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CONVERGENCE RATES FOR THE STRONG LAW OF LARGE NUMBERS UNDER ASSOCIATION

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Abstract: We prove convergence rates for the Strong Laws of Large Numbers (SLLN) for associated variables which are arbitrarily close to the optimal rates for independent variables. A first approach is based on exponential inequalities, a usual tool for this kind of problems. Following the optimization efforts of several authors, we improve the rates derived from exponential inequalities to $\frac{\log^2 n}{n^{1/2}}$. A more recent approach tries to use maximal inequalities together with moment inequalities. We prove a new maximal order inequality of order 4 for associated variables, using a telescoping argument. This inequality is then used to prove a SLLN convergence rate arbitrarily close to $\frac{\log^{1/4} n}{n^{1/2}}$.

Keywords: association, convergence rates, exponential inequalities, maximal inequalities.
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1. Introduction

Characterizing convergence rates in the Strong Law of large Numbers (SLLN) has been studied under various dependence structures. For associated variables several recent results have proved almost optimal rates, that is, rates that are arbitrarily close to the best rates for independent variables. The first SLLN for associated variables seems to have been proved by Newman and Wright [8], Newman [9] and Birkel [2] under suitable decrease rates on the covariance structure. These results gave no indication about convergence rates as there was no exponential inequality available for this type of dependence. Some moment inequalities were proved in Birkel [1], Shao and Yu [14] and Shao [13], but the main focus of these references was on central limit problems. Some convergence rates for the SLLN for associated variables appeared in Ioannides and Roussas [10], where the first exponential inequality for this dependence structure was proved, assuming the variables to be bounded and using a block decomposition and coupling. This was later extended for general associated variables in Oliveira [11]. The convergence

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rate derived was of order $\frac{\log^{5/3} n}{n^{1/3}}$, assuming the covariances to decrease geometrically. A convergence rate of the same order had appeared in Yang [17] published in Chinese. The same method of approach was used in Sung [15] to prove the rate $\frac{\log n}{n^{1/3}}$ and in Xing and Yang [16] who were able to optimize the proof to reach the rate $\frac{\log^2 n\,(\log\log n)^{1/2}}{n^{1/2}}$, both assuming geometrically decreasing covariances. Another approach to this problem is based on moment inequalities, enabling results for polynomially decreasing covariances. This approach emerged while studying the consistency of the kernel density estimator with almost optimal rates proved in Masry [6]. The approach based on exponential inequalities to the convergence of this estimator was used in Henriques and Oliveira [5], with an almost optimal rate with geometrically decreasing covariances but a milder assumption on the distributions. More recently, using moment inequalities together with a maximal approach Yang, Su and Yu [18] proved the almost optimal rate $\frac{(\log n)^{1/2}(\log\log n)^{\delta/2}}{n^{1/2}}$, where $\delta > 1$, under an assumption on the covariance that allows for polynomial decrease.

In this paper we start by showing that the approach based on exponential inequalities may be improved to derive the almost optimal SLLN rate $\frac{(\log n)^2}{n^{1/2}}$, improving the rate derived by Xing and Yang [16]. Using a different approach, we find conditions under which the better convergence rate $\frac{(\log n)^{1/4+\delta}}{n^{1/2}}$, $\delta > 0$, holds. The assumptions made allow for polynomial decreasing covariances with an extra assumption on conditional moments. The proof of this result is based on a maximal inequality of order 4 for associated variables, proved by using a telescoping series argument (see Garsia [4] or Dedecker and Rio [3] and Peligrad and Utev [12] for similar contexts).

Given a sequence of random variables X_n, $n \geq 1$, denote $S_n = X_1 + \cdots + X_n$. The random variables are associated if, for any $m \in \mathbb{N}$ and any two real-valued coordinatewise nondecreasing functions f and g it holds

$$\mathrm{Cov}\Big(f\left(X_1, \ldots, X_m\right), g\left(X_1, \ldots, X_m\right)\Big) \geq 0,$$

whenever this covariance exists.

In the framework of association, it is usual to state conditions on the covariance structure in terms of decrease rate of

$$u(n) = \sum_{j=n+1}^{\infty} \mathrm{Cov}(X_1, X_j).$$

Throughout this paper we will denote by c a generic positive constant, which may take different values in each appearance.

2. Using exponential inequalities

Given an associated sequence of random variables X_n, $n \geq 1$, and $c > 0$ define

$$X_{c,i} = -c\mathbb{I}_{(-\infty,-c)}(X_i) + X_i\mathbb{I}_{[-c,c]}(X_i) + c\mathbb{I}_{(c,+\infty)}(X_i), \tag{1}$$

where $\mathbb{I}_A$ represents the characteristic function of the set A. Notice that, for each $n \geq 1$ fixed, all these variables are monotone transformations of the initial variables X_n, so they are associated.

Consider a sequence of natural numbers p_n such that, for each $n \geq 1$, $p_n < \frac{n}{2}$ and define r_n as the greatest integer less or equal to $\frac{n}{2p_n}$. Define then, for $n \geq 1$ and $j = 1, \ldots, 2r_n$,

$$\begin{aligned} Y_{c,j,n} &= \sum_{l=(j-1)p_n+1}^{jp_n} \Big(X_{c,l} - \mathrm{E}(X_{c,l}) \Big) \\ Z_{c,n,od} &= \sum_{j=1}^{r_n} Y_{c,2j-1,n}, \quad Z_{c,n,ev} = \sum_{j=1}^{r_n} Y_{c,2j,n}, \end{aligned} \tag{2}$$

We start by proving un upper bound for second order moments of partial sums.

Lemma 2.1. *Let $c > 0$ and $S_{c,n} = \sum_{i=1}^{n} (X_{c,i} - \mathrm{E}(X_{c,i}))$. Assume that random the variables X_n, $n \geq 1$, are associated, stationary and $u(0) < \infty$. Then $\mathrm{E}(S_{c,n}^2) \leq 2nc^*$, where $c^* \geq c^2 + u(0)$.*

Proof: Using the stationarity, it follows easily that

$$\mathrm{E}(S_{c,n}^2) = n\mathrm{Var}(X_{c,1}) + 2\sum_{j=1}^{n-1}(n-j)\mathrm{Cov}(X_{c,1}, X_{c,j+1}) \leq 2nc^2 + 2nu(0) \leq 2nc^*,$$

since $\mathrm{Cov}(X_{c,1}, X_{c,j+1}) \leq \mathrm{Cov}(X_1, X_{j+1})$. ∎

This inequality will help improving the bound in Lemma 3.1 of Oliveira [11], as stated below.

Lemma 2.2. *Assume the variables X_n, $n \geq 1$, are associated, stationary and $u(0) < \infty$. If $d_n > 1$ and $0 < \lambda < \frac{d_n - 1}{d_n} \frac{1}{2c\, p_n}$, then*

$$\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{c,2j-1,n}}\right) \leq \exp\left(\lambda^2 n c^* d_n\right),$$

and the same bound holds for $\prod_{j=1}^{r_n} \mathrm{E}\left(e^{\lambda Y_{c,2j,n}}\right)$.

Proof: Remembering that $|Y_{c,2j-1,n}| \leq 2cp_n$, it follows, using a Taylor expansion and Lemma 2.1, that, for each $j = 1, \ldots, r_n$,

$$\begin{aligned}
\mathrm{E}\left(e^{\lambda Y_{c,2j-1,n}}\right) &\leq 1 + \lambda^2 \mathrm{E}(Y_{c,2j-1,n}^2) \sum_{k=2}^{\infty} (2c\lambda p_n)^{k-2} \\
&\leq 1 + \lambda^2 2 p_n c^* \frac{1}{1 - 2c\lambda p_n} \\
&\leq \exp\left(2\lambda^2 p_n c^* d_n\right),
\end{aligned}$$

according to the assumptions on d_n and λ. Finally, remember we chose the sequences such that $2r_n p_n \leq n$. ■

Theorem 2.3. *Assume the variables X_n, $n \geq 1$, are centered, associated, stationary and $u(0) < \infty$. Let $d_n > 1$ and assume that*

$$\exists \delta > 3: \sup_{|t| \leq \delta} \mathrm{E}(e^{tX_1}) \leq M < \infty, \tag{3}$$

$$\varepsilon_n^2 = \frac{648\alpha\, d_n \log^3 n}{n}, \quad 3 < \alpha \leq \delta, \tag{4}$$

$$\frac{n d_n \log n}{p_n^2} \longrightarrow \infty, \tag{5}$$

$$\frac{d_n}{(d_n - 1)^2} \frac{p_n^2 \log n}{n} < \frac{1}{2\alpha}, \tag{6}$$

$$\frac{1}{d_n \log n} \exp\left(\frac{\alpha}{2} \frac{n \log n}{d_n}\right)^{1/2} u(p_n) \leq C_0\,. \tag{7}$$

Then, for sufficiently large n,

$$\mathrm{P}\left(\frac{1}{n}|S_n| \geq \varepsilon_n\right) \leq \left(4\left(1 + \frac{\alpha C_0}{4}\right) - \frac{M}{162\alpha^3} \frac{n^2}{d_n \log^3 n}\right) e^{-\alpha \log n}. \tag{8}$$

Proof: We retrace the proofs in Henriques and Oliveira [5] and Oliveira [11] to prove an exponential inequality for the sum of the bounded variables $X_{c,n}$, using the block decomposition of the sum $\sum_{i=1}^{n}(X_{c,i}-\mathrm{E}(X_{c,i}))$, with blocks defined by (2). Next, Lemma 4.1 of [11] is used to handle the sum of the unbounded variables $(X_i-c)\mathbb{I}_{(c,+\infty)}(X_i)$ and $(X_i+c)\mathbb{I}_{(-\infty,-c)}(X_i)$.

Consider a sequence c_n for the truncation in (1). First, we obtain an exponential inequality for the sum of these bounded variables. Use then the arguments of the proof of Theorem 3.6 in [11] or of Theorems 1 or 2 in [5] to obtain, applying our Lemma 2.2,

$$\mathrm{P}\left(\frac{1}{n}\left|\sum_{i=1}^{n}(X_{c_n,i}-\mathrm{E}(X_{c_n,i}))\right|>\frac{\varepsilon_n}{3}\right)\leq 4(1+C_1)\exp\left(-\frac{1}{18^2}\frac{n\varepsilon_n^2}{c_n^* d_n}\right), \tag{9}$$

where c_n^* relates to c_n according to Lemma 2.1, if

$$\varepsilon_n<\frac{9(d_n-1)c_n^*}{p_n c_n}, \tag{10}$$

in order to meet the requirement on λ of Lemma 2.2, and also if

$$\frac{n\varepsilon_n^2}{648(c_n^*)^2 d_n^2}\exp\left(\frac{n\varepsilon_n c_n}{18 c_n^* d_n}\right)u(p_n)\leq C_1, \tag{11}$$

$$\frac{n\varepsilon_n}{p_n c_n}\longrightarrow\infty. \tag{12}$$

Use now Lemma 4.1 of [11], with $t=\alpha$ and $c_n=\log n$, to get, under (10), (11), (12) and (3),

$$\mathrm{P}\left(\frac{1}{n}|S_n|>\varepsilon_n\right)\leq\frac{4Mn\exp(-\alpha\log n)}{\alpha^2\varepsilon_n^2}+4(1+C_1)\exp\left(-\frac{1}{18^2}\frac{n\varepsilon_n^2}{c_n^* d_n}\right).$$

Finally, choose $c_n^*=2\log^2 n$ and insert (4) into the last inequality and into conditions (10), (11) and (12). ■

Assuming a geometric decrease rate of the covariances $\mathrm{Cov}(X_1,X_n)$, the previous theorem yields a convergence rate for the strong law of large numbers near the best possible convergence rate in the independent setting.

Corollary 2.4. *Assume the variables X_n, $n\geq 1$, are centered, associated, stationary and that $u(n)\leq\rho_0\rho^n$, where $\rho_0>0$ and $\rho\in(0,1)$. Assume further that (3) holds true. Then, $\frac{1}{n}S_n$ converges almost surely to zero with rate $\frac{\log^2 n}{n^{1/2}}$.*

Proof: Using the assumption on $u(n)$, condition (7) follows from

$$\log\left(\frac{1}{d_n \log n}\right) + \left(\frac{\alpha}{2}\,\frac{n\log n}{d_n}\right)^{1/2} + p_n \log\rho \le c.$$

This suggests the choice $p_n \sim \left(\frac{n\log n}{d_n}\right)^{1/2}$. Now, in order satisfy (6), choose $d_n \sim \log n$. Then, also (5) and (7) are verified. Additionally, $\varepsilon_n \sim \frac{\log^2 n}{n^{1/2}}$ and the upper bound in (8) defines a convergent series provided that $\alpha > 3$. ■

3. Maximal inequalities and convergence rates

In this section we will use maximal inequalities to characterize convergence rates in the SLLN. This approach has been used in Yang, Su and Yu [18] to prove some almost optimal convergence rates. We will use a maximal inequality due to Newman and Wright [7] to improve the convergence rate obtained in the previous section, as well as relax the assumption of geometric decrease rate of the covariances. Next we will prove a new maximal inequality for associated variables and use it to derive a faster convergence rate, even closer to the optimal rate.

Lemma 3.1 (Newman and Wright [7])**.** *Let X_n, $n \ge 1$, be centered, associated and square integrable variables. Then*

$$\mathrm{E}\left(\max_{i=1,\dots,n} |S_i|^2\right) \le 2\mathrm{E}S_n^2.$$

Theorem 3.2. *Let X_n, $n \ge 1$, be centered, associated, square integrable and stationary variables, such that $u(0) < \infty$. Then $\frac{1}{n}S_n$ converges almost surely to zero with rate $\frac{(\log n)^{1/2+\delta}}{n^{1/2}}$.*

Proof: Define $\varepsilon_n = \frac{(\log n)^{1/2+\delta}}{n^{1/2}}$, for some $\delta > 0$. According to the arguments of the proof of Theorem 2.1 in Yang, Su and Yu [18], it is enough to prove that $\max_{j=1,\dots,2^k} \frac{|S_j|}{2^k \varepsilon_{2^k}} \to 0$ almost surely. To prove this we use the Markov inequality and the maximal inequality of Lemma 3.1 to find

$$\mathrm{P}\left(\max_{j=1,\dots,2^k} |S_j| > 2^k \varepsilon_{2^k}\eta\right) \le \frac{2\mathrm{E}(S_{2^k}^2)}{2^{2k}\varepsilon_{2^k}^2\eta^2}.$$

The arguments used in the proof of Lemma 2.1 yield $\mathrm{E}(S_n)^2 \leq c\,n$, so that we get

$$\mathrm{P}\left(\max_{j=1,\dots,2^k} |S_j| > 2^k \varepsilon_{2^k} \eta\right) \leq c \frac{2^k}{2^{2k}\varepsilon_{2^k}^2} \leq c(k\log 2)^{-(1+2\delta)},$$

replacing the ε_n by the above choice. So the almost sure convergence to zero of $\max_{j=1,\dots,2^k} \frac{|S_j|}{2^k \varepsilon_{2^k}}$ follows by the Borel-Cantelli Lemma and the theorem is proved. ∎

We will now establish a maximal inequality of order 4 for associated random variables, which enables an improvement of the convergence rate proved in the previous theorem, at the cost of a stronger assumption on the covariance decrease rate. For that define, for each $n \geq 1$, $M_n^+ = \max(0, S_1, \dots, S_n)$ and $M_n^- = \max(0, -S_1, \dots, -S_n)$. We start by writing the telescoping representation:

$$(M_n^+)^4 = \sum_{k=1}^n \left[(M_k^+)^4 - (M_{k-1}^+)^4\right], \tag{13}$$

as done in Garsia [4]. The terms in the summation are obviously nonnegative and may be rewritten as

$$\left((M_k^+) - (M_{k-1}^+)\right) \times \left((M_k^+) + (M_{k-1}^+)\right) \times \left((M_k^+)^2 + (M_{k-1}^+)^2\right).$$

It is now evident that each term in (13) is nonnegative if and only if $S_k > M_{k-1}^+$. But then $M_k^+ = S_k$ and $(M_k^+)^2 = S_k^2$. Thus, it follows that

$$(M_k^+)^4 - (M_{k-1}^+)^4 \leq 2S_k^2 \left[(M_k^+)^2 - (M_{k-1}^+)^2\right].$$

As every term is nonnegative, we find that

$$(M_n^+)^4 \leq 2S_n^2 (M_n^+)^2 - 2\sum_{k=1}^n (M_{k-1}^+)^2 \left(S_k^2 - S_{k-1}^2\right).$$

Notice further that $2S_n^2(M_n^+)^2 \leq \frac{1}{2}(M_n^+)^4 + 2S_n^4$. Inserting this on the previous expression, we easily derive that

$$(M_n^+)^4 \leq 4S_n^4 - 4\sum_{k=1}^n (M_{k-1}^+)^2 \left(S_k^2 - S_{k-1}^2\right). \tag{14}$$

Writing $\left(S_k^2 - S_{k-1}^2\right) = -\left((S_n^2 - S_k^2) - (S_n^2 - S_{k-1}^2)\right)$, we may rewrite (14) as

$$(M_n^+)^4 \leq 4S_n^4 - 4\sum_{k=1}^{n}\left((M_k^+)^2 - (M_{k-1}^+)^2\right)\left(S_n^2 - S_k^2\right). \tag{15}$$

Taking now mathematical expectations, we get

$$\mathrm{E}(M_n^+)^4 \leq 4\mathrm{E}(S_n^4) - 4\sum_{k=1}^{n}\mathrm{E}\left[\left((M_k^+)^2 - (M_{k-1}^+)^2\right)\left(S_n^2 - S_k^2\right)\right]. \tag{16}$$

We may now repeat the arguments with M_n^-. Putting the two inequalities together yields, with $M_n = \max(|S_1|, \ldots, |S_n|)$,

$$\begin{aligned}\mathrm{E}(M_n^4) &\leq 8\mathrm{E}(S_n^4)\\ &-4\sum_{k=1}^{n}\mathrm{E}\Big[\left[\left((M_k^+)^2 - (M_{k-1}^+)^2\right) + \left((M_k^-)^2 - (M_{k-1}^-)^2\right)\right]\left(S_n^2 - S_k^2\right)\Big].\end{aligned} \tag{17}$$

Lemma 3.3. *Let X_n, $n \geq 1$, be random variables with finite moments of order 4. If, for every $i < j$, $\mathrm{E}(X_iX_j|\sigma(X_1, \ldots, X_i)) \geq 0$, then $\mathrm{E}(M_n^4) \leq 8\mathrm{E}(S_n^4)$.*

Proof: The mathematical expectation in the summation on (17) is equal to

$$\mathrm{E}\left[\Big[\left((M_k^+)^2 - (M_{k-1}^+)^2\right) + \left((M_k^-)^2 - (M_{k-1}^-)^2\right)\Big]\mathrm{E}\Big[\left(S_n^2 - S_k^2\right)|\sigma(X_1, \ldots, X_k)\Big]\right].$$

The term in square brackets outside the inner mathematical expectation is clearly nonnegative. As for the conditional expectation, we rewrite it as

$$2\sum_{i=1}^{k}\sum_{j=k+1}^{n}\mathrm{E}\Big[X_iX_j|\sigma(X_1, \ldots, X_i)\Big] + \mathrm{E}\Big[(S_n - S_k)^2|\sigma(X_1, \ldots, X_k)\Big],$$

which, under the assumption made, is also nonnegative. The lemma now follows readily from (17). ∎

Theorem 3.4. *Let X_n, $n \geq 1$, be centered, associated and stationary random variables such that $\mathrm{E}(X_1^r) < \infty$, for some $r > 4$, $u(n) \leq cn^{-1}$ and $\mathrm{E}(X_iX_j|\sigma(X_1, \ldots, X_i)) \geq 0$, for every $i < j$. Then $\frac{1}{n}S_n$ converges almost surely to zero with rate $\frac{(\log n)^{1/4+\delta}}{n^{1/2}}$, where $\delta > 0$.*

Proof: Let $\varepsilon_n = \frac{(\log n)^{1/4+\delta}}{n^{1/2}}$, for some $\delta > 0$. Following the arguments of the proof of Theorem 3.2, but applying now the maximal inequality of Lemma 3.3, we find

$$\mathrm{P}\left(\max_{j=1,\ldots,2^k} |S_j| > 2^k \varepsilon_{2^k}\eta\right) \leq \frac{4\mathrm{E}(S_{2^k}^4)}{2^{4k}\varepsilon_{2^k}^4\eta^4}.$$

Now, we use Theorem 4.2 from Shao and Yu [14], where we take f to be the identity function, $p=4$, $\theta=1$ and $\varepsilon=1$, to obtain $\mathrm{E}(S_n^4) \leq c\, n^2$. We then have

$$\mathrm{P}\left(\max_{j=1,\ldots,2^k} |S_j| > 2^k \varepsilon_{2^k}\eta\right) \leq c\frac{2^{2k}}{2^{4k}\varepsilon_{2^k}^4} \leq c(k\log 2)^{-(1+4\delta)},$$

replacing the ε_n by the above choice, which concludes the proof. ■

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