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THE SET OF INVARIANT STAR PRODUCTS ON NON-DEGENERATE TRIANGULAR LIE BIALGEBRAS OBTAINED IN THE ETINGOF-KAHZDAN QUANTIZATION THEORY

CARLOS MORENO AND JOANA TELES

ABSTRACT: Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ be a non-degenerate triangular finite dimensional Lie bialgebra over $\mathbb{R}$. When a Lie associator Φ is fixed and for any formal solution $r_\hbar = r_1 + \hbar r_2 + \cdots \in (\mathfrak{a} \wedge \mathfrak{a})[[\hbar]]$ of Yang Baxter Equation (YBE), the theory of Etingof-Kazhdan allows us to obtain an Invariant Star Product (ISP), $\tilde{J}_{r_\hbar} \in (\mathcal{U}\mathfrak{a} \otimes \mathcal{U}\mathfrak{a})[[\hbar]]$ on $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$. Every ISP F determines an element $r_\hbar$ such that F and $\tilde{J}_{r_\hbar}$ are equivalent. We define in this way a bijective mapping between the set of classes of ISPS modulo equivalence and the co-homology space $\hbar H^2(\mathfrak{a})[[\hbar]]$.

KEYWORDS: Quantization of Lie bialgebras, Star Products, Quantized Universal Enveloping algebras.
AMS SUBJECT CLASSIFICATION (2000): 16W30; 17B62; 17B37; 46L65; 53D55.

1. Introduction

1) Let $(\mathfrak{a}, [,]_{\mathfrak{a}})$ be the Lie algebra of a Lie group $\mathbf{G}$. Let $(\mathbf{G}, \beta_1)$ be a connected and simply connected Lie group endowed with an invariant symplectic structure $\beta_1 \in \mathfrak{a}^* \wedge \mathfrak{a}^*$ and let $r_1 \in \mathfrak{a} \wedge \mathfrak{a}$ be the corresponding solution to the Yang-Baxter-Equation (YBE) on $(\mathfrak{a}, [,]_{\mathfrak{a}})$. Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c\, r_1)$ be the corresponding non-degenerate triangular Lie bialgebra. In [5] Drinfeld obtains all the Invariant Star Products (ISPS) on $(\mathbf{G}, \beta_1)$ (equivalently on $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c\, r_1)$) and a theorem showing that under equivalence [10] the classifying set of all those ISPS is the Chevalley space $\beta_1 + \hbar \cdot \sum H^2(\mathfrak{a}, \mathbb{R})[[\hbar]]$.

2) The aim of this paper is to obtain a classification theorem for all the ISPS on any non-degenerate triangular finite dimensional Lie bialgebra over $\mathbb{R}$ when they are obtained following the Etingof-Kazhdan [9] theory of quantization of Lie bialgebras. This theory is very different from the one considered in [5] but the classifying set for the set of ISPS is again $\beta_1 + \hbar \cdot \sum H^2(\mathfrak{a}, \mathbb{R})[[\hbar]]$. In case we consider the Knizhnik-Zamolodchikov associator [7], over the field

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$\mathbb{C}$, we have seen [22] that the ISP $\tilde{J}_{r_1}$ obtained here and the ISP $F(r_1)$ obtained by Drinfeld in [5] coincide modulo $\hbar^3$. In two particular cases of dimensions 2 and 4 for $\mathfrak{a}$, $\tilde{J}_{r_1}$ and $F(r_1)$ coincide modulo $\hbar^4$.

3) The adjoint representation of $\mathbf{G}$ induces a representation on the Chevalley complex $H^*(\mathfrak{a}, \mathbb{R})$ which is trivial. This classical theorem contains the idea for the proof of our classification theorem 6.11. We may compare this with the proofs in [19, 20] for the similar theorem in the quantization context of [5].

We present here the proofs of most of the results obtained.

4) Similar results as in the Abstract can be obtained on a *non-degenerate triangular deformation Lie bialgebra*, [6, 7], $(\mathfrak{a}_t \equiv \mathfrak{a} \otimes_{\mathbb{K}} \mathbb{K}[[t]], [,]_{\mathfrak{a}_t}, \varepsilon_{\mathfrak{a}_t} = d_c(t) r_1(t))$ over $\mathbb{K}[[t]]$ of the non-degenerate triangular Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ over a field of characteristic zero $\mathbb{K}$. In this case we need to observe that *a)* $\mathbb{K}[[t]]$ *is a local ring and a Principal Ideal Domain; b) the symmetric algebra of the* $\mathbb{K}[[t]]$*-module* $\mathfrak{a}[[t]]$ *is the algebra of divided powers* $\Gamma(\mathfrak{a}_t)$, *in the sense of* [3], *over the* $\mathbb{K}[[t]]$*-module* $\mathfrak{a}_t$; *c) this symmetric algebra is isomorphic to the algebra of symmetric tensors* $TS(\mathfrak{a}[[t]])$[4]; *d) the Hochschild cohomology* $H^*(\mathcal{U}\mathfrak{a}[[t]])$ *of the coalgebra* $\mathcal{U}\mathfrak{a}[[t]]$ *over* $\mathbb{K}[[t]]$ *is* $\bigwedge(\mathfrak{a}[[t]])$ [3].

2. Some notations

1) Definitions and notations are those of [5, 7, 15, 16]. *A finite dimensional Lie bialgebra over* $\mathbb{R}$ *is denoted by the symbol* $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$ *where* $\varepsilon_{\mathfrak{a}}$ *is a 1-cocycle in the Chevalley cohomology on* $(\mathfrak{a}, [,]_{\mathfrak{a}})$ *with respect to the* $ad_{\mathfrak{a}}$*- representation. When it is quasitriangular we write* $\varepsilon_{\mathfrak{a}} = d_c r_1$, *where* d_c *is the coboundary in the above cohomology,* $r_1 \in \mathfrak{a} \otimes \mathfrak{a}$ *is a solution to CYBE,* $[r_1, r_1] = 0$ *on* $(\mathfrak{a}, [,]_{\mathfrak{a}})$ *and* $r_1 + \sigma(r_1)$ *is* $ad_{\mathfrak{a}}$*-invariant where* σ *is the permutation* (12). *In case* r_1 *is skew-symmetric, it is a triangular Lie bialgebra and if moreover* $det(r_1) \neq 0$ *we call it a non-degenerate triangular Lie bialgebra. If* $(\mathfrak{a}, [,], \varepsilon_{\mathfrak{a}})$ *is a Lie bialgebra we denote its quasitriangular double Lie bialgebra as the set* $(\mathfrak{a} \oplus \mathfrak{a}^*, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*} = d_c r)$, *where* $r \in (\mathfrak{a} \oplus \mathfrak{a}^*)^{\otimes 2}$ *is the invariant canonical element* [2]. *The element* $\Omega = r + \sigma(r)$ *is symmetric and* $ad_{\mathfrak{a}\oplus\mathfrak{a}^*}$*-invariant.*

2) The symbol $\left(\mathfrak{a}[[\hbar]], [,]_{\mathfrak{a}[[\hbar]]}\right)$ *will denote the Lie algebra over the ring* $\mathbb{R}[[\hbar]]$ *obtained from the Lie algebra* $(\mathfrak{a}, [,]_{\mathfrak{a}})$ *over* $\mathbb{R}$ *by the extension of scalars* $\mathbb{R} \to \mathbb{R}[[\hbar]]$. $(\mathfrak{a} \oplus \mathfrak{a}^*)[[\hbar]]$ *will denote the Lie algebra (bialgebra) over* $\mathbb{R}[[\hbar]]$ *which is the extension of the Lie bialgebra* $(\mathfrak{a} \oplus \mathfrak{a}^*, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*} = d_c r)$ *over* $\mathbb{R}$. *It is obvious that* $\Omega = r + \sigma(r) \in (\mathfrak{a} \oplus \mathfrak{a}^*)^{\otimes 2}[[\hbar]]$ *is* $ad_{\mathfrak{a}[[\hbar]]}$ *invariant* *and that these Lie-algebras over* $\mathbb{R}[[\hbar]]$ *are deformations algebras* [7] *of their corresponding Lie-algebras over* $\mathbb{R}$.

3) Let $r_t = r_1 + r_2 t + r_3 t^2 + \cdots \in \mathfrak{a} \wedge \mathfrak{a}$ *be an analytic function on a neighborhood of* $0 \in \mathbb{R}$ *defining a non-degenerate,* $rank(r_1) = dim\, \mathfrak{a}$, *solution of the CYBE* $[r_t, r_t] = 0$ *on the Lie algebra* $(\mathfrak{a}, [,]_{\mathfrak{a}})$ *over* $\mathbb{R}$. *For each* t, $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_t)$ *is then a non-degenerate triangular Lie bialgebra on* MR. *The symbol* $\left(\mathfrak{a} \oplus \mathfrak{a}^*_{r_t}, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}} = d_c\, r\right)$ *will denote the corresponding quasitriangular double Lie bialgebra on* $\mathbb{R}$. *Let* $\mu_{r_t} : \wedge^2\mathfrak{a} \longrightarrow \wedge^2\mathfrak{a}^*$ *be the linear isomorphism defined by the Poisson cocycle* r_t. *It induces an isomorphism between Poisson and Chevalley cohomology spaces* [14]. *Let* $(\mathbf{G}, \beta_t)$ *be the corresponding connected and simply-connected Lie group endowed with the invariant symplectic structure* $\beta_t = \mu_{r_t}(r_t)$.

4) Let $r_\hbar = r_1 + r_2\hbar + r_3\hbar^2 + \cdots \in (\mathfrak{a}\wedge\mathfrak{a})[[\hbar]]$ *be an invertible element solution of the CYBE* $[r_\hbar, r_\hbar] = 0$ *on* $\left(\mathfrak{a}[[\hbar]], [,]_{\mathfrak{a}[[\hbar]]}\right)$. *The set* $\left(\mathfrak{a}[[\hbar]], [,]_{\mathfrak{a}[[\hbar]]}, \varepsilon_{\mathfrak{a}[[\hbar]]} = d_c r_\hbar)\right)$ *will denote the corresponding triangular non-degenerate Lie bialgebra over* $\mathbb{R}[[\hbar]]$ *and its quasitriangular double Lie bialgebra will be denoted by* $\left((\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar})[[\hbar]], [,]_{(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar})},\ \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar}} = d_c r\right)$. *In this situation, let* $\mu_{r_\hbar}$ *be the isomorphism similar to* μ_{r_t} *in 3).*

5) We fix [9] *a Lie associator* $\Phi = exp\, P(\hbar t_{12}, \hbar t_{23})$ *over* $\mathbb{R}$.

3. Finite dimensional Etingof-Kazhdan quantization theory.

3.1. Quantization of the pair $(\mathfrak{a} \oplus \mathfrak{a}^*; r)$**.** *From theorem A" in* [7], *we can deduce the following:*

Theorem 3.1. [7] *Let* $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$, $(\mathfrak{a} \oplus \mathfrak{a}^*, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*} = d_c r)$ *and* $\Omega = r + \sigma(r)$ *be as in section 2, 1). Let* $\Phi = exp\, P(\hbar\Omega_{12}, \hbar\Omega_{23}) \in \mathcal{U}(\mathfrak{a} \oplus \mathfrak{a}^*)^{\otimes 3}[[\hbar]]$. *Let* $(\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*)[[\hbar]], \cdot, \Delta_0, \epsilon_0, S_0)$ *be the usual Hopf universal enveloping algebra. Write* $\Phi = \sum_i X_i \otimes Y_i \otimes Z_i$ *and* $c = \sum_i X_i \cdot S_0(Y_i) \cdot Z_i$. *Then the set*

$$\left(\mathcal{U}(\mathfrak{a} \oplus \mathfrak{a}^*)[[\hbar]]; \cdot; 1; \Delta_0; \epsilon_0; \Phi; S_0; \alpha = c^{-1}; \beta = 1; R = e^{\frac{\hbar}{2}\Omega}\right)$$

is a quasitriangular quasi-Hopf QUE-algebra whose classical limit [7] *is* $(\mathfrak{a} \oplus \mathfrak{a}^*; \Omega)$.

The existence of the antipode follows from Theorem 1.6 in [7]. *From Propositions 1.1 and 1.3 in* [7] *and from* [11] *it can be taken,* [22], *as the triple* $(S_0; \alpha = c^{-1}; \beta = 1)$.

To quantize the pair $(\mathfrak{a}\oplus\mathfrak{a}^*; r)$ is to obtain a quasitriangular-Hopf QUE-algebra over $\mathbb{R}[[\hbar]]$ such that its classical limit is that pair. A main theorem in this direction is the following (see also [8]*):*

Theorem 3.2. [9] *Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}})$, $(\mathfrak{a}\oplus\mathfrak{a}^*, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*} = d_c r)$ and $\Omega = r + \sigma(r)$ be as in section 2, 1). Let $M_\pm$ be the $\mathfrak{a}\oplus\mathfrak{a}^*$-modules with one generator $1_\pm$ and defined as follows: $M_+ = \mathcal{U}\mathfrak{a}^*\cdot 1_+$, $\mathcal{U}\mathfrak{a}\cdot 1_+ = 0$ and $M_- = \mathcal{U}\mathfrak{a}\cdot 1_-$, $\mathcal{U}\mathfrak{a}^*\cdot 1_- = 0$. Then*

1) The equalities $i_\pm(1_\pm) = 1_\pm\otimes 1_\pm$ define unique $\mathfrak{a}\oplus\mathfrak{a}^*$-module morphisms $i_\pm : M_\pm \longrightarrow M_\pm\otimes M_\pm$.

2) The equality $\phi(1) = 1_+\otimes 1_-$ defines a unique $\mathfrak{a}\oplus\mathfrak{a}^*$-module morphism $\phi : \mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*) \longrightarrow M_+\otimes M_-$. ϕ is an isomorphism.

3) There exists an element $J = \sum u_i\otimes v_i \in (\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*)[[\hbar]])^{\hat{\otimes}2}$ with $(id\otimes\epsilon_0)J = 1 = (\epsilon_0\otimes id)J$ such that when twisting [6, 7] *the quasitriangular quasi-Hopf algebra of theorem 3.1 via J^{-1} one obtains a quasitriangular Hopf QUE-algebra, $(\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*)[[\hbar]]; \cdot; 1; \Delta; \epsilon_0; S; R)$, which is a quantization of pair $(\mathfrak{a}\oplus\mathfrak{a}^*; r)$. The element J is*

$$J = (\phi^{-1}\otimes\phi^{-1})\left(\Phi^{-1}_{1,2,34}\circ\Phi_{2,3,4}\circ\sigma_{23}\circ e^{\frac{\hbar}{2}\Omega_{23}}\circ\Phi^{-1}_{2,3,4}\circ\Phi_{1,2,34}(i_+\otimes i_-)(\phi(1))\right),$$

and when writing $Q = \sum S_0(u_i)\cdot v_i$, $u\in\mathfrak{a}\oplus\mathfrak{a}^*$ it is

$$\Delta(u) = J^{-1}\cdot\Delta_0(u)\cdot J, \quad S(u) = Q^{-1}\cdot S_0(u)\cdot Q, \quad R = \sigma(J^{-1})\cdot e^{\frac{\hbar}{2}\Omega}\cdot J$$

and Φ verifies the following equalities

$$\Phi\cdot(\Delta_0\otimes id)(J)\cdot(J\otimes 1) = (1\otimes\Delta_0)(J)\cdot(1\otimes J), \quad R = 1\otimes 1 + \hbar r \mod \hbar^2.$$

This quasitriangular Hopf QUE-algebra will be denoted by $A_{(\mathfrak{a}\oplus\mathfrak{a}^*)[[\hbar]],\Omega,J^{-1}}$.

3.2. Quantization of quasitriangular Lie bialgebras. *1) Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ be a quasitriangular Lie bialgebra. Consider the following linear isomorphisms $\theta^+, \theta^- : \mathfrak{a}\otimes\mathfrak{a} \longrightarrow Hom(\mathfrak{a}^*, \mathfrak{a})$ defined by*

$$\theta^+(x\otimes y)(x^*) = x\cdot x^*(y) \qquad \theta^-(x\otimes y)(x^*) = x^*(x)\cdot y.$$

Write $\mathfrak{a}_\pm = Im\,\theta^\pm(r_1) \subseteq \mathfrak{a}$. The notion of the rank of r_1 allows us to assert:

a) the subspaces $Im\,\theta^\pm(r_1) = \mathfrak{a}_\pm$ associated with $r_1\in\mathfrak{a}\otimes\mathfrak{a}$ are canonically isomorphic to $(Im\,\theta^\mp(r_1))^* \equiv \mathfrak{a}^*_\mp$.

b) $r_1\in\mathfrak{a}_+\otimes\mathfrak{a}_-$ and $r_1 = \sum_i a_i\otimes b_i$, where $a_i\in\mathfrak{a}_+$ and $b_i\in\mathfrak{a}_-$, $\forall\ i = \{1, 2, \cdots, rank(r_1)\}$.

c) *the mapping* $\chi_{r_1} : \mathfrak{a}_+^* \to \mathfrak{a}_-$ *defined by* $\chi_{r_1}(x^*) = \sum_i x^*(a_i) \cdot b_i$ *is an isomorphism.*

As r_1 *is a solution of the CYBE,* $\mathfrak{a}_+$ *and* $\mathfrak{a}_-$ *are Lie subalgebras of* $\mathfrak{a}$*, a result from* [12]*.*

Using the isomorphism χ_{r_1} *it is possible to define a Lie algebra structure on* $\mathfrak{a}_+^*$ *by* $[,]^*_{\mathfrak{a}_+} = \chi_{r_1}^{-1} \circ [,]_{\mathfrak{a}_-} \circ (\chi_{r_1} \otimes \chi_{r_1})$*. Then we have*

Theorem 3.3. *The Lie algebra structures* $[,]_{\mathfrak{a}_+^*}$ *on* $\mathfrak{a}_+^*$ *and* $[,]_{\mathfrak{a}_+}$ *on* $\mathfrak{a}_+$ *are compatible in the sense of Drinfeld (see* [15, 16]*). The mapping* $\varepsilon_{\mathfrak{a}_+} = \phi^t : \mathfrak{a}_+ \longrightarrow \mathfrak{a}_+ \otimes \mathfrak{a}_+$ *where* $\phi(\xi_1 \otimes \xi_2) = [\xi_1 ; \xi_2]_{\mathfrak{a}_+^*}$ *is then a 1-cocycle on* $(\mathfrak{a}_+, [,]_{\mathfrak{a}_+})$*. The set* $(\mathfrak{a}_+, [,]_{\mathfrak{a}_+}, \varepsilon_{\mathfrak{a}_+} = \phi^t)$ *is a Lie bialgebra whose quasitriangular double Lie bialgebra is* $(\mathfrak{a}_+ \oplus \mathfrak{a}_+^*, [,]_{\mathfrak{a}_+ \oplus \mathfrak{a}_+^*}, \varepsilon_{\mathfrak{a}_+ \oplus \mathfrak{a}_+^*} = d_c r_+)$*, where* r_+ *is the invariant canonical element.*

From [12, 9] *it follows*

Proposition 3.4. *The mapping* $\tilde{\pi} : \mathfrak{a}_+ \oplus \mathfrak{a}_+^* \longrightarrow \mathfrak{a}$*, defined as* $\tilde{\pi}(x; \xi) = x + \chi_{r_1}(\xi)$ *is a Lie-bialgebra-morphism. That is, a Lie-algebra morphism verifying* $d_c r_1 \circ \tilde{\pi} = (\tilde{\pi} \otimes \tilde{\pi}) \circ d_c r_+$*. Moreover* $(\tilde{\pi} \otimes \tilde{\pi}) r_+ = r_1$*. The symbol* $\tilde{\pi}$ *will also denote the unique algebra morphism* $\tilde{\pi} : \mathcal{U}(\mathfrak{a}_+ \oplus \mathfrak{a}_+^*) \longrightarrow \mathcal{U}(\mathfrak{a})$ *defined by the Lie algebra morphism* $\tilde{\pi}$*.*

2) *From theorems 3.1, 3.3* and *proposition 3.4 it is possible to obtain a quantization of the pair* $(\mathfrak{a}, r_1 + \sigma(r_1))$*. With the obvious notations we have*

Theorem 3.5. *Let* $(\mathfrak{a}_+ \oplus \mathfrak{a}_+^*, [,]_{\mathfrak{a}_+ \oplus \mathfrak{a}_+^*}, \varepsilon_{\mathfrak{a}_+ \oplus \mathfrak{a}_+^*} = d_c\, r_+)$ *be the quasitriangular Lie bialgebra in theorem 3.3. Let* $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, 1, \Delta_{\mathfrak{a}}, \epsilon_{\mathfrak{a}}, S_{\mathfrak{a}})$ *be the usual Hopf universal enveloping algebra. Let*

$$\left(\mathcal{U}(\mathfrak{a}_+ \oplus \mathfrak{a}_+^*)[[\hbar]], \cdot, 1, \Delta_0^+, \epsilon_0^+, \Phi^+, S_0^+, \alpha^+ = (c^+)^{-1}, \beta^+ = 1, R_0^+ = e^{\frac{\hbar}{2}\Omega_+}\right),$$

be the quasitriangular quasi-Hopf algebra in theorem 3.1 whose classical limit is the pair $(\mathfrak{a}_+ \oplus \mathfrak{a}_+^*, \Omega_+)$*. Then we have the equalities :* $(\tilde{\pi} \otimes \tilde{\pi}) \circ \Delta_0^+ = \Delta_{\mathfrak{a}} \circ \tilde{\pi}$*;* $\tilde{\pi} \circ S_0^+ = S_{\mathfrak{a}} \circ \tilde{\pi}$ *and putting* $\tilde{\Phi}^+ = (\tilde{\pi} \otimes \tilde{\pi})\Phi^+$*,* $R_{\mathfrak{a}} = (\tilde{\pi} \otimes \tilde{\pi}) R_0^+$*, the set* $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, 1, \Delta_{\mathfrak{a}}, \epsilon_{\mathfrak{a}}, \tilde{\Phi}^+, S_{\mathfrak{a}}, \tilde{\alpha} = (\tilde{\pi}(c^+))^{-1}, \tilde{\beta} = 1, R_{\mathfrak{a}})$ *is a quasitriangular quasi-Hopf QUE-algebra whose classical limit is the pair* $(\mathfrak{a}, r_1 + \sigma(r_1))$*.*

The above results allow us to obtain a quantization of the pair $(\mathfrak{a}, r_1)$*. With the obvious notations the result can be stated as follows* [9] *(see also* [22]*):*

Theorem 3.6. *Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ be a quasitriangular Lie bialgebra. Let $(\mathfrak{a}_+ \oplus \mathfrak{a}_+^*, [,]_{\mathfrak{a}_+\oplus\mathfrak{a}_+^*}, \varepsilon_{\mathfrak{a}_+\oplus\mathfrak{a}_+^*} = d_c\, r_+)$ be the quasitriangular Lie bialgebra in theorem 3.3. Let $(\mathcal{U}(\mathfrak{a}_+ \oplus \mathfrak{a}_+^*)[[\hbar]], \cdot, 1; \varepsilon_0^+; \Delta^+, S^+, R^+)$ be the quasitriangular Hopf QUE algebra obtained as in theorem 3.2, 3). Write $\tilde{J}^+_{r_1} = (\tilde{\pi} \otimes \tilde{\pi})J^+$. We have the equality*

$$\tilde{\Phi}^+_{1,2,3} \cdot (\Delta_{\mathfrak{a}} \otimes id)\tilde{J}^+_{r_1} \cdot (\tilde{J}^+_{r_1} \otimes 1) = (id \otimes \Delta_{\mathfrak{a}})\tilde{J}^+_{r_1} \cdot (1 \otimes \tilde{J}^+_{r_1}).$$

Write again $\tilde{J}_{r_1} = \sum p_i \otimes q_i$; $a \in \mathcal{U}\mathfrak{a}$, $\quad \tilde{Q} = \sum S_{\mathfrak{a}}(p_i) \cdot q_i$. The set $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, 1,$ $\tilde{\Delta}, \tilde{\epsilon} = \epsilon_{\mathfrak{a}}, \tilde{S}, \tilde{R})$ where

$$\tilde{\Delta}(a) = (\tilde{J}^+_{r_1})^{-1} \cdot \Delta_{\mathfrak{a}}(a) \cdot \tilde{J}^+_{r_1}; \epsilon_{\mathfrak{a}}; \tilde{R} = (\tilde{\pi} \otimes \tilde{\pi})\, R^+; \tilde{S}(a) = \tilde{Q}^{-1} \cdot S_{\mathfrak{a}}(a) \cdot \tilde{Q}$$

is a quasitriangular Hopf QUE-algebra which is a quantization of the pair $(\mathfrak{a}; r_1)$, and has been obtained by a twist, [6] *, via the element $(\tilde{J}^+_{r_1})^{-1}$ from the quasitriangular quasi-Hopf algebra in theorem 3.5.*

4. Quantization of non-degenerate triangular Lie bialgebras

1) In case of a non-degenerate triangular Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$, we have rank$(r_1) = dim\, \mathfrak{a}$, $r_1 \in \wedge^2(\mathfrak{a})$. Also $\mathfrak{a}_+ = \mathfrak{a}_- = \mathfrak{a}$ as Lie algebras, $(\tilde{\pi} \otimes \tilde{\pi})\Omega_+ = (\tilde{\pi} \otimes \tilde{\pi})(r_1 + \sigma(r_1)) = 0$ and we get the equality $\varepsilon_{\mathfrak{a}_+} = \varepsilon_{\mathfrak{a}} = d_c r_1$.

Definition 4.1. [1, 5, 19, 17]*) An ISP on a non-degenerate triangular Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ is any element $F = \sum_0^\infty F_k \cdot \hbar^k \in \mathcal{U}\mathfrak{a}^{\otimes 2}[[\hbar]]$ verifying the following equalities:*

$$\begin{array}{ll} 1) & (\epsilon_{\mathfrak{a}} \otimes id)F = (id \otimes \epsilon_{\mathfrak{a}})F = 1 \otimes 1; \\ 2) & F - \sigma(F) = r_1 \hbar \, mod\, \hbar^2; \\ 3) & (\Delta_{\mathfrak{a}} \otimes 1)F \cdot (F \otimes 1) = (1 \otimes \Delta_{\mathfrak{a}})F \cdot (1 \otimes F). \end{array}$$

Theorem 3.6 allows us to obtain an ISP on any non-degenerate triangular Lie bialgebra. In fact it allows us to obtain, modulo equivalence, all of them, as we will see in section 6.

Theorem 4.2. *Suppose in Theorem 3.6 that* $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$ *is a non-degenerate triangular Lie bialgebra. Then*

$$\tilde{\Phi} = (\tilde{\pi} \otimes \tilde{\pi} \otimes \tilde{\pi})\Phi = 1 \otimes 1 \otimes 1.$$

1) $(\epsilon_{\mathfrak{a}} \otimes id)\tilde{J}_{r_1} = (id \otimes \epsilon_{\mathfrak{a}})\tilde{J}_{r_1} = 1 \otimes 1.$

2) $\tilde{J}_{r_1} = 1 \otimes 1 + \frac{1}{2} r_1 \hbar + \cdots$

3) $(\Delta_{\mathfrak{a}} \otimes 1)\tilde{J}_{r_1} \cdot (\tilde{J}_{r_1} \otimes 1) = (1 \otimes \Delta_{\mathfrak{a}})\tilde{J}_{r_1} \cdot (1 \otimes \tilde{J}_{r_1}).$

4) $\tilde{R} = (\tilde{\pi} \otimes \tilde{\pi})R = \sigma(\tilde{J}_{r_1}^{-1}) \cdot (1 \otimes 1) \cdot \tilde{J}_{r_1} = 1 \otimes 1 + r_1 \hbar + \cdots$

In particular $\tilde{J}_{r_1}$ *is an ISP on* $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1.)$
The triangular Hopf QUE algebra $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, 1, \tilde{\Delta}, \tilde{\epsilon} = \epsilon_{\mathfrak{a}}, \tilde{S}, \tilde{R})$*, denoted by* $A_{\mathfrak{a}[[\hbar]], \tilde{J}_{r_1}^{-1}}$*, which is obtained by a twist via* $\tilde{J}_{r_1}^{-1}$ *from the trivial triangular Hopf QUE algebra* $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, 1, \Delta_{\mathfrak{a}}, \epsilon_{\mathfrak{a}}, S_{\mathfrak{a}}, R_{\mathfrak{a}} = 1 \otimes 1)$ *is a quantization of the pair* $(\mathfrak{a}; r_1)$.

2) The above proposition shows how to obtain an ISP on $(\mathbf{G}, \beta_t)$ *as in 3) Section 2. The following proposition will show that if we put* $\hbar$ *in place of* t *in this star product we have again an ISP but this time on the Lie group* $(\mathbf{G}, \beta_1)$*. In this way we don't get all the ISPS on* $(\mathbf{G}, \beta_1)$ *but if we now replace the above* $r_\hbar$ *coming from* r_t *by any element in* $(\mathfrak{a} \wedge \mathfrak{a})[[\hbar]]$ *of the form* $r_\hbar = r_1 + \cdots \in (\mathfrak{a} \wedge \mathfrak{a})[[\hbar]]$ *we obtain, up to equivalence, all the (ISP's) on* $(\mathbf{G}, \beta_1)$.

Proposition 4.3. *Let* $\Big((\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar})[[\hbar]], [,]_{\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar}}, \varepsilon_{\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar}} = d_c r\Big)$ *and* $r_\hbar \in (\mathfrak{a} \wedge \mathfrak{a})[[\hbar]]$ *be as in Section 2, 4) . For any* $N \in \mathbb{N}$ *there exists an analytic function in a neighborhood of* $t = 0$, $r_t^N = r_1 + r_2 t + r_3 t^2 + \cdots \in \mathfrak{a} \wedge \mathfrak{a}$ *and a solution of YBE on* $(\mathfrak{a}, [,]_{\mathfrak{a}})$*, such that when replacing* t *by* $\hbar$ *in the above series expansion* r_t^N *and* $r_\hbar$ *coincide up to order* N.

Proof:
In the Poisson cohomology on the Lie algebra $(\mathfrak{a}[[\hbar]], [,]_{\mathfrak{a}[[\hbar]]})$ *the element* $\beta_\hbar = \beta_1 + \beta_2 \hbar + \beta_3 \hbar^2 + \cdots \in (\mathfrak{a}^* \wedge \mathfrak{a}^*)[[\hbar]]$ *defined as* $\beta_\hbar = \mu_{r_\hbar}(r_\hbar)$ *is a Chevalley 2-cocycle. The polynomial* $\beta_t^N = \beta_1 + \beta_2 t + \beta_3 t^2 + \cdots + \beta_{N-1} t^N \in \mathfrak{a}^* \wedge \mathfrak{a}^*$ *is also a Chevalley 2-cocycle on* $(\mathfrak{a}, [,]_{\mathfrak{a}})$*. The corresponding element* $r_t^N = r_1 + r_2 t + \cdots$ *satisfies YBE on* $(\mathfrak{a}, [,]_{\mathfrak{a}})$ *and when we replace* $\hbar$ *in place of* t *it coincides with* $r_\hbar$ *up to term* N. ■

3) From theorem 3.1 we may deduce, in the obvious notations,

Theorem 4.4. *Let* $\left(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}} = d_c\, r\right)$ *over* $\mathbb{R}$ *be as in Section 2, 3). The set*

$$(\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})[[\hbar]], \cdot_t, 1, \Delta^t_0, \epsilon^t_0, \Phi_{r_t}, S^t_0, \alpha^t = c_t^{-1}, \beta^t = 1, R^t_0 = e^{\frac{\hbar}{2}\Omega_t}),$$

where $c_t = \sum_i X_i \cdot_t S^t_0(Y_i) \cdot_t Z_i$, *with* $\Phi_{r_t} = \sum_i X^t_i \otimes Y^t_i \otimes Z^t_i$, *is a quasitriangular quasi-Hopf QUE algebra over* $\mathbb{R}[[\hbar]]$ *with the pair* $(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}, \Omega_t)$ *as its classical limit.*

From theorem 4.4 and proposition 4.3 we can prove

Theorem 4.5. *Let* $\left(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar}[[\hbar]], [,]_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar}}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar}} = d_c\, r\right)$ *over* $\mathbb{R}[[\hbar]]$ *be as in Section 2, 4). The set*

$$\left(\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar})[[\hbar]], \cdot_\hbar, 1, \Delta^\hbar_0, \epsilon_0, \Phi_{r_\hbar}, S^\hbar_0, \alpha^\hbar = c_\hbar^{-1}, \beta^\hbar = 1, R^\hbar_0 = e^{\frac{\hbar}{2}\Omega_\hbar}\right)$$

is then a quasitriangular quasi-Hopf QUE algebra over $\mathbb{R}[[\hbar]]$, *where* $c_\hbar = \sum_i X^\hbar_i \cdot_\hbar S^\hbar_0(Y^\hbar_i) \cdot_\hbar Z^\hbar_i$ *with* $\Phi_{r_\hbar} = \sum_i X^\hbar_i \otimes Y^\hbar_i \otimes Z^\hbar_i$ *and* $S^\hbar_0$ *is the antipode of* $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar}[[\hbar]])$.

Proof:
This theorem follows from Theorem A" in [7]. *In view of the next sections we want to obtain it from theorem 4.4 by quantizing first the Lie groups* $(\mathbf{G}, \beta_t)$. *All the elements in the above set in the theorem are well defined with the corresponding meanings on* $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar})[[\hbar]]$ *and can be seen as those obtained from the corresponding ones in theorem 4.4 if we use the full-meaning trick of putting* $\hbar$ *in place of* t. *To prove that this set defines a quasitriangular quasi-Hopf QUE algebra over* $\mathbb{R}[[\hbar]]$ *we need to prove the equalities which define this structure* [6, 7]. *These equalities are satisfied in the case of* r_t. *This means that for each one of them and when an ordered basis is used in* $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})$ *an infinite set of polynomials in the components of* r_t *are zero. But the set of these components is characterized just by the algebraic equations characterizing a solution of YBE. As a consequence we can see that the corresponding equalities are also satisfied if we replace everywhere* t *by* $\hbar$, *and of course also in the products of elements in the above basis, that is* r_t *by the corresponding solution* $r_\hbar$ *of YBE on* $(\mathfrak{a}[[\hbar]], [,]_{\mathfrak{a}[[\hbar]]})$. *Then applying Proposition 4.3 we get the theorem.* ∎

4) *Theorem 3.2 allows us to write*

Theorem 4.6. *Let* $\left(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t},[,]_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}},\varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}}=d_c\,r\right)$ *over* $\mathbb{R}$ *be as in Section 2, 3). Write* $J_{r_t}=\sum_i u_i\otimes v_i\in\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})^{\otimes^2}[[\hbar]]$ *where*

$$J_{r_t}=(\phi_t^{-1}\otimes\phi_t^{-1})\left((\Phi_{r_t}^{-1})_{1,2,34}\circ(\Phi_{r_t})_{2,3,4}\circ\sigma_{23}\circ e^{\frac{\hbar}{2}(\Omega_t)_{23}}\circ(\Phi_{r_t}^{-1})_{2,3,4}\circ\right.$$
$$\left.\circ(\Phi_{r_t})_{1,2,34}\circ\circ(i_+\otimes i_-)\circ\phi_t(1)\right)$$

is the corresponding element to the one introduced in theorem 3.2, part 3). In the present case we have written $\phi_t:\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})\longrightarrow M_+^{r_t}\otimes M_-^{r_t}$ *for the morphism* ϕ. *The element* J_{r_t} *satisfies the equalities* $(id\otimes\epsilon_0^t)J_{r_t}=1=(\epsilon_0^t\otimes id)J_{r_t}$ *and when twisting the quasitriangular quasi-Hopf QUE-algebra in theorem 4.4 via* $J_{r_t}^{-1}$ *one obtains a quasitriangular Hopf QUE-algebra* $\left(\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})[[\hbar]],\cdot_t,1,\Delta_t,\epsilon_t\equiv\epsilon_0^t,S_t,R_t\right)$ *over* $\mathbb{R}[[\hbar]]$ *whose classical limit is the pair* $(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t};r_t)$ *and which will be denoted by* $A_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}[[\hbar]],\Omega_t,J^{-1}_{r_t}}$. *If we write* $Q_t=\sum_i S_0^t(u_i)\cdot_t v_i$ *the above defining elements are*

$$\Delta_t(u)=J_{r_t}^{-1}\cdot_t\Delta_0^t(u)\cdot_t J_{r_t};\; S_t(u)=Q_t^{-1}\cdot_t S_0^t(u)\cdot_t Q_t;\, R_t=\sigma(J_{r_t})^{-1}\cdot_t e^{\frac{\hbar}{2}\Omega_t}\cdot_t J_{r_t},$$

and Φ_{r_t} *satisfies the following equalities*

$$\Phi_{r_t}\cdot_t(\Delta_0^t\otimes id)(J_{r_t})\cdot_t(J_{r_t}\otimes 1)=(1\otimes\Delta_0^t)(J_{r_t})\cdot_t(1\otimes J_{r_t});\, R_t=1\otimes 1+\hbar r_t\mod\hbar^2.$$

The next theorem can be proved from theorem 4.6 in a similar way as theorem 4.5. See also Lemma 4.8.

Theorem 4.7. *The set* $(\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar})[[\hbar]],\cdot_\hbar,1,\Delta_\hbar,\epsilon_0,S_\hbar,R_\hbar)$ *is a quasitriangular Hopf QUE algebra over* $\mathbb{R}[[\hbar]]$. *Its defining elements are the following*

$$\Delta_\hbar(a)=J_{r_\hbar}^{-1}\cdot_\hbar\Delta_0^\hbar(a)\cdot_\hbar J_{r_\hbar},\quad S_\hbar(a)=Q_\hbar^{-1}\cdot_\hbar S_0^\hbar(a)\cdot_\hbar Q_\hbar,$$
$$R_\hbar=\sigma(J_{r_\hbar}^{-1})\cdot_\hbar e^{\frac{\hbar}{2}\Omega}\cdot_\hbar J_{r_\hbar},$$

where $Q_\hbar=\sum_i S_0^\hbar(p_i)\cdot_\hbar q_i$, $J_{r_\hbar}=\sum_i p_i\otimes q_i$, $a\in\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar})[[\hbar]]$ *and* $\Phi_{r_\hbar}$ *satisfies the following equalities*

$$\Phi_{r_\hbar}\cdot_\hbar(\Delta_0^\hbar\otimes id)(J_{r_\hbar})\cdot_\hbar(J_{r_\hbar}\otimes 1)=(1\otimes\Delta_0^\hbar)(J_{r_\hbar})\cdot_\hbar(1\otimes J_{r_\hbar});\, R_\hbar=1\otimes 1+\hbar r_\hbar\mod\hbar^2.$$

This quasitriangular Hopf QUE algebra over $\mathbb{R}[[\hbar]]$ *is therefore obtained from the quasitriangular quasi-Hopf algebra in theorem 4.5 via the element* $J_{r_\hbar}^{-1}\in\widehat{\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar})}\hat{\otimes}\widehat{\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar})}$. *Its classical limit is the quasitriangular Lie bialgebra* $(\mathfrak{a}\oplus\mathfrak{a}^*_{r_1},[,]_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_1}},\varepsilon=d_c r)$. *We denote it by* $A_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar},\Omega,J^{-1}_{r_\hbar}}$.

5) The following two lemmas will be needed.

Choose an ordered basis $\{e_a\}$ in $\mathfrak{a}$, and its dual basis $\{e^a\}$ in $\mathfrak{a}^*$. Then we can construct ordered bases in $\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}$, $\mathcal{U}\mathfrak{a}$, $\mathcal{U}\mathfrak{a}^*$, $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})^{\otimes 2}$. We can prove

Lemma 4.8. [22] *Let $r_t = r_1 + r_2 t + r_3\, t^2 + \cdots \in \mathfrak{a}\wedge\mathfrak{a}$ be as in Section 2, 3). The element $J_{r_t}\in\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})^{\otimes 2}[[\hbar]]$ in Theorem 4.6 can be written as*

$$J_{r_t} = 1\otimes 1 + \frac{1}{2} r\,\hbar + \sum_{k\geq 2}\left(r_t^{i_1 j_1}\dots r_t^{i_{l(k)} j_{l(k)}} Q_{i_1,\dots,i_{l(k)},j_1,\dots,j_{l(k)},k}\right)\hbar^k,$$

where $Q_{i_1,\dots,i_{l(k)},j_1,\dots,j_{l(k)},k}\in\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})^{\otimes 2}$ is a (finite) linear combination of tensor products of elements in the above ordered basis. r_t is manifested in every element of the ordered basis through the product in $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})$ but it does not appear in the coefficients defining $Q_{i_1,\dots,i_{l(k)},j_1,\dots,j_{l(k)},k}$.

Proof:

From the expression of J on theorem 3.2 and because $\Phi = 1\otimes 1\otimes 1 + O(\hbar^2)$ we have

$$\begin{aligned}
J_{r_t} &= (\phi_t^{-1}\otimes\phi_t^{-1})\left((1+\frac{\hbar}{2}\Omega_{23})(1_+\otimes 1_-\otimes 1_+\otimes 1_-)\right)\ mod\,\hbar^2\\
&= \Big((\phi_t^{-1}\otimes\phi_t^{-1})\,(1_+\otimes 1_-\otimes 1_+\otimes 1_-) + \\
&\quad + \frac{1}{2}(\phi_t^{-1}\otimes\phi_t^{-1})\,(1_+\otimes r_{12}(1_-\otimes 1_+)\otimes 1_-)\,\hbar\Big)\ mod\,\hbar^2\\
&= \left(1\otimes 1+\frac{1}{2}r\hbar\right)\ mod\,\hbar^2.
\end{aligned}$$

In the expression for J_{r_t} the coefficient of $\hbar^k$, for $k\geq 2$, is an element in $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})\otimes\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})$ which depends on r_t (by the brackets on Φ_{r_t} and by the products on $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})$), so is of the form

$$r_t^{i_1 j_1}\dots r_t^{i_{l(k)} j_{l(k)}} Q_{i_1,\dots,i_{l(k)},j_1,\dots,j_{l(k)},k},$$

where $Q_{i_1,\dots,i_{l(k)},j_1,\dots,j_{l(k)},k}\in\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})\otimes\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})$ only depends on r_t through the products on the enveloping algebra $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})$ in the ordered basis. ∎

As the product in $\mathcal{U}\mathfrak{a}$ is independent of r_t, applying $\tilde\pi_t\otimes\tilde\pi_t$ to J_{r_t} and then putting $t=\hbar$, we can also prove

Lemma 4.9. [22] *Let $r_t = r_1 + r_2 t + r_3\, t^2 + \cdots \in \wedge^2(\mathfrak{a})$ be as in Section 2, 3). Write $r_l = r_l^{ab}\, e_a\otimes e_b$, $r_l^{ab} + r_l^{ba} = 0$, $l = 1,2,3,\cdots$. Let $\tilde\pi_t$ be the Lie* *bialgebra morphism defined in proposition 3.4 and define* $\tilde{J}_{r_t} = (\tilde{\pi}_t \otimes \tilde{\pi}_t) J_{r_t}$, $\tilde{J}_{r_\hbar} = (\tilde{J}_{r_t}) \mid_{t\to\hbar} \in (\mathcal{U}\mathfrak{a})^{\otimes 2}[[\hbar]]$. *Write it as a formal power series in* $\hbar$. *The element* $\tilde{J}_{r_\hbar}$ *can be written as*

$$\tilde{J}_{r_\hbar} = 1 \otimes 1 + \frac{1}{2} r_1 \hbar + \sum_{R=2}^{\infty} \Bigg(\frac{1}{2} r_R + \sum_{i_2, j_2, \ldots i_R, j_R} \Bigg(\sum_{A_{i_2,j_2}(R), \ldots, A_{i_R j_R}(R)}$$

$$(r_1^{i_2 j_2})^{A_{i_2 j_2}(R)} \ldots (r_{R-1}^{i_R j_R})^{A_{i_R j_R}(R)} H_{i_2, \ldots, i_R, j_2, \ldots, j_R, A_{i_2 j_2}(R), \ldots, A_{i_R j_R}(R), R} \Bigg) \Bigg) \hbar^R,$$

where $H_{i_2,\ldots,i_R,j_2,\ldots,j_R,A_{i_2j_2}(R),\ldots,A_{i_Rj_R}(R),R} \in (\mathcal{U}\mathfrak{a})^{\otimes 2}$ *is a (finite) linear combination of tensor products of elements in the above ordered basis and it is independent of* r_l, $l = 1, 2, 3, \ldots$

6) From lemma 4.9 we obtain the following proposition and corollary which will be applied in the next section.

Proposition 4.10. [22] *Let* $r_t = r_1 + r_2 t + r_3 t^2 + \cdots \in \Lambda^2(\mathfrak{a})$ *be as in Section 2, 3). Let* $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_t)$ *be the non-degenerate triangular Lie bialgebra defined by* r_t. *Let* $\Big(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, \tilde{\Delta}_t, \tilde{S}_t, \tilde{R}_t\Big)$ *be the triangular Hopf QUE-algebra whose classical limit is the pair* $(\mathfrak{a}; r_t)$ *and was obtained in theorem 4.2 from the usual triangular Hopf algebra* $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, \Delta_{\mathfrak{a}}, S_{\mathfrak{a}}, R_{\mathfrak{a}} = 1)$ *by a twist via* $(\tilde{J}_{r_t})^{-1}$. *Consider, as before, the element* $\tilde{J}_{r_\hbar} = (\tilde{J}_{r_t}) \mid_{t\to\hbar} \in \mathcal{U}\mathfrak{a}[[\hbar]] \hat{\otimes} \mathcal{U}\mathfrak{a}[[\hbar]]$. *Then the following equalities hold:*

(a) $(\Delta_{\mathfrak{a}} \otimes id) \tilde{J}_{r_\hbar} \cdot_\hbar (\tilde{J}_{r_\hbar} \otimes 1) = (1 \otimes \Delta_{\mathfrak{a}}) \tilde{J}_{r_\hbar} \cdot_\hbar (id \otimes \tilde{J}_{r_\hbar})$;

(b) $\tilde{J}_{r_\hbar} = 1 \otimes 1 + \frac{1}{2} r_1 \, \hbar + 0(\hbar^2)$.

Let us define $\Delta(a) = (\tilde{J}_{r_\hbar})^{-1} \cdot_\hbar \Delta_{\mathfrak{a}}(a) \cdot_\hbar \tilde{J}_{r_\hbar}$, $R = (\sigma \tilde{J}_{r_\hbar})^{-1} \cdot_\hbar (1 \otimes 1) \cdot_\hbar \tilde{J}_{r_\hbar}$ *and* $S(a) = Q^{-1} \cdot_\hbar S_{\mathfrak{a}}(a) \cdot_\hbar Q$, *where* $Q = \sum S_{\mathfrak{a}}(a_i) \cdot_\hbar b_i$, $\tilde{J}_{r_\hbar} = \sum a_i \otimes b_i$; $a_i, b_i \in \mathcal{U}\mathfrak{a}[[\hbar]]$. *The set* $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot_\hbar, \Delta, S, R)$ *is then a triangular Hopf QUE algebra obtained twisting the usual triangular Hopf algebra* $(\mathcal{U}\mathfrak{a}[[\hbar]], \cdot, \Delta_{\mathfrak{a}}, S_{\mathfrak{a}}, R_{\mathfrak{a}} = 1 \otimes 1)$ *via the element* $(\tilde{J}_{r_\hbar})^{-1} \in \mathcal{U}\mathfrak{a}[[\hbar]] \hat{\otimes} \mathcal{U}\mathfrak{a}[[\hbar]]$. *We write it as* $A_{\mathfrak{a}[[\hbar]], (\tilde{J}_{r_\hbar})^{-1}}$.

Corollary 4.11. *Let* $r'_t = r_1 + r_2 t + r_3 t^2 + \cdots + r_{k-1} t^{k-2} + (r_k + s_k) t^{k-1} + \cdots \in \Lambda^2(\mathfrak{a})$ *be another element. Let* $\tilde{J}_{r'_\hbar}$ *be the star product determined by* r'_t *in the similar way as* $\tilde{J}_{r_\hbar}$ *was from* r_t. *Then* $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r'_\hbar}$ *coincide up to order* $k-1$ *and*

$$\Big(\tilde{J}_{r'_\hbar}\Big)_k - \Big(\tilde{J}_{r_\hbar}\Big)_k = \frac{1}{2} s_k.$$

5. ISP F determines $r_\hbar \in (\mathfrak{a} \otimes \mathfrak{a})[[\hbar]]$. $\tilde{J}^{\Phi}_{r_\hbar}$ and F are equivalent.

Let $F \in \mathcal{U}\mathfrak{a}[[\hbar]]\hat{\otimes}\mathcal{U}\mathfrak{a}[[\hbar]]$ be an ISP on the non-degenerate triangular Lie bialgebra $(\mathfrak{a}, [,]_\mathfrak{a}, \varepsilon_\mathfrak{a} = d_c r_1)$. Let $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ be the triangular Hopf QUE algebra obtained by a twist via F^{-1} from the trivial triangular Hopf QUE algebra $(\mathcal{U}(\mathfrak{a})[[\hbar]], \cdot, 1, \Delta_\mathfrak{a}, S_\mathfrak{a}, R_\mathfrak{a} = 1 \otimes 1)$. It is then a quantization of the pair $(\mathfrak{a}, r_1)$.

The following proposition does not depend on any specific context of quantization [19] *but only on the notion of deformation of associative algebras* [10]*, the fact that Hochschild cohomology spaces of coalgebra $(\mathcal{U}\mathfrak{a}, \mathbb{R})$ are $H^k(\mathcal{U}\mathfrak{a}) = \Lambda^k\mathfrak{a}$, $k \in \mathbb{N}$,* [3]*, and the Hochschild cohomological* [18] *interpretation of the Quantum Yang Baxter equation.*

Proposition 5.1. [18] *Let $F = \sum_i^\infty F_i \hbar^i$ and $F' = \sum_i^\infty F'_i \hbar^i$ be ISP on $(\mathfrak{a}, [,]_\mathfrak{a}, \varepsilon_\mathfrak{a} = d_c r_1)$. Let $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ and $A_{\mathfrak{a}[[\hbar]],F'^{-1}}$ be as before in this section. Suppose that F and F' coincide up to order k, i.e. $F'_l = F'_l$, $\quad l = 1, 2, \cdots, k$. Then: a) there exist $h_{k+1} \in \mathfrak{a} \wedge \mathfrak{a}$ and $E_{k+1} \in \mathcal{U}\mathfrak{a}$ such that $F'_{k+1} - F_{k+1} = h_{k+1} + d_H E_{k+1}$ where d_H is the coboundary operator in the Hochschild cohomology of $\mathcal{U}\mathfrak{a}$; b) h_{k+1} is not only a Hochschild 2-cocycle but also a Poisson 2-cocycle relative to the invariant Poisson structure defined by the element $r_1 \in \mathfrak{a} \wedge \mathfrak{a}$.*

Again, the above Hochschild cohomology spaces and proposition 5.1 play a central role in the proof of the next theorem. In the context of quantification in [9] *the next theorem corresponds to a main theorem by Drinfeld in the context of quantification in* [5] *and in* [19, 20] *there is a proof of this Drinfeld theorem. See the References in* [20] *for a similar theorem about Star Products on general symplectic manifolds and* [13] *on Poisson manifolds.*

Theorem 5.2. *Fix a Lie associator Φ. Let $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ be as defined at the beginning of this section. We have:*

(a) There exist elements $r_\hbar = r_1 + r_2\hbar + r_3\hbar^2 + \cdots \in (\wedge^2\mathfrak{a})[[\hbar]]$ and $E^{r_\hbar} = 1 + E_1^{r_\hbar}\hbar + \cdots + E_n^{r_\hbar}\hbar^n + \cdots \in \mathcal{U}\mathfrak{a}[[\hbar]]$ such that

$$F = \Delta_\mathfrak{a}((E^{r_\hbar})^{-1}) \cdot_\hbar \tilde{J}^{\Phi}_{r_\hbar} \cdot_\hbar (E^{r_\hbar} \otimes E^{r_\hbar});$$

i.e., F and $\tilde{J}^{\Phi}_{r_\hbar}$ are equivalent ISPS over the non-degenerate triangular Lie bialgebra $(\mathfrak{a}, [,]_\mathfrak{a}, \varepsilon_\mathfrak{a} = d_c r_1)$.

(b) The triangular Hopf QUE algebras $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ *and* $A_{\mathfrak{a}[[\hbar]],\left(\tilde{J}^{\Phi}_{r_\hbar}\right)^{-1}}$ *are isomorphic.*

Proof:
(a) We construct an analytic function

$$r_t(n) = r_1 + \sum_{l=2}^{\infty} r_l(n)t^{l-1} \in \wedge^2(\mathfrak{a})$$

which is a non-degenerate solution of CYBE and we prove that $\tilde{J}^{\Phi}_{r_\hbar(n)}$ *and* F *are equivalent at order* n, $n \in \mathbb{N}$, *by induction on the order of equivalence. The results comes from that equivalence.*

(i) Let $\tilde{J}_{r_\hbar(1)}$ *be the star product obtained from the E-K quantization (proposition 4.10), determined by the CYBE solution* $r_t(1) = r_1$.

As $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ *is a quantization of the given Lie bialgebra* $R = 1\otimes 1 + r_1\hbar + \ldots$, *and because it is the twist of the usual triangular Hopf QUE algebra* ($R_{\mathfrak{a}} = 1 \otimes 1$) *we have*

$$F_1 - \sigma F_1 = r_1 \tag{5.1}$$

and, by construction,

$$(\tilde{J}_{r_\hbar(1)})_1 = \frac{1}{2} r_1. \tag{5.2}$$

The associative property at order 1 for F *and* $\tilde{J}_{r_\hbar(1)}$ *is* $d_H F_1 = 0$ *and* $d_H(\tilde{J}_{r_\hbar(1)})_1 = 0$, d_H *being the Hochschild cohomology operator. Therefore,* $d_H((\tilde{J}_{r_\hbar(1)})_1 - F_1) = 0$, *i.e.,* $(\tilde{J}_{r_\hbar(1)})_1 - F_1$ *is a Hochschild* 2*-cocycle. The* 2*-cocycle condition implies that there exist* $h_1 \in \mathfrak{a} \wedge \mathfrak{a}$ *and* $E_1 \in \mathcal{U}\mathfrak{a}$ *such that*

$$(\tilde{J}_{r_\hbar(1)})_1 - F_1 = h_1 + d_H E_1. \tag{5.3}$$

On the other hand, from (5.1) and (5.2), and because $(\tilde{J}_{r_\hbar(1)})_1$ *is skew-symmetric, we conclude that* $F_1 - (\tilde{J}_{r_\hbar(1)})_1 = \sigma(F_1 - (\tilde{J}_{r_\hbar(1)})_1)$, *i.e.,* $F_1 - (\tilde{J}_{r_\hbar(1)})_1$ *is symmetric. As* h_1 *is the skew-symmetric part in (5.3), we get* $h_1 = 0$ *and*

$$(\tilde{J}_{r_\hbar(1)})_1 - F_1 = d_H E_1. \tag{5.4}$$

$\tilde{J}_{r_\hbar(1)}$ *and* F *are equivalent to order 1* ($E^{r_\hbar(1)} = 1 + E_1\hbar$).

(ii) $\tilde{J}_{r_\hbar(1)}$ *and* F *being equivalent to order 1, we know (Gerstenhaber* [10]*) that* $(\tilde{J}_{r_\hbar(1)})_2 - F_2 + G_2(E_1, (\tilde{J}_{r_\hbar(1)})_1, F_1)$ *is a Hochschild* 2*-cocycle. Then, there*

exist $h_2 \in \mathfrak{a} \wedge \mathfrak{a}$ and $E_2 \in \mathcal{U}\mathfrak{a}$ such that

$$(\tilde{J}_{r_\hbar(1)})_2 - F_2 + G_2(E_1, (\tilde{J}_{r_\hbar(1)})_1, F_1) = \frac{1}{2} h_2 + d_H E_2. \tag{5.5}$$

Put $E^{r_\hbar(2)} = 1 + E_1\hbar + E_2\hbar^2 \in \mathcal{U}\mathfrak{a}[[\hbar]]$ and consider the star product, equivalent to F, $F^{'(1)} = \Delta_{\mathfrak{a}}((E^{r_\hbar(2)})^{-1}) \cdot F \cdot (E^{r_\hbar(2)} \otimes E^{r_\hbar(2)})$.

At first order, the equivalence condition is

$$F_1^{'(1)} - F_1 = d_H E_1, \tag{5.6}$$

and, from (5.4), we conclude that $F_1^{'(1)} = (\tilde{J}_{r_\hbar(1)})_1$.

At order 2, the equivalence can be written in the form

$$F_2^{'(1)} - F_2 + G_2(E_1, F_1^{'(1)}, F_1) = d_H E_2 \tag{5.7}$$

but, as $F_1^{'(1)} = (\tilde{J}_{r_\hbar(1)})_1$, we get $G_2(E_1, F_1^{'(1)}, F_1) = G_2(E_1, (\tilde{J}_{r_\hbar(1)})_1, F_1)$ and, comparing (5.5) with (5.7), we obtain

$$F_2^{'(1)} = (\tilde{J}_{r_\hbar(1)})_2 - \frac{1}{2} h_2. \tag{5.8}$$

As $(\tilde{J}_{r_\hbar(1)})_1 = F_1^{'(1)}$, h_2 is a Poisson 2-cocycle ([19]), so that $\beta_2 = \mu_{r_1}(h_2)$ is an invariant De Rham (or Chevalley) 2-cocycle, where μ_{r_1} is the isomorphism defined before. We consider $\beta_1 = \mu_{r_1}(r_1) \in \mathfrak{a}^* \wedge \mathfrak{a}^*$ (equivalent to $(r_1)^{ab}(\beta_1)_{ac} = \delta_c^b$) and define $\beta_t(2) = \beta_1 + \beta_2 t$ ($d_c\beta_t = 0$ and β_1 is non-degenerate). Let us define $r_t(2) = r_1(2) + \sum_{k\geq 2} r_k(2)t^{k-1} \in \mathfrak{a} \wedge \mathfrak{a}[[t]]$, by $\mu_{r_t(2)}^{-1}(\beta_t(2)) = r_t(2)$. Then, $r_1(2) = r_1 = r_1(1)$ and $r_2(2) = -\mu_{r_1}^{-1}(\beta_2) = -h_2$ and $[r_t(2), r_t(2)] = 0$, $r_t(2)$ is a solution of the CYBE.

Let $\tilde{J}_{r_\hbar(2)}$ be the star product determined following Etingof-Kazhdan by the element $r_t(2)$, after $t = \hbar$, as in proposition 4.10. We have, also from proposition 4.10, that $(\tilde{J}_{r_\hbar(2)})_1 = (\tilde{J}_{r_\hbar(1)})_1$ and $(\tilde{J}_{r_\hbar(2)})_2 = (\tilde{J}_{r_\hbar(1)})_2 + \frac{1}{2} r_2(2) = (\tilde{J}_{r_\hbar(1)})_2 - \frac{1}{2} h_2$. Thus, $(\tilde{J}_{r_\hbar(2)})_1 = F_1^{'(1)} = (\tilde{J}_{r_\hbar(1)})_1$, and from (5.8), we have $(\tilde{J}_{r_\hbar(2)})_2 = F_2^{'(1)}$. Replacing these equalities in the expressions (5.6) and (5.7), we get

$$\begin{aligned} &(\tilde{J}_{r_\hbar(2)})_1 - F_1 = d_H E_1 \\ &(\tilde{J}_{r_\hbar(2)})_2 - F_2 + G_2(E_1, (\tilde{J}_{r_\hbar(2)})_1, F_1) = d_H E_2. \end{aligned}$$

So, F is equivalent to $\tilde{J}_{r_\hbar(2)}$, to order 2, $\tilde{J}_{r_\hbar(2)}$ being the star product determined by $r_t(2)$, with $\mu_{r_t(2)}^{-1}(\beta_t(2)) = r_t(2)$ and $\beta_t(2) = \beta_1 + \mu_{r_1}(h_2)t$.

(iii) Using the induction hypothesis, and following the same steps, we conclude the proof.

(b) this part follows from part a) and [7] *page 841, Remark 2.* ■

As a consequence we have the following isomorphisms (see also [21]*):*

Corollary 5.3. *Let Φ, Φ' be two Lie associators. Let $A_{\mathfrak{a}[[\hbar]],F^{-1}}$ be given as in the theorem. Let $r_\hbar$, $r'_\hbar \in (\wedge^2\mathfrak{a})[[\hbar]]$ the elements respectively determined in the theorem by the pairs $(\Phi; A_{\mathfrak{a}[[\hbar]],F^{-1}})$ and $(\Phi'; A_{\mathfrak{a}[[\hbar]],F^{-1}})$. Then we have*

$$A_{\mathfrak{a}[[\hbar]],F^{-1}} \overset{isom}{\approx} A_{\mathfrak{a}[[\hbar]],\left(\tilde{J}^{\Phi}_{r_\hbar}\right)^{-1}} \overset{isom}{\approx} A_{\mathfrak{a}[[\hbar]],\left(\tilde{J}^{\Phi'}_{r_\hbar}\right)^{-1}}$$

6. Invariant star products on $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_1)$

1) We now develop what we wrote in 3) at the Introduction. We need the following proposition:

Proposition 6.1. [22] *Let Γ be a set. Let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon = d_c r_s)$ be a non-degenerate triangular Lie bialgebra, $s \in \Gamma$. Let $\varphi^1_s : \mathfrak{a} \longrightarrow \mathfrak{a}$ a Lie algebra isomorphisms $\forall\, s \in \Gamma$. Let r'_s be the element in $\mathfrak{a}\wedge\mathfrak{a}$ defined by $r'_s = (\varphi^1_s \otimes \varphi^1_s) r_s$.*

a) The set $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon'_{\mathfrak{a}} = d_c r'_s)$ is a non-degenerate triangular Lie bialgebra.

b) The transposed map $(\varphi^1_s)^{\mathrm{t}} : \mathfrak{a}^*_{r'_s} \longrightarrow \mathfrak{a}^*_{r_s}$ is a Lie algebra isomorphism.

c) The pair $(\varphi^1_s; \varphi^2_s = ((\varphi^1_s)^{\mathrm{t}})^{-1})$, $s \in \Gamma$, defines a Lie bialgebra isomorphism between the Lie bialgebra $\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}$ (the classical double of the Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon_{\mathfrak{a}} = d_c r_s)$) and the Lie bialgebra $\mathfrak{a} \oplus \mathfrak{a}^*_{r'_s}$ (the classical double of the Lie bialgebra $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon'_{\mathfrak{a}} = d_c r'_s)$). Furthermore, this isomorphism sends the canonical element r into itself.

Corollary 6.2. [22] *a) Under the hypothesis of the proposition let $\beta_s = \mu_{r_s}(r_s)$, $\beta'_s = \mu_{r'_s}(r'_s)$. Then $(\varphi^2_s \otimes \varphi^2_s)\beta_s = \beta'_s$.*

b) Conversely, let $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon = d_c r_s)$ and $(\mathfrak{a}, [,]_{\mathfrak{a}}, \varepsilon'_{\mathfrak{a}} = d_c r'_s)$ be non-degenerate triangular Lie bialgebras. Let β_s and β'_s be as in a). Let $\varphi^1_s : \mathfrak{a} \longrightarrow \mathfrak{a}$ be a Lie algebra isomorphism and $\varphi^2_s = ((\varphi^1_s)^{\mathrm{t}})^{-1}$. Suppose that $(\varphi^2_s \otimes \varphi^2_s)\beta_s = \beta'_s$. Then, $(\varphi^1_s \otimes \varphi^1_s) r_s = r'_s$.

2) Let $(\mathfrak{a}, [,]_{\mathfrak{a}})$ be a Lie algebra over $\mathbb{R}$. *Consider the following Lie-algebra isomorphisms: $\varphi^1_t = \exp(t \cdot ad_{X_t})$ where $X_t = X_1 + X_2 t + X_3 t^2 + \cdots \in \mathfrak{a}$ is an analytic function in a neighborhood of* $0 \in \mathbb{R}$. *Then $\varphi^2_t = \exp(-t \cdot ad^{\mathrm{t}}_{X_t}) =$* $\exp(t \cdot ad^*_{X_t})$. *Our interest is in the map $\varphi^2_t \otimes \varphi^2_t =$* $\exp(ad^*_{tX_t})^{\otimes^2}$. *We have $\varphi^2_t \otimes \varphi^2_t =$* $\exp(ad^*_{tX_t} \otimes 1 + 1 \otimes ad^*_{tX_t})$ *and then*

Proposition 6.3. *Let $\beta_t = \beta_1 + \beta_2 t + \beta_3 t^2 + \cdots \in \wedge^2(\mathfrak{a}^*)$ be an analytic function in a neighborhood of $0 \in \mathbb{R}$ defining a non-degenerate 2-cocycle $\forall\, t$. $\beta_1, \beta_2, \cdots \in \wedge^2(\mathfrak{a}^*)$ are then 2-cocycles and β_1 is non-degenerate. Let X_t be as before. Then we obtain*

$$\exp(ad^*_{tX_t})^{\otimes^2}(\beta_t) = \exp(ad^*_{tX_t} \otimes 1 + 1 \otimes ad^*_{tX_t})(\beta_t) = \beta_t + d_c\alpha_t,$$

where $\alpha_t = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \cdots \in \mathfrak{a}^*$ is an analytic function in a neighborhood of $0 \in \mathbb{R}$ given by

$$\alpha_k = \sum_{j=1}^{k} \Big(\sum_{i=1}^{k-j+1} \frac{1}{i!} \sum_{\substack{a_1+\cdots+a_i=k-j+1 \\ a_1,\ldots,a_i \geq 1}} \big((-1)^{i+1}(i_{X_{a_1}}\beta_j) \cdot ad_{X_{a_2}} \cdots ad_{X_{a_i}}\big)\Big).$$

Proof:

The first terms of the series are:

$$\exp(ad^*_{tX_t} \otimes 1 + 1 \otimes ad^*_{tX_t})(\beta_1) = \beta_1 + (1 \otimes ad^*_{X_1} + ad^*_{X_1} \otimes 1)(\beta_1)t +$$
$$+ \left((1 \otimes ad^*_{X_2} + ad^*_{X_2} \otimes 1) + \frac{1}{2!}(1 \otimes ad^*_{X_1} + ad^*_{X_1} \otimes 1)^2 \right)(\beta_1)t^2 + \cdots ,$$

because

$$\exp(ad^*_{tX_t} \otimes 1 + 1 \otimes ad^*_{tX_t}) =$$
$$= 1 \otimes 1 + \sum_{p \geq 1} \left(\sum_{i=1}^{p} \frac{1}{i!} \sum_{\substack{a_1+\cdots+a_i=p \\ a_1,\ldots,a_i \geq 1}} \prod_{j=1}^{i} (1 \otimes ad^*_{X_{a_j}} + ad^*_{X_{a_j}} \otimes 1) \right) t^p. \qquad (6.9)$$

For any $e_a, e_b \in \mathfrak{a}$ elements in a basis, we have

$$\begin{aligned}
\big\langle (1 \otimes ad^*_{X_1} + ad^*_{X_1} \otimes 1)(\beta_1); e_a \otimes e_b \big\rangle &= -\langle \beta_1; (1 \otimes ad_{X_1} + ad_{X_1} \otimes 1)e_a \otimes e_b \rangle \\
&= -\langle \beta_1; e_a \otimes [X_1, e_b] + [X_1, e_a] \otimes e_b \rangle \\
&= -X_1^i C^t_{ib}(\beta_1)_{at} - X_1^i C^k_{ia}(\beta_1)_{kb} \\
&= X_1^i C^k_{ab}(\beta_1)_{ki} \\
&= (-i_{X_1}\beta_1)([e_a, e_b]) \\
&= \langle d_c(i_{X_1}\beta_1); e_a \otimes e_b \rangle ,
\end{aligned}$$

where we used that β_1 is a 2-cocycle ($\beta_1([x,y],z) + \beta_1([y,z],x) + \beta_1([z,x],y) = 0$, $x, y, z \in \mathfrak{a}$), C^k_{ab} are the structure constants of the Lie algebra $\mathfrak{a}$ in a basis $\{e_i\}$ and $d_c\alpha(e_a \otimes e_b) = -\alpha([e_a, e_b])$.

So, we obtain

$$(1 \otimes ad^*_{X_1} + ad^*_{X_1} \otimes 1)(\beta_1) = d(i_{X_1}\beta_1), \tag{6.10}$$

for any cocycle β_1 and any $X_1 \in \mathfrak{a}$.

With a similar computation, for any $X_1, X_2 \in \mathfrak{a}$, we get

$$(1 \otimes ad^*_{X_2} + ad^*_{X_2} \otimes 1) \cdot (1 \otimes ad^*_{X_1} + ad^*_{X_1} \otimes 1)(\beta_1) = d_c\left(-(i_{X_1}\beta_1) \cdot ad_{X_2}\right). \tag{6.11}$$

By (6.10) and (6.11), we have

$$\begin{aligned}\exp(ad^*_{tX} \otimes 1 + 1 \otimes ad^*_{tX})(\beta_1) &= \beta_1 + d_c(i_{X_1}\beta_1)t + \\ &+ \left(d_c(i_{X_2}\beta_1) + \frac{1}{2!}d_c(-(i_{X_1}\beta_1) \cdot ad_{X_1})\right)t^2 + \dots\end{aligned}$$

By induction it is possible to prove that, for any n, $X_1, X_2, \dots, X_n \in \mathfrak{a}$ and any cocycle β_1,

$$\begin{aligned}(1 \otimes ad^*_{X_n} + ad^*_{X_n} \otimes 1) \cdots (1 \otimes ad^*_{X_2} + ad^*_{X_2} \otimes 1) \cdot (1 \otimes ad^*_{X_1} + ad^*_{X_1} \otimes 1)\beta_1 = \\ = d_c\left((-1)^{n+1}(i_{X_1}\beta_1) \cdot ad_{X_2} \cdot \dots \cdot ad_{X_{n-1}} \cdot ad_{X_n}\right).\end{aligned}$$

With this result, and from (6.9), it is clear that

$$\exp(ad^*_{tX_t} \otimes 1 + 1 \otimes ad^*_{tX_t})(\beta_1) = \beta_1 + d_c\gamma_t,$$

where $\gamma_t = \gamma_1 t + \gamma_2 t^2 + \dots$ is the 1-cochain given by

$$\gamma_k = \sum_{i=1}^{k} \frac{1}{i!} \sum_{\substack{a_1+\dots+a_i=k \\ a_1,\dots,a_i \geq 1}} \left((-1)^{i+1}(i_{X_{a_1}}\beta_1) \cdot ad_{X_{a_2}} \cdots ad_{X_{a_i}}\right).$$

Since, for each t, $X_t \in \mathfrak{a}$ and $\exp(ad_{tX_t}) : \mathfrak{a} \longrightarrow \mathfrak{a}$, we have $\exp(ad^*_{tX_t}) : \mathfrak{a}^* \longrightarrow \mathfrak{a}^*$, i.e., $\beta_1 + d_c\gamma_t \in \mathfrak{a}^*$, so $d_c\gamma_t \in \mathfrak{a}^*$, for each t.

Applying this result to the cocycles $\beta_1, \beta_2, \beta_3, \dots,$ we get the expression given. ∎

A converse of proposition 6.3 is:

Proposition 6.4. *Let $\beta_t = \beta_1 + \beta_2 t + \beta_3 t^2 + \dots \in \wedge^2(\mathfrak{a}^*)$ as in proposition 6.3. Let $\alpha_t = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 + \dots \in \mathfrak{a}^*$ be an analytic function. Define $\beta'_t = \beta_t + d_c\alpha_t$. Then, there exists a unique $X_t = X_1 + X_2 t + X_3 t^2 + \dots \in \mathfrak{a}$, analytic,* *such that* $\exp(ad^*_{tX_t})^{\otimes^2}(\beta_t) = \beta'_t$*, where* $X_p = -(\omega_p \otimes 1)r_1 = -\chi_{r_1}(\omega_p)$*,* $r_1 = \mu^{-1}_{r_1}(\beta_1)$*,* $\omega_1 = \alpha_1$ *and, for* $p \geq 2$*,*

$$\begin{aligned}\omega_p =&\alpha_p - \sum_{i=2}^{p} \frac{1}{i!} \sum_{\substack{a_1+\cdots+a_i=p\\ a_1,\ldots,a_i\geq 1}} \big((-1)^{i+1}(i_{X_{a_1}}\beta_1)\cdot ad_{X_{a_2}}\cdots ad_{X_{a_i}}\big)\\ &- \sum_{j=2}^{p}\Big(\sum_{i=1}^{p-j+1}\frac{1}{i!}\sum_{\substack{a_1+\cdots+a_i=p-j+1\\ a_1,\ldots,a_i\geq 1}}\big((-1)^{i+1}(i_{X_{a_1}}\beta_j)\cdot ad_{X_{a_2}}\cdots ad_{X_{a_i}}\big)\Big).\end{aligned}$$

Proof:

By the previous theorem, we know that, for any $X_t \in \mathfrak{a}$ *in the above form, we have*

$$\exp(ad^*_{tX_t})^{\otimes^2}(\beta_t) = \beta_t + d_c\alpha_t,$$

where $\alpha_t = \alpha_1 t + \alpha_2 t^2 + \cdots \in \mathfrak{a}^*$ *is the* 1*-cochain of theorem 6.3. The question now is to know if it is possible to determine* $X_t = X_1 + X_2 t + X_3 t^2 + \ldots$ *in such a way that this* 1*-cochain* α_t *is equal to the* 1*-cochain given* $\alpha_t = \alpha_1 t + \alpha_2 t^2 + \ldots$*. The elements* $X_1, X_2, \cdots \in \mathfrak{a}$ *must satisfy*

$$\begin{aligned}\alpha_1 &= i_{X_1}\beta_1\\ \alpha_2 &= i_{X_2}\beta_1 - \frac{1}{2!}(i_{X_1}\beta_1)\cdot ad_{X_1} + i_{X_1}\beta_2.\end{aligned}$$

Since β_1 *is non-degenerate, the first equality has a (unique) solution*

$$X_1 = -(\alpha_1 \otimes 1)r_1 = -\chi_{r_1}(\alpha_1),$$

where $r_1 = \mu^{-1}_{r_1}(\beta_1)$ *and* χ_{r_1} *is the map defined before.*

From the second equality, X_1 *is known and* β_1 *being non-degenerate, we obtain*

$$X_2 = -\chi_{r_1}\left(\alpha_2 + \frac{1}{2!}(i_{X_1}\beta_1)\cdot ad_{X_1} - i_{X_1}\beta_2\right).$$

Suppose now that we know $X_1, X_2, \ldots, X_{p-1}$*. Putting* α_p *equal to the expression given in theorem 6.3 with* $k = p$ *and separating some terms of the* *sum, we obtain*

$$\alpha_p = i_{X_p}\beta_1 + \sum_{i=2}^{p} \frac{1}{i!} \sum_{\substack{a_1+\cdots+a_i=p \\ a_1,\dots,a_i\geq 1}} \left((-1)^{i+1}(i_{X_{a_1}}\beta_1)\cdot ad_{X_{a_2}}\cdots ad_{X_{a_i}}\right) +$$
$$+\sum_{j=2}^{p}\left(\sum_{i=1}^{p-j+1}\frac{1}{i!}\sum_{\substack{a_1+\cdots+a_i=p-j+1 \\ a_1,\dots,a_i\geq 1}}\left((-1)^{i+1}(i_{X_{a_1}}\beta_j)\cdot ad_{X_{a_2}}\cdots ad_{X_{a_i}}\right)\right). \qquad (6.12)$$

Consider the 1-cochain

$$\omega_p = \alpha_p - \sum_{i=2}^{p} \frac{1}{i!} \sum_{\substack{a_1+\cdots+a_i=p \\ a_1,\dots,a_i\geq 1}} \left((-1)^{i+1}(i_{X_{a_1}}\beta_1)\cdot ad_{X_{a_2}}\cdots ad_{X_{a_i}}\right)$$
$$-\sum_{j=2}^{p}\left(\sum_{i=1}^{p-j+1}\frac{1}{i!}\sum_{\substack{a_1+\cdots+a_i=p-j+1 \\ a_1,\dots,a_i\geq 1}}\left((-1)^{i+1}(i_{X_{a_1}}\beta_j)\cdot ad_{X_{a_2}}\cdots ad_{X_{a_i}}\right)\right), \quad p\geq 2.$$

We may say that, knowing $X_1, X_2, \dots, X_{p-1}$, ω_p is determined.

Again, as β_1 is non-degenerate, we can compute X_p, using ω_p and the equality (6.12):

$$X_p = -\chi_{r_1}(\omega_p) = -(\omega_p \otimes 1)r_1.$$

By the bijectivity between β'_t and X_t is is easy to see that if one of them is convergent so is the other. ■

3) We need to relate the Etingof-Kazhdan quantization of classical doubles with the isomorphisms between these doubles. Even if the following proposition could be expected its proof is not trivial.

Proposition 6.5. [22] *Let $(\mathfrak{a}, [,]_\mathfrak{a}, \varepsilon_\mathfrak{a} = d_c r_s)$ and $(\mathfrak{a}, [,]_\mathfrak{a}, \varepsilon'_\mathfrak{a} = d_c r'_s)$ be non-degenerate triangular Lie bialgebras, $s \in \Gamma$, whose quasitriangular double Lie-bialgebras are respectively $(\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}} = d_c r)$ and $(\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}, [,]_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}}, \varepsilon_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}} = d_c r)$. Let $(\varphi^1_s;\psi_s) : \mathfrak{a}\oplus\mathfrak{a}^*_{r_s} \longrightarrow \mathfrak{a}\oplus\mathfrak{a}^*_{r'_s}$ be a Lie algebra isomorphism such that $\varphi^1_s : \mathfrak{a} \longrightarrow \mathfrak{a}$ and $\psi_s : \mathfrak{a}^*_{r_s} \longrightarrow \mathfrak{a}^*_{r'_s}$ are Lie algebra isomorphisms. Let $\tilde{\varphi}^1_s$, $\tilde{\psi}_s$ be the extensions of φ^1_s and ψ_s to homomorphisms $\mathcal{U}\mathfrak{a}[[\hbar]] \longrightarrow \mathcal{U}\mathfrak{a}[[\hbar]]$ and $\mathcal{U}\mathfrak{a}^*_{r_s}[[\hbar]] \longrightarrow \mathcal{U}\mathfrak{a}^*_{r'_s}[[\hbar]]$ respectively. Let $X \in \mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_s})^{\otimes 2}$. Let ϕ_{r_s} be the $\mathfrak{a}\oplus\mathfrak{a}^*_{r_s}$-module isomorphism from $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_s})$ to $M^{r_s}_+ \otimes M^{r_s}_-$ such that*

$\phi_{r_s}\left(1_{\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_s})}\right) = 1^{r_s}_{+} \otimes 1^{r_s}_{-}$ *and similarly for* $\phi_{r'_s}$.*(See the definition of* ϕ *in theorem 3.2). Then we have*

$$\phi^{-1}_{r'_s}\left[\left((\tilde{\varphi}^1_s;\tilde{\psi}_s)^{\otimes^2}(X)\right)\cdot(1^{r'_s}_{+}\otimes 1^{r'_s}_{-})\right] = \left((\tilde{\varphi}^1_s;\tilde{\psi}_s)\circ\phi^{-1}_{r_s}\right)(X\cdot(1^{r_s}_{+}\otimes 1^{r_s}_{-})).$$

If in the expression of J *given in theorem 3.2 we take into account the above proposition and also that a Lie associator determines* $\Phi = e^{P(\hbar\Omega_{12},\hbar\Omega_{23})}$ *we arrive* [22] *to:*

Proposition 6.6. *Hypotheses are as in the above proposition and suppose moreover that* $(\varphi^1_s;\psi_s)\otimes(\varphi^1_s;\psi_s)\,\Omega = \Omega$. *Denote by* $J_{r'_s}$ *and* J_{r_s} *the corresponding elements in theorem 3.2. Then we have the equality* $J_{r'_s} = (\tilde{\varphi}^1_s;\tilde{\psi}_s)^{\otimes^2} J_{r_s}$. *In particular this proposition is valid for the Lie bialgebra isomorphism* $(\varphi^1_s;\varphi^2_s)$ *constructed in proposition 6.1 and those in propositions 6.3 and 6.4 .*

We can also prove the following:

Proposition 6.7. *Let* $(\mathfrak{a},[,]_{\mathfrak{a}},\varepsilon_{\mathfrak{a}} = d_c r_s)$ *and* $(\mathfrak{a},[,]_{\mathfrak{a}},\varepsilon'_{\mathfrak{a}} = d_c r'_s)$ *be non-degenerate triangular Lie bialgebras,* $s \in \Gamma$. *Let* $\varphi^1_s : \mathfrak{a} \longrightarrow \mathfrak{a}$ *be a Lie algebra isomorphism such that* $r'_s = (\varphi^1_t \otimes \varphi^1_s) r_s$ *and let* $(\varphi^1_s;\varphi^2_s)$ *be the Lie bialgebra isomorphism between the corresponding classical doubles constructed in proposition 6.1. Then we have*

$$\tilde{\pi}'_s \circ (\varphi^1_s;\varphi^2_s) = \varphi^1_s \circ \tilde{\pi}_s,$$

where $\tilde{\pi}$ *is defined in Proposition 3.4.*

4) Using propositions 6.1, 6.6, 6.7 and corollary 6.2 we can prove:

Theorem 6.8. *Let* $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r'_\hbar}$ *be invariant star products on a non-degenerate triangular Lie bialgebra over* $\mathbb{R}$, $(\mathfrak{a},[,]_{\mathfrak{a}},\varepsilon_{\mathfrak{a}} = d_c r_1)$, *and determined as in lemma 4.9 and propositions 4.10, respectively by non-degenerate skew-symmetric solutions* $r_\hbar = r_1 + \cdots$ *and* $r'_\hbar = r_1 + \cdots$ *of YBE on Lie algebra over* $\mathbb{R}[[\hbar]]$, $\left(\mathfrak{a}[[\hbar]],[,]_{\mathfrak{a}[[\hbar]]}\right)$. *Let* $\mu_{r_\hbar}(r_\hbar) = \beta_\hbar = \beta_1 + \beta_2\hbar + \cdots \in (\mathfrak{a}^* \wedge \mathfrak{a}^*)[[\hbar]]$ *and* $\mu_{r'_\hbar}(r'_\hbar) = \beta'_\hbar = \beta_1 + \beta'_2\hbar + \cdots \in (\mathfrak{a}^* \wedge \mathfrak{a}^*)[[\hbar]]$. *Suppose that the cocycles* $\beta_\hbar = \mu_{r_\hbar}(r_\hbar) = \beta_1 + \beta_2\hbar + \ldots$ *and* $\beta'_\hbar = \mu_{r'_\hbar}(r'_\hbar) = \beta_1 + \beta'_2\hbar + \ldots$ *belong to the same cohomological class, i.e., that* $\beta'_\hbar = \beta_\hbar + d_c\alpha_\hbar$ *for some* 1*-cochain* $\alpha_\hbar = \alpha_1\hbar + \alpha_2\hbar^2 + \cdots \in \mathfrak{a}^*[[\hbar]]$. *Then,* $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r'_\hbar}$ *are equivalent ISP.*

Proof:

The elements $\tilde{J}_{r_t}$ *and* $\tilde{J}_{r'_t}$ *are defined by*

$$\tilde{J}_{r_t} = (\tilde{\pi}_t \otimes \tilde{\pi}_t) J_{r_t}, \quad \tilde{J}_{r'_t} = (\tilde{\pi}'_t \otimes \tilde{\pi}'_t) J_{r'_t},$$

where J_{r_t} and $J_{r'_t}$ are the elements in $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r_t})^{\otimes^2}[[\hbar]]$ and $\mathcal{U}(\mathfrak{a}\oplus\mathfrak{a}^*_{r'_t})^{\otimes^2}[[\hbar]]$, respectively, defined in theorem 3.2.

Since $\beta_t,\beta'_t\in\wedge^2(\mathfrak{a}^*)$ belong to the same cohomological class, by theorem 6.4, there exists a $X_t=X_1+X_2t+\cdots\in\mathfrak{a}$ such that

$$\exp(ad^*_{tX_t})^{\otimes^2}(\beta_t)\equiv(\varphi^2_t)^{\otimes^2}\beta_t=\beta'_t.$$

By proposition 6.2, $\varphi^1_t=((\exp(ad^*_{tX_t}))^{-1})^{\mathfrak{t}}=\exp(ad_{tX_t})$ is a Lie algebra isomorphism $\mathfrak{a}\longrightarrow\mathfrak{a}$ such that $(\varphi^1_t\otimes\varphi^1_t)r_t=r'_t$, where $r_t=\mu^{-1}_{r_t}(\beta_t)$ and $r'_t=\mu^{-1}_{r'_t}(\beta'_t)$. By proposition 6.1, $(\varphi^1_t;\varphi^2_t)$ is a Lie bialgebra isomorphism between $\mathfrak{a}\oplus\mathfrak{a}^*_{r_t}$ and $\mathfrak{a}\oplus\mathfrak{a}^*_{r'_t}$ such that $(\varphi^1_t;\varphi^2_t)^{\otimes^2}r=r$, where r is the canonical element in $(\mathfrak{a}\oplus\mathfrak{a}^*)\otimes(\mathfrak{a}\oplus\mathfrak{a}^*)$. So, $(\varphi^1_t;\varphi^2_t)^{\otimes^2}\Omega=\Omega$.

Then, we have

$$\tilde{J}_{r'_t}=(\tilde{\pi}'_t\otimes\tilde{\pi}'_t)J_{r'_t}=(\tilde{\varphi}^1_t\otimes\tilde{\varphi}^1_t)\tilde{J}_{r_t},$$

using propositions 6.6 and 6.7.

Putting $t=\hbar$, we obtain $\tilde{J}_{r'_\hbar}=(\tilde{\varphi}^1_\hbar\otimes\tilde{\varphi}^1_\hbar)\tilde{J}_{r_\hbar}$, or, equivalently, $\tilde{J}^{-1}_{r'_\hbar}=(\tilde{\varphi}^1_\hbar\otimes\tilde{\varphi}^1_\hbar)\tilde{J}^{-1}_{r_\hbar}$. The map $\varphi^1_\hbar=\exp(ad_{\hbar X_\hbar}):\mathfrak{a}[[\hbar]]\longrightarrow\mathfrak{a}[[\hbar]]$ is obviously a morphism (in fact, an isomorphism) of Lie algebras over $\mathbb{R}[[\hbar]]$ (where the Lie algebra structure on $\mathfrak{a}[[\hbar]]$ is the trivial one of $(\mathfrak{a},\mathbb{R})$ by extension of the ring of the scalars from $\mathbb{R}$ to $\mathbb{R}[[\hbar]]$) and, considering $u=1$ and the last equality, we may apply proposition 3.9 in [6].

Then, the extension $\tilde{\varphi}^1_\hbar:\mathcal{U}(\mathfrak{a}[[\hbar]])\longrightarrow\mathcal{U}(\mathfrak{a}[[\hbar]])$ is a morphism of triangular Hopf QUE algebras $A_{\mathfrak{a}[[\hbar]],\tilde{J}^{-1}_{r_\hbar}}\longrightarrow A_{\mathfrak{a}[[\hbar]],\tilde{J}^{-1}_{r'_\hbar}}$. The pairs $(\varphi^1_\hbar,1)$ and (λ_1,u_1) determine the same morphism if, and only if, $\lambda_1=\exp(-\hbar\,ad_v)\circ\varphi^1_\hbar$ and $u_1=e^{\hbar v}$, for some $v\in\mathfrak{a}[[\hbar]]$. Putting $v=X_t\mid_{t=\hbar}$, we obtain $\lambda_1=1$, $u_1=e^{\hbar X_\hbar}$ and

$$\tilde{J}^{-1}_{r'_\hbar}=(u_1\otimes u_1)\cdot_\hbar\tilde{J}^{-1}_{r_\hbar}\cdot_\hbar\Delta_{\mathfrak{a}}(u_1)^{-1}.$$

This last equality is equivalent to

$$\tilde{J}_{r'_\hbar}=\Delta_{\mathfrak{a}}(u_1)\cdot_\hbar\tilde{J}_{r_\hbar}\cdot_\hbar(u_1^{-1}\otimes u_1^{-1}),$$

and u_1^{-1} defines an equivalence between the invariant star products $\tilde{J}_{r_\hbar}$ and $\tilde{J}_{r'_\hbar}$ on $(\mathfrak{a},[,]_{\mathfrak{a}},\varepsilon_{\mathfrak{a}}=d_cr_1)$. ∎

Before proving the converse result we need the following lemma.

Lemma 6.9. *Suppose that in theorem 6.8*

$$\beta_\hbar = \mu_{r_\hbar}(r_\hbar) = \beta_1 + \beta_2\hbar + \cdots + \beta_{R-1}t^{R-2} + \beta_R\hbar^{R-1} + \ldots$$
$$\beta'_\hbar = \mu_{r'_\hbar}(r'_\hbar) = \beta_1 + \beta_2\hbar + \cdots + \beta_{R-1}\hbar^{R-2} + (\beta_R + d_c\alpha_{R-1})\hbar^{R-1} + \ldots,$$

where α_{R-1} is a 1-cochain. This means that $\beta_\hbar$ and $\beta'_\hbar$ are equal except in the term of order $R-1$. Then, $\tilde{J}_{r_\hbar}$ and $\tilde{J}_{r'_\hbar}$ are equivalent,

$$\tilde{J}_{r'_\hbar} = \Delta_{\mathfrak{a}}(E)^{-1} \cdot_\hbar \tilde{J}_{r_\hbar} \cdot_\hbar (E \otimes E),$$

and the element $E = 1 + E_1\hbar + E_2\hbar^2 + \cdots + E_{R-1}\hbar^{R-1} + \ldots$ which defines this equivalence verifies

$$E_1 = 0, \quad E_2 = 0, \quad \ldots \quad , \quad E_{R-2} = 0, \quad E_{R-1} = \chi_{r_1}(\alpha_{R-1}) = \mu_{r_1}^{-1}(\alpha_{R-1}).$$

Proof:

Consider the elements $\beta_\hbar$ and *$\beta'_\hbar$ obeying the above conditions. One of the steps of the proof of theorem 6.8 is to find the element $X_t = X_1 + X_2 t + X_3 t^2 + \cdots + X_{R-1}t^{R-2} + \ldots$ of theorem 6.4. For the elements β_t* and *β'_t* considered, *this element will be*

$$X_1 = 0, \quad X_2 = 0, \quad \ldots \quad , \quad X_{R-2} = 0, \quad X_{R-1} = -\chi_{r_1}(\alpha_{R-1}).$$

Then, the element E will be, in view of the same proof, the following:

$$E = u_1^{-1} = (e^{\hbar X_\hbar})^{-1} = 1 + 0\hbar + \cdots + 0\hbar^{R-2} + \chi_{r_1}(\alpha_{R-1})\hbar^{R-1} + \cdots .$$

∎

Lemma 6.9 and Hochschild cohomology properties allow us to prove

Theorem 6.10. *Let $\tilde{J}_{r_\hbar}$ and $\tilde{J}_{r'_\hbar}$ as in a) Theorem 6.8. Suppose $\tilde{J}_{r_\hbar}$ and $\tilde{J}_{r'_\hbar}$ are equivalent. Then, $\beta_\hbar$ and $\beta'_\hbar$ belong to the same cohomological class, i.e., there exists a formal 1-cochain $\alpha_\hbar = \alpha_1\hbar + \alpha_2\hbar^2 + \cdots \in \mathfrak{a}^*[[\hbar]]$ such that $\beta'_\hbar = \beta_\hbar + d_c\alpha_\hbar$.*

Proof:

If $\tilde{J}_{r_\hbar}$ and *$\tilde{J}_{r'_\hbar}$ are equivalent ISP, there exists $E^{(1)} = 1 + E_1^{(1)}\hbar + E_2^{(1)}\hbar^2 + \cdots \in \mathcal{U}\mathfrak{a}[[\hbar]]$ such that*

$$\tilde{J}_{r'_\hbar} = \Delta_{\mathfrak{a}}(E^{(1)})^{-1} \cdot_\hbar \tilde{J}_{r_\hbar} \cdot_\hbar (E^{(1)} \otimes E^{(1)}).$$

At order 1, this equivalence may be written as $(\tilde{J}_{r'_\hbar})_1 - (\tilde{J}_{r_\hbar})_1 = d_H E_1^{(1)}$, where d_H is the Hochschild cohomology operator and $E_1^{(1)} \in \mathcal{U}\mathfrak{a}$. But, in this case,

$(\tilde{J}_{r'_\hbar})_1 = (\tilde{J}_{r_\hbar})_1 = \frac{1}{2}r_1$. *So, we have* $d_H E_1^{(1)} = 0$, *which implies that* $E_1^{(1)} \in \mathfrak{a}$. *Therefore,* $\Delta_{\mathfrak{a}}(E_1^{(1)}) = E_1^{(1)} \otimes 1 + 1 \otimes E_1^{(1)}$.

At order 2, there exists $E_2^{(1)} \in \mathcal{U}\mathfrak{a}$ *such that*

$$(\tilde{J}_{r'_\hbar})_2 - (\tilde{J}_{r_\hbar})_2 + G_2(E_1^{(1)}, (\tilde{J}_{r'_\hbar})_1, (\tilde{J}_{r_\hbar})_1) = d_H E_2^{(1)}, \tag{6.13}$$

where, for $k = 2, 3, \dots,$

$$\begin{aligned} &G_k(E_1, \dots, E_{k-1}, F'_1, \dots, F'_{k-1}, F_1, \dots, F_{k-1}) = \\ &= \sum_{\substack{i+j=k \\ i,j\geq 1}} \big(\Delta_{\mathfrak{a}}(E_i) \cdot F'_j - F_i \cdot (1 \otimes E_j) - F_i \cdot (E_j \otimes 1)\big) - \sum_{\substack{i+j=k \\ i,j\geq 1}} (E_i \otimes 1) \cdot (1 \otimes E_j) - \\ &\qquad - \sum_{\substack{i+j+l=k \\ i,j,l\geq 1}} F_i \cdot (E_j \otimes 1) \cdot (1 \otimes E_l). \end{aligned}$$

Since $(\tilde{J}_{r'_\hbar})_1 = (\tilde{J}_{r_\hbar})_1 = \frac{1}{2}r_1$, *we have*

$$\begin{aligned} &G_2(E_1^{(1)}, (\tilde{J}_{r'_\hbar})_1, (\tilde{J}_{r_\hbar})_1) = \\ &\quad = \Delta_{\mathfrak{a}}(E_1^{(1)}) \cdot \frac{1}{2}r_1 - \frac{1}{2}r_1 \cdot (1 \otimes E_1^{(1)} + E_1^{(1)} \otimes 1) - E_1^{(1)} \otimes E_1^{(1)} \\ &\quad = -\frac{1}{2} d_P^{r_1} E_1^{(1)} - E_1^{(1)} \otimes E_1^{(1)}. \end{aligned}$$

The skew-symmetric part of $G_2(E_1^{(1)}, (\tilde{J}_{r'_\hbar})_1, (\tilde{J}_{r_\hbar})_1)$ *is* $-\frac{1}{2}d_P^{r_1}E_1^{(1)}$, *where* $d_P^{r_1}$ *is the Poisson cohomology operator. We know that* $(\tilde{J}_{r'_\hbar})_2 - (\tilde{J}_{r_\hbar})_2 = \frac{1}{2}(r'_2 - r_2)$. *So, the equality between the skew-symmetric parts of both sides of (6.13) leads to*

$$r'_2 = r_2 + d_P^{r_1} E_1^{(1)}. \tag{6.14}$$

Let $\beta_t = \mu_{r_t}(r_t) = \beta_1 + \beta_2 t + \dots$ *and* $\beta'_t = \beta_t^{(1)} = \mu_{r'_t}(r'_t) = \beta_1 + \beta'_2 t + \beta'_3 t^2 + \dots$. *Using (6.14), we get* $\beta'_2 - \beta_2 = -\mu_{r_1}(r'_2 - r_2) = -\mu_{r_1}(d_P^{r_1} E_1^{(1)}) = d_c(\mu_{r_1}(E_1^{(1)}))$ *(the last equality, using relation* $\mu \circ (-d_P) = d_c \circ \mu$*). This means that there exists a 1-cochain* $\alpha_1 = \mu_{r_1}(E_1^{(1)})$ *such that* $\beta'_2 = \beta_2 + d_c\alpha_1$.

Consider now the following elements:

$$\beta_t^{(2)} = \beta_1 + \beta_2 t + \beta'_3 t^2 + \beta'_4 t^3 + \dots \quad \text{and} \quad r_t^{(2)} = \mu^{-1}_{r_t^{(2)}}(\beta_t^{(2)}).$$

The elements $\beta_t^{(2)}$ *and* β'_t *belong to the same formal cohomological class. By theorem 6.8, the star products* $\tilde{J}_{r_\hbar^{(2)}}$ *and* $\tilde{J}_{r'_\hbar}$ *are equivalent, and the element*

$u^{(2)}$ *which defines the equivalence*

$$\tilde{J}_{r'_\hbar} = \Delta_{\mathfrak{a}}(u^{(2)})^{-1} \cdot \tilde{J}_{r^{(2)}_\hbar} \cdot (u^{(2)} \otimes u^{(2)}),$$

equivalent also to

$$\tilde{J}_{r^{(2)}_\hbar} = \Delta_{\mathfrak{a}}(u^{(2)}) \cdot \tilde{J}_{r'_\hbar} \cdot ((u^{(2)})^{-1} \otimes (u^{(2)})^{-1})$$

satisfies (by the previous lemma)

$$u^{(2)} = 1 + \mu_{r_1}^{-1}(\alpha_1)\hbar + \cdots = 1 + E_1^{(1)}\hbar + \cdots .$$

But, as $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r'_\hbar}$ *are also equivalent, we obtain*

$$\begin{aligned} \tilde{J}_{r^{(2)}_\hbar} &= \Delta_{\mathfrak{a}}(u^{(2)}) \cdot \Delta_{\mathfrak{a}}(E^{(1)})^{-1} \cdot \tilde{J}_{r_\hbar} \cdot (E^{(1)} \otimes E^{(1)}) \cdot ((u^{(2)})^{-1} \otimes (u^{(2)})^{-1}) \\ &= \Delta_{\mathfrak{a}}(E^{(1)} \cdot (u^{(2)})^{-1})^{-1} \cdot \tilde{J}_{r_\hbar} \cdot ((E^{(1)} \cdot (u^{(2)})^{-1}) \otimes (E^{(1)} \cdot (u^{(2)})^{-1})), \end{aligned}$$

i.e., the element $E^{(2)} = E^{(1)} \cdot (u^{(2)})^{-1} \in \mathcal{U}\mathfrak{a}[[\hbar]]$ *defines an equivalence between* $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r^{(2)}_\hbar}$ *at any order, and we may compute the first terms of* $E^{(2)}$*:*

$$E^{(2)} = 1 + (E_1^{(1)} - E_1^{(1)})\hbar + E_2^{(2)}\hbar^2 + \cdots = 1 + 0\hbar + E_2^{(2)}\hbar^2 + \cdots .$$

Since $r_t^{(2)} = r_1 + r_2 t + r_3^{(2)} t^2 + \ldots$ *and* $r_t = r_1 + r_2 t + r_3 t^2 + \ldots$*, we have*

$$\left(\tilde{J}_{r^{(2)}_\hbar}\right)_1 = \left(\tilde{J}_{r_\hbar}\right)_1, \ \left(\tilde{J}_{r^{(2)}_\hbar}\right)_2 = \left(\tilde{J}_{r_\hbar}\right)_2 \text{ and } \left(\tilde{J}_{r^{(2)}_\hbar}\right)_3 - \left(\tilde{J}_{r_\hbar}\right)_3 = \frac{1}{2}(r_3^{(2)} - r_3).$$

But $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r^{(2)}_\hbar}$ *are equivalent. So, they are equivalent at order 2. This means that there exists an element* $E_2^{(2)} \in \mathcal{U}\mathfrak{a}$ *such that*

$$\left(\tilde{J}_{r^{(2)}_\hbar}\right)_2 - \left(\tilde{J}_{r_\hbar}\right)_2 + G_2\left(E_1^{(2)}, \left(\tilde{J}_{r^{(2)}_\hbar}\right)_1, \left(\tilde{J}_{r_\hbar}\right)_1\right) = d_H E_2^{(2)}.$$

Since $E_1^{(2)} = 0$*, we have* $G_2\left(E_1^{(2)}, \left(\tilde{J}_{r^{(2)}_\hbar}\right)_1, \left(\tilde{J}_{r_\hbar}\right)_1\right) = 0$ *and so* $d_H E_2^{(2)} = 0$*, which implies that* $E_2^{(2)}$ *belongs to* $\mathfrak{a}$*.*

The equivalence at order 3 means that there exists $E_3^{(2)} \in \mathcal{U}\mathfrak{a}$ *such that*

$$\left(\tilde{J}_{r^{(2)}_\hbar}\right)_3 - \left(\tilde{J}_{r_\hbar}\right)_3 + G_3\left(E_1^{(2)}, E_2^{(2)}, \left(\tilde{J}_{r^{(2)}_\hbar}\right)_1, \left(\tilde{J}_{r_\hbar}\right)_1, \left(\tilde{J}_{r^{(2)}_\hbar}\right)_2, \left(\tilde{J}_{r_\hbar}\right)_2\right) = d_H E_3^{(2)}. \tag{6.15}$$

Since $E_1^{(2)} = 0$ *and* $\left(\tilde{J}_{r_\hbar^{(2)}}\right)_1 = \left(\tilde{J}_{r_\hbar}\right)_1$, *we obtain*

$$G_3\left(E_1^{(2)}, E_2^{(2)}, \left(\tilde{J}_{r_\hbar^{(2)}}\right)_1, \left(\tilde{J}_{r_\hbar}\right)_1, \left(\tilde{J}_{r_\hbar^{(2)}}\right)_2, \left(\tilde{J}_{r_\hbar}\right)_2\right) =$$
$$= \Delta_{\mathfrak{a}}(E_2^{(2)}) \cdot \left(\tilde{J}_{r_\hbar}\right)_1 - \left(\tilde{J}_{r_\hbar}\right)_1 \cdot (1 \otimes E_2^{(2)}) - \left(\tilde{J}_{r_\hbar}\right)_1 \cdot (E_2^{(2)} \otimes 1).$$

The skew-symmetric part of this element is

$$AG_3\left(E_1^{(2)}, E_2^{(2)}, \left(\tilde{J}_{r_\hbar^{(2)}}\right)_1, \left(\tilde{J}_{r_\hbar}\right)_1, \left(\tilde{J}_{r_\hbar^{(2)}}\right)_2, \left(\tilde{J}_{r_\hbar}\right)_2\right) = -\frac{1}{2} d_P^{r_1} E_2^{(2)}.$$

The equality of the skew-symmetric parts of both sides of (6.15) leads to

$$r_3^{(2)} = r_3 + d_P^{r_1} E_2^{(2)}.$$

Thus,

$$\beta_3' - \beta_3 = \beta_3^{(2)} - \beta_3 = -\mu_{r_1}(r_3^{(2)} - r_3) = -\mu_{r_1}(d_P^{r_1} E_2^{(2)}) = d_c(\mu_{r_1}(E_2^{(2)})).$$

This means that there exists a 1*-cochain* $\alpha_2 = \mu_{r_1}(E_2^{(2)})$ *such that*

$$\beta_3' = \beta_3 + d_c \alpha_2.$$

Using the induction hypothesis, and following the same steps, we conclude the proof. ■

Combining the last two theorems we obtain the following result, similar in Etingof-Kazhdan quantization theory to the one by Drinfeld in [5].

Theorem 6.11. *Let* $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r'_\hbar}$ *be as in a) Theorem 6.8. Then,* $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r'_\hbar}$ *are equivalent ISP if, and only if,* $\beta_\hbar$ *and* $\beta'_\hbar$ *belong to the same formal cohomological class. In other words,* $\tilde{J}_{r_\hbar}$ *and* $\tilde{J}_{r'_\hbar}$ *are equivalent ISP if, and only if, there exists a formal* 1*-cochain* $\alpha_\hbar = \alpha_1 \hbar + \alpha_2 \hbar^2 + \dots$ *such that* $\beta'_\hbar = \beta_\hbar + d_c \alpha_\hbar$.

Theorem 6.11 and Remark 2) in page 841 of [7] *allow us to obtain*

Theorem 6.12. *Two triangular Hopf QUE-algebras* $A_{\mathfrak{a}[[\hbar]], \tilde{J}^{-1}_{r_\hbar}}$ *and* $A_{\mathfrak{a}[[\hbar]], \tilde{J}^{-1}_{r'_\hbar}}$ *are isomorphic if, and only if, there exists an isomorphism of Lie algebras* $\lambda : \mathfrak{a}[[\hbar]] \longrightarrow \mathfrak{a}[[\hbar]]$ *over* $MR[[\hbar]]$ *such that* $(\lambda^2 \otimes \lambda^2)\beta_\hbar$ *and* $\beta'_\hbar$ *belong to the same cohomological class where* $\beta_\hbar = \mu_{r_\hbar}(r_\hbar)$, $\beta'_\hbar = \mu_{r'_\hbar}(r'_\hbar)$ *and* $\lambda^2 = (\lambda^{-1})^{\mathfrak{t}}$.

Proof:

If the pair (λ, u) defines an isomorphism between the triangular Hopf QUE algebras (see proposition 3.9 in [6]*), in particular, the map $Ad(u)\circ\tilde{\lambda}$ satisfies the following equality*

$$(Ad(u)\circ\tilde{\lambda})\otimes(Ad(u)\circ\tilde{\lambda})R^{r_\hbar}=R^{r'_\hbar}.$$

We know that $R^{r_\hbar}=1+r_1\hbar+O(\hbar^2)$ and $R^{r'_\hbar}=1+r_1\hbar+O(\hbar^2)$. Computing the coefficients of $\hbar$ at both sides of last equality, we obtain

$$(\lambda\otimes\lambda)r_1=r_1 \; mod\, \hbar.$$

So, $(\lambda\otimes\lambda)(r_\hbar)=r_1+r''_2\hbar+r''_3\hbar^2+\cdots$. Denote this element by $r''_\hbar$ (it is also a solution of CYBE). We know also that

$$\tilde{J}^{-1}_{r'_\hbar}=(u\otimes u)\cdot(\tilde{\lambda}\otimes\tilde{\lambda})\left(\tilde{J}^{-1}_{r_\hbar}\right)\cdot\Delta_{\mathfrak{a}}(u)^{-1}. \tag{6.16}$$

Consider the invariant star product $\tilde{J}_{r''_\hbar}$ defined through the E-K quantization. By proposition 6.6, since $(\lambda\otimes\lambda)(r_\hbar)=r''_\hbar$, we have

$$(\lambda,\lambda^2)^{\otimes^2}J_{r_\hbar}=J_{r''_\hbar},$$

where $\lambda^2=(\lambda^{-1})^{\mathfrak{t}}$. Using proposition 6.7, we obtain

$$(\tilde{\lambda}\otimes\tilde{\lambda})\tilde{J}_{r_\hbar}=\tilde{J}_{r''_\hbar}.$$

Taking inverses at both sides of (6.16), we obtain

$$\begin{aligned}\tilde{J}_{r'_\hbar} &= \Delta_{\mathfrak{a}}(u)\cdot(\tilde{\lambda}\otimes\tilde{\lambda})\left(\tilde{J}_{r_\hbar}\right)\cdot(u^{-1}\otimes u^{-1})\\ &= \Delta_{\mathfrak{a}}(u)\cdot\tilde{J}_{r''_\hbar}\cdot(u^{-1}\otimes u^{-1}).\end{aligned}$$

This means that $\tilde{J}_{r'_\hbar}$ and $\tilde{J}_{r''_\hbar}$ are equivalent star products. By theorem 6.10, $\beta'_\hbar=\mu_{r'_\hbar}(r'_\hbar)$ and $\beta''_\hbar=\mu_{r''_\hbar}(r''_\hbar)$ belong to the same formal cohomological class. Since $(\lambda\otimes\lambda)r_\hbar=r''_\hbar$, by proposition 6.2, we get $\beta''_\hbar=(\lambda^{-1})^{\mathfrak{t}}\otimes(\lambda^{-1})^{\mathfrak{t}}\beta_\hbar=(\lambda^2\otimes\lambda^2)\beta_\hbar$ and we conclude that $\beta'_\hbar$ and $(\lambda^2\otimes\lambda^2)\beta_\hbar$ belong to the same formal cohomological class.

If $((\varphi^1)^{-1})^{\mathfrak{t}}\otimes((\varphi^1)^{-1})^{\mathfrak{t}}\beta_\hbar=\beta''_\hbar$ and $\beta'_\hbar$ belong to the same cohomological class, then, by theorem 6.8, $\tilde{J}_{r''_\hbar}$ and $\tilde{J}_{r'_\hbar}$ are equivalent star products, where $r''_\hbar=\mu^{-1}_{r''_\hbar}(\beta''_\hbar)$. This means that there exists an element $u\in\mathcal{U}\mathfrak{a}[[\hbar]]$, $u\equiv 1 \; mod\, \hbar$, such that

$$\tilde{J}_{r'_\hbar}=\Delta_{\mathfrak{a}}(u)^{-1}\cdot\tilde{J}_{r''_\hbar}\cdot(u\otimes u),$$

equivalent also to

$$\tilde{J}_{r'_\hbar}^{-1} = (u^{-1} \otimes u^{-1}) \cdot \tilde{J}_{r''_\hbar}^{-1} \cdot \Delta_{\mathfrak{a}}(u). \tag{6.17}$$

Since $(\varphi^1 \otimes \varphi^1) r_\hbar = r''_\hbar$ *(see proposition 6.2), we have, by propositions 6.6 and 6.7,*

$$\tilde{J}_{r''_\hbar} = (\tilde{\varphi}^1 \otimes \tilde{\varphi}^1) \tilde{J}_{r_\hbar}.$$

Substituting this in (6.17), we obtain

$$\tilde{J}_{r'_\hbar}^{-1} = (u^{-1} \otimes u^{-1}) \cdot (\tilde{\varphi}^1 \otimes \tilde{\varphi}^1) \left(\tilde{J}_{r_\hbar}^{-1} \right) \cdot \Delta_{\mathfrak{a}}(u).$$

Considering $\lambda_1 = \varphi^1$ *and* $u_1 = u^{-1}$*, the pair* (λ_1, u_1) *defines an isomorphism between* $A_{\mathfrak{a}[[\hbar]], \tilde{J}_{r'_\hbar}^{-1}}$ *and* $A_{\mathfrak{a}[[\hbar]], \tilde{J}_{r_\hbar}^{-1}}$ *(see proposition 3.9 in* [6]*).* ■

5) From the above results and Remark 2) in page 841 of [7] *we may also prove :*

Proposition 6.13. *Let* $A_{\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar}, \Omega, J_{r_\hbar}^{-1}}$ *and* $A_{\mathfrak{a} \oplus \mathfrak{a}^*_{r'_\hbar}, \Omega, J_{r'_\hbar}^{-1}}$ *be quasitriangular Hopf QUE algebras over* $\mathbb{R}[[\hbar]]$ *which are quantizations, as in theorem 4.7, of the quasitriangular Lie bialgebra* $(\mathfrak{a} \oplus \mathfrak{a}^*_{r_1}, [,]_{\mathfrak{a} \oplus \mathfrak{a}^*_{r_1}}, \varepsilon_{\mathfrak{a} \oplus \mathfrak{a}^*_{r_1}} = d_c r)$*. Let* $\beta_\hbar = \mu_{r_\hbar}(r_\hbar) = \beta_1 + \beta_2 \hbar + \dots$ *and* $\beta'_\hbar = \mu_{r'_\hbar}(r'_\hbar) = \beta_1 + \beta'_2 \hbar + \dots$*. If* $\beta_\hbar$ *and* $\beta'_\hbar$ *belong to the same cohomological class, then* $A_{\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar}, \Omega, J_{r_\hbar}^{-1}}$ *and* $A_{\mathfrak{a} \oplus \mathfrak{a}^*_{r'_\hbar}, \Omega, J_{r'_\hbar}^{-1}}$ *are isomorphic.*

Proof:

If β_t *and* β'_t *belong to the same cohomological class, by theorem 6.4, there exists a unique element* $X_t = X_1 + X_2 t + \dots \in \mathfrak{a}$*, such that*

$$\exp(ad^*_{tX_t})^{\otimes^2} \beta_t = \beta'_t.$$

Using proposition 6.2 with $\varphi_t^2 = \exp(ad^*_{tX_t})$*, there exists an isomorphism* $\varphi_t^1 = ((\varphi_t^2)^{\mathrm{t}})^{-1} : \mathfrak{a} \longrightarrow \mathfrak{a}$ *of Lie algebras such that* $(\varphi_t^1 \otimes \varphi_t^1) r_t = r'_t$*. By proposition 6.1, this pair* $(\varphi_t^1; \varphi_t^2)$ *defines an isomorphism from the Lie bialgebra* $\mathfrak{a} \oplus \mathfrak{a}^*_{r_t}$ *to the Lie bialgebra* $\mathfrak{a} \oplus \mathfrak{a}^*_{r'_t}$ *and sends the canonical element* r *of the vector space* $(\mathfrak{a} \oplus \mathfrak{a}^*)^{\otimes^2}$ *to itself. Thus,* $(\varphi_t^1; \varphi_t^2)$ *also will send* Ω *into* Ω*, where* $\Omega = r_{12} + r_{21}$*. Applying now proposition 6.6, we have*

$$J_{r'_t} = (\tilde{\varphi}_t^1; \tilde{\varphi}_t^2)^{\otimes^2} J_{r_t},$$

where $\tilde{\varphi}_t^1$ *and* $\tilde{\varphi}_t^2$ *are extensions of* φ_t^1 *and* φ_t^2 *to homomorphisms* $\mathcal{U}\mathfrak{a}[[\hbar]] \longrightarrow \mathcal{U}\mathfrak{a}[[\hbar]]$ *and* $\mathcal{U}\mathfrak{a}^*_{r_t}[[\hbar]] \longrightarrow \mathcal{U}\mathfrak{a}^*_{r'_t}[[\hbar]]$*, respectively. Putting* $t = \hbar$ *it is clear* *that* $(\varphi^1_\hbar, \varphi^2_\hbar)$ *defines an isomorphism* $\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar} \longrightarrow \mathfrak{a} \oplus \mathfrak{a}^*_{r'_\hbar}$ *and the elements* $J_{r'_\hbar} = J_{r'_t} \mid_{t=\hbar}$, $J_{r_\hbar} = J_{r_t} \mid_{t=\hbar}$. *We obtain the equality*

$$J_{r'_\hbar} = (\tilde{\varphi}^1_\hbar; \tilde{\varphi}^2_\hbar)^{\otimes^2} J_{r_\hbar}.$$

Using an analogous proposition to proposition 3.9 in [6] *with* $\lambda = (\varphi^1_\hbar; \varphi^2_\hbar)$ *and* $u = 1$, *we conclude that the map*

$$\widetilde{(\varphi^1_\hbar; \varphi^2_\hbar)} = (\tilde{\varphi}^1_\hbar; \tilde{\varphi}^2_\hbar) : \hat{\mathcal{U}}(\mathfrak{a} \oplus \mathfrak{a}^*_{r_\hbar}) \longrightarrow \hat{\mathcal{U}}(\mathfrak{a} \oplus \mathfrak{a}^*_{r'_\hbar})$$

is an isomorphism $A_{\mathfrak{a}\oplus\mathfrak{a}^*_{r_\hbar}, \Omega, J^{-1}_{r_\hbar}} \longrightarrow A_{\mathfrak{a}\oplus\mathfrak{a}^*_{r'_\hbar}, \Omega, J^{-1}_{r'_\hbar}}$. ■

About the converse of this proposition, we have [22] *some examples of isomorphisms where* $[\beta_\hbar] \neq [\beta'_\hbar]$.

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Carlos Moreno
Departamento de Física Teórica II, Universidad Complutense, E-28040 Madrid, Spain.

Joana Teles
CMUC, Department of Mathematics, University of Coimbra, Coimbra, Portugal
E-mail address: `jteles@mat.uc.pt`