m} \in \sigma_+(A)$; $\alpha_{r+1}, \alpha_{r+2}, \cdots, \alpha_{r+s-m'} \in \sigma_-(A)$, and $\sigma_0(A) = \{\lambda_0\}$. Let Y_{λ_0} in (10) be a Krein space of type (m, m'), let J' be defined in (11), and let A' be the restriction of A to Y_{λ_0}. Then the following holds:*

(i) $W_C^J(A)$ is a right half-line $(M, +\infty)$, or $[M, +\infty)$, for some $M \in \mathbb{R}$ if and only if $W_+^{J'}(A') = (\lambda_0, +\infty)$ and $\max \sigma_-(A) \le \lambda_0 \le \min \sigma_+(A)$.

(ii) $W_C^J(A)$ is a left half-line $(-\infty, M)$, or $(-\infty, M]$, for some $M \in \mathbb{R}$ if and only if $W_+^{J'}(A') = (-\infty, \lambda_0)$ and $\max \sigma_+(A) \le \lambda_0 \le \min \sigma_-(A)$,

Proof. We prove the *only if part* of (i). We assume that $W_C^J(A)$ is a right half-line. Since A' is a non-scalar restriction of A, from Theorem 3.1 it follows that $W_+^{J'}(A')$ is also a right half-line $(\lambda_0, +\infty)$. More precisely, from Theorem 3.1 and by the method of its proof we can show that $W_+^{J'}(A')$ is neither the whole line or the left half-line. We show that $\alpha_{r+1} \le \lambda_0 \le \alpha_{r-m}$. If $\alpha_1 \le \alpha_{r+s-m'}$ and $\alpha_{r-m} < \alpha_{r+1}$, it can be easily seen that $W_C^J(A)$ contains a left half-line, which is impossible. If $[\alpha_1, \alpha_{r-m}] \cap [\alpha_{r+1}, \alpha_{r+s-m'}] \ne \emptyset$ and $\alpha_1 \ne \alpha_{r+s-m'}$, $\alpha_{r+1} \ne \alpha_{r-m}$, then $W_C^J(A)$ is the whole real line ([5]). Thus, $\alpha_{r+1} \le \alpha_{r-m}$. If $\alpha_{r-m} < \lambda_0$, then $\Re(z) = \lambda_0 \Im(z)$ is the unique support line of $W(J'A' + i\,J')$ passing through 0. There exists a point $[A'\xi, \xi] + i[\xi, \xi]$ with $[\xi, \xi] < 0$, $\langle \xi, \xi \rangle = 1$, near the line $\Re(z) = \lambda_0 \Im(z)$. Moreover, there exists a vector η with $\langle \eta, \eta \rangle = 1$ and $[A\eta, \eta] + i[\eta, \eta] = t(\alpha_{r-m} + i)$ for some $0 < t \leq 1$. The line segment joining the points $[A'\xi, \xi] + i[\xi, \xi]$ and $t(\alpha_{r-m} + i)$ is contained in $W(JA + iJ)$ due to its convexity, and hence $W_+^J(A)$ contains a left half-line, which is impossible. If $\alpha_{r+1} > \lambda_0$, analogous arguments hold, and the result follows.

We prove the *if part* of (i). By a translation and a dilation, we may assume that $c_{r+1} \leq 0 \leq c_r$. Observe that A is the direct sum of A' in the hypothesis and

$$A'' = U \operatorname{diag}(\alpha_1, \dots, \alpha_{r-m}, \alpha_{r+1}, \dots, \alpha_{r+s-m'})\, U^{-1},$$

for some $U \in U(r - m, s - m')$. For an orthogonal basis $\{\xi_1, \dots, \xi_{r+s}\}$ with $[\xi_1, \xi_1] = \cdots = [\xi_r, \xi_r] = 1$, $[\xi_{r+1}, \xi_{r+1}] = \cdots = [\xi_{r+s}, \xi_{r+s}] = -1$, the expression

$$\sum_{j=1}^{r} c_j [A\xi_j, \xi_j] - \sum_{j=1}^{s} c_{r+j} [A\xi_{r+j}, \xi_{r+j}]$$

has the lower bound

$$(\sum_{j=1}^{r} c_j) M_1 + (\sum_{j=1}^{s} c_{r+j}) M_2,$$

being $W_+^J(A) \subset [M_1, +\infty) = [\lambda_0, +\infty)$ and $W_-^J(A) \subset (-\infty, M_2] = (-\infty, \lambda_0]$.

By the hypothesis, $[A\xi_j, \xi_j] \geq \lambda_0, j = 1, \cdots, r$, and $[A\xi_{j+r}, \xi_{j+r}] \geq -\lambda_0, j = 1, \cdots, s$, being $W_+^J(A) = (\lambda_0, +\infty)$ if $\alpha_{r-m} > \lambda_0$ and $W_+^J(A) = [\lambda_0, +\infty)$ if $\alpha_{r-m} = \lambda_0$. Thus,

$$\sum_{j=1}^{r} c_j [A\xi_j, \xi_j] - \sum_{j=1}^{s} c_{r+j} [A\xi_{r+j}, \xi_{r+j}] \geq [\sum_{j=1}^{r} c_j + \sum_{j=1}^{s} c_{r+j}] \lambda_0,$$

and (i) easily follows.

The assertion (ii) is similarly proved. ∎

Remark. We notice that an analogous theorem to Theorem 4.1 involving $W_-^{J'}(A')$ is valid, with the adequate adaptations.

Theorem 4.2. *Let C be a non-scalar J-Hermitian and J-unitarily diagonalizable matrix whose eigenvalues satisfy (2). Let A be a J-Hermitian matrix acting on $\mathbb{C}^{r+s}$. The following conditions are mutually equivalent:*

(i) $W_+^J(A)$ *is a (right) half-line.*
(ii) $W_C^J(A)$ *is a (right) half-line.*

Proof. By Theorem 4.1, the condition (i) implies the condition (ii). We prove the converse by contradiction. Suppose that $W_+^J(A)$ is the whole real line. By Proposition 3.1, some restriction B of A acting on a 2 or 3-dimensional space satisfies certain condition ensuring that $W_C^J(A)$ is the real line. By Theorem 4.1 $W_+^J(B) = \mathbb{R}$, which is a contradiction. The assumption that $W_+^J(A)$ is a left half-line also leads to contradiction. ■

We recall that the *discriminant* D_h of a monic polynomial

$$h(\lambda) = \lambda^{m+m'} + a_1\lambda^{m+m'-1} + \ldots + a_{m+m'}$$

is defined as the *resultant*, or the *Sylvester determinant*, of $h(\lambda)$ and $h'(\lambda)$. The monic polynomial h has a multiple root in $\mathbb{C}$ if and only if D_h vanishes.

Theorem 4.3. *Let J', A', A, C be defined as in Theorem 4.1. If $W_+^{J'}(A') = (\lambda_0, +\infty)$ and $\alpha_{r+1} \leq \lambda_0 \leq \alpha_{r-m}$, a lower bound to the endpoint of the half-line $W_C^J(A)$ is given by*

$$L = \sum_{p=m+1}^{r} c_p\alpha_{r-p+1} + \sum_{p=1}^{s-m'} c_{r+p}\alpha_{r+s-m'-p+1} + (\sum_{j=1}^{m} c_j + \sum_{j=r+s-m'+1}^{r+s} c_j)\lambda_0. \quad (16)$$

Proof. Let $\{\zeta_1, \ldots, \zeta_m, \zeta_{m+1}, \ldots, \zeta_{m+m'}\}$ be an orthogonal basis of the Krein space Y_{λ_0} of type (m, m') satisfying $[\zeta_1, \zeta_1] = \cdots = [\zeta_m, \zeta_m] = 1, [\zeta_{m+1}, \zeta_{m+1}] = \cdots = [\zeta_{m+m'}, \zeta_{m+m'}] = -1$. Let J' and K be the linear operators defined on Y_{λ_0} by $J'(\zeta_j) = \zeta_j$, $1 \leq j \leq m$; $J'(\zeta_k) = -\zeta_k$, $m+1 \leq k \leq m+m'$, and $K(\zeta_j) = j\,\zeta_j$, $1 \leq j \leq m+m'$, respectively. For a sequence of decreasing positive numbers (ϵ_n), $n \in \mathbb{N}$, converging to 0 with $\epsilon_1 < 1$, define:

$$A(\epsilon_n) = A^{'}(\epsilon_n) \oplus A^{''}(\epsilon_n),$$

$$A^{'}(\epsilon_n) = A' + \beta\epsilon_n^{m+m'}J' + \gamma_n\epsilon_n^{m+m'}K,$$

$$A^{''}(\epsilon_n) = U\,\mathrm{diag}(\alpha_1 + 3\epsilon_n, \ldots, \alpha_{r-m} + 3\epsilon_n, \alpha_{r+1} - 3\epsilon_n, \ldots, \alpha_{r+s-m'} - 3\epsilon_n)\,U^{-1},$$

where $U \in U(r-m, s-m')$, and β, γ_n are nonnegative real numbers chosen as follows. Let $\beta \in (0,1)$ be such that every eigenvalue λ' of $A' + \beta\epsilon_n^{m+m'}J'$

satisfies $|\lambda' - \lambda_0| < \epsilon_n$, $n = 1, 2, \ldots$. By the inequality (K.5) in [19], p. 334, we have

$$|\lambda' - \lambda_0| \leq 2(m + m' + 1)^2 (M + 1)\, \delta_n^{1/(m+m')},$$

where

$$\delta_n = \frac{\beta}{M} \epsilon_n^{m+m'}, \;\; M = \max\{|a'_{ij}| : 1 \leq i, j \leq m + m'\}.$$

Hence $\delta_n^{1/(m+m')} \leq \epsilon_n \, (\frac{\beta}{M})^{1/(m+m')}$, and taking $0 < \beta < M$ sufficiently small, it follows that $|\lambda' - \lambda_0| < \epsilon_n$.

Now, choose $\gamma_n \in [0, 1)$ so that:
(1) every eigenvalue of $A'(\epsilon_n)$ has algebraic multiplicity 1;
(2) every eigenvalue λ'' of $A'(\epsilon_n)$ satisfies $|\lambda'' - \lambda'| < \epsilon_n$ for some eigenvalue λ' of $A' + \beta\epsilon_n^{m+m'} J'$;
(3) the origin does not belong to $W(JA'(\epsilon_n) + i\, J)$.

If every eigenvalue of $A' + \beta\epsilon_n^{m+m'} J'$ has algebraic multiplicity 1, we can choose $\gamma_n = 0$, and every point z of

$$\Gamma = W(J'(A' + \beta\epsilon_n^{m+m'} J') + i\, J') = \beta\epsilon_n^{m+m'} + W(JA' + iJ') \qquad (17)$$

satisfies $|z| \geq \beta\epsilon_n^{m+m'} / \sqrt{1 + \lambda_0^2} > 0$.

Suppose that there exists an eigenvalue of $A' + \beta\epsilon_n^{m+m'} J'$ with algebraic multiplicity greater than or equal to 2. The discriminant $D(t)$ of the polynomial $g(\lambda, t) = \det(\lambda I_{m+m'} - (1 - t)(A' + \beta\epsilon_n^{m+m'} J') - t\, K)$, with respect to λ satisfies $D(0) = 0$, $D(1) \neq 0$, because every eigenvalue of K is simple. Thus, $D(t)$ is a non-zero polynomial in t, and so it has a finite number of roots. Every point z' of $W(J'A'(\epsilon_n) + i\, J')$ satisfies the inequality

$$|z - z'| = |\gamma_n \epsilon_n^{m+m'} \langle JK\xi, \xi\rangle| \leq \gamma_n \epsilon_n^{m+m'} \lambda_{\max}(K) = (m + m')\gamma_n \epsilon_n^{m+m'},$$

for some point $z \in \Gamma$ and some unit vector ξ, $\lambda_{\max}(K)$ denoting the largest eigenvalue of the matrix K. Hence, we can choose $\gamma_n > 0$ so that the asserted conditions are satisfied. Thus, $W_+^{J'}(A'(\epsilon_n))$ is a half-line and by Theorem 2.1 of [5] all the eigenvalues of $A'(\epsilon_n)$ are real.

By Proposition 2.1, $A'(\epsilon_n)$ is J-unitarily diagonalizable, and we assume that its eigenvalues are non-increasingly ordered

$$\lambda_m(\epsilon_n), \cdots, \lambda_1(\epsilon_n) \in \sigma_+(A'(\epsilon_n)); \;\; \lambda_{m+m'}(\epsilon_n), \cdots, \lambda_{m+1}(\epsilon_n) \in \sigma_-(A'(\epsilon_n))$$

and

$$\lambda_{m+1}(\epsilon_n) < \lambda_0 < \lambda_m(\epsilon_n).$$

By Theorem 1.1 of [5], every point t_n of $W_C^J(A(\epsilon_n))$ satisfies

$$t_n \geq \sum_{j=1}^{r-m} c_{m+j}(\alpha_{r+1-j-m} + 3\epsilon_n) + \sum_{j=1}^{s-m'} c_{r+j}(\alpha_{r+s-m'-j+1} - 3\epsilon_n)$$

$$+ \sum_{j=1}^{m} c_j \lambda_{m-j+1}(\epsilon_n) + \sum_{j=1}^{m'} c_{r+s-j+1}\lambda_{m+j}(\epsilon_n).$$

In an orthogonal basis $\{\xi_1, \ldots, \xi_r, \xi_{r+1}, \ldots, \xi_{r+s}\}$ with $[\xi_1, \xi_1] = \ldots = [\xi_r, \xi_r] = 1$, $[\xi_{r+1}, \xi_{r+1}] = \cdots = [\xi_{r+s}, \xi_{r+s}] = -1$, every point $t \in W_C^J(A)$ is expressed as

$$t = \sum_{j=1}^{r} c_j [A\xi_j, \xi_j] - \sum_{j=r+1}^{r+s} c_j [A\xi_j, \xi_j].$$

Moreover, $[A\xi_j, \xi_j] = \lim_{n\to\infty}[A(\epsilon_n)\xi_j, \xi_j]$, $j = 1, \ldots, r+s$. This implies that L is a lower bound. ■

5. $W_C^J(A)$ for A being a J-Hermitian nilpotent matrix

In this Section, we characterize $W_C^J(A)$ for J-Hermitian nilpotent matrices A. A matrix A is said to have *nilpotency index* N_0 if $N_0 = \min\{n \in \mathbb{N} : A^n = 0\}$.

In Theorem 4.3 we proved that L in (16) is a lower bound of $W_C^J(A)$, under the hypothesis $W_+^{J'}(A') = (\lambda_0, +\infty)$. Later, we shall show that L is the greatest lower bound. Since the restriction of $A - \lambda_j I$ to X_{λ_j} is a J-Hermitian nilpotent operator, the study of $W_C^J(A)$, $W_+^J(A)$ and $W_-^J(A)$ for a J-Hermitian nilpotent matrix A is of interest. So, we are lead to investigate $W_+^J(A)$ for J-Hermitian nilpotent matrices.

The following lemma is used in the proof of Theorem 5.1.

Lemma 5.1. *Let A be a J-Hermitian matrix acting on a Krein space $\mathbb{C}^6$ of type $(3,3)$ with $J = I_3 \oplus -I_3$. If there exists a vector $\xi \in \mathbb{C}^6$ satisfying $A^2\xi \neq 0$ and $A^3\xi = 0$, then at least one of the following conditions holds:*

(i) There exists a non-degenerate subspace X of type (2,1), or (1,2), for which the restriction B of $P_X A P_X$ to X satisfies $B^2 \neq 0$ and $B^3 = 0$;

(ii) The matrix A is nilpotent with nilpotency index less than or equal to 4 and satisfies $W_+^J(A) = \mathbb{R}$.

Proof. The hypothesis implies that $[A^2\xi, A^2\xi] = [A\xi, A^3\xi] = 0$. Thus, $A^2\xi$ has zero J-norm and so we may assume that $A^2\xi = (0,0,1,0,0,1)^T$, $A\xi = (0, a_2, a, 0, b, a)^T$, where $a \in \mathbb{R}$ and $b > 0$. Let $A = (\alpha_{ij})$, with $\Re\alpha_{ij} = a_{ij}$ and $\Im\alpha_{ij} = b_{ij}$. By the relations $A^\sharp = A$, $A^3\xi = 0$ and $A^2\xi \neq 0$, the matrix A is represented in the form:

$$\begin{pmatrix} a_{11} & -b\alpha_{15} & \alpha_{13} & \alpha_{14} & \alpha_{15} & -\alpha_{13} \\ -b\,\overline{\alpha_{15}} & -b^2a_{55} & 1-b^2a_{55}-i\,bb_{56} & b\overline{\alpha_{45}} & ba_{55} & -1+b^2a_{55}+ibb_{56} \\ \overline{\alpha_{13}} & 1-b^2a_{55}+i\,bb_{56} & a_{33} & \alpha_{46}-ib_{46} & ba_{55}-ib_{56} & -a_{33} \\ -\overline{\alpha_{14}} & -b\alpha_{45} & -a_{46}-ib_{46} & a_{44} & \alpha_{45} & \alpha_{46} \\ -\overline{\alpha_{15}} & -ba_{55} & -ba_{55}-ib_{56} & \overline{\alpha_{45}} & a_{55} & ba_{55}+ib_{56} \\ \overline{\alpha_{13}} & 1-b^2a_{55}+ibb_{56} & a_{33} & \overline{\alpha_{46}} & ba_{55}-ib_{56} & -a_{33} \end{pmatrix}.$$

Case 1. The restriction B_0 of $P_{X_0}AP_{X_0}$ to the space

$$X_0 = \{(0, x_2, x_3, 0, x_5, x_6)^T : (x_2, x_3, x_5, x_6) \in \mathbb{C}^4\}$$

is:

$$B_0 = \begin{pmatrix} -b^2a_{55} & 1-b^2a_{55}-i\,bb_{56} & ba_{55} & -1+b^2a_{55}+ibb_{56} \\ 1-b^2a_{55}+i\,bb_{56} & a_{33} & ba_{55}-ib_{56} & -a_{33} \\ -ba_{55} & -ba_{55}-ib_{56} & a_{55} & ba_{55}+ib_{56} \\ 1-b^2a_{55}+ibb_{56} & a_{33} & ba_{55}-ib_{56} & -a_{33} \end{pmatrix},$$

and its characteristic polynomial is $\det(\lambda I_4 - B_0) = (\lambda - (1-b^2)a_{55})\lambda^3$.

If $a_{55}(1-b^2) \neq 0$, the minimal polynomial of B_0 is $\lambda^3(\lambda - (1-b^2)a_{55})$, and the space $X = \{\xi \in \mathbb{C}^4 : B^3\xi = 0\}$ is 3-dimensional. By the theory of Krein spaces [7], X is non-degenerate and satisfies the condition (i) of the lemma.

If $a_{55} = 0$, the characteristic polynomial of B_0 is λ^4 and so B_0 is nilpotent. Moreover, some computations yield that if $1 - (1-b^2)b_{56}^2 \neq 0$, then the minimal polynomial of B_0 is λ^3. The non-neutral vector $\zeta = (b_{56}, 0, b\,b_{56} - i, 0)^T$ is an eigenvector of B_0 associated with the eigenvalue 0. Then the space $X = \{\xi \in \mathbb{C}^4 : [\xi, \zeta] = 0\}$ is non-degenerate and satisfies the condition (i) of the lemma.

If $a_{55} = 0$ and $(1-b^2)b_{56}^2 = 1$, the minimal polynomial of B_0 is λ^2, and

$$\det(\lambda I_6 - A) = \lambda^3\,[\lambda^3 - (a_{11}+a_{44})\lambda^2 + (a_{11}a_{44} + |\alpha_{14}|^2 + (|\alpha_{15}|^2 - |\alpha_{45}|^2)(1-b^2))\lambda$$
$$+(1-b^2)(a_{11}|\alpha_{45}|^2 - a_{44}|\alpha_{15}|^2 + 2a_{45}(a_{14}a_{15} + b_{14}b_{15}) + 2b_{45}(a_{14}b_{15} - b_{14}a_{15}))].$$

If $H = a_{11}|\alpha_{45}|^2 - a_{44}|\alpha_{15}|^2 + 2a_{45}(a_{14}a_{15} + b_{14}b_{15}) + 2b_{45}(a_{14}b_{15} - b_{14}a_{15})$ does not vanish, then by direct computations we conclude that λ^3 is a factor of the minimal polynomial of A. In this case, the space $X = \{\xi \in \mathbb{C}^6 : A^3\xi = 0\}$ satisfies the condition (i) of the lemma.

Case 2. We consider the case $H = 0$ under the assumption $(1-b^2)b_{56}^2 = 1$ (and so $1 - b^2 \neq 0$).

The restriction of $P_{X_2}AP_{X_2}$ to the non-degenerate space

$$X_2 = \{(0, x_2, x_3, x_4, x_5, x_6)^T \in \mathbb{C}^6 : x_j \in \mathbb{C}, 2 \leq j \leq 6\}$$

is:

$$B_2 = \begin{pmatrix} 0 & 1 - i\,bb_{56} & b\overline{\alpha_{45}} & 0 & -1 + ibb_{56} \\ 1 + i\,bb_{56} & a_{33} & a_{46} - ib_{46} & -ib_{56} & -a_{33} \\ -b\alpha_{45} & -a_{46} - ib_{46} & a_{44} & \alpha_{45} & \alpha_{46} \\ 0 & -ib_{56} & \overline{\alpha_{45}} & 0 & ib_{56} \\ 1 + ibb_{56} & a_{33} & \overline{\alpha_{46}} & -ib_{56} & -a_{33} \end{pmatrix},$$

and the restriction of $P_{X_3}AP_{X_3}$ to the non-degenerate space

$$X_3 = \{(x_1, x_2, x_3, 0, x_5, x_6)^T \in \mathbb{C}^6 : x_j \in \mathbb{C}, j = 1, 2, 3, 5, 6)\}$$

is

$$B_3 = \begin{pmatrix} a_{11} & -b\alpha_{15} & \alpha_{13} & \alpha_{15} & -\alpha_{13} \\ -b\,\overline{\alpha_{15}} & -b^2a_{55} & 1 - b^2a_{55} - i\,bb_{56} & ba_{55} & -1 + b^2a_{55} + ibb_{56} \\ \overline{\alpha_{13}} & a1 - b^2a_{55} + i\,bb_{56} & a_{33} & ba_{55} - ib_{56} & -a_{33} \\ -\overline{\alpha_{15}} & -ba_{55} & -ba_{55} - ib_{56} & a_{55} & ba_{55} + ib_{56} \\ \overline{\alpha_{13}} & 1 - b^2a_{55} + ibb_{56} & a_{33} & ba_{55} - ib_{56} & -a_{33} \end{pmatrix}.$$

The characteristic polynomials of B_2 and B_3 are $\det(\lambda I_5 - B_2) = \lambda^3\,(\lambda^2 - a_{44}\lambda - (1-b^2)|\alpha_{45}|^2)$ and $\det(\lambda I_5 - B_3) = \lambda^3\,(\lambda^2 - a_{11}\lambda + (1-b^2)|\alpha_{15}|^2)$, respectively. If $\alpha_{45} \neq 0$, the minimal polynomial of B_2 coincides with its characteristic polynomial. In this case, $X = \{\xi \in X_2 : B_2^3\xi = 0\}$ satisfies the condition (i) of the lemma. If $\alpha_{15} \neq 0$, the minimal polynomial of B_3 coincides with its characteristic polynomial, and $X = \{\xi \in X_3 : B_3^3\xi = 0\}$ satisfies the condition (i) of the lemma.

Case 3. Let us assume that we have simultaneously: $H = 0$, $\alpha_{15} = 0$ and $\alpha_{45} = 0$. If $|\alpha_{14}|^2 + a_{11}a_{44} \neq 0$, then $A^2\,(A^2 - (a_{11} + a_{44})A + (|\alpha_{14}|^2 + a_{11}a_{44})I) = 0$. Thus, the minimal polynomial of A is $\lambda^2\,(\lambda^2 - (a_{11} + a_{44})\lambda + |\alpha_{14}|^2 + a_{11}a_{44})$

and does not exist a vector $\xi \in \mathbb{C}^6$ satisfying $A^3\xi = 0$ and $A^2\xi \neq 0$, which is a contradiction. Hence, $|\alpha_{14}|^2 + a_{11}a_{44} = 0$. Consider the restriction of $P_{X_4}AP_{X_4}$ to the non-degenerate space $X_4 = \{(x_1, 0, x_3, x_4, x_5, x_6)^T \in \mathbb{C}^6 : x_j \in \mathbb{C},\ j = 1, 3, 4, 5, 6\}$:

$$B_4 = \begin{pmatrix} a_{11} & \alpha_{13} & \alpha_{14} & 0 & -\alpha_{13} \\ \overline{\alpha_{13}} & a_{33} & \overline{\alpha_{46}} & -ib_{56} & -a_{33} \\ -\overline{\alpha_{14}} & -\alpha_{46} & a_{44} & 0 & \alpha_{46} \\ 0 & -ib_{56} & 0 & 0 & ib_{56} \\ \overline{\alpha_{13}} & a_{33} & \overline{\alpha_{46}} & -ib_{56} & -a_{33} \end{pmatrix},$$

and the restriction $B_{4,0}$ of $P_{4,0}AP_{4,0}$ to the non-degenerate space $X_{4,0} = \{(x_1, 0, x_3, x_4, 0, x_6)^T \in \mathbb{C}^6 : x_j \in \mathbb{C}, j = 1, 3, 4, 6\}$:

$$B_{4,0} = \begin{pmatrix} a_{11} & \alpha_{13} & \alpha_{14} & -\alpha_{13} \\ \overline{\alpha_{13}} & a_{33} & \overline{\alpha_{46}} & -a_{33} \\ -\overline{\alpha_{14}} & -\alpha_{46} & a_{44} & \alpha_{46} \\ \overline{\alpha_{13}} & a_{33} & \overline{\alpha_{46}} & -a_{33} \end{pmatrix}.$$

Since $|\alpha_{14}|^2 + a_{11}a_{44} = 0$, the characteristic polynomials of B_4 and $B_{4,0}$ are $\lambda^4 (\lambda - (a_{11} + a_{44}))$ and $\lambda^3 (\lambda - (a_{11} + a_{44}))$, respectively. Assuming that $a_{11} + a_{44} \neq 0$, one of the following conditions holds:

(1) the minimal polynomial of B_4 coincides with its characteristic polynomial.
(2) the minimal polynomial of $B_{4,0}$ coincides with its characteristic polynomial.

The condition (2) is equivalent to the existence of a vector $\xi \in \mathbb{C}^4$ satisfying $B_{4,0}\xi = (0, 1, 0, 1)^T$, that is, $a_{11}\alpha_{46} - a_{13}\alpha_{14} - b_{13}b_{14} + ia_{14}b_{13} \neq 0$, or equivalently, $a_{11}\overline{\alpha_{46}}(k + i) + \overline{\alpha_{13}}(k - i)\sqrt{-a_{11}a_{44}} \neq 0$ for $a_{14} = (1 - k^2)/(1 + k^2)\sqrt{-a_{11}a_{44}}$ and $b_{14} = (2k)/(1 + k^2)\sqrt{-a_{11}a_{44}}$. If (2) holds, then the non-degenerate space $X = \{\eta \in \mathbb{C}^4 : B_{4,0}^3\eta = 0\}$ satisfies the asserted condition.

Suppose that (1) holds, and (2) does not hold. Let η_0 be a non-zero non-neutral vector of X_4 satisfying $B_4\eta_0 = (a_{11} + a_{44})\eta_0$. We consider the restriction $B_{4,5}$ of $P_{4,5}B_4P_{4,5}$ to the nondegenerate space $X_{4,5} = \{\xi \in X_4 : [\xi, \eta_0] = 0\}$. Then, the 4-dimensional J-Hermitian nilpotent matrix $B_{4,5}$ satisfies $B_{4,5}^3 = 0$ and $B_{4,5}^2 \neq 0$. Thus, its Jordan canonical form is necessarily of the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

If the 2-dimensional subspace $Y = \{\xi \in X_{4,5} : B_{4,5}\xi = 0\}$ satisfies $[\xi, \eta] = 0$ for every $\xi, \eta \in Y$, then there exists an orthogonal basis $\xi_1, \xi_2, \xi_3, \xi_4$ with $[\xi_1, \xi_1] = [\xi_2, \xi_2] = 1, [\xi_3, \xi_3] = [\xi_4, \xi_4] = -1$, and $\{\xi_1 + \xi_3, \xi_2 + \xi_4\}$ is a basis of Y. In this basis, $B_{4,5}$ is represented by

$$\begin{pmatrix} a_{11} & a_{12} & -a_{11} & -a_{12} \\ \overline{a_{12}} & a_{22} & -\overline{a_{12}} & -a_{22} \\ a_{11} & a_{12} & -a_{11} & -a_{12} \\ \overline{a_{12}} & a_{22} & -\overline{a_{12}} & -a_{22} \end{pmatrix},$$

which implies $B_{4,5}^2 = 0$, contradicting that $B_{4,5}^2 \neq 0$. Hence, the space Y is non-neutral, there exists a non-neutral vector $\eta_1 \in Y$ and the non-degenerate space $X = \{\xi \in X_{45} : [\xi, \eta_1] = 0\}$ satisfies the asserted condition.

Case 4. If $|\alpha_{14}|^2 + a_{11}a_{44} = 0$ and $a_{11} + a_{44} = 0$, then A is a J-Hermitian nilpotent matrix satisfying $A^4 = 0$, and is represented by

$$A = \begin{pmatrix} a_{11} & 0 & \alpha_{13} & \alpha_{14} & 0 & -\alpha_{13} \\ 0 & 0 & 1 - ib\, b_{56} & 0 & 0 & -1 + ib\, b_{56} \\ \overline{\alpha_{13}} & 1 + ib\, b_{56} & a_{33} & \overline{\alpha_{46}} & -ib\, b_{56} & -a_{33} \\ -\overline{\alpha_{14}} & 0 & -\alpha_{46} & -a_{11} & 0 & \alpha_{46} \\ 0 & 0 & -ib_{56} & 0 & 0 & ib_{56} \\ \overline{\alpha_{13}} & 1 + ib\, b_{56} & a_{33} & \overline{\alpha_{46}} & -ib_{56} & -a_{33} \end{pmatrix}.$$

The restriction of $P_Y A P_Y$ to

$$Y = \{(x_1, x_2, x_3, 0, 0, 0) \in \mathbb{C}^6 : x_j \in \mathbb{C}, j = 1, 2, 3\}$$

is the Hermitian matrix

$$B = \begin{pmatrix} a_{11} & 0 & \alpha_{13} \\ 0 & 0 & 1 - ib\, b_{56} \\ \overline{\alpha_{13}} & 1 + ib\, b_{56} & a_{33} \end{pmatrix}.$$

Since $b_{56} \neq 0$, we conclude that $[-\epsilon, \epsilon] \subset W(B) \subset W_+^J(A)$.

By considering the restriction C of $P_Z A P_Z$ to $Z = \{(0,0,0,y_1,y_2,y_3) \in \mathbb{C}^6 : x_j \in \mathbb{C}, j = 1,2,3\}$ we conclude, by similar arguments to those used for Y, that $[-\epsilon,\epsilon] \subset W(C) \subset W_-^J(A)$.

Thus, Proposition 2.5 implies that $W_+^J(A) = \mathbb{R}$ and the condition (ii) holds. ∎

Theorem 5.1. *Let A be a J-Hermitian nilpotent matrix acting on a Krein space $\mathbb{C}^{r+s}$ of type (r,s) with $A^2 \neq 0$, $A^{N_0} = 0$ for some $N_0 \geq 3$. Then $W_+^J(A)$ is the whole real line.*

Proof. Let A be a J-Hermitian nilpotent matrix with nilpotency index $N_0 \geq 3$. Then there exists a non-zero vector ξ such that $A^{N_0-1}\xi \neq 0$ and $A^{N_0}\xi = 0$. By replacing ξ by $A^{N_0-3}\xi$, we obtain a vector ξ such that $A^2\xi \neq 0$ and $A^3\xi = 0$. Moreover, $[A\xi, A^2\xi] = 0$.

Let X_1 be the non-degenerate space spanned by $\{\xi, A\xi, A^2\xi, J\xi, JA\xi, JA^2\xi\}$, which may be isometrically imbedded into a Krein space of type $(3,3)$. If A is replaced by $\tilde{A} = A \oplus \mathrm{diag}(0,0,\ldots,0)$, where $\mathrm{diag}(0,0,\ldots,0)$ acts on a Krein space of type (m,m'), then $W_+^J(\tilde{A}) = W_+^J(A)$, and the nilpotent matrix $\tilde{A}$ also satisfies $\tilde{A}^2 \neq 0$. By replacing A by some suitable $\tilde{A}$, the Krein space on which A acts contains a Krein space X_2 of type $(3,3)$. We show that the restriction A' of $P_{X_2} A P_{X_2}$ to X_2 satisfies $W_+^{J'}(A') = \mathbb{R}$, where J' is the restriction of J to X_2.

If the condition (ii) of Lemma 5.1 holds, the result follows. If the condition (i) holds, then for the there mentioned B we have $W_+^{J'}(B) \subset W_+^J(A)$. Since $B^3 = 0$ and $B^2 \neq 0$, the J-Hermitian matrix B is irreducible. By similar arguments to those used in the proof of Lemma 3.1, we conclude that certain scalar multiplication of B is J-unitarily similar to the matrix in (15). Since this matrix has imaginary eigenvalues, by Theorem 2.1 of [5] $W_+^J(A)$ is the whole real line. ∎

The following corollary is a consequence of Theorem 5.1, Theorem 2.1, and Theorem 4.1.

Corollary 5.1. *If a J-Hermitian nilpotent matrix A satisfies $W_+^J(A) \subset [0,+\infty)$, or $W_+^J(A) \subset (-\infty,0]$, then $A^2 = 0$.*

This corollary is refined in the following.

Theorem 5.2. *Let A be a J-Hermitian nilpotent matrix on a Krein space $\mathbb{C}^{r+s}$ of type (r, s) satisfying $A^2 = 0$ and $W_+^J(A) \subset [0, +\infty)$. Then there exist non-degenerate subspaces Y and Z of $\mathbb{C}^{r+s}$ satisfying the following conditions:*

(1) $\mathbb{C}^{r+s} = Y \oplus Z,\ \ [\eta, \zeta] = 0,\ \forall \eta \in Y,\ \forall \zeta \in Z,\ \ A(Y) = \{0\},\ \ A(Z) \subset Z;$

(2) The restriction of $P_Z A P_Z$ to Z is J-unitarily similar to the direct sum of J-Hermitian matrices of size 2

$$\begin{pmatrix} a_j & -a_j \\ a_j & -a_j \end{pmatrix}$$

acting on a Krein space of type (1,1) for some $a_j > 0$, $j = 1, 2, \ldots, m$.

Proof. We set $X = A(\mathbb{C}^{r+s})$. From the canonical form of J-Hermitian matrices, the equation $[A\xi, A\xi] = 0$ clearly holds for every $\xi \in \mathbb{C}^{r+s}$, and X is a neutral subspace of $\mathbb{C}^{r+s}$. Hence, there exists an orthogonal system $\{\xi_1, \ldots, \xi_m, \xi_{m+1}, \ldots, \xi_{2m}\}$ of vectors in $\mathbb{C}^{r+s}$ satisfying $[\xi_1, \xi_1] = \cdots = [\xi_m, \xi_m] = 1, [\xi_{m+1}, \xi_{m+1}] = \cdots = [\xi_{2m}, \xi_{2m}] = -1$, and

$$X = \{x_1(\xi_1 + \xi_{m+1}) + \ldots + x_m(\xi_m + \xi_{2m}) : x_1, \ldots, x_m \in \mathbb{C}\}.$$

We may replace J by $J' = UJU^{-1}$ for a suitable $U \in U(r, s)$ satisfying $J'(\xi_j) = \xi_j,\ J'(\xi_k) = -\xi_k$ for $1 \leq j \leq m,\ m + 1 \leq k \leq 2m$. Let

$$Y = \{\xi \in \mathbb{C}^{r+s} : [\xi, \eta] = [\xi, J'(\eta)] = 0,\ \forall \eta \in X\},$$

that is,

$$Y = \{\xi \in \mathbb{C}^{r+s} : [\xi, \xi_j] = 0,\ 1 \leq j \leq 2m\}.$$

Then Y is a non-degenerate subspace of $\mathbb{C}^{r+s}$ such that $\mathbb{C}^{r+s} = X \oplus J'(X) \oplus Y$. Thus, the equation $[A\xi, \eta] = 0$ holds for every $\xi \in Y, \eta \in \mathbb{C}^{r+s}$, and so $A\xi = 0$.

Thus, we may assume that $\mathbb{C}^{r+s} = Z = X \oplus J'(X)$. Since $A(\xi_j + \xi_{m+j}) = 0$, $1 \leq j \leq m$, the matrix A is represented by

$$A = \begin{pmatrix} A_1 & -A_1 \\ A_1 & -A_1 \end{pmatrix}$$

for some $m \times m$ Hermitian matrix. By choosing a suitable $m \times m$ unitary matrix V, and replacing A by $(V \oplus V)A(V^* \oplus V^*)$ we may assume that A_1 is a real diagonal matrix. Since $W_+^J(A) \subset [0, +\infty)$, the matrix A_1 is positive semi-definite and having in mind that $A(\mathbb{C}^{r+s}) = X$, A_1 is positive definite. ■

We observe that by Theorem 5.2, the indices m, m' in Theorem 4.1 necessarily satisfy $m = m'$.

Theorem 5.3. *Let C be a non-scalar J-Hermitian and J-unitarily diagonalizable matrix whose eigenvalues satisfy (2). Let A be a J-Hermitian matrix acting on $\mathbb{C}^{r+s}$ satisfying $(A-\lambda_0 I_{r+s})^{r+s} = 0$ and $W_+^J(A) \subset [\lambda_0, +\infty)$. Then $W_C^J(A)$ is the closed (or open) right half-line $[M, +\infty)$ (or $(M, +\infty)$) with endpoint $M = \lambda_0(c_1 + \ldots + c_r + c_{r+1} + \ldots + c_{r+s})$.*

Proof. Since we have already proved that $M = \lambda_0(c_1 + \ldots + c_{r+s})$ is a lower bound of $W_C^J(A)$, we show that M is the greatest lower bound. We may assume that $\lambda_0 = 0$, that is, A is nilpotent. For J' defined in (11), $C' = \operatorname{diag}(c_1, c_2)$ with $c_1 > c_2$, and the matrix S of size 2 with first and second rows equal to $(1, -1)$, we have

$$W_{C'}^{J'}(S) = (c_1 - c_2) W_+^{J'}(S).$$

Taking into account that

$$[S(\cosh t, \sinh t)^T, (\cosh t, \sinh t)^T] = e^{-2t} \to 0$$

as $t \to +\infty$, the result follows from Theorem 5.2. ∎

The following theorem is a consequence of Theorem 4.3 and Theorem 5.2.

Theorem 5.4. *Let C be a non-scalar J-Hermitian and J-unitarily diagonalizable matrix with eigenvalues satisfying (2). Let A be a J-Hermitian matrix acting on $\mathbb{C}^{r+s}$ with real spectrum such that $\sigma_0(A)$ defined in (7) is the singleton $\{\lambda_0\}$. Consider $\alpha_1, \alpha_2, \cdots, \alpha_{r-m} \in \sigma_+(A)$, $\alpha_{r+1}, \alpha_{r+2}, \cdots, \alpha_{r+s-m} \in \sigma_-(A)$, nonincreasingly ordered, and $\alpha_{r+1} \leq \lambda_0 \leq \alpha_{r-m}$. For Y_{λ_0} defined in (10) with $\lambda = \lambda_0$, let the restriction A' of A to Y_{λ_0} satisfy $W_+^{J'}(A') = (\lambda_0, +\infty)$. Suppose that Y_{λ_0} is a Krein space of type (m, m). Then $W_C^J(A)$ is the right half-line $[M, +\infty)$ (or $(M, +\infty)$) with endpoint*

$$M = \sum_{j=m+1}^{r} c_j \alpha_{r-j+1} + \sum_{j=r+1}^{r+s-m} c_j \alpha_{2r+s-m+1-j} + \Big(\sum_{j=1}^{m} c_j + \sum_{j=r+s-m+1}^{r+s} c_j\Big)\lambda_0.$$

The following corollary is an obvious consequence of Theorem 5.4, and generalizes Corollary 1.2 of [5].

Corollary 5.2. (Richter-Mirsky inequality of indefinite type) *Under the conditions of Theorem 5.4, we have*

$$\mathrm{Tr}(CA) \geq \sum_{j=m+1}^{r} c_j \alpha_{r-j+1} + \sum_{j=r+1}^{r+s-m} c_j \alpha_{r+s-m+1-j} + (\sum_{j=1}^{m} c_j + \sum_{j=r+s-m+1}^{r+s} c_j)\lambda_0.$$

The following corollary is an easy consequence of Theorem 5.4 and extends Corollary 3.2 of [5].

Corollary 5.3. (Raleigh-Ritz inequalities of indefinite type) *For A under the conditions of Theorem 5.4, the following holds:*

(1) If $X_{\lambda_0} = \{0\}$, $\xi \in \mathbb{C}^{r+s}$ and $\xi^* J \xi = 1$, then $\xi^* J A \xi \geq \alpha_r$.

(2) If $X_{\lambda_0} \neq \{0\}$, $\xi \in \mathbb{C}^{r+s}$ and $\xi^* J \xi = 1$, then $\xi^* J A \xi \geq \lambda_0$ ($\xi^* J A \xi > \lambda_0$) if $\alpha_{r-m} = \lambda_0$ ($\alpha_{r-m} > \lambda_0$).

The following corollary follows from Theorem 5.4 by letting

$$C = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0).$$

For simplicity of the notation, we change the convention for the spectrum of the J-Hermtian matrix A.

Corollary 5.4. (Indefinite version of Ky Fan's inequalities) *For A under the conditions of Theorem 5.4 and with $\alpha_{r-m+1} = \cdots = \alpha_r = \alpha_{r+1} = \cdots = \alpha_{r+m} = \lambda_0$, $\sigma_+(A) = \{\alpha_1 \geq \cdots \geq \alpha_{r-m}\}, \sigma_-(A) = \{\alpha_{r+m+1} \geq \cdots \geq \alpha_{r+s}\}$, the following holds:*

(1) If $\leq k \leq r$ then

$$\sum_{j=r-k+1}^{r} \alpha_j \leq \sum_{j=1}^{k} x_j^* J A x_j$$

for all $x_j \in \mathbb{C}^{r+s}$ such that $x_j^* J x_l = \delta_{jl}, j = 1, \cdots, r$.

(2) If $r + 1 \leq k \leq r + s$ then

$$\sum_{j=1}^{r} \alpha_j + \sum_{j=2r+s-k+1}^{r+s} \alpha_j \leq \sum_{j=1}^{r} x_j^* J A x_j - \sum_{j=r+1}^{k} x_j^* J A x_j$$

for all $x_j \in \mathbb{C}^{r+s}$ such that $x_j^* J x_l = \delta_{jl}, j = 1, \cdots, r$, $x_j^* J x_l = -\delta_{jl}, j = r + 1, \cdots, r + s$.

The following corollary follows from Theorem 5.4 by letting

$$C = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$$

and taking $\{x_1, \ldots, x_r, x_{r+1}, \ldots, x_{r+s}\}$ as a rearrangement of the canonical orthogonal basis of $(\mathbf{C}^{r+s}, [\cdot, \cdot])$.

Corollary 5.5. (Indefinite version of Schur's inequalities) *Consider $A = (a_{ij})$ under the conditions of Theorem 5.4, and let $a'_{11} \geq \cdots \geq a'_{rr}$ and $a'_{r+1,r+1} \geq \cdots \geq a'_{r+s,r+s}$ be a rearragement of the diagonal entries $a_{11}, \cdots, a_{rr}$ and $a_{r+1,r+1}, \cdots, a_{r+s,r+s}$, respectively. Then, with $\alpha_{r-m+1} = \cdots = \alpha_r = \alpha_{r+1} = \cdots = \alpha_{r+m} = \lambda_0$, $\sigma_+(A) = \{\alpha_1 \geq \cdots \geq \alpha_{r-m}\}, \sigma_-(A) = \{\alpha_{r+m+1} \geq \cdots \geq \alpha_{r+s}\}$, the following holds:*

(1) If $\leq k \leq r$, then

$$\sum_{j=r-k+1}^{r} \alpha_j \leq \sum_{j=r-k+1}^{r} a'_{jj}.$$

(2) If $r+1 \leq k \leq r+s$, then

$$\sum_{j=1}^{r} \alpha_j + \sum_{j=2r+s-k+1}^{r+s} \alpha_j \leq \sum_{j=1}^{r} a'_{jj+} \sum_{j=2r+s-k+1}^{r+s} a'_{jj}$$

with equality for $k = r + s = n$.

Final Remarks. In [5], indefinite versions of classical results of Schur, Ky-Fan and Rayleigh-Ritz were stated for J-Hermitian matrices, $J = I_r \oplus -I_{n-r}$, $0 < r < n$, under the condition that all eigenvalues correspond to eigenvectors with non-vanishing norm. The results on $W_C^J(A)$ obtained in this paper allow the removal of that restriction. Theorem 5.4 generalizes the fundamental theorem of [5] (Theorem 1.1) and leads to the immediate generalization of all the results in Section 3.3 of [5].

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