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AKIVIS SUPERALGEBRAS AND SPECIALITY

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In honour of Ivan Shestakov on the occasion of his 60th birthday

Abstract: In this paper we define Akivis superalgebra and study enveloping superalgebras for this class of algebras, proving an analogous of the PBW Theorem.

Lie and Malcev superalgebras are examples of Akivis superalgebras. For these particular superalgebras, we describe the connection between the classical enveloping superalgebras and the corresponding generalized concept defined in this work.

Keywords: Lie superalgebras; Malcev superalgebras; Enveloping superalgebras.
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1. Introduction

Definition 1. The supervector space $M = M_0 \oplus M_1$ is called an *Akivis superalgebra* if it is endowed with two operations:

- a bilinear superanticommutative map $[,]$ that induces on M a structure of superalgebra;
- a trilinear map A, compatible with the gradation (i.e, $A(M_\alpha, M_\beta, M_\gamma) \subseteq M_{\alpha+\beta+\gamma}$, all $\alpha, \beta, \gamma \in Z_2$), satisfying the following identity:

$$[[x,y],z] + (-1)^{\alpha(\beta+\gamma)}[[y,z],x] + (-1)^{\gamma(\beta+\alpha)}[[z,x],y] =$$

$$A(x,y,z) + (-1)^{\alpha(\beta+\gamma)}A(y,z,x) + (-1)^{\gamma(\beta+\alpha)}A(z,x,y) -$$
$$-(-1)^{\alpha\beta}A(y,x,z) - (-1)^{\alpha(\beta+\gamma)+\beta\gamma}A(z,y,x) - (-1)^{\gamma\beta}A(x,z,y),$$

for homogeneous elements $x \in M_\alpha, y \in M_\beta, z \in M_\gamma$, all $\alpha, \beta, \gamma \in Z_2$.

This superalgebra will be denoted in this work by $(M, [,], A)$, or simply M, if no confusion arises.

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This definition is a generalization of the notion of Akivis algebra presented by I. Shestakov in [8]. In fact, the even part of an Akivis superalgebra is an Akivis algebra. Akivis algebras were introduced by M. A. Akivis in [1] as local algebras of local analytic loops.

In this paper, we consider Akivis superalgebras over a field K of characteristic different from 2 and 3. It is our aim to study the enveloping superalgebra of an Akivis superalgebra and to prove an analogous of the PBW Theorem. Our approach is similar to the one used in [8] but, as it is expected, the superization of the results imply more elaborated calculations and arguments. This is particularly evident in the definition of the superalgebra $\tilde{V}(M)$ studied in Section 5.

Given a superalgebra W with multiplication $(,)$, we will denote by W^- the superalgebra with underlying supervector space W and multiplication $[,]$ given by $[x,y] = (x,y) - (-1)^{\alpha\beta}(y,x)$ for homogeneous elements $x \in W_\alpha$, $y \in W_\beta$ (and extended by linearity to every element of W). It is known that if W is an associative superalgebra then the superalgebra W^- is a Lie superalgebra, and if W is an alternative superalgebra then W^- is a Malcev superalgebra. Standard calculations show that if W is any superalgebra, the superalgebra W^- is an Akivis superalgebra for the trilinear map $A(x,y,z) = (xy)z - x(yz)$. This superalgebra will be denoted in this work by W^A.

We recall that a superalgebra S is said to be special if it is isomorphic to U^- for some superalgebra U.

It is well known that every Lie superalgebra is isomorphic to a superalgebra S^-, where S is an associative superalgebra. In fact, for any Lie superalgebra L, let $T(L)$ denote the associative tensor superalgebra of the vector space L, and consider its bilateral ideal I generated by the homogeneous elements $x \otimes y - (-1)^{\alpha\beta} y \otimes x - [x,y]$, all $x \in M_\alpha, y \in M_\beta, \alpha, \beta \in Z_2$. Then the associative superalgebra $T(L)/I$ is the universal enveloping superalgebra of L and L is isomorphic to a subsuperalgebra of $(T(L)/I)^-$.

It is an open problem to know if a Malcev superalgebra is isomorphic to S^- for some alternative superalgebra S. This problem was solved only partially in [3] and [4]. There it was shown that, in some cases, a Malcev algebra is isomorphic to a subalgebra of $Nat(T)^-$ for an algebra T, where $Nat(T)$ denotes the generalized alternative nucleous of T.

In this work, we prove that an Akivis superalgebra M defined over a field K of characteristic different from 2 and 3 is isomorphically embedded in $\tilde{U}(M)^A$, where $\tilde{U}(M)$ is its enveloping superalgebra. So M is special.

2. Examples of Akivis superalgebras

Lie superalgebras and more generally Malcev superalgebras are Akivis superalgebras. For the first class, we consider the trilinear map A to be the zero map and, for the second class, we take $A(x,y,z) = 1/6\, SJ(x,y,z)$. Here $SJ(x,y,z)$ denotes the superjacobian

$$SJ(x,y,z) = [[x,y],z] + (-1)^{\alpha(\beta+\gamma)}[[y,z],x] + (-1)^{\gamma(\beta+\alpha)}[[z,x],y],$$

of the homogeneous elements $x \in M_\alpha, y \in M_\beta, z \in M_\gamma$, $(\alpha, \beta, \gamma \in Z_2)$.

Next, we give two examples of Akivis superalgebras which are not included in these classes.

Consider the algebra of octonions O as the algebra obtained by Cayley-Dickson Process from the quaternions Q, with the Z_2 gradation

$$O_0 = Q =< 1, e_1, e_2, e_3 > \quad \text{and} \quad O_1 = e_4 Q =< e_4, e_5, e_6, e_7 > .$$

The multiplication table of the Akivis superalgebra O^A is shown below. Note that the even part of this superalgebra is the simple Lie algebra $sl(2,K) =$ $< e_1, e_2, e_3 >$ together with $1 \in Z(O)$.

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	0	0	0	0	0	0	0	0
e_1	0	0	$-2e_3$	$2e_2$	$-2e_5$	$2e_4$	$2e_7$	$-2e_6$
e_2	0	$2e_3$	0	$-2e_1$	$-2e_6$	$-2e_7$	$2e_4$	$2e_5$
e_3	0	$-2e_2$	$2e_1$	0	$-2e_7$	$2e_6$	$-2e_5$	$2e_4$
e_4	0	$2e_5$	$2e_6$	$2e_7$	-2	0	0	0
e_5	0	$-2e_4$	$2e_7$	$-2e_6$	0	-2	0	0
e_6	0	$-2e_7$	$-2e_4$	$2e_5$	0	0	-2	0
e_7	0	$2e_6$	$-2e_5$	$-2e_4$	0	0	0	-2

This superalgebra is an Akivis superalgebra that is neither a Lie nor a Malcev superalgebra. Indeed, we have that $SJ(e_3, e_7, e_2) \neq 0$ and $((e_4e_2)e_3)e_5 - ((e_2e_3)e_5)e_4 \neq (e_4e_3)(e_2e_5)$.

The second class of examples can be obtained using antiassociative superalgebras (nonassociative Z_2-graded quasialgebras). Consider D a division algebra and n, m natural numbers. In [3] the authors studied the superalgebra $\widetilde{Mat}_{n,m}(D)$ of the $(n+m)\times(n+m)$ matrices over D, with the chess-board Z_2-grading

$$\widetilde{Mat}_{n,m}(D)_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a \in Mat_n(D),\, b \in Mat_m(D) \right\},$$
$$\widetilde{Mat}_{n,m}(D)_1 = \left\{ \begin{pmatrix} 0 & v \\ w & 0 \end{pmatrix} : v \in Mat_{n\times m}(D),\, w \in Mat_{m\times n}(D) \right\}$$

and with multiplication given by

$$\begin{pmatrix} a_1 & v_1 \\ w_1 & b_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & v_2 \\ w_2 & b_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + v_1w_2 & a_1v_2 + v_1b_2 \\ w_1a_2 + b_1w_2 & -w_1v_2 + b_1b_2 \end{pmatrix}.$$

This superalgebra is antiassociative, with even part isomorphic to $Mat_n(D)\times Mat_m(D)$.
$\widetilde{Mat}_{n,m}(D)^A$ is an Akivis superalgebra that is neither a Lie nor a Malcev superalgebra. In fact, if $E_{i,j}$ denotes the $(n+m)\times(n+m)$ matrix with (ij) entry equal to 1 and all the other entries equal to 0, then

$$SJ(E_{1,n+1}, E_{n+1,1}, E_{1,n+1}) \neq 0$$

and

$$2((E_{1,n+1}E_{n+1,1})E_{1,n+1})E_{n+1,1} - ((E_{n+1,1}E_{1,n+1})E_{n+1,1})E_{1,n+1} \neq E^2_{1,n+1}E^2_{n+1,1}.$$

In the case $m = n = 1$, $\tilde{M}at_{1,1}(D)^A$ has abelian even part and multiplication given by the following table, where $a = E_{11}, b = E_{11} - E_{22}, x = E_{12}, y = E_{21}$:

	a	b	x	y
a	0	0	x	$-y$
b	0	0	$2x$	$-2y$
x	$-x$	$-2x$	0	b
y	y	$2y$	b	0

3. Enveloping superalgebra of an Akivis superalgebra

In this section we construct and study the universal enveloping superalgebra of an Akivis superalgebra.

Given the Akivis superalgebras $(M, [,], A)$ and $(N, [,]', A')$, by an Akivis homomorphism we mean a superalgebra homomorphism of degree 0, $f : M \to N$, such that, for all $x, y, z \in M$, $f(A(x, y, z)) = A'(f(x), f(y), f(z))$.

Definition 2. *Let M be an Akivis superalgebra. A pair $(\tilde{U}, \iota)$ is an universal enveloping superalgebra of M if $\tilde{U}$ is a superalgebra and $\iota : M \to \tilde{U}^A$ is an Akivis homomorphism satisfying the following condition: given any superalgebra W and any Akivis homomorphism $\theta : M \to W^A$, there is a unique superalgebra homomorphism of degree 0, $\tilde{\theta} : \tilde{U} \to W$, such that $\theta = \tilde{\theta}\iota$.*

In a similar way to the one used in the classical case for Lie superalgebras we can prove the following:

Proposition 1. *1) The universal enveloping superalgebra of an Akivis superalgebra is unique up to isomorphism;*

2) Let M be an Akivis superalgebra and $(\tilde{U}, \iota)$ its universal enveloping superalgebra. Then the superalgebra $\tilde{U}$ is generated by $\iota(M)$ and K;

3)Consider two Akivis superalgebras M_1 and M_2 with universal enveloping superalgebras $(\tilde{U}_1, \iota_1)$ and $(\tilde{U}_2, \iota_2)$, respectively. If there is an Akivis homomorphism $\phi : M_1 \to M_2$ then there is a superalgebra homomorphism $\tilde{\phi} : \tilde{U}_1 \to \tilde{U}_2$ such that $\tilde{\phi}\iota_1 = \iota_2\phi$.

4)Let M be an Akivis superalgebra with universal enveloping superalgebra $(\tilde{U}, \iota)$. Let I be a graded ideal in M and let J be the graded ideal of $\tilde{U}$ generated by $\iota(I)$. If $m \in M$, the map $\lambda : m + I \to \iota(m) + J$ is an Akivis homomorphism of M/I in $(\tilde{U}/J)^A$ and $(\tilde{U}/J, \lambda)$ is the universal enveloping superalgebra of M/I.

Next we will construct the universal enveloping superalgebra of the Akivis superalgebra $(M, [,], A)$. We start by considering the nonassociative Z-graded tensor algebra of M

$$\tilde{T}(M) = \oplus_{n \in Z} \tilde{T}^n(M),$$

where $\tilde{T}^n(M) = 0$, if $n < 0$, $\tilde{T}^0(M) = K$, $\tilde{T}^1(M) = M$ and $\tilde{T}^n(M) = \oplus_{i=1}^{n-1} \tilde{T}^i(M) \otimes \tilde{T}^{n-i}(M)$, for $n \geq 2$, with multiplication defined by $xy = x \otimes y$.

$\tilde{T}(M)$ is a superalgebra with the Z_2-gradation

$$\tilde{T}(M) = (\oplus_{n\geq 0}\tilde{T}^n(M)_0) \oplus (\oplus_{n\geq 1}\tilde{T}^n(M)_1),$$

where

$$\tilde{T}^n(M)_\gamma = \oplus_{i=1}^{n-1} \oplus_{\alpha+\beta=\gamma} (\tilde{T}^i(M)_\alpha \otimes \tilde{T}^{n-i}(M)_\beta),\ \gamma \in Z_2.$$

Let I be the Z_2-graded ideal of $\tilde{T}(M)$ generated by the homogeneous elements

$$x \otimes y - (-1)^{\alpha\beta} y \otimes x - [x,y] \quad \text{and} \quad (x \otimes y) \otimes z - x \otimes (y \otimes z) - A(x,y,z),$$

for $x \in M_\alpha, y \in M_\beta, z \in M_\gamma$. The quocient algebra $\tilde{U}(M) = \tilde{T}(M)/I$ is a superalgebra with the natural Z_2-gradation induced by the graded ideal I. Consider the map $\iota : M \to \tilde{T}(M) \to \tilde{U}(M)$ obtained by the composition of the canonical injection with the quocient map. It is obvious that ι is an Akivis homomorphism between M and $\tilde{U}(M)^A$.

Proposition 2. *The superalgebra $(\tilde{T}(M)/I, \iota)$ is the universal enveloping superalgebra of the Akivis superalgebra M.*

Proof: Given any superalgebra W and an Akivis homomorphism $f : M \to W^A$, we need to prove that there is a unique homomorphism $\tilde{f} : \tilde{U}(M) \to W$ such that $f = \tilde{f}\iota$.

Using the universal property of tensor products it is easy to see that there is a unique superalgebra homomorphism of degree 0, $f^* : \tilde{T}(M) \to W$, such that $f^*(m) = f(m)$, for all $m \in M$. The fact that f is an Akivis homomorphism implies that $I \subseteq \ker f^*$. Hence, there is an homomorphism of superalgebras of degree 0, $\tilde{f} : \tilde{U}(M) \to W$ such that $\tilde{f}(\iota(m)) = f^*(m) = f(m)$, all $m \in M$. As $\tilde{U}(M)$ is generated by K and $\iota(M)$, the unicity of $\tilde{f}$ follows. ■

4. Enveloping superalgebras of Lie and Malcev superalgebras

In [4], Pérez-Izquierdo studied enveloping algebras of Sabinin algebras, showing that these generalize the classical notions of enveloping algebras for the particular cases of Lie algebras and Malcev algebras. In this section, we will study the connections between the classical definitions and the definition of enveloping superalgebras of Lie and Malcev superalgebras considered as Akivis superalgebras.

Given a Lie superalgebra L, denote by $(\tilde{U}(L), \iota)$ its enveloping superalgebra as an Akivis superalgebra and by $(U(L), \sigma)$ its classical universal enveloping superalgebra. Clearly σ is an Akivis homomorphism from L to $U(L)^A$. So there is a unique homomorphism of superalgebras $\tilde{\sigma}$ such that $\tilde{\sigma}\iota = \sigma$. As $U(L)$ is generated by K and $\sigma(L)$, $\tilde{\sigma}$ is surjective. So $U(L)$ is an epimorphic image of $\tilde{U}(L)$ by $\tilde{\sigma}$, i. e.,

$$U(L) \simeq \tilde{U}(L)/\ker \tilde{\sigma} .$$

Suppose now that N is a Malcev superalgebra. Superizing the theory exposed in [5], we can naturally define the classical enveloping superalgebra of N as the superalgebra $(\tilde{T}(N)/\tilde{I}, \tilde{\iota})$, where $\tilde{I}$ is the graded ideal of $\tilde{T}(N)$ generated by the homogeneous elements

$$a\otimes b-(-1)^{\alpha\beta}b\otimes a-[a,b],\ \ (a,x,y)+(-1)^{\alpha\gamma}(x,a,y),\ \ (x,a,y)+(-1)^{\alpha\xi}(x,y,a),$$

all $a \in N_\alpha$, $b \in N_\beta$, $x \in \tilde{T}(N)_\gamma$, $y \in \tilde{T}(N)_\xi$, and where $(x, y, z) = (xy)z - x(yz)$ denotes the usual associator. The map $\tilde{\iota}$ is the composition of the canonical injection from N to $\tilde{T}(N)$ with the canonical epimorphism $\mu : \tilde{T}(N) \to \tilde{T}(N)/\tilde{I}$.

We will prove that $\tilde{T}(N)/\tilde{I}$ is an epimorphic image of $\tilde{U}(N)$. For this, we show that $I \subseteq \tilde{I}$ and so μ gives rise to the epimorphism $\tilde{\mu} : \tilde{T}(N)/I \to \tilde{T}(N)/\tilde{I}$ defined by $\tilde{\mu}(n+I) = n+\tilde{I}$, all $n \in \tilde{T}(N)$. To see that $I \subseteq \tilde{I}$ notice that, for all homogeneous elements $x \in N_\gamma, y \in N_\xi, z \in N_\alpha$, we have that

$$SJ(x, y, z) - 3(xy)z =$$
$$(-1)^{\alpha(\gamma+\xi)}[(z, x, y) + (-1)^{\alpha\gamma}(x, z, y)] - (-1)^{\alpha\xi}[(x, z, y) + (-1)^{\alpha\xi}(x, y, z)]$$

is an element of $\tilde{I}$ and so $SJ(x, y, z) - 3(xy)z + SJ(y, z, x) - 3(yz)x \in \tilde{I}$. Therefore, $SJ(x, y, z) \in \tilde{I}$ and $(x, y, z) \in \tilde{I}$. As the generators of I are in $\tilde{I}$, we can conclude that $I \subseteq \tilde{I}$.

5. Speciality of Akivis superalgebras

The canonical filtration of $\tilde{T}(M)$, $\tilde{T}_0(M) \subseteq \tilde{T}_1(M) \subseteq \cdots$, where $\tilde{T}_0(M) = K$, and $\tilde{T}_n(M) = \oplus_{i=0}^{n}\tilde{T}^i(M)$, $n > 0$, gives rise to the canonical filtration of $\tilde{U}(M)$, $\tilde{U}_0(M) \subseteq \tilde{U}_1(M) \subseteq \cdots$, where $\tilde{U}_n(M) = \tilde{T}_n(M) + I$.

Associated with this filtration there is the Z-graded superalgebra,

$$gr\tilde{U}(M) = \oplus_{n\in Z}(gr\tilde{U}(M))_n,$$

where $(gr\tilde{U}(M))_n = 0$, if $n < 0$, $(gr\tilde{U}(M))_0 = K$,

$$(gr\tilde{U}(M))_n = \tilde{U}_n(M)/\tilde{U}_{n-1}(M),$$

for $n \geq 1$, with multiplication given by

$$(a + \tilde{U}_{i-1}(M))(b + \tilde{U}_{j-1}(M)) = ab + \tilde{U}_{i+j-1}(M)$$

for $a \in \tilde{U}_i(M)$ and $b \in \tilde{U}_j(M)$. For simplicity we identify $(gr\tilde{U}(M))_1$ with $\iota(M)$.

Now consider the classical tensor algebra $T(M)$ of M, that is naturally a Z-graded associative superalgebra,

$$T(M) = \oplus_{n \in Z} T^n(M)$$

where $T^n(M) = 0$ if $n < 0$, $T^0(M) = K$, $T^n(M) = M \otimes M \otimes \cdots \otimes M$ (n times) if $n > 0$. Let J be the ideal of $T(M)$ generated by the homogeneous elements $x \otimes y - (-1)^{\alpha\beta} y \otimes x$, all $x \in M_\alpha, y \in M_\beta$, $\alpha, \beta \in Z_2$. The associative Z-graded quotient superalgebra $S(M) = T(M)/J$ is called the supersymmetric superalgebra of M. Note that as associative Z-graded algebra the homogeneous spaces of $S(M)$ are $S^n(M) = T^n(M) + J$ and as a superalgebra we have

$$S(M)_\alpha = \oplus_{n \in Z}(\oplus_{\alpha_1 + \cdots + \alpha_n = \alpha}(M_{\alpha_1} \otimes \cdots \otimes M_{\alpha_n} + J)).$$

Since the generators of J lie in $T^2(M)$, we identify $S^0(M)$ with K and $S^1(M)$ with M.

We will now construct from $S(M)$ a nonassociative superalgebra $V(M)$ which will play in this work the role that the symmetric algebra plays in the classical case of the PBW Theorem.

We define the Z-graded supervector space $V(M) = \oplus_{n \in Z} V^n(M)$, where the subspaces $V^n(M)$ are defined by

$$V^n(M) = S^n(M), \text{ if } n \leq 3, \text{ and } V^n(M) = \oplus_{i=1}^{n-1}(V^i(M) \otimes V^{n-i}(M)), \text{ if } n > 3.$$

We turn $V(M)$ into a superalgebra by defining the multiplication for homogeneous elements $v_i \in V^i(M), v_j \in V^j(M)$ by $v_i.v_j = v_i v_j$ if $i + j \leq 3$ and $v_i.v_j = v_i \otimes v_j$ if $i + j > 3$ (where juxtaposition of elements means the product of these elements in $S(M)$).

Lemma 1. *The superalgebra $V(M)$ is the enveloping Akivis superalgebra of the trivial Akivis superalgebra with underlying vector space M, i.e., $V(M) \cong \tilde{T}(M)/I^*$ where I^* is the ideal of $\tilde{T}(M)$ generated by the homogeneous elements $x \otimes y - (-1)^{\alpha\beta} y \otimes x$, $(x \otimes y) \otimes z - x \otimes (y \otimes z)$, all $x \in M_\alpha, y \in M_\beta, z \in M_\gamma$.*

Proof: The inclusion map $\pi : M \to V(M)^A$ is an Akivis homomorphism. Therefore, as $\tilde{T}(M)/I^*$ is the enveloping superalgebra of the trivial Akivis superalgebra obtained from M, there is a superalgebra map of degree 0, $\tilde{\pi} : \tilde{T}(M)/I^* \to V(M)$ such that $\tilde{\pi}(m + I^*) = m$, all $m \in M$.

Notice that in $\tilde{T}(M)/I^*$ we have

$$x \otimes y - (-1)^{\alpha\beta} y \otimes x + I^* = 0$$

and

$$(x \otimes y) \otimes z - x \otimes (y \otimes z) + I^* = 0,$$

for homogeneous elements x, y, z. Hence, using the universal property of the tensor products, we may define linear maps

$$\rho_1 : M \to \tilde{T}(M)/I^*,\ \rho_2 : M \otimes M \to \tilde{T}(M)/I^*,\ \rho_3 : M \otimes M \otimes M \to \tilde{T}(M)/I^*$$

by $\rho_i(a) = a + I^*$, $i = 1, 2, 3$. Clearly, $J \cap M \otimes M \subseteq \ker \rho_2$ and $J \cap M \otimes M \otimes M \subseteq \ker \rho_3$. Therefore, there are linear maps $\tilde{\rho}_i : V^i(M) \to \tilde{T}(M)/I^*$ defined by $\tilde{\rho}_i(a + J) = a + I^*$, i=1, 2, 3. Using once more the universal property of tensor products and induction, one can extend these maps to an homomorphism of superalgebras of degree 0, $\tilde{\rho} : V(M) \to \tilde{T}(M)/I^*$ satisfying $\tilde{\rho}(m) = m + I^*$ for all $m \in M$. The maps $\tilde{\rho}$ and $\tilde{\pi}$ are inverse of each other. So $\tilde{\pi}$ is an isomorphism. ∎

Lemma 2. *There exists an epimorphism of Z-graded superalgebras*

$$\tilde{\tau} : \tilde{T}(M)/I^* \to gr\tilde{U}(M),$$

of degree 0*, such that* $\tilde{\tau}(m + I^*) = \iota(m)$*, all* $m \in M$*.*

Proof: Consider the natural epimorphism of Z-graded algebras $\tau : \tilde{T}(M) \to gr\tilde{U}(M)$ given by $\tau(a) = (a + I) + \tilde{U}_{n-1}(M)$, for each $a \in \tilde{T}^n(M)$. Notice that, since we identify $(gr\tilde{U}(M))_1$ with $\iota(M) = M + I$, then $\tau(m) = \iota(m)$, for all $m \in M$. To see that τ preserves the gradation, recall that for $\alpha \in Z_2$, $\tilde{T}(M)_\alpha = \oplus_{n \geq 0} \tilde{T}^n(M)_\alpha$ and

$$\begin{aligned}(gr\tilde{U}(M))_\alpha &= \oplus_{n\geq 0}(\tilde{U}_n(M)/\tilde{U}_{n-1}(M))_\alpha \\ &= \oplus_{n\geq 0}(\oplus_{i=0}^{n}(\tilde{T}^i(M)_\alpha + I) + \tilde{U}_{n-1}(M)).\end{aligned}$$

So if $a \in \tilde{T}^n(M)_\alpha$, then $\tau(a) = (a+I) + \tilde{U}_{n-1}(M) \in (\tilde{T}^n(M)_\alpha + I) + \tilde{U}_{n-1}(M)$. Hence

$$\tau(\tilde{T}(M)_\alpha) = \sum_{n\geq 0} \tau(\tilde{T}^n(M)_\alpha) \subseteq \oplus_{n\geq 0}(\tilde{T}^n(M)_\alpha + I) + \tilde{U}_{n-1}(M)) \subseteq (gr\tilde{U}(M))_\alpha,$$

as desired. Now for any $x \in M_\alpha, y \in M_\beta, z \in M_\gamma$, consider $\bar{x} = \iota(x)$, $\bar{y} = \iota(y)$, $\bar{z} = \iota(z) \in (gr\tilde{U}(M))_1$. Then, in $gr\tilde{U}(M)$ there holds

$$\bar{x}\bar{y}-(-1)^{\alpha\beta}\bar{y}\bar{x} = (\iota(x)\iota(y)-(-1)^{\alpha\beta}\iota(y)\iota(x))+\tilde{U}_1(M) = \iota([x,y])+\tilde{U}_1(M) = 0;$$

$$\begin{aligned}(\bar{x}\bar{y})\bar{z} - \bar{x}(\bar{y}\bar{z}) &= (\iota(x)\iota(y))\iota(z) - \iota(x)(\iota(y)\iota(z)) + \tilde{U}_2(M) \\ &= \iota(A(x,y,z)) + \tilde{U}_2(M) \\ &= 0.\end{aligned}$$

This implies that $I^* \subseteq \operatorname{Ker}\tau$. Therefore, there is an epimorphism of Z-graded superalgebras $\tilde{\tau} : \tilde{T}(M)/I^* \to gr\tilde{U}(M)$ such that $\tilde{\tau}(m+I^*) = \tau(m) = \iota(m)$, for all $m \in M$. ■

From the two previous lemmas, we know that the composite map

$$\tilde{\tau}\tilde{\pi}^{-1} : V(M) \to gr\tilde{U}(M)$$

is an epimorphism of superalgebras satisfying $\tilde{\tau}\tilde{\pi}^{-1}(m) = \iota(m)$ for all $m \in M$. It is our aim, to prove that this epimorphism is in fact an isomorphism. For this we need to endow $V(M)$ with a convenient superalgebra structure.

Let $\{e_r, r \in \Delta\}$ be a basis of M indexed by the totally ordered set $\Delta = \Delta_0 \cup \Delta_1$ satisfying the following: $\{e_r : r \in \Delta_\alpha\}$ is a basis of M_α, $\alpha = 0, 1$, and $r < s$ if $r \in \Delta_0$, $s \in \Delta_1$. It is well known that, in these conditions,

$$\{e_{r_1}e_{r_2} : r_1 \leq r_2, \text{ and } r_1 < r_2 \text{ if } r_1, r_2 \in \Delta_1\}$$

is a basis of $V^2(M)$ and

$$\{e_{r_1}e_{r_2}e_{r_3} : r_1 \leq r_2 \leq r_3, \text{ and } r_p < r_{p+1} \text{ if } r_p, r_{p+1} \in \Delta_1\}$$

is a basis of $V^3(M)$. In the supervector space $V(M)$ we define a new multiplication denoted by $*$ in the following way: if $a \in V^i(M)$, $b \in V^j(M)$, then

$$a * b = a \otimes b \text{ if } i + j > 3;$$

if $i + j \leq 3$ the multiplication is defined on the basis elements by the following identities (for simplicity we use $\bar{r}, \bar{s}, \bar{k}$ to denote the degrees of e_r, e_s, e_k, respectively):

$$e_r * e_s = \begin{cases} e_r e_s, & \text{if } r \leq s \ \text{ and } \ r \neq s \ \text{ if } \ r \in \Delta_1; \\ 1/2[e_r, e_r], & \text{if } r = s \in \Delta_1; \\ (-1)^{\bar{r}\bar{s}} e_s e_r + [e_r, e_s], & \text{if } r > s. \end{cases}$$

$$(e_r e_s) * e_k =$$

$$= \begin{cases} e_r e_s e_k, & \text{if } r \leq s \leq k \ \text{ and } \ k \neq s \ \text{ if } \ s \in \Delta_1; \\ A(e_r, e_s, e_s) + 1/2 e_r * [e_s, e_s], & \text{if } r < s = k \ \text{ and } \ s \in \Delta_1; \\ (-1)^{\bar{k}\bar{s}} e_r e_k e_s + e_r * [e_s, e_k] + & \\ +A(e_r, e_s, e_k) - (-1)^{\bar{s}\bar{k}} A(e_r, e_k, e_s), & \text{if } r \leq k < s \ \text{ and } \ r \neq k \ \text{ if } \ r \in \Delta_1; \\ -1/2[e_r, e_r] * e_s + e_r * [e_s, e_r] + & \\ +A(e_r, e_s, e_r) + A(e_r, e_r, e_s), & \text{if } r = k < s \ \text{ and } \ r \in \Delta_1; \\ (-1)^{\bar{k}(\bar{r}+\bar{s})} e_k e_r e_s + & \\ +(-1)^{\bar{k}\bar{s}} [e_r, e_k] * e_s + e_r * [e_s, e_k] - & \\ -(-1)^{\bar{s}\bar{k}} A(e_r, e_k, e_s) + A(e_r, e_s, e_k), & \text{if } k < r \leq s. \end{cases}$$

$$e_r * (e_s e_k) =$$

$$= \begin{cases} e_r e_s e_k - A(e_r, e_s, e_k), & \text{if } r \leq s \leq k \ \text{ and } \ r \neq s \ \text{ if } \ s \in \Delta_1; \\ 1/2[e_r, e_r] * e_k - A(e_r, e_r, e_k), & \text{if } r = s < k \ \text{ and } \ r \in \Delta_1; \\ (-1)^{\bar{r}\bar{s}} e_s e_r e_k - A(e_r, e_s, e_k) + & \\ +[e_r, e_s] * e_k, & \text{if } s < r \leq k \ \text{ and } \ r \neq k \ \text{ if } \ r \in \Delta_1; \\ 1/2(-1)^{\bar{r}\bar{s}} e_s * [e_r, e_r] + [e_r, e_s] * e_r - & \\ -A(e_r, e_s, e_r) + (-1)^{\bar{r}\bar{s}} A(e_s, e_r, e_r), & \text{if } s < r = k \ \text{ and } \ r \in \Delta_1; \\ (-1)^{\bar{r}(\bar{k}+\bar{s})} e_s e_k e_r + & \\ +(-1)^{\bar{r}\bar{s}} e_s * [e_r, e_k] + [e_r, e_s] * e_k + & \\ +(-1)^{\bar{r}\bar{s}} A(e_s, e_r, e_k) - A(e_r, e_s, e_k) - & \\ -(-1)^{(\bar{s}+\bar{k})\bar{r}} A(e_s, e_k, e_r), & \text{if } s \leq k < r. \end{cases}$$

Note that if we consider a basis element $e_p e_q$ of $V^2(M)$ we always assume $p \leq q$ and $p \neq q$ if $p, q \in \Delta_1$.

With this multiplication $V(M)$ becomes a superalgebra that will be denoted by $\tilde{V}(M)$.

Lemma 3. *There is an homomorphism of superalgebras $\hat{\epsilon} : \tilde{U}(M) \to \tilde{V}(M)$, of degree 0, satisfying $\hat{\epsilon}(\iota(m)) = m$, for all $m \in M$.*

Proof: Denote the operations in the Akivis superalgebra $\tilde{V}(M)^A$ by

$$< x, y >= x * y - (-1)^{\alpha\beta} y * x\,, \quad < x, y, z >= (x * y) * z - x * (y * z),$$

for $x \in M_\alpha$, $y \in M_\beta$, $z \in M_\gamma$.

We start by proving that the inclusion map $\epsilon : M \to \tilde{V}(M)^A$ is an homomorphism of Akivis superalgebras. For this, it is enough to show that

$$[e_r, e_s] =< e_r, e_s > \quad \text{and} \quad A(e_r, e_s, e_k) =< e_r, e_s, e_k >\,,$$

for the basis elements e_r, e_s, e_k considered above. It is quite simple to see that the first of these two inequalities holds. For the second one, we have to consider several cases. Here we present only four of them, being the other cases similar.

(1) $r = s = k \in \Delta_1$:

$$< e_r, e_r, e_r >= (e_r * e_r) * e_r - e_r * (e_r * e_r) = 1/2([e_r, e_r] * e_r - e_r * [e_r, e_r]).$$

As $[e_r, e_r] \in M_0$, we have $[e_r, e_r] = \sum_{t \in \Delta_0} \alpha_t e_t$ (sum with finite support), for scalars $\alpha_t \in K$. Therefore, as $t < r$ for any $t \in \Delta_0$, there holds

$$[e_r, e_r] * e_r = \sum_{t \in \Delta_0} \alpha_t e_t * e_r = \sum_{t \in \Delta_0} \alpha_t e_t e_r.$$

In a similar way, we see that

$$e_r * [e_r, e_r] = \sum_{t \in \Delta_0} \alpha_t (e_t e_r + [e_r, e_t]) = \sum_{t \in \Delta_0} \alpha_t e_t e_r + [e_r, [e_r, e_r]].$$

Hence

$$< e_r, e_r, e_r >= -1/2[e_r, [e_r, e_r]] = 1/2[[e_r, e_r], e_r].$$

On the other hand, from the definition of Akivis superalgebra, we have that

$$SJ(e_r, e_r, e_r) = 3[[e_r, e_r], e_r] = 6A(e_r, e_r, e_r).$$

Thus $< e_r, e_r, e_r >= A(e_r, e_r, e_r)$.

(2) $r \leq k < s$ and $r \neq k$ if $r \in \Delta_1$:

$$\begin{aligned}
< e_r, e_s, e_k > &= (e_r e_s) * e_k - e_r * ((-1)^{\bar{s}\bar{k}} e_k e_s + [e_s, e_k]) \\
&= (-1)^{\bar{s}\bar{k}} e_r e_k e_s + A(e_r, e_s, e_k) - (-1)^{\bar{s}\bar{k}} A(e_r, e_k, e_s) + \\
&\quad + e_r * [e_s, e_k] - e_r * [e_s, e_k] - (-1)^{\bar{s}\bar{k}} e_r e_k e_s + (-1)^{\bar{s}\bar{k}} A(e_r, e_k, e_s) \\
&= A(e_r, e_s, e_k).
\end{aligned}$$

(3) $r = k < s$ and $r \in \Delta_1$:

$$\begin{aligned}
< e_r, e_s, e_r > &= -1/2[e_r, e_r] * e_s + e_r * [e_s, e_r] + A(e_r, e_s, e_r) + A(e_r, e_r, e_s) + \\
&\quad + e_r * (e_r e_s) - e_r * [e_s, e_r] \\
&= A(e_r, e_s, e_r) + A(e_r, e_r, e_s) - 1/2[e_r, e_r] * e_s + 1/2[e_r, e_r] * e_s - \\
&\quad - A(e_r, e_r, e_s) \\
&= A(e_r, e_s, e_r).
\end{aligned}$$

(4) $k < s < r$:

$$\begin{aligned}
< e_r, e_s, e_k > &= (-1)^{\bar{s}\bar{r}} (e_s e_r) * e_k + [e_r, e_s] * e_k - (-1)^{\bar{s}\bar{k}} e_r * (e_k e_s) - e_r * [e_s, e_k] \\
&= (-1)^{(\bar{s}+\bar{k})\bar{r}} [e_s, e_k] * e_r + (-1)^{\bar{s}\bar{r}} e_s * [e_r, e_k] + [e_r, e_s] * e_k - \\
&\quad - e_r * [e_s, e_k] - (-1)^{(\bar{s}+\bar{r})\bar{k}} e_k * [e_r, e_s] - (-1)^{\bar{s}\bar{k}} [e_r, e_k] * e_s + \\
&\quad + (-1)^{\bar{s}\bar{r}} A(e_s, e_r, e_k) - (-1)^{(\bar{s}+\bar{k})\bar{r}} A(e_s, e_k, e_r) + \\
&\quad + (-1)^{\bar{s}\bar{k}+(\bar{s}+\bar{k})\bar{r}} A(e_k, e_s, e_r) - (-1)^{(\bar{s}+\bar{r})\bar{k}} A(e_k, e_r, e_s) + \\
&\quad + (-1)^{\bar{s}\bar{k}} A(e_r, e_k, e_s) \\
&= \text{ (since } [x, y] =< x, y >, \text{ all } x, y \in M) \\
&\quad SJ(e_r, e_s, e_k) - ((-1)^{\bar{s}\bar{r}} A(e_s, e_r, e_k) - (-1)^{(\bar{s}+\bar{k})\bar{r}} A(e_s, e_k, e_r) + \\
&\quad + (-1)^{\bar{s}\bar{k}+(\bar{s}+\bar{k})\bar{r}} A(e_k, e_s, e_r) - (-1)^{(\bar{s}+\bar{r})\bar{k}} A(e_k, e_r, e_s) + \\
&\quad + (-1)^{\bar{s}\bar{k}} A(e_r, e_k, e_s)) \\
&= A(e_r, e_s, e_k) \text{ (by the definition of Akivis superalgebra).}
\end{aligned}$$

As ϵ is an Akivis homomorphism, from the definition of enveloping superalgebra, there is an homomorphism of superalgebras of degree 0, $\hat{\epsilon} : \tilde{U}(M) \to \tilde{V}(M)$ satisfying $\hat{\epsilon}(\iota(m)) = m$, all $m \in M$. ■

Theorem 1. *The Z-graded superalgebras $V(M)$ and $gr\tilde{U}(M)$ are isomorphic.*

Proof: The superalgebra $\tilde{V}(M)$ has a natural filtration defined by the sequence of subspaces $\tilde{V}_n(M) = \oplus_{i=0}^n V^i(M)$. So we may consider the associated Z-graded superalgebra $gr\tilde{V}(M)$. (As usual we identify $(gr\tilde{V}(M))_1$ with M). Since the map $\hat{\epsilon}$, considered in the previous lemma, is an homomorphism of superalgebras, we have $\hat{\epsilon}(\tilde{U}_n(M)) \subseteq \tilde{V}_n(M)$. So, we may define $\tilde{\epsilon} : gr\tilde{U}(M) \to gr\tilde{V}(M)$ by $\tilde{\epsilon}(a_i + \tilde{U}_{i-1}(M)) = \hat{\epsilon}(a_i) + \tilde{V}_{i-1}(M)$. This map is an homomorphism of Z-graded superalgebras of degree 0 and satisfies $\tilde{\epsilon}(\iota(m)) = m$, all $m \in M$.

We now return to the algebra $V(M)$. For $n \geq 1$, the map $\mu_n : V^n(M) \to (gr\tilde{V}(M))_n$ defined by $\mu_n(v) = v + \tilde{V}_{n-1}(M)$ is an isomorphism of vector spaces. So taking $\mu_0 = Id_K$,

$$\mu = \oplus_{n\geq 0}\mu_n : V(M) \to gr\tilde{V}(M)$$

is an isomorphism of Z- graded vector spaces. Looking at the formulas which define the multiplication in $\tilde{V}(M)$, it is easy to see that this is in fact an isomorphism of Z-graded superalgebras. The composite homomorphisms

$$\mu^{-1}\tilde{\epsilon} : gr\tilde{U}(M) \to gr\tilde{V}(M) \to V(M)$$

and

$$\tilde{\tau}\tilde{\pi}^{-1} : V(M) \to \tilde{T}(M)/I^* \to gr\tilde{U}(M)$$

(recall the preceeding lemmas) are inverse of each other. So the result follows. ∎

The following results are immediate consequence of this theorem.

Corollary 1. *The canonical map $\iota : M \to \tilde{U}(M)$ is injective.*

Corollary 2. *Any Akivis superalgebra, defined over a field of characteristic different from 2 and 3, is special.*

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