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COMPLEX HIGH ORDER TODA AND VOLTERRA LATTICES

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ABSTRACT: Given a solution of a high order Toda lattice we construct a one parameter family of new solutions. In our method, we use a set of Bäcklund transformations in such a way that each new generalized Toda solution is related to a generalized Volterra solution.

KEYWORDS: Toda lattice, Volterra lattice, sequences of polynomials, Bäcklund transformation.
AMS SUBJECT CLASSIFICATION (2000): Primary 37F05; Secondary 33C45.

1. Introduction

In [7] was studied the construction of a solution of the *Toda lattice*

$$\left.\begin{array}{rcl} \dot{\alpha}_n(t) & = & \lambda_{n+1}^2(t) - \lambda_n^2(t) \\ \dot{\lambda}_{n+1}(t) & = & \dfrac{\lambda_{n+1}(t)}{2}\left[\alpha_{n+1}(t) - \alpha_n(t)\right] \end{array}\right\}, \quad n \in \mathbb{Z}, \tag{1}$$

from another given solution, considering sequences $\{\alpha_n(t)\,,\,\lambda_n(t)\}\,,\, n \in \mathbb{Z}$, of real functions (here and in what following, the dot means differentiation with respect to $t \in \mathbb{R}$). Both solutions of (1) were linked to each other by a *Bäcklund transformation* (or *Miura transformation*)

$$\lambda_{n+1}^2(t) = \gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\,, \quad \alpha_n(t) = \gamma_{2n-1}^2(t) + \gamma_{2n}^2(t) + C\,, \quad n \in \mathbb{Z}, \tag{2a}$$

$$\widetilde{\lambda}_{n+1}^2(t) = \gamma_{2n+1}^2(t)\gamma_{2n+2}^2(t)\,, \quad \widetilde{\alpha}_n(t) = \gamma_{2n}^2(t) + \gamma_{2n+1}^2(t) + C\,, \quad n \in \mathbb{Z}\,, \tag{2b}$$

with $C = 0$, where $\{\gamma_n(t)\}$ is a solution of the *Volterra lattice*

$$\dot{\gamma}_{n+1}(t) = \gamma_{n+1}(t)\left(\gamma_{n+2}(t) - \gamma_n(t)\right)\,, \quad n \in \mathbb{Z}\,. \tag{3}$$

In [3] the authors generalize this analysis to the case of Toda lattices where $\alpha_n(t)$ and $\lambda_n(t)$ are complex functions of $t \in \mathbb{R}$. A semi-infinite lattice, i.e., (1) with $n \in \mathbb{N} = \{1, 2, \ldots\}$, was studied. Moreover, for each solution

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of the Toda lattice, a family of new solutions was obtained for this lattice, each one associated with a different solution of the Volterra lattice (3) by a Bäcklund transformation like (2a)-(2b), where it is possible that $C \neq 0$.

In this paper, we generalize the analysis given in [7] and [3] to the kind of Toda and Volterra lattices studied in [1] and [2]. For this purpose we consider the family $\{J(t)\}$, $t \in \mathbb{R}$, of tridiagonal infinite matrices given by

$$J(t) = \begin{pmatrix} \alpha_1(t) & \lambda_2(t) & & \\ \lambda_2(t) & \alpha_2(t) & \lambda_3(t) & \\ & \lambda_3(t) & \alpha_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad t \in \mathbb{R}, \tag{4}$$

where $\{\alpha_n(t)\}$, $\{\lambda_n(t)\}$ are two sequences of complex functions with real variable $t \in \mathbb{R}$.

In all the following we assume $\lambda_n(t) \neq 0$, $n \in \mathbb{N}$, $t \in \mathbb{R}$. In the sequel we assume $p \in \mathbb{N}$ fixed. Here and in the following, for each $m = 0, 1, \ldots$ and for any matrix $M(t)$, we denote by $M^m_{i,j}(t)$, $i, j = 0, 1, \ldots$, the entry in the $(i+1)$-th row and $(j+1)$-th column of matrix $(M(t))^m$. In particular, $M^0_{i,j}(t) = \delta_{ij}$ are the entries of the identity matrix $I = (M(t))^0$. Also, we replace $M^1_{i,j}(t)$ by $M_{i,j}(t)$. Also, we denote by $J_n(t)$ the finite submatrix formed by the first n rows and columns of $J(t)$.

Definition 1. We say that $\{J(t)\}$, $t \in \mathbb{R}$, is a *solution of the high order Toda lattice*, or a *generalized Toda solution*, if we have

$$\left.\begin{array}{rcl} \dot{J}_{n,n}(t) & = & J_{n,n+1}(t) J^p_{n,n+1}(t) - J_{n-1,n}(t) J^p_{n-1,n}(t) \\ \dot{J}_{n,n+1}(t) & = & \frac{1}{2} J_{n,n+1}(t) \left[J^p_{n+1,n+1}(t) - J^p_{n,n}(t) \right] \end{array}\right\}, \quad n = 0, 1, \ldots. \tag{5}$$

In the same way, consider the family $\{\Gamma(t)\}$, $t \in \mathbb{R}$, of infinite matrices,

$$\Gamma(t) = \begin{pmatrix} 0 & \gamma_2(t) & & \\ \gamma_2(t) & 0 & \gamma_3(t) & \\ & \gamma_3(t) & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \quad t \in \mathbb{R}. \tag{6}$$

Definition 2. We say that $\{\Gamma(t)\}$, $t \in \mathbb{R}$, is a *solution of the high order Volterra lattice*, or a *generalized Volterra solution*, if we have

$$\dot{\Gamma}_{n-1,n}(t) = \frac{1}{2} \Gamma_{n-1,n}(t) \left[(\Gamma^2(t) + CI)^p_{n,n} - (\Gamma^2(t) + CI)^p_{n-1,n-1} \right], \quad n \in \mathbb{N}, \tag{7}$$

for some $C \in \mathbb{C}$.

Note that, for $p = 1$ and $n \in \mathbb{N}$, (5) and (7) coincide, respectively, with (1) and (3).

The main result of our work is the following.

Theorem 1. *Let $\{J(t)\}$, $t \in \mathbb{R}$, be a generalized Toda solution. Let $C \in \mathbb{C}$ be such that*

$$\det(J_n(t) - CI_n) \neq 0 \tag{8}$$

for each $n \in \mathbb{N}$ and for all $t \in \mathbb{R}$. Then there exists $\{\Gamma(t)\}$, $t \in \mathbb{R}$, generalized Volterra solution, and there exists another generalized Toda solution $\{\widetilde{J}(t)\}$, $t \in \mathbb{R}$, with

$$\widetilde{J}(t) = \begin{pmatrix} \widetilde{\alpha}_1(t) & \widetilde{\lambda}_2(t) & & & \\ \widetilde{\lambda}_2(t) & \widetilde{\alpha}_2(t) & \widetilde{\lambda}_3(t) & & \\ & \widetilde{\lambda}_3(t) & \widetilde{\alpha}_3(t) & \ddots & \\ & & \ddots & \ddots & \end{pmatrix}, \quad t \in \mathbb{R},$$

such that (2a)-(2b) *hold.*
Moreover, for each C in the above conditions, we have that $\{\widetilde{\lambda}_{n+1}(t)\}$, $\{\widetilde{\alpha}_n(t)\}$, $\{\gamma_n^2(t)\}$, $n \in \mathbb{N}$, are the unique sequences verifying (2a) *and* (2b).

The main tool in the proof of Theorem 1 is the sequence of monic polynomials $\{P_n(t,z)\}$, $n \in \mathbb{N}$, associated with the matrix $J(t)$ for each $t \in \mathbb{R}$ (see (4)). These polynomials are generated by the three-term recurrence relation

$$\left.\begin{array}{l} P_{n+1}(t,z) = (z - \alpha_{n+1}(t))P_n(t,z) - \lambda_{n+1}^2(t)P_{n-1}(t,z)\,, \quad n = 0,1,\ldots \\ P_{-1}(t,z) \equiv 0\,, \quad P_0(t,z) \equiv 1\,. \end{array}\right\} \tag{9}$$

In the following result we determinate a necessary and sufficient condition over $\{P_n(t,z)\}$, $n \in \mathbb{N}$, in order to the coefficients of (9) define a generalized Toda solution.

Theorem 2. *With the above notation, $\{J(t)\}$, $t \in \mathbb{R}$, is a generalized Toda solution if and only if*

$$\dot{P}_n(t,z) = -\sum_{j=1}^{p} J^p_{n,n-j}(t)\lambda_{n-j+2}(t)\cdots\lambda_{n+1}(t)P_{n-j}(t,z)\,, \quad t \in \mathbb{R}, \tag{10}$$

for each $n \in \mathbb{N}$ and all $z \in \mathbb{C}$.

Let

$$W(z) = \sum_{k\geq 0} \frac{w_k}{z^{k+1}}$$

be a formal power series at $z=\infty$ and let $f_n(z) = P_n^{(1)}(z)/P_n(z)$ be the n-diagonal Padé approximant of $W(z)$, $n \in \mathbb{N}$. It is well known that the sequences of polynomials $\{P_n(z)\}$ and $\{P_n^{(1)}(z)\}$ verify the same three-term recurrence relation, whose coefficients define a tridiagonal matrix J. Moreover, we have

$$w_k = \langle J^k e_0, e_0\rangle = e_0^T J^k e_0\,, \quad k=0,1,\ldots\,, \quad e_0 = (1,0,\ldots)^T\,,$$

and, in a formal sense,

$$W(z) = \langle (zI-J)^{-1} e_0, e_0\rangle$$

for z in the resolvent set of J. In [2] was established that, when the entries of J are bounded and depend on $t\in\mathbb{R}$, then $\{J(t)\}$ is a generalized Toda solution if and only if

$$\dot{w}_k(t) = w_k(t)w_p(t) - w_{k+p}(t)\,, \quad k\in\mathbb{N}\,, \tag{11}$$

holds. From this and Theorem 2 we have the following consequence.

Corollary 1. *Let $\{P_n(t,z)\}$ be the sequence of monic polynomials defined by* (9)*. Assume that, for each $t\in\mathbb{R}$, there exists $M(t)\in\mathbb{R}_+$ such that*

$$\sup_{n\in\mathbb{N}} \{|\alpha_n(t)|\,,\, |\lambda_n(t)|\} \leq M(t)\,.$$

Then $\{P_n(t,z)\}$ verify (10) *if and only if the sequence of moments associated with $W(z)$, $\{w_n\}$, verify* (11)*.*

We present the proof of Theorem 2 in Section 2. In Section 3 we analyze the Bäcklund transformation (2a)-(2b) under the perspective of sequences of polynomials generated by a three term recurrence relation. The rest of the paper, Section 4, is devoted to prove Theorem 1.

2. Proof of Theorem 2

Firstly, we shall show that (10) is a necessary condition. Assume that $\{J(t)\}$, $t\in\mathbb{R}$, is a generalized Toda solution. The system (5) was described in [1] as representation in Lax pair

$$\dot{J}(t) = [J(t), A(t)]\,, \tag{12}$$

where $[J(t), A(t)] = J(t)A(t) - A(t)J(t)$ is the commutator of $J(t)$ and $A(t)$, being for all $t \in \mathbb{R}$

$$A(t) = \frac{1}{2}\begin{pmatrix} 0 & -J^p_{0,1}(t) & \cdots & -J^p_{0,p}(t) & 0 & \\ J^p_{0,1}(t) & 0 & -J^p_{1,2}(t) & \cdots & -J^p_{1,p+1}(t) & \ddots \\ \vdots & \ddots & \ddots & \ddots & & \ddots \\ J^p_{0,p}(t) & & & & & \\ 0 & J^p_{1,p+1}(t) & & & & \\ & & & & & \\ & \ddots & \ddots & & & \end{pmatrix}, \quad (13)$$

i.e., $A(t)$ is a skew-symmetric $(2p+1)$-diagonal matrix whose low triangular part coincides with $(J(t))^p$. Thus, the structure of $A(t)$ depends on the fixed number $p \in \mathbb{N}$.

If we define

$$\widehat{p}_n(t,z) := \frac{P_n(t,z)}{\lambda_2(t)\cdots\lambda_{n+1}(t)}, \quad n \in \mathbb{N}, \quad (\widehat{p}_0(t,z) \equiv 1\,,\, \widehat{p}_{-1}(t,z) \equiv 0)$$

then, since (9), it is easy to prove that for each $t \in \mathbb{R}$ the sequence $\{\widehat{p}_n(t,z)\}$, $n = 0, 1, \dots$, verifies

$$\lambda_{n+1}(t)\widehat{p}_{n-1}(t,z) + (\alpha_{n+1}(t) - z)\widehat{p}_n(t,z) + \lambda_{n+2}(t)\widehat{p}_{n+1}(t,z) = 0\,. \quad (14)$$

We are going to prove

$$\dot{\widehat{p}}_n(t,z) = -\sum_{j=1}^{p} J^p_{n,n-j}(t)\widehat{p}_{n-j}(t,z) + \frac{\left(J^p_{0,0}(t) - J^p_{n,n}(t)\right)\widehat{p}_n(t,z)}{2}\,. \quad (15)$$

For this purpose, we can rewrite (14) as

$$(J(t) - zI)\,\mathcal{P}(t,z) = (0,0,\dots)^T\,, \quad (16)$$

where we understand $\mathcal{P}(t,z) := (\widehat{p}_0(t,z)\,,\,\widehat{p}_1(t,z),\dots)^T$ as a sequence. Taking derivatives in (16), and taking into account (12) and again (16), we obtain

$$\begin{aligned} \dot{J}(t)\mathcal{P}(t,z) &+ (J(t) - zI)\,\dot{\mathcal{P}}(t,z) \\ &= (J(t)A(t) - A(t)J(t))\,\mathcal{P}(t,z) + (J(t) - zI)\,\dot{\mathcal{P}}(t,z) \\ &= (J(t) - zI)\left(A(t)\mathcal{P}(t,z) + \dot{\mathcal{P}}(t,z)\right) = 0\,. \end{aligned} \quad (17)$$

Taking $n = 0, 1, \ldots$ successively in (14) we can see that the only solutions of $(J(t) - zI)\, X = (0, 0, \ldots)^T$ are the sequences $X = \mu \mathcal{P}(t, z)\,,\ \mu = \mu(t) \in \mathbb{C}$. Therefore, because of (17), we have

$$A(t)\mathcal{P}(t, z) + \dot{\mathcal{P}}(t, z) = \mu \mathcal{P}(t, z) \tag{18}$$

for some $\mu \in \mathbb{C}$. We understand (18), in a formal sense, as a set of relations. In particular, from the first of these relations we deduce

$$\mu = \mu \widehat{p}_0(t, z) = -\frac{1}{2} \sum_{j=1}^{p} J^p_{0,j}(t) \widehat{p}_j(t, z) \tag{19}$$

(see (13)). From the $(n + 1)$-th row of (18) we obtain for all $n = 0, 1, \ldots$

$$\begin{aligned} &\dot{\widehat{p}}_n(t, z) \\ &\quad = -\frac{1}{2} \sum_{j=1}^{p} J^p_{n,n-j}(t) \widehat{p}_{n-j}(t, z) + \frac{1}{2} \sum_{j=1}^{p} J^p_{n,n+j}(t) \widehat{p}_{n+j}(t, z) + \mu \widehat{p}_n(t, z)\,. \end{aligned} \tag{20}$$

Moreover, since (16) we have

$$(J^p(t) - z^p I)\, \mathcal{P}(t, z) = (0, 0, \ldots)^T$$

or, what is the same, for all $n = 0, 1, \ldots$

$$\begin{aligned} \sum_{j=1}^{p} J^p_{n,n-j}(t) \widehat{p}_{n-j}(t, z) + J^p_{n,n}(t) \widehat{p}_n(t, z) + \sum_{j=1}^{p} J^p_{n,n+j}(t) \widehat{p}_{n+j}(t, z) \\ = z^p \widehat{p}_n(t, z)\,, \end{aligned} \tag{21}$$

where $J^p_{n,k}(t) = 0$ for $k < 0$. Due to (20) and (21),

$$\dot{\widehat{p}}_n(t, z) = -\sum_{j=1}^{p} J^p_{n,n-j}(t) \widehat{p}_{n-j}(t, z) + \mu \widehat{p}_n(t, z) + \frac{1}{2} \left(z^p - J^p_{n,n}(t) \right) \widehat{p}_n(t, z)\,. \tag{22}$$

On the other hand, taking $n = 0$ in (21), and from (19) we obtain

$$\mu = -\frac{1}{2} \left(z^p - J^p_{0,0}(t) \right)$$

and replacing this expression in (22) we arrive to (15).

Now, using (5),

$$\dot{\lambda}_i(t) = \frac{1}{2} \lambda_i(t) \left(J^p_{i-1,i-1}(t) - J^p_{i-2,i-2}(t) \right)\,, \quad i \in \mathbb{N}\,.$$

Then,

$$\frac{d}{dt}\left(\lambda_2(t)\cdots\lambda_{n+1}(t)\right)=\frac{1}{2}\lambda_2(t)\cdots\lambda_{n+1}(t)\left(J^p_{n,n}(t)-J^p_{0,0}(t)\right)\,.$$

Substituting this expression and (15) in

$$\dot{P}_n(t,z)=\frac{d}{dt}\left(\lambda_2(t)\cdots\lambda_{n+1}(t)\right)\widehat{p}_n(t,z)+\lambda_2(t)\cdots\lambda_{n+1}(t)\,\dot{\widehat{p}}_n(t,z)$$

we arrive to (10).

Conversely, we shall to prove that (10) is a sufficient condition to obtain a solution of the high order Toda lattice. Let $\{P_n(t,z)\}$ be the sequence of polynomials given by (9) and assume that (10) is verified. We want to prove that the sequence $\{\alpha_n(t),\, \lambda_n^2(t)\}$, given in (9), defines a matrix $J(t)$ such that (5) is verified.

Taking derivatives with respect to t in (9),

$$\begin{aligned}\dot{P}_{n+1}(t,z)=-\dot{\alpha}_{n+1}(t)P_n(t,z)+(z-\alpha_{n+1}(t))\dot{P}_n(t,z)&\\ -2\lambda_{n+1}(t)\dot{\lambda}_{n+1}(t)P_{n-1}(t,z)-\lambda^2_{n+1}(t)\dot{P}_{n-1}(t,z)\,.&\end{aligned}$$

From this and (10),

$$\begin{aligned}&-\sum_{j=1}^{p}J^p_{n+1,n-j+1}(t)\lambda_{n-j+3}(t)\cdots\lambda_{n+2}(t)P_{n-j+1}(t,z)\\ &\quad=-(z-\alpha_{n+1}(t))\sum_{j=0}^{p-1}J^p_{n,n-j-1}(t)\lambda_{n-j+1}(t)\cdots\lambda_{n+1}(t)P_{n-j-1}(t,z)\\ &\qquad\quad-\dot{\alpha}_{n+1}(t)P_n(t,z)-2\lambda_{n+1}(t)\dot{\lambda}_{n+1}(t)P_{n-1}(t,z)\\ &\qquad\quad+\lambda^2_{n+1}(t)\sum_{j=1}^{p}J^p_{n-1,n-j-1}(t)\lambda_{n-j+1}(t)\cdots\lambda_n(t)P_{n-j-1}(t,z)\,.\end{aligned}\tag{23}$$

Comparing the coefficients of z^n in (23),

$$-J^p_{n+1,n}(t)\lambda_{n+2}(t)=-\dot{\alpha}_{n+1}(t)-J^p_{n,n-1}(t)\lambda_{n+1}(t)\,,\quad n\in\mathbb{N}\,.\tag{24}$$

Moreover, $J_{m,m+1}(t)=\lambda_{m+2}(t)\,,\ J_{m,m}(t)=\alpha_{m+1}(t)\,,\ m=0,1,\ldots,$ being $(J(t))^p$ a symmetric matrix. Thus, (24) is the first part of (5).

Now, comparing the coefficients of z^{n-1} in (23),

$$
\begin{aligned}
&- J^p_{n+1,n}(t)\lambda_{n+2}(t)\gamma_{n,n-1}(t) - J^p_{n+1,n-1}(t)\lambda_{n+1}(t)\lambda_{n+2}(t) \\
&\quad = -\dot{\alpha}_{n+1}(t)\gamma_{n,n-1}(t) + J^p_{n,n-1}(t)\alpha_{n+1}(t)\lambda_{n+1}(t) - J^p_{n,n-2}(t)\lambda_n(t)\lambda_{n+1}(t) \\
&\qquad\qquad - 2\lambda_{n+1}(t)\dot{\lambda}_{n+1}(t) - J^p_{n,n-1}(t)\lambda_{n+1}(t)\gamma_{n-1,n-2}(t)\,,
\end{aligned}
$$

where $\gamma_{m,m-1}(t)$ is for each m the coefficient of z^{m-1} in $P_m(t,z)$. Taking into account (24) and take common factor $\lambda_{n+1}(t)$,

$$
\begin{aligned}
- J^p_{n+1,n-1}(t)\lambda_{n+2}(t) &= J^p_{n,n-1}(t)\left(\gamma_{n,n-1}(t) - \gamma_{n-1,n-2}(t)\right) \\
&\quad + \alpha_{n+1}(t)J^p_{n,n-1}(t) - \lambda_n(t)J^p_{n,n-2}(t) - 2\dot{\lambda}_{n+1}(t)\,. \qquad (25)
\end{aligned}
$$

On the other hand, from (9) it is easy to deduce that

$$
\gamma_{m,m-1}(t) = -\sum_{j=1}^{m} \alpha_j(t)\,, \quad m = n-1, n\,.
$$

Then, substituting this expression in (25) we obtain

$$
\begin{aligned}
2\dot{\lambda}_{n+1}(t) &= -\alpha_n(t)J^p_{n,n-1}(t) \\
&\quad + \alpha_{n+1}(t)J^p_{n,n-1}(t) - \lambda_n(t)J^p_{n,n-2}(t) + \lambda_{n+2}(t)J^p_{n+1,n-1}(t)\,. \qquad (26)
\end{aligned}
$$

Moreover, $J^{p+1}_{n-1,n}(t)$ is obtained by multiplying the n-th row of $(J(t))^p$ and the $(n+1)$-th column of $J(t)$, i.e.,

$$
J^{p+1}_{n-1,n}(t) = \lambda_{n+1}(t)J^p_{n-1,n-1}(t) + \alpha_{n+1}(t)J^p_{n-1,n}(t) + \lambda_{n+2}(t)J^p_{n-1,n+1}(t)\,. \qquad (27)
$$

Also, because of the symmetry of matrix $(J(t))^{p+1}$, we have that $J^{p+1}_{n-1,n}(t) = J^{p+1}_{n,n-1}(t)$ is obtained by multiplying the $(n+1)$-th row of $(J(t))^p$ and the n-th column of $J(t)$, this is,

$$
J^{p+1}_{n-1,n}(t) = \lambda_n(t)J^p_{n,n-2}(t) + \alpha_n(t)J^p_{n,n-1}(t) + \lambda_{n+1}(t)J^p_{n,n}(t)\,. \qquad (28)
$$

Comparing (27) and (28),

$$
\begin{aligned}
&\alpha_{n+1}(t)J^p_{n-1,n}(t) + \lambda_{n+2}(t)J^p_{n-1,n+1}(t) \\
&\qquad = \lambda_n(t)J^p_{n,n-2}(t) + \alpha_n(t)J^p_{n,n-1}(t) + \lambda_{n+1}(t)\left[J^p_{n,n}(t) - J^p_{n-1,n-1}(t)\right]\,.
\end{aligned}
$$

From this and (26) we arrive to

$$
2\dot{\lambda}_{n+1}(t) = \lambda_{n+1}(t)\left[J^p_{n,n}(t) - J^p_{n-1,n-1}(t)\right]\,,
$$

which is the second part of (5). ■

3. Bäcklund transformation and sequences of polynomials

Given a family of tridiagonal matrices $\{J(t)\}$, $t \in \mathbb{R}$, as in (4), we consider the sequence $\{P_n(t,z)\}$ of polynomials defined in (9). The well-known Favard's Theorem states that, if $\lambda_n(t) \neq 0$ for $n = 2, 3, \dots$, then the sequence $\{P_n(t,z)\}$ is orthogonal with respect to some quasi-definite moment functional.

Let $C \in \mathbb{C}$ be such that $P_n(t,C) \neq 0$, $n \in \mathbb{N}$, $t \in \mathbb{R}$. We consider, also, the sequence of monic polynomials $\{Q_n^{(C)}(t,z)\}$ defined by

$$Q_n^{(C)}(t,z) = \frac{P_{n+1}(t,z) - \dfrac{P_{n+1}(t,C)}{P_n(t,C)} P_n(t,z)}{z - C}\,, \quad n = 0, 1, \dots\,, \quad t \in \mathbb{R}\,. \tag{29}$$

In [5] was proved that, if $\{P_n(t,z)\}$ is the sequence of orthogonal polynomials with respect to a quasi-definite functional, then $\{Q_n^{(C)}(t,z)\}$ is also orthogonal with respect to another quasi-definite functional and verify, as consequence, a three-terms recurrence relation. Moreover, the relationship between the coefficients of both recurrence relations was established. In our case, this fact holds true, because we are assuming $\lambda_n(t) \neq 0$, $n = 2, 3, \dots$, $t \in \mathbb{R}$. We summarize the situation in the following auxiliary result (cf. [5, Theorems 7.1, 4.1, 4.2]).

Lemma 1. *The polynomials $\{Q_n^{(C)}(t,z)\}$ satisfies the three-term recurrence relation*

$$\left.\begin{array}{l} Q_n^{(C)}(t,z) = (z - \widetilde{\alpha}_n(t)) Q_{n-1}^{(C)}(t,z) - \widetilde{\lambda}_n^2(t) Q_{n-2}^{(C)}(t,z)\,, \quad n \in \mathbb{N} \\ Q_{-1}^{(C)} \equiv 0\,, \quad Q_0^{(C)} \equiv 1 \end{array}\right\},$$

being

$$\left.\begin{array}{rcl} \widetilde{\alpha}_n(t) & = & \dfrac{P_{n+1}(t,C)}{P_n(t,C)} + \alpha_{n+1}(t) - \dfrac{P_n(t,C)}{P_{n-1}(t,C)} \\ \widetilde{\lambda}_n^2(t) & = & \lambda_n^2(t) \dfrac{P_{n-2}(t,C) P_n(t,C)}{P_{n-1}^2(t,C)} \end{array}\right\}, \quad n \in \mathbb{N}\,, \quad t \in \mathbb{R}\,. \tag{30}$$

In a similar way as in [5, Theorem 9.1, p. 46], we define the sequence of complex functions $\{\gamma_n(t)\}$, $n \in \mathbb{N}$, as

$$\gamma_1(t) = 0\,, \quad \gamma_{2n}^2(t) = -\frac{P_n(t,C)}{P_{n-1}(t,C)}\,, \quad \gamma_{2n+1}^2(t) = -\lambda_{n+1}^2(t) \frac{P_{n-1}(t,C)}{P_n(t,C)}\,. \tag{31}$$

Note that we only define $\gamma_n^2(t)$, $n \in \mathbb{N}$. In fact, it is possible to find more than one sequence $\{\gamma_n(t)\}$ in the above conditions.

We remark that $\{\gamma_n(t)\}$, $n \in \mathbb{N}$, depends on C. However, we don't make explicit this dependence for brevity.

The sequence $\{\gamma_n^2(t)\}$ is associated with $\{S_n^{(C)}(t,z)\}$, $n \in \mathbb{N}$, which is another sequence of polynomials verifying the recurrence relation

$$\left.\begin{array}{l} S_n^{(C)}(t,z) = zS_{n-1}^{(C)}(t,z) - \gamma_n^2(t) S_{n-2}^{(C)}(t,z)\,, \quad n \in \mathbb{N} \\ S_{-1}^{(C)} \equiv 0\,, \quad S_0^{(C)} \equiv 1 \end{array}\right\}.$$

From (30) and (31) we immediately deduce (2a) and (2b). In other words, the coefficients of recurrence relations defining $\{P_n(t,z)\}$ and $\{Q_n^{(C)}(t,z)\}$ are linked by a Bäcklund transformation associated with the coefficients $\{\gamma_n^2(t)\}$ of the recurrence relation for $\{S_n^{(C)}(t,z)\}$.

4. Proof of Theorem 1

We shall dedicate the rest of the work to prove Theorem 1. Therefore, we take a generalized Toda solution $\{J(t)\}$, $t \in \mathbb{R}$, and $C \in \mathbb{C}$ verifying (8). Given the sequence $\{P_n(t,z)\}$, $n \in \mathbb{N}$, defined in (9), it is well know that

$$P_n(t,z) = \det\left(zI_n - J_n(t)\right)\,, \quad n \in \mathbb{N}\,, \quad t \in \mathbb{R}\,.$$

Then, from (8) we have $P_n(t,C) \neq 0$, $n \in \mathbb{N}$, $t \in \mathbb{R}$, and, consequently, we can define the sequence of monic polynomials $\{Q_n^{(C)}(t,z)\}$ as in (29).

Now, we are going to prove that $\{\widetilde{\alpha}_n(t)\}$, $\{\widetilde{\lambda}_n(t)\}$, $\{\gamma_n^2(t)\}$ defined in (30) and (31) are the unique sequences verifying (2a) and (2b). Define

$$J^{(1)}(t) = \begin{pmatrix} \alpha_1(t) & \lambda_2^2(t) & & \\ 1 & \alpha_2(t) & \lambda_3^2(t) & \\ & 1 & \alpha_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{32}$$

It is well known, and easy to prove, that

$$\det\left(J_n^{(1)}(t) - CI_n\right) = \det\left(J_n(t) - CI_n\right)\,, \quad t \in \mathbb{R}\,,$$

for any $n \in \mathbb{N}$. Thus, since (8) we know that it is possible to obtain a lower triangular infinite matrix $L(t)$ and an upper triangular infinite matrix $U(t)$ such that

$$J^{(1)}(t) - CI = L(t)U(t) \tag{33}$$

(see [6, Theorem 1, p. 35]). Moreover, taking the value 1 in all the diagonal entries of $U(t)$, the matrices $L(t)$ and $U(t)$ are uniquely determined in (33). But it is obvious that (2a) can be expressed as (33) for

$$L(t)=\begin{pmatrix}\gamma_2^2(t) & & & \\ 1 & \gamma_4^2(t) & & \\ & \ddots & \ddots & \end{pmatrix}, \quad U(t)=\begin{pmatrix}1 & \gamma_3^2(t) & & \\ & 1 & \gamma_5^2(t) & \\ & & \ddots & \ddots\end{pmatrix}. \tag{34}$$

Thus, the sequence $\{\gamma_n^2(t)\}$, $n\in\mathbb{N}$, is uniquely determined by (2a) and, consequently, $\{\widetilde{\lambda}_n(t)\}$, $\{\widetilde{\alpha}_n(t)\}$, $n\in\mathbb{N}$, are the unique sequences given in (2b).

To complete the proof of Theorem 1 we need to prove that the families of matrices $\{\widetilde{J}(t)\}$, with entries defined by (2b),

$$\widetilde{J}(t):=\begin{pmatrix}\widetilde{\alpha}_1(t) & \widetilde{\lambda}_2(t) & & \\ \widetilde{\lambda}_2(t) & \widetilde{\alpha}_2(t) & \widetilde{\lambda}_3(t) & \\ & \widetilde{\lambda}_3(t) & \widetilde{\alpha}_3(t) & \ddots \\ & & \ddots & \ddots\end{pmatrix}, \quad t\in\mathbb{R},$$

and $\{\Gamma(t)\}$, $t\in\mathbb{R}$, defined as in (6), are solutions of, the high order Toda lattice and of the high order Volterra lattice, respectively.

Taking into account (2a) and (2b) we can see

$$\Gamma^2(t)+CI=\begin{pmatrix}\alpha_1(t) & 0 & \lambda_2(t) & 0 & & \\ 0 & \widetilde{\alpha}_1(t) & 0 & \widetilde{\lambda}_2(t) & 0 & \\ \lambda_2(t) & 0 & \alpha_2(t) & 0 & \lambda_3(t) & \ddots \\ 0 & \widetilde{\lambda}_2(t) & 0 & \widetilde{\alpha}_3(t) & \ddots & \ddots \\ & \ddots & \ddots & \ddots & & \end{pmatrix}, \quad t\in\mathbb{R}. \tag{35}$$

The matrix $\Gamma^2(t)+CI$ is the key for understanding the connection between the initial generalized Toda solution $\{J(t)\}$ and the new solution $\{\widetilde{J}(t)\}$ obtained in Theorem 1. In fact, $\Gamma^2(t)+CI$ is a bridge between the matrices $J(t)$ and $\widetilde{J}(t)$. In the same way, for each $m\in\mathbb{N}$ the matrix $\left(\Gamma^2(t)+CI\right)^m$ interlaces $(J(t))^m$ and $\left(\widetilde{J}(t)\right)^m$, as we show in the following result.

Lemma 2. *For each $m \in \mathbb{N}$ and $j, k = 0, 1, \ldots,$*

$$\left(\Gamma^2(t) + CI\right)^m_{j,k} = \begin{cases} 0 & , \ j+k \ odd \\ J^m_{\frac{j}{2},\frac{k}{2}}(t) & , \ j,k \ even \\ \widetilde{J}^m_{\frac{j-1}{2},\frac{k-1}{2}}(t) & , \ j,k \ odd \end{cases} \tag{36}$$

Proof: We proceed by induction. (36) is obvious for $m = 1$ (see (35)). Let us assume that (36) holds for $m \in \mathbb{N}$. The entry $\left(\Gamma^2(t) + CI\right)^{m+1}_{j,k}$ of $\left(\Gamma^2(t) + CI\right)^{m+1}$ is obtained multiplying the $(j+1)$-th row of $\left(\Gamma^2(t) + CI\right)^m$ by the $(k+1)$-th column of $\Gamma^2(t) + CI$. Then, if $j + k$ is odd, we can see that this product is zero (see (35) and (36)). On the contrary, when j and k are even (odd, respectively), then there are only entries of $J(t)^m$ and $J(t)$ ($\widetilde{J}(t)^m$ and $\widetilde{J}(t)$, respectively) in the product and the result follows. ■

For checking (7), firstly we study the evolution of $\{\gamma_{2n}(t)\}$, $n \in \mathbb{N}$. Taking derivatives in (31),

$$\dot{\gamma}_{2n}(t) = \frac{1}{2}\gamma_{2n}(t)\left(\frac{\dot{P}_n(t,C)}{P_n(t,C)} - \frac{\dot{P}_{n-1}(t,C)}{P_{n-1}(t,C)}\right), \ n \in \mathbb{N}. \tag{37}$$

On the other hand, from (10) we deduce

$$\frac{\dot{P}_n(t,C)}{P_n(t,C)} = -\sum_{j=1}^{p} J^p_{n,n-j}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)}, \quad n \in \mathbb{N}. \tag{38}$$

In the next lemma we see that the above expression can be written in terms of diagonal and sub-diagonal entries of matrices $(J(t))^i$, $i = 0, 1, \ldots$.

Lemma 3. *For any $m, n \in \mathbb{N}$ we have*

$$-\sum_{j=1}^{m} J^m_{n,n-j}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)} = \gamma^2_{2n+1}(t)B^{(m)}_{n,n}(t) + \lambda_{n+1}(t)B^{(m)}_{n,n-1}(t), \tag{39}$$

being

$$B^{(m)}(t) := C^{m-1}I + C^{m-2}J(t) + \cdots + (J(t))^{m-1}. \tag{40}$$

Proof: Let $n, m \in \mathbb{N}$ be fixed. Let us define

$$S(n,m,t) := -\sum_{j=1}^{m} J^m_{n,n-j}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)}, \quad t \in \mathbb{R}. \tag{41}$$

Using (31),

$$S(n,1,t) = -J_{n,n-1}(t)\frac{\widehat{p}_{n-1}(t,C)}{\widehat{p}_n(t,C)} = \gamma^2_{2n+1}(t)\,. \tag{42}$$

Moreover, multiplying the $(i+1)$-th row of $(J(t))^m$ and the $(j+1)$-th column of $J(t)$ we obtain for all $i,j=0,1,\ldots,$

$$J^{m+1}_{i,j}(t) = J^m_{i,j-1}(t)\lambda_{j+1}(t) + J^m_{i,j}(t)\alpha_{j+1}(t) + J^m_{i,j+1}(t)\lambda_{j+2}(t)\,. \tag{43}$$

Therefore, substituting i by n and j by $n-j$ in (43),

$$\begin{aligned} &S(n,m+1,t) \\ &= -\sum_{j=1}^{m-1} J^m_{n,n-j-1}(t)\lambda_{n-j+1}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)} - \sum_{j=1}^{m} J^m_{n,n-j}(t)\alpha_{n-j+1}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)} \\ &\qquad - \sum_{j=1}^{m+1} J^m_{n,n-j+1}(t)\lambda_{n-j+2}(t)\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)}\,, \end{aligned} \tag{44}$$

where we used the fact that, since $(J(t))^m$ is a $(2m+1)$-diagonal matrix, in the first term we have $J^m_{n,n-j-1}(t)=0$ for $j=m,m+1$ and, in the second term, we have $J^m_{n,n-j}(t)=0$ for $j=m+1$. Computing the right hand side of (44),

$$\begin{aligned} S(n,m+1,t) &= -J^m_{n,n-1}(t)\frac{\alpha_n(t)\widehat{p}_{n-1}(t,C)+\lambda_n(t)\widehat{p}_{n-2}(t,C)}{\widehat{p}_n(t,C)} \\ &\quad - J^m_{n,n}(t)\lambda_{n+1}(t)\frac{\widehat{p}_{n-1}(t,C)}{\widehat{p}_n(t,C)} - \sum_{j=2}^{m} J^m_{n,n-j}(t) \\ &\quad \times \frac{\lambda_{n-j+2}(t)\widehat{p}_{n-j+1}(t,C)+\alpha_{n-j+1}(t)\widehat{p}_{n-j}(t,C)+\lambda_{n-j+1}(t)\widehat{p}_{n-j-1}(t,C)}{\widehat{p}_n(t,C)}\,. \end{aligned}$$

Using (14) in the last relation,

$$\begin{aligned} S(n,m+1,t) &= -J^m_{n,n-1}(t)\frac{C\widehat{p}_{n-1}(t,C)-\lambda_{n+1}(t)\widehat{p}_n(t,C)}{\widehat{p}_n(t,C)} \\ &\qquad - J^m_{n,n}(t)\lambda_{n+1}(t)\frac{\widehat{p}_{n-1}(t,C)}{\widehat{p}_n(t,C)} - \sum_{j=2}^{m} J^m_{n,n-j}(t)\frac{C\widehat{p}_{n-j}(t,C)}{\widehat{p}_n(t,C)}\,. \end{aligned}$$

Thus,

$$S(n,m+1,t)=CS(n,m,t)+\lambda_{n+1}(t)J^m_{n,n-1}(t)+\gamma^2_{2n+1}J^m_{n,n}(t)\,,\quad m\in\mathbb{N}\,. \tag{45}$$

From this and (42) we arrive to (39). To see this fact, proceed by induction. Assume that (39) holds for a certain $m\in\mathbb{N}$, i.e.,

$$S(n,m,t)=\gamma^2_{2n+1}(t)B^{(m)}_{n,n}(t)+\lambda_{n+1}(t)B^{(m)}_{n,n-1}(t)\,.$$

Due to (45),

$$\begin{aligned}&S(n,m+1,t)\\&\quad=\gamma^2_{2n+1}CB^{(m)}_{n,n}(t)+\lambda_{n+1}(t)CB^{(m)}_{n,n-1}(t)+\lambda_{n+1}(t)J^m_{n,n-1}(t)+\gamma^2_{2n+1}J^m_{n,n}(t)\\&\quad=\gamma^2_{2n+1}\left(CB^{(m)}_{n,n}(t)+J^m_{n,n}(t)\right)+\lambda_{n+1}(t)\left(CB^{(m)}_{n,n-1}(t)+J^m_{n,n-1}(t)\right)\,.\end{aligned}$$

Note that, by (40),

$$B^{(m+1)}(t)=CB^{(m)}(t)+(J(t))^m\;.$$

Then, (39) is verified. ■

The next lemma describes the relation between the ratios given in the left hand side of (38) and the matrix $\left(\Gamma^2(t)+CI\right)^p$.

Lemma 4. *For any $m,n\in\mathbb{N}$ we have*

$$\begin{aligned}&\sum_{j=1}^{m}\left(\Gamma^2(t)+CI\right)^m_{2n-2,2n-2j-2}\frac{\widehat{p}_{n-j-1}(t,C)}{\widehat{p}_{n-1}(t,C)}\\&\qquad\qquad-\sum_{j=1}^{m}\left(\Gamma^2(t)+CI\right)^m_{2n,2n-2j}\frac{\widehat{p}_{n-j}(t,C)}{\widehat{p}_{n}(t,C)}\\&\qquad\qquad\qquad=\left(\Gamma^2(t)+CI\right)^m_{2n-1,2n-1}-\left(\Gamma^2(t)+CI\right)^m_{2n-2,2n-2}\,.\end{aligned}$$

Proof: We use the notation given in (41). Taking into account Lemma 2, we want to prove

$$S(n,m,t)-S(n-1,m,t)=\widetilde{J}^m_{n-1,n-1}(t)-J^m_{n-1,n-1}(t)\,,\quad n,m\in\mathbb{N}\,. \tag{46}$$

We proceed by induction on m.

For $m=1$, (46) is reduced to

$$\gamma^2_{2n+1}(t)-\gamma^2_{2n-1}(t)=\widetilde{\alpha}_1(t)-\alpha_1(t)\,,$$

and this is true because of (2a) and (2b).

Suppose that (46) holds for $m \in \mathbb{N}$. Then, using (45),

$$\begin{aligned} S(n,m+1,t) - S(n-1,m+1,t) = C\left(\widetilde{J}^{m}_{n-1,n-1}(t) - J^{m}_{n-1,n-1}(t)\right) \\ + \lambda_{n+1}(t)J^{m}_{n,n-1}(t) + \gamma^2_{2n+1}J^{m}_{n,n}(t) - \lambda_n(t)J^{m}_{n-1,n-2}(t) - \gamma^2_{2n-1}J^{m}_{n-1,n-1}(t). \quad (47) \end{aligned}$$

Moreover, taking $i = j = n-1$ in (43), obtaining there $\lambda_n(t)J^{m}_{n-1,n-2}(t)$, and substituting in (47),

$$\begin{aligned} S(n,m+1,t) - S(n-1,m+1,t) = -J^{m+1}_{n-1,n-1}(t) + C\widetilde{J}^{m}_{n-1,n-1}(t) \\ + \gamma^2_{2n}(t)J^{m}_{n-1,n-1}(t) + 2\lambda_{n+1}(t)J^{m}_{n,n-1}(t) + \gamma^2_{2n+1}(t)J^{m}_{n,n}(t)\,. \quad (48) \end{aligned}$$

On the other hand, if we define

$$\widetilde{J}^{(1)}(t) = \begin{pmatrix} \widetilde{\alpha}_1(t) & \widetilde{\lambda}^2_2(t) & & \\ 1 & \widetilde{\alpha}_2(t) & \widetilde{\lambda}^2_3(t) & \\ & 1 & \widetilde{\alpha}_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

and $L(t)$, $U(t)$ are the matrices defined in (34), then in a formal sense we can verify (cf. [4])

$$\widetilde{J}^{(1)}(t) - CI = U(t)L(t)\,. \quad (49)$$

This is, we understand (49) as a set of relations given by the rows of these matrices.

Given the diagonal matrices $A(t) := \operatorname{diag}\{1\,, \lambda_2(t)\,, \lambda_2(t)\lambda_3(t)\,, \ldots\}$ and $\widetilde{A}(t) := \operatorname{diag}\{1\,, \widetilde{\lambda}_2(t)\,, \widetilde{\lambda}_2(t)\widetilde{\lambda}_3(t)\,, \ldots\}$, $t \in \mathbb{R}$ it is easy to show

$$J(t) = A(t)J^{(1)}(t)A(t)^{-1}\,, \quad \widetilde{J}(t) = \widetilde{A}(t)\widetilde{J}^{(1)}(t)\widetilde{A}(t)^{-1}$$

(see (32)) where we represent by $A(t)^{-1}$ (respectively $\widetilde{A}(t)^{-1}$) the diagonal matrix whose entries are $1, \lambda_2(t)^{-1}\,, \lambda_2(t)^{-1}\lambda_3(t)^{-1}, \ldots$ (respectively $1, \widetilde{\lambda}_2(t)^{-1}$, $\widetilde{\lambda}_2(t)^{-1}\widetilde{\lambda}_3(t)^{-1}\,, \ldots$). Thus, for all $k = 0, 1, \ldots$

$$\left(J(t)\right)^k A(t) = A(t)\left(J^{(1)}(t)\right)^k\,, \quad \left(\widetilde{J}(t)\right)^k \widetilde{A}(t) = \widetilde{A}(t)\left(\widetilde{J}^{(1)}(t)\right)^k\,. \quad (50)$$

Moreover, from (33) and (49) we have

$$J^{(1)}(t)L(t) = (L(t)U(t) + CI)L(t) = L(t)(U(t)L(t) + CI) = L(t)\widetilde{J}^{(1)}(t)$$

and, in general, it is easy to see that

$$\left(J^{(1)}(t)\right)^k L(t) = L(t)\left(\widetilde{J}^{(1)}(t)\right)^k , \quad k \in \mathbb{N}.$$

From this and (50),

$$\begin{aligned} (J(t))^k \left(A(t)L(t)\widetilde{A}(t)^{-1}\right) &= \left(A(t)\left(J^{(1)}(t)\right)^k\right) L(t)\widetilde{A}(t)^{-1} \\ = A(t)\left(L(t)\left(\widetilde{J}^{(1)}(t)\right)^k\right)\widetilde{A}(t)^{-1} &= \left(A(t)L(t)\widetilde{A}(t)^{-1}\right)\left(\widetilde{J}(t)\right)^k \quad , \qquad k \in \mathbb{N}, \end{aligned}$$

and so, taking $R(t) := A(t)L(t)\widetilde{A}(t)^{-1}$ we have proved the relations

$$(J(t))^k R(t) = R(t)\left(\widetilde{J}(t)\right)^k , \quad k = 0, 1, \dots , \quad t \in \mathbb{R}, \tag{51}$$

between the powers of $J(t)$ and $\widetilde{J}(t)$. Therefore, since

$$R(t) = \begin{pmatrix} \gamma_2^2(t) & 0 & & \\ \lambda_2(t) & \dfrac{\lambda_2(t)\gamma_4^2(t)}{\widetilde{\lambda}_2(t)} & \ddots & \\ 0 & \dfrac{\lambda_2(t)\lambda_3(t)}{\widetilde{\lambda}_2(t)} & \dfrac{\lambda_2(t)\lambda_3(t)\gamma_6^2(t)}{\widetilde{\lambda}_2(t)\widetilde{\lambda}_3(t)} & \\ & \ddots & \ddots & \ddots \end{pmatrix}$$

it is possible to find the relationship between the entries of the matrices given in (51). For $k = m$, the entry of the $(i+1)$-th row and the $(j+1)$-th column is obtained by multiplying the corresponding row and column of each matrix, i.e.,

$$\begin{aligned} &((J(t))^m R(t))_{i,j} \\ &\quad = J^m_{i,j}(t)\frac{\lambda_2(t)\cdots\lambda_{j+1}(t)}{\widetilde{\lambda}_2(t)\cdots\widetilde{\lambda}_{j+1}(t)}\gamma^2_{2j+2}(t) + J^m_{i,j+1}(t)\frac{\lambda_2(t)\cdots\lambda_{j+2}(t)}{\widetilde{\lambda}_2(t)\cdots\widetilde{\lambda}_{j+1}(t)} , \quad t \in \mathbb{R}, \end{aligned} \tag{52}$$

and, in a similar way,

$$\begin{aligned} &\left(R(t)\left(\widetilde{J}(t)\right)^m\right)_{i,j} \\ &\quad = \widetilde{J}^m_{i-1,j}(t)\frac{\lambda_2(t)\cdots\lambda_{i+1}(t)}{\widetilde{\lambda}_2(t)\cdots\widetilde{\lambda}_i(t)} + \widetilde{J}^m_{i,j}(t)\frac{\lambda_2(t)\cdots\lambda_{i+1}(t)}{\widetilde{\lambda}_2(t)\cdots\widetilde{\lambda}_{i+1}(t)}\gamma^2_{2i+2}(t) , \quad t \in \mathbb{R}. \end{aligned} \tag{53}$$

In particular, taking $i = j = n-1$ in (52) and (53), from (51) we obtain

$$J^m_{n-1,n-1}(t)\gamma^2_{2n}(t) + J^m_{n-1,n}(t)\lambda_{n+1}(t) \\ = \widetilde{J}^m_{n-2,n-1}(t)\widetilde{\lambda}_n(t) + \widetilde{J}(t)^m_{n-1,n-1}\gamma^2_{2n}(t) \quad (54)$$

and, taking $i = n$, $j = n-1$,

$$J^m_{n,n-1}(t)\gamma^2_{2n}(t) + J^m_{n,n}(t)\lambda_{n+1}(t) \\ = \widetilde{J}^m_{n-1,n-1}(t)\widetilde{\lambda}_{n+1}(t) + \widetilde{J}^m_{n,n-1}(t)\frac{\lambda_{n+1}(t)}{\widetilde{\lambda}_{n+1}(t)}\gamma^2_{2n+2}(t)\,.$$

Multiplying by $\lambda_{n+1}(t)\widetilde{\lambda}^2_{n+1}(t)$ the last equality, taking into account (2a) and (2b), and simplifying the obtained result,

$$J^m_{n,n-1}(t)\lambda_{n+1}(t) + J^m_{n,n}(t)\gamma^2_{2n+1}(t) \\ = \widetilde{J}^m_{n-1,n-1}(t)\gamma^2_{2n+1}(t) + \widetilde{J}^m_{n,n-1}(t)\widetilde{\lambda}_{n+1}(t)\,. \quad (55)$$

Adding (54) and (55),

$$J^m_{n-1,n-1}(t)\gamma^2_{2n}(t) + 2J^m_{n-1,n}(t)\lambda_{n+1}(t) + J^m_{n,n}(t)\gamma^2_{2n+1}(t) \\ = \widetilde{J}^m_{n-2,n-1}(t)\widetilde{\lambda}_n + (\widetilde{\alpha}_n(t) - C)\widetilde{J}^m_{n-1,n-1}(t) + \widetilde{J}^m_{n,n-1}(t)\widetilde{\lambda}_{n+1}(t)\,.$$

From this and

$$\widetilde{J}^{m+1}_{n-1,n-1}(t) = \widetilde{J}^m_{n-1,n-2}(t)\widetilde{\lambda}_n(t) + \widetilde{J}^m_{n-1,n-1}(t)\widetilde{\alpha}_n(t) + \widetilde{J}^m_{n-1,n}(t)\widetilde{\lambda}_{n+1}(t)$$

(see (43)) we arrive to

$$C\widetilde{J}^m_{n-1,n-1}(t) + J^m_{n-1,n-1}(t)\gamma^2_{2n}(t) + 2J^m_{n-1,n}(t)\lambda_{n+1}(t) + J^m_{n,n}(t)\gamma^2_{2n+1}(t) \\ = \widetilde{J}^{m+1}_{n-1,n-1}(t)$$

which, substituted in the right hand side of (48), conduces to

$$S(n, m+1, t) - S(n-1, m+1, t) = \widetilde{J}^{m+1}_{n-1,n-1}(t) - J^{m+1}_{n-1,n-1}(t)\,,$$

as we needed to prove. ■

Due to lemmas 2 and 4, (37) and (38), we know that (7) is verified for any odd number $n \in \mathbb{N}$. Moreover, taking derivatives in $\lambda^2_{n+1}(t) = \gamma^2_{2n}(t)\gamma^2_{2n+1}(t)$

(see (5) and (2a)),

$$
\begin{aligned}
2\lambda_{n+1}(t)\dot{\lambda}_{n+1}(t) &= \lambda_{n+1}^2(t)\left(J_{n,n}^p(t) - J_{n-1,n-1}^p(t)\right) \\
&= \gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\left[\left(\Gamma^2(t)+CI\right)_{2n-1,2n-1}^p - \left(\Gamma^2(t)+CI\right)_{2n-2,2n-2}^p\right] \\
&\qquad + 2\gamma_{2n}^2(t)\gamma_{2n+1}(t)\dot{\gamma}_{2n+1}(t)\,.
\end{aligned}
$$

Then, dividing by $\lambda_{n+1}^2(t) = \gamma_{2n}^2(t)\gamma_{2n+1}^2(t)$, we obtain

$$
\begin{aligned}
2\frac{\dot{\gamma}_{2n+1}(t)}{\gamma_{2n+1}(t)} &= J_{n,n}^p(t) - J_{n-1,n-1}^p(t) - \left(\Gamma^2(t)+CI\right)_{2n-1,2n-1}^p \\
&\quad + \left(\Gamma^2(t)+CI\right)_{2n-2,2n-2}^p = \left(\Gamma^2(t)+CI\right)_{2n,2n}^p - \left(\Gamma^2(t)+CI\right)_{2n-1,2n-1}^p\,,
\end{aligned}
$$

this is, (7) holds also when n is an even number. Thus, we conclude that $\Gamma(t)$ is a generalized Volterra solution.

Finally, we need to prove that $\widetilde{J}(t)$ is a generalized Toda solution. For this purpose, we want to verify (5) when we substitute J by $\widetilde{J}$. Firstly, we take into consideration the first part of (30), i.e.,

$$\widetilde{\alpha}_n(t) = \alpha_{n+1}(t) - \gamma_{2n+2}^2(t) + \gamma_{2n}^2(t)\,.$$

Then, since (5) and (7),

$$
\begin{aligned}
\dot{\widetilde{\alpha}}_n(t) = \dot{\widetilde{J}}_{n-1,n-1}(t) &= \dot{\alpha}_{n+1}(t) - 2\gamma_{2n+2}(t)\dot{\gamma}_{2n+2}(t) + 2\gamma_{2n}(t)\dot{\gamma}_{2n}(t) \\
&= \left[J_{n,n+1}(t)J_{n,n+1}^p(t) - J_{n-1,n}(t)J_{n-1,n}^p(t)\right] \\
&\quad - \Gamma_{2n,2n+1}^2(t)\left[\left(\Gamma^2(t)+CI\right)_{2n+1,2n+1}^p - \left(\Gamma^2(t)+CI\right)_{2n,2n}^p\right] \\
&\quad + \Gamma_{2n-2,2n-1}^2(t)\left[\left(\Gamma^2(t)+CI\right)_{2n-1,2n-1}^p - \left(\Gamma^2(t)+CI\right)_{2n-2,2n-2}^p\right]\,.
\end{aligned}
$$

Therefore, by Lemma 2,

$$
\begin{aligned}
\dot{\widetilde{\alpha}}_n(t) &= \left[\lambda_{n+2}(t)J_{n,n+1}^p(t) + \gamma_{2n+2}^2(t)J_{n,n}^p(t)\right] \\
&\qquad - \left[\lambda_{n+1}(t)J_{n-1,n}^p(t) + \gamma_{2n}^2(t)J_{n-1,n-1}^p(t)\right] \\
&\qquad\qquad + \left[-\gamma_{2n+2}^2(t)\widetilde{J}_{n,n}^p(t) + \gamma_{2n}^2(t)\widetilde{J}_{n-1,n-1}^p(t)\right]\,.
\end{aligned}
$$

Using (54) (with $m = p$),

$$\dot{\widetilde{\alpha}}_n(t) = \left[\widetilde{J}^p_{n-1,n}(t)\widetilde{\lambda}_{n+1}(t) + \widetilde{J}^p_{n,n}(t)\gamma^2_{2n+2}(t)\right]$$
$$- \left[\widetilde{J}^p_{n-2,n-1}(t)\widetilde{\lambda}_n(t) + \widetilde{J}^p_{n-1,n-1}(t)\gamma^2_{2n}(t)\right] - \widetilde{J}^p_{n,n}(t)\gamma^2_{2n+2}(t) + \widetilde{J}^p_{n-1,n-1}(t)\gamma^2_{2n}(t)$$
$$= \widetilde{J}^p_{n-1,n}(t)\widetilde{\lambda}_{n+1}(t) - \widetilde{J}^p_{n-2,n-1}(t)\widetilde{\lambda}_n(t)\,,$$

which is the first relation in (5).

On the other hand, taking derivatives in $\widetilde{\lambda}^2_{n+1}(t) = \gamma^2_{2n+1}(t)\gamma^2_{2n+2}(t)$ and dividing the obtained result by $\widetilde{\lambda}^2_{n+1}(t)$ we arrive to

$$\frac{\dot{\widetilde{\lambda}}_{n+1}(t)}{\widetilde{\lambda}_{n+1}(t)} = \frac{\dot{\gamma}_{2n+1}(t)}{\gamma_{2n+1}(t)} + \frac{\dot{\gamma}_{2n+2}(t)}{\gamma_{2n+2}(t)}\,. \tag{56}$$

Taking into account (7) (substituting there n by $2n$ and $2n+1$, successively), from (56) we obtain

$$\frac{\dot{\widetilde{\lambda}}_{n+1}(t)}{\widetilde{\lambda}_{n+1}(t)} = \frac{\dot{\Gamma}_{2n-1,2n}(t)}{\Gamma_{2n-1,2n}(t)} + \frac{\dot{\Gamma}_{2n,2n+1}(t)}{\Gamma_{2n,2n+1}(t)}$$
$$= \frac{1}{2}\left[\left(\Gamma^2(t) + CI\right)^p_{2n,2n} - \left(\Gamma^2(t) + CI\right)^p_{2n-1,2n-1}\right]$$
$$+ \frac{1}{2}\left[\left(\Gamma^2(t) + CI\right)^p_{2n+1,2n+1} - \left(\Gamma^2(t) + CI\right)^p_{2n,2n}\right]$$
$$= \frac{1}{2}\left[\left(\Gamma^2(t) + CI\right)^p_{2n+1,2n+1} - \left(\Gamma^2(t) + CI\right)^p_{2n-1,2n-1}\right]\,.$$

From this and (36),

$$\dot{\widetilde{\lambda}}_{n+1}(t) = \frac{1}{2}\widetilde{\lambda}_{n+1}(t)\left(\widetilde{J}^p_{n,n} - \widetilde{J}^p_{n-1,n-1}\right)\,,$$

what is the second relation of (5). ■

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