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MATRIX SYLVESTER EQUATIONS IN THE THEORY OF ORTHOGONAL POLYNOMIALS ON THE UNIT CIRCLE

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ABSTRACT: In this paper we characterize sequences of polynomials on the unit circle, orthogonal with respect to a Hermitian linear functional such that its corresponding Carathéodory function satisfies a Riccati differential equation with polynomial coefficients, in terms of matrix Sylvester differential equations. Furthermore, under certain conditions, we give a representation of such sequences in terms of semi-classical orthogonal polynomials on the unit circle. For the particular case of semi-classical orthogonal polynomials on the unit circle, a characterization in terms of first order differential systems is established.

KEYWORDS: Carathéodory function, matrix Riccati differential equations, matrix Sylvester differential equations, semi-classical functionals, measures on the unit circle.
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1. Introduction

A regular Hermitian linear functional defined in the linear space of Laurent polynomials with complex coefficients is said to be *Laguerre-Hahn* if the corresponding Carathéodory function, F, satisfies a Riccati differential equation with polynomial coefficients

$$zAF' = BF^2 + CF + D\,,\ A \not\equiv 0\,. \tag{1}$$

The corresponding sequence of orthogonal polynomials is said to be *Laguerre--Hahn*. We shall refer to the set of all such functionals (respectively, sequences of orthogonal polynomials) as the *Laguerre-Hahn class on the unit circle* (see [3]).

We remark that, analogously to the real line (see [10, 13, 14] for a study of the Laguerre-Hahn class on the set of functionals defined in the linear space of real polynomials), the Laguerre-Hahn class on the unit circle includes the *Laguerre-Hahn affine class* on the unit circle, which corresponds to the case

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$B=0$ in (1), and the *semi-classical class* on the unit circle, which corresponds to the case $B=0$ and C,D specific polynomials depending on A,B in (1) (see [2, 4]). Other examples of Laguerre-Hahn sequences can be found in [3].

In this paper we give a characterization of Laguerre-Hahn orthogonal polynomials on the unit circle in terms of matrix Sylvester differential equations. Let u be a Hermitian Laguerre-Hahn functional such that the corresponding Carathéodory function satisfies (1). We establish the equivalence between (1) and the following matrix Sylvester differential equations for $Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix}$ and $\mathcal{Q}_n = [-Q_n \;\; Q_n^*]^T$, where T denotes the transpose matrix,

$$\begin{cases} zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \\ zA\mathcal{Q}_n' = \left(\mathcal{B}_n + (BF + C/2)\, I\right) \mathcal{Q}_n \,, n \in \mathbb{N}\,, \end{cases} \tag{2}$$

where $\{\phi_n\}$, $\{\Omega_n\}$, $\{Q_n\}$ are, respectively, the sequence of orthogonal polynomials with respect to u, the corresponding sequence of polynomials of the second kind, and the sequence of functions of the second kind; $\mathcal{B}_n$ and $\mathcal{C}$ are matrices of order two with polynomial elements (see Theorem 4). As a consequence of the referred equivalence, a characterization for semi-classical orthogonal polynomials on the unit circle in terms of first order differential systems is obtained (see Theorem 5). Moreover, the equivalence between (1) and (2) allow us to give $\{Y_n\}$ in terms of the solutions of two linear differential systems, $zA\mathcal{L}' = \mathcal{C}\mathcal{L}$ and $zA\mathcal{P}_n' = \mathcal{B}_n\mathcal{P}_n$, as $Y_n = \mathcal{P}_n\mathcal{L}^{-1}\,, \forall n \geq 1$ (see Theorem 6). Furthermore, under certain conditions, we obtain $\{Y_n\}$ defined in terms of sequences of semi-classical orthogonal polynomials on the unit circle (see Theorem 8).

This paper is organized as follows. In section 2 we give the definitions and state the basic results which will be used in the forthcoming sections. In section 3 we establish a characterization theorem for functionals in the Laguerre-Hahn class: we establish the equivalence between (1) and the matrix Sylvester differential equations (2). In section 4 we establish a characterization for semi-classical orthogonal polynomials on the unit circle in terms of first order differential systems. In section 5 we solve the system of matrix Sylvester differential equations obtained in section 3. Furthermore, taking into account the characterization of semi-classical orthogonal polynomials on the unit circle previously obtained, we determine a representation for its solution in terms of sequences of semi-classical orthogonal polynomials on the unit circle. Finally, in section 6, an example is presented.

2. Preliminary results

Let $\Lambda = \text{span}\{z^k : k \in \mathbb{Z}\}$ be the space of Laurent polynomials with complex coefficients. Given a linear functional $u : \Lambda \to \mathbb{C}$ and the sequence of moments $(c_n)_{n\in\mathbb{Z}}$ of u, $c_n = \langle u, \xi^{-n}\rangle$, $n \in \mathbb{Z}$, $c_0 = 1$, define the minors of the Toeplitz matrix $\Delta = (c_n)_{n\in\mathbb{N}}$, by

$$\Delta_{-1} = 1\,,\ \Delta_0 = c_0\,,\ \Delta_k = \begin{vmatrix} c_0 & \cdots & c_k \\ \vdots & & \vdots \\ c_{-k} & \cdots & c_0 \end{vmatrix}\,,\ k \in \mathbb{N}\,.$$

The linear functional u is *Hermitian* if $c_{-n} = \overline{c}_n, \forall n \in \mathbb{N}$, and *regular* (*positive definite*) if $\Delta_n \neq 0$ ($\Delta_n > 0$), $\forall n \in \mathbb{N}$.

In this work we shall consider linear functionals that are Hermitian and positive definite. We will use the notation $\mathcal{R}^+$ to denote this set of functionals.

It is known that if $u \in \mathcal{R}^+$, then u has an integral representation defined in terms of a probability measure, μ, with infinite support on the unit circle $\mathbb{T} = \{e^{i\theta} : \theta \in [0, 2\pi[\,\}$, i.e.,

$$\left\langle u, e^{in\theta}\right\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta} d\mu(\theta)\,,\ n \in \mathbb{Z}\,.$$

The corresponding sequence of orthogonal polynomials, called *orthogonal polynomials on the unit circle* (with respect to μ), is then defined by

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_n(e^{i\theta})\overline{\phi}_m(e^{-i\theta})\,d\mu(\theta) = h_n\delta_{n,m}\,,\ h_n \neq 0\,,\ n, m \in \mathbb{N}\,.$$

If each ϕ_n is monic, then $\{\phi_n\}$ will be called a *monic orthogonal polynomial sequence*, and will be denoted by MOPS.

Given a measure μ, the function F defined by

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z}\,d\mu(\theta) \tag{3}$$

is a *Carathéodory function*, i.e., is an analytic function on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such that $F(0) = 1$ and $\Re e(F) > 0$ for $|z| < 1$. The converse result also holds, since any Carathéodory function has a representation (3) for a unique probability measure μ on $\mathbb{T}$ (see, for example, [16]).

Given a sequence of monic polynomials $\{\phi_n\}$ orthogonal with respect to μ, the *associated polynomials of the second kind* are given by

$$\Omega_0(z) = 1,\ \Omega_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\left(\phi_n(e^{i\theta}) - \phi_n(z)\right)\, d\mu(\theta)\,,\ \forall n\in\mathbb{N}\,,$$

and the *functions of the second kind* associated with $\{\phi_n\}$ are given by

$$Q_n(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\phi_n(e^{i\theta})\, d\mu(\theta)\,,\ n=0,1,\ldots$$

Following the ideas of [9], if we define

$$Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix},\ \mathcal{Q}_n = \begin{bmatrix} -Q_n \\ Q_n^* \end{bmatrix},\ \forall n\in\mathbb{N}\,, \tag{4}$$

with $p^*(z) = z^n\overline{p}(1/z)$, where n is the degree of the polynomial p, and $Q_n^*(z) = z^n\overline{Q}(1/z)$, then the recurrence relations satisfied by $\{\phi_n\}$ and $\{\Omega_n\}$ can be written in the matrix form as given in the following theorem.

Theorem 1 (cf. [7, 8, 15]). *Let F be a Carathéodory function, $\{\phi_n\}$, $\{\Omega_n\}$, $\{Q_n\}$, respectively, the corresponding MOPS on the unit circle, the sequence of associated polynomials of the second kind, and the sequence of the functions of the second kind. Let $\{Y_n\}$ and $\{\mathcal{Q}_n\}$ be the sequences defined in (4). Then, the following relations hold, $\forall n\in\mathbb{N}$,*

$$Y_n = \mathcal{A}_n Y_{n-1},\ \ \mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix}, \tag{5}$$

$$\mathcal{Q}_n = Y_n \begin{bmatrix} F \\ -1 \end{bmatrix}, \tag{6}$$

with $a_n = \phi_n(0)$, $Y_0 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathcal{Q}_0 = \begin{bmatrix} -F \\ -F \end{bmatrix}$.
Moreover, $\forall n\in\mathbb{N}$,

$$\phi_n^*(z)\Omega_n(z) + \phi_n(z)\Omega_n^*(z) = 2h_n z^n\,, \tag{7}$$

$$\phi_n^*(z)Q_n(z) + \phi_n(z)Q_n^*(z) = 2h_n z^n\,, \tag{8}$$

with $h_n = \prod_{k=1}^n (1-|a_k|^2)$.

Let $H_0(z) = \sum_{j=0}^{+\infty} b_j z^j,\ |z|<1,\ \ H_\infty(z) = \sum_{j=0}^{+\infty} b_j z^{-j}\,,\ |z|>1$. We will write $H_0(z) = \mathcal{O}(z^k)$ or $H_\infty(z) = \mathcal{O}(z^{-k})$ if $b_0 = \cdots = b_{k-1} = 0\,,\ k\in\mathbb{N}$.

Corollary 1. *Let $\{\phi_n\}$ be a MOPS on the unit circle and $\{Q_n\}$ be the corresponding sequence of functions of the second kind. Then, $\forall n \in \mathbb{N}$,*

$$Q_n(z) = 2h_n z^n + \mathcal{O}(z^{n+1}),\ |z| < 1\,,$$
$$Q_n(z) = 2a_{n+1}h_n z^{-1} + \mathcal{O}(z^{-2}),\ |z| > 1\,,$$

with $a_{n+1} = \phi_{n+1}(0)$, $h_n = \prod_{k=1}^{n}(1-|a_k|^2)$.

Corollary 2. *Let $\{\phi_n\}$ be a MOPS on the unit circle and $\{\Omega_n\}$ be the corresponding sequence of associated polynomials of the second kind. Then, the following holds:*
a) If there exists $k \in \mathbb{N}$ such that $\phi_k(\alpha) = \Omega_k(\alpha) = 0$, then $\alpha = 0$;
b) If there exists $k \in \mathbb{N}$ such that $\phi_k(\alpha) = Q_k(\alpha) = 0$, then $\alpha = 0$.

Theorem 2 (Geronimus, [6])**.** *Given a sequence of complex numbers (a_n) satisfying $|a_n| < 1$, $\forall n \in \mathbb{N}$, let $\{\phi_n\}$ and $\{\Omega_n\}$ be the sequences of polynomials defined by the recurrence relation (5), and let F be the corresponding Carathéodory function. Then, the sequence defined for $n \geq 1$, by*

$$\frac{\Omega_n^*(z)}{\phi_n^*(z)} = 1 + \frac{-2\overline{a}_1 z \,\big|}{\big|\, 1+\overline{a}_1 z} - \frac{\frac{\overline{a}_2}{\overline{a}_1} z(1-|a_1|^2)\,\big|}{\big|\, 1+\frac{\overline{a}_2}{\overline{a}_1} z} - \cdots - \frac{\frac{\overline{a}_{n+1}}{\overline{a}_n} z(1-|a_n|^2)\,\big|}{\big|\, 1+\frac{\overline{a}_{n+1}}{\overline{a}_n} z}\,,$$

converges uniformly to F, on compact subsets of $\mathbb{D}$.

Definition 1 (cf. [17])**.** Let μ be a measure given by $d\mu = w\,d\theta + \sum_{k=1}^{N}\lambda_k\delta_k$, $K \in \mathbb{N}$. μ is *semi-classical* if there exist polynomials A, C such that the absolutely continuous part of μ, w, satisfies

$$\frac{w'(z)}{w(z)} = \frac{C(z)}{zA(z)}\,. \tag{9}$$

The corresponding sequence of orthogonal polynomials is called *semi-classical.*

Lemma 1 (cf. [2, 4])**.** *A measure μ defined by $d\mu = w\,d\theta + \sum_{k=1}^{N}\lambda_k\delta_k$ is semi-classical and its absolutely continuous part satisfies (9), if and only if the corresponding Carathéodory function F satisfies*

$$zA(z)F'(z) = C(z)F(z) + C_3(z)\,,$$

with $C_3(z) = -zA'(z) - 2\displaystyle\sum_{k=2}^{\deg(A)}\frac{A^{(k)}(z)}{k!}\int_0^{2\pi} 2e^{i\theta}(e^{i\theta}-z)^{k-2}d\mu(\theta)\,.$

3. Characterization in terms of matrix Sylvester differential equations

Hereafter, I denotes the identity matrix of order two.

Theorem 3. *Let F be a Carathéodory function and $\{Y_n\}$ and $\{\mathcal{Q}_n\}$ the corresponding sequences defined by (4). The following statements are equivalent:*
a) F satisfies the differential equation with polynomial coefficients

$$zAF' = BF^2 + CF + D\,; \tag{10}$$

b) $\{Y_n\}$ and $\{\mathcal{Q}_n\}$ satisfy the Sylvester differential equations

$$zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \tag{11}$$

$$zA\mathcal{Q}_n' = (\mathcal{B}_n + (BF + C/2)\,I)\,\mathcal{Q}_n\,,\ n \in \mathbb{N}\,, \tag{12}$$

where $\mathcal{B}_n$ are matrices of bounded degree polynomials,

$$\mathcal{B}_n = \begin{bmatrix} l_n^1 & -\Theta_n^1 \\ -\Theta_n^2 & l_n^2 \end{bmatrix}, \tag{13}$$

and

$$\mathcal{C} = \begin{bmatrix} C/2 & -D \\ B & -C/2 \end{bmatrix}. \tag{14}$$

Proof: *a)* $\Rightarrow$ *b)*.
Let F satisfy (10). Firstly we obtain (11). This will be done dividing the proof in two parts: in the first part we deduce the equations

$$\begin{cases} zA\Omega_n' = (l_n^1 + C/2)\Omega_n - D\phi_n + \Theta_n^1\Omega_n^* \\ zA\phi_n' = (l_n^1 - C/2)\phi_n + B\Omega_n - \Theta_n^1\phi_n^* \end{cases} \tag{15}$$

and in the second part we deduce the equations

$$\begin{cases} zA(\Omega_n^*)' = (l_n^2 + C/2)\Omega_n^* + D\phi_n^* + \Theta_n^2\Omega_n \\ zA(\phi_n^*)' = (l_n^2 - C/2)\phi_n^* - B\Omega_n^* - \Theta_n^2\phi_n \end{cases} \tag{16}$$

with polynomials $l_n^1, l_n^2, \Theta_n^1, \Theta_n^2$ whose degrees do not depend on n. Then we will write these two systems of equations in the matrix form (11), with $\mathcal{B}_n$ and $\mathcal{C}$ given by (13) and (14), respectively.

Part 1. If we substitute $F = \frac{Q_n}{\phi_n} - \frac{\Omega_n}{\phi_n}$ (cf. (6)) in $zAF' = BF^2 + CF + D$ we obtain

$$zA\left(\frac{Q_n}{\phi_n} - \frac{\Omega_n}{\phi_n}\right)' = B\left(\frac{Q_n}{\phi_n} - \frac{\Omega_n}{\phi_n}\right)^2 + C\left(\frac{Q_n}{\phi_n} - \frac{\Omega_n}{\phi_n}\right) + D\,,$$

i.e.,

$$\begin{aligned} zA\left(\frac{Q_n}{\phi_n}\right)' - B\frac{Q_n}{\phi_n}\left(\frac{Q_n}{\phi_n} - 2\frac{\Omega_n}{\phi_n}\right) - C\frac{Q_n}{\phi_n} \\ = zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D\,. \end{aligned}$$

Therefore we have

$$\left\{zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D\right\}\phi_n^2 = \tilde{\Theta}_n \tag{17}$$

with

$$\tilde{\Theta}_n = \left\{zA\left(\frac{Q_n}{\phi_n}\right)' - B\frac{Q_n}{\phi_n}\left(\frac{Q_n}{\phi_n} - 2\frac{\Omega_n}{\phi_n}\right) - C\frac{Q_n}{\phi_n}\right\}\phi_n^2\,.$$

From (17) it follows that $\tilde{\Theta}_n$ is a polynomial. From the asymptotic expansion of Q_n in $|z| < 1$ (see Corollary 1), and since the left side of (17) is a polynomial, we get

$$\tilde{\Theta}_n(z) = z^n\tilde{\Theta}_n^1(z)\,,$$

with $\tilde{\Theta}_n^1$ a polynomial. From the asymptotic expansion of Q_n in $|z| > 1$ (see Corollary 1) it follows that $\tilde{\Theta}_n^1$ has bounded degree,

$$\deg(\tilde{\Theta}_n^1) = \max\{\deg(zA) - 2, \deg(B) - 1, \deg(C) - 1\},\ \forall n \in \mathbb{N}\,.$$

Thus, (17) becomes

$$\left\{zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D\right\}\phi_n^2 = z^n\tilde{\Theta}_n^1\,.$$

Using (7) in previous equation we obtain

$$\left\{zA\left(\frac{\Omega_n}{\phi_n}\right)' + B\left(\frac{\Omega_n}{\phi_n}\right)^2 - C\left(\frac{\Omega_n}{\phi_n}\right) + D\right\}\phi_n^2 = \Theta_n^1(\phi_n\Omega_n^* + \Omega_n\phi_n^*)\,,$$

where $\Theta_n^1 = \tilde{\Theta}_n^1/(2h_n)$.

Consequently, $\forall n \in \mathbb{N}$,

$$\left\{ zA\Omega_n' - \frac{C}{2}\Omega_n + D\phi_n - \Theta_n^1\Omega_n^* \right\} \phi_n = \left\{ zA\phi_n' + \frac{C}{2}\phi_n - B\Omega_n + \Theta_n^1\phi_n^* \right\} \Omega_n.$$

We distinguish the following cases (see Corollary 2):
a) ϕ_n and Ω_n have no common roots, $\forall n \in \mathbb{N}$, i.e., $\phi_n(0) \neq 0, \forall n \in \mathbb{N}$;
b) there exists a finite number of indexes $k \in \mathbb{N}$ such that ϕ_k and Ω_k have common roots, i.e., $\phi_k(0) = \Omega_k(0) = 0$ for a finite number of k's;
c) there exists $n_0 > 1$ such that $\phi_n(0) = 0$, $\forall n \geq n_0$.

Case a) If ϕ_n and Ω_n have no common roots, $\forall n \in \mathbb{N}$, then we conclude that there exists a polynomial l_n^1 such that

$$\begin{cases} zA\phi_n' + \frac{C}{2}\phi_n - B\Omega_n + \Theta_n^1\phi_n^* = l_n^1\phi_n \\ zA\Omega_n' - \frac{C}{2}\Omega_n + D\phi_n - \Theta_n^1\Omega_n^* = l_n^1\Omega_n\,,\ \forall n \in \mathbb{N}\,, \end{cases} \tag{18}$$

and we obtain (15). Moreover, l_n^1 has bounded degree,

$$\deg(l_n^1) = \max\{\deg(zA) - 1, \deg(C), \deg(B)\},\ \forall n \in \mathbb{N}\,.$$

Case b) We first assume that $\phi_1(0) \neq 0, \ldots, \phi_{k-1}(0) \neq 0$, and k is the first index such that $\phi_k(0) = 0$. So, ϕ_n and Ω_n have no common roots for $n = 1, \ldots, k-1$. From case a), equations (18) hold for $n = 1, \ldots, k-1$. Now we write (18) to $k-1$ and multiply by z, to obtain

$$\begin{cases} z^2A\phi_{k-1}' + \frac{C}{2}z\phi_{k-1} - Bz\Omega_{k-1} + z\Theta_{k-1}^1\phi_{k-1}^* = l_{k-1}^1 z\phi_{k-1} \\ z^2A\Omega_{k-1}' - \frac{C}{2}z\Omega_{k-1} + Dz\phi_{k-1} - z\Theta_{k-1}^1\Omega_{k-1}^* = l_n^1 z\Omega_{k-1}\,. \end{cases}$$

By substituting

$$\phi_k(z) = k\phi_{k-1}(z),\ \phi_k^*(z) = \phi_{k-1}^*(z),\ z\phi_{k-1}'(z) = \phi_k'(z) - \phi_{k-1}(z)$$

and

$$\Omega_k(z) = z\Omega_{k-1}(z),\ \Omega_k^*(z) = \Omega_{k-1}^*(z),\ z\Omega_{k-1}'(z) = \Omega_k'(z) - \Omega_{k-1}(z)$$

in previous equations, it follows that

$$\begin{cases} zA\phi_k' + \frac{C}{2}\phi_k - B\Omega_k + z\Theta_{k-1}^1\phi_k^* = \left(l_{k-1}^1 + A\right)\phi_k \\ zA\Omega_k' - \frac{C}{2}\Omega_k + D\phi_k - z\Theta_{k-1}^1\Omega_k^* = l_n^1\Omega_k\,, \end{cases}$$

and we obtain (15) to $n = k$ with $l_k^1 = l_{k-1}^1 + A$ and $\Theta_k^1 = z\Theta_{k-1}^1$.

Furthermore, if $\phi_{k+1}(0)=\cdots=\phi_{k+k_0}(0)=0$, $\phi_{k+k_0+1}(0)\neq 0$ to some $k_0\in\mathbb{N}$, then, using the same method as before, the differential relations (15) are obtained for $n=k+1,\ldots,k+k_0$, with

$$l_n^1=l_{k-1}^1+(n-k+1)A\,,\ \ \Theta_n^1=z^{n-k+1}\Theta_{k-1}^1\,,\ \ n=k+1,\ldots,k+k_0\,.$$

Case c) If $\phi_n(0)=0$, $\forall n\geq n_0$, then ϕ_n and Ω_n are polynomials of the Bernstein-Szegő type,

$$\phi_n(z)=z^{n-n_0+1}\phi_{n_0-1}(z)\,,\ \ \Omega_n(z)=z^{n-n_0+1}\Omega_{n_0-1}(z)\,.$$

Applying the same method as before, we conclude that equations (15) hold, $\forall n\in\mathbb{N}$, and, for $n\geq n_0$, l_n^1 and Θ_n^1 are given by

$$l_n^1=l_{n_0-1}+(n-n_0+1)A\,,\ \Theta_n^1=z^{n-n_0+1}\Theta_{n_0-1}^1\,.$$

Part 2. If we substitute $F=\dfrac{\Omega_n^*}{\phi_n^*}-\dfrac{Q_n^*}{\phi_n^*}$ (cf. (6)) in $zAF'=BF^2+CF+D$ and proceed as in part one, we obtain (16) with

$$\deg(l_n^2)=\max\{\deg(zA)-1,\deg(B),\deg(C)\},\ \forall n\in\mathbb{N}\,.$$

Finally, equations (15) and (16) can be presented in the matrix form (11).

We now obtain (12). Taking derivatives on $Q_n=\Omega_n+\phi_nF$ and $Q_n^*=\Omega_n^*-\phi_n^*F$ (cf. (6)) we obtain

$$\begin{aligned}zAQ_n'&=zA\Omega_n'+zA\phi_n'F+zAF'\phi_n\,,\\ zA(Q_n^*)'&=zA(\Omega_n^*)'-zA(\phi_n^*)'F-zAF'\phi_n^*\,.\end{aligned}$$

Using (15) and (16), respectively, in previous equations, (12) follows.

$b)\Rightarrow a)$.

Taking into account (6), $\mathcal{Q}_n=Y_n\begin{bmatrix}F\\-1\end{bmatrix}$, $\forall n\in\mathbb{N}$, we see that (12) is equivalent to

$$zAY_n'\begin{bmatrix}F\\-1\end{bmatrix}+zAY_n\begin{bmatrix}F'\\0\end{bmatrix}=\mathcal{B}_nY_n\begin{bmatrix}F\\-1\end{bmatrix}+(BF+C/2)Y_n\begin{bmatrix}F\\-1\end{bmatrix}\,.$$

From (11) it follows that

$$(\mathcal{B}_nY_n-Y_n\mathcal{C})\begin{bmatrix}F\\-1\end{bmatrix}+zAY_n\begin{bmatrix}F'\\0\end{bmatrix}=\mathcal{B}_nY_n\begin{bmatrix}F\\-1\end{bmatrix}+(BF+C/2)Y_n\begin{bmatrix}F\\-1\end{bmatrix}\,,$$

i.e.,

$$Y_n\left(zA\begin{bmatrix}F'\\0\end{bmatrix}-\mathcal{C}\begin{bmatrix}F\\-1\end{bmatrix}\right)=(BF+C/2)Y_n\begin{bmatrix}F\\-1\end{bmatrix}\,.$$

Taking into account that Y_n is regular, then we obtain

$$zA\begin{bmatrix} F' \\ 0 \end{bmatrix} - \mathcal{C}\begin{bmatrix} F \\ -1 \end{bmatrix} = (BF + C/2)\begin{bmatrix} F \\ -1 \end{bmatrix} .$$

Since $\mathcal{C}$ is given by (14), $zAF' = BF^2 + CF + D$ follows. ■

Remark . Hereafter we will say that the matrices $\mathcal{B}_n$ are associated with the equation $zAF' = BF^2 + CF + D$.

The following formula for $\mathrm{tr}(\mathcal{B}_n)$ was given in [12] for a particular case of a semi-classical sequence of orthogonal polynomials on the unit circle.

Corollary 3. *Under the conditions of the previous theorem, the matrices* $\mathcal{B}_n$ *given by (13) satisfy*

$$zA\mathcal{A}_n' = \mathcal{B}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{B}_{n-1} \, , \; n \geq 2 \, , \tag{19}$$

$$\mathrm{tr}(\mathcal{B}_n) = nA \, , \; n \in \mathbb{N} \, , \tag{20}$$

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_1) - A\sum_{k=1}^{n-1} l_{k,2} \, , \; n \geq 2 \, , \tag{21}$$

where $\mathrm{tr}(\mathcal{B}_n)$ *and* $\det(\mathcal{B}_n)$ *denote, respectively, the trace and the determinant of* $\mathcal{B}_n$ *, and*

$$\det(\mathcal{B}_1) = A\left(2zA\overline{a}_1 - h_1(D+B) + C(|a_1|^2+1)\right)/(2\,h_1) + BD - C^2/4 \, , \tag{22}$$

$a_1 = \phi_1(0), \; h_1 = 1 - |a_1|^2$.

Proof: To obtain (19) we take derivatives on $Y_n = \mathcal{A}_nY_{n-1}$ and substitute $Y_n' = \mathcal{A}_n'Y_{n-1} + \mathcal{A}_nY_{n-1}'$ in (11), $zAY_n' = \mathcal{B}_nY_n - Y_n\mathcal{C}$. Therefore, we get

$$zA\mathcal{A}_n'Y_{n-1} + zA\mathcal{A}_nY_{n-1}' = \mathcal{B}_nY_n - Y_n\mathcal{C} \, .$$

Using (11) with $n-1$ in previous equation we get

$$zA\mathcal{A}_n'Y_{n-1} + \mathcal{A}_n\left(\mathcal{B}_{n-1}Y_{n-1} - Y_{n-1}\mathcal{C}\right) = \mathcal{B}_nY_n - Y_n\mathcal{C} \, .$$

Using the recurrence relation (5) we obtain

$$zA\mathcal{A}_n'Y_{n-1} + \mathcal{A}_n\left(\mathcal{B}_{n-1}Y_{n-1} - Y_{n-1}\mathcal{C}\right) = \mathcal{B}_n\mathcal{A}_nY_{n-1} - \mathcal{A}_nY_{n-1}\mathcal{C} \, ,$$

i.e.,

$$zA\mathcal{A}_n'Y_{n-1} = \left(\mathcal{B}_n\mathcal{A}_n - \mathcal{A}_n\mathcal{B}_{n-1}\right)Y_{n-1} \, .$$

Since Y_n is regular, for all $n \in \mathbb{N}$ and $z \neq 0$, we obtain (19).

To deduce (20) we use equations (15) and (16),

$$\begin{cases} zA\phi_n' + C/2\phi_n - B\Omega_n + \Theta_{n,1}\phi_n^* = l_{n,1}\phi_n \\ zA\Omega_n' - C/2\Omega_n + D\phi_n - \Theta_{n,1}\Omega_n^* = l_{n,1}\Omega_n \\ zA(\Omega_n^*)' - C/2\Omega_n^* - D\phi_n^* - \Theta_{n,2}\Omega_n = l_{n,2}\Omega_n^* \\ zA(\phi_n^*)' + C/2\phi_n^* + B\Omega_n^* + \Theta_{n,2}\phi_n = l_{n,2}\phi_n^* \,. \end{cases}$$

If we multiply previous equations by Ω_n^*, ϕ_n^*, ϕ_n and Ω_n, respectively, we obtain, after summing,

$$zA\left(\phi_n'\Omega_n^* + \phi_n(\Omega_n^*)' + (\phi_n^*)'\Omega_n + \phi_n^*\Omega_n'\right) = (l_{n,1} + l_{n,2})\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)\,,$$

i.e.,

$$zA\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)' = (l_{n,1} + l_{n,2})\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)\,.$$

Thus,

$$zA\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)' = \operatorname{tr}(\mathcal{B}_n)\left(\phi_n\Omega_n^* + \phi_n^*\Omega_n\right)\,.$$

If we use (7) in previous equation then we get (20).

We now establish (21). From (19) we obtain, for $n \geq 2$,

$$\det(\mathcal{B}_n\mathcal{A}_n) = \det(zA\mathcal{A}_n' + \mathcal{A}_n\mathcal{B}_{n-1})\,.$$

Taking into account that $\mathcal{B}_n$ is given by (13) and $\mathcal{A}_n = \begin{bmatrix} z & a_n \\ \overline{a}_n z & 1 \end{bmatrix}$, we obtain

$$\det(\mathcal{B}_n)\det(\mathcal{A}_n) = z(1 - |a_n|^2)\left(\det(\mathcal{B}_{n-1}) + A\,l_{n-1,2}\right),\ \forall n \geq 2\,.$$

Since $\det(\mathcal{A}_n) = z(1 - |a_n|^2)$, then the last equation is equivalent, if $z \neq 0$, to

$$\det(\mathcal{B}_n) = \det(\mathcal{B}_{n-1}) + A\,l_{n-1,2}\,,\ \forall n \geq 2\,.$$

Consequently, we obtain (21). Moreover, if we compute $\det(\mathcal{B}_1)$ by taking $n = 1$ in (11), we obtain (22). ■

Remark . (19) is equivalent to the following equations, for all $n \in \mathbb{N}$,

$$\begin{cases} a_n l_{n,1} - \Theta_{n,1} = -z\Theta_{n-1,1} + a_n l_{n-1,2} \\ z l_{n,1} - \overline{a}_n z\Theta_{n,1} = z l_{n-1,1} - a_n\Theta_{n-1,2} + zA \\ -z\Theta_{n,2} + \overline{a}_n z l_{n,2} = \overline{a}_n z l_{n-1,1} - \Theta_{n-1,2} + \overline{a}_n zA \\ -a_n\Theta_{n,2} + l_{n,2} = -\overline{a}_n z\Theta_{n-1,1} + l_{n-1,2}\,. \end{cases} \tag{23}$$

4. A characterization for semi-classical orthogonal polynomials on the unit circle

The following lemma can be found in [5].

Lemma 2. *Let X and M be matrix functions of order two such that $X' = M\,X$. Then,*

$$(\det(X))' = \operatorname{tr}(M)\,\det(X)\,. \tag{24}$$

Next theorem is a generalization of a result for semi-classical orthogonal polynomials on the real line established in [11], by Magnus. Moreover, it shows that the necessary condition given in [2] for a MOPS on the unit circle to be semi-classical is also sufficient.

Theorem 4. *Let $\{\phi_n\}$ be a MOPS on the unit circle with respect to a measure μ whose absolutely continuous part is denoted by w, $\{Q_n\}$ be the sequence of functions of the second kind, and $\tilde{Y}_n = \begin{bmatrix} \phi_n & Q_n/w \\ \phi_n^* & -Q_n^*/w \end{bmatrix}$, $\forall n \geq 1$. Then, μ is semi-classical and w satisfies*

$$w(z) = K e^{\int_{z_1}^{z} \frac{C(t)}{tA(t)} dt}\,,\ K \in \mathbb{C}\,, \tag{25}$$

if, and only if, $\tilde{Y}_n$ satisfy

$$zA\tilde{Y}_n' = (\mathcal{B}_n - C/2\ I)\tilde{Y}_n,\ \forall n \in \mathbb{N}\,, \tag{26}$$

where $\mathcal{B}_n$ is the matrix associated with the equation

$$zAF' = CF + D\,, \tag{27}$$

satisfied by the corresponding Carathéodory function.

Proof: If w satisfies $w'/w = C/(zA)$, then the corresponding F satisfies (27) (see [2, 4]).

From Theorem 3 the following two equations hold,

$$zA \begin{bmatrix} Q_n'/w \\ -(Q_n^*)'/w \end{bmatrix} = (\mathcal{B}_n + C/2\,I) \begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}\,, \tag{28}$$

$$zA \begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}' = (\mathcal{B}_n - C/2\,I) \begin{bmatrix} \phi_n \\ \phi_n^* \end{bmatrix}\,. \tag{29}$$

Moreover, as

$$w'/w = C/(zA),$$

we obtain

$$zA\begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}' = zA\begin{bmatrix} Q_n'/w \\ -(Q_n^*)'/w \end{bmatrix} - C\,I\begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}. \tag{30}$$

If we substitute (28) in (30) we get

$$zA\begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}' = (\mathcal{B}_n - C/2\,I)\begin{bmatrix} Q_n/w \\ -Q_n^*/w \end{bmatrix}. \tag{31}$$

Finally, from (29) and (31), the differential system (26) follows.

We now prove the converse.

If $\tilde{Y}_n = \begin{bmatrix} \phi_n & Q_n/w \\ \phi_n^* & -Q_n^*/w \end{bmatrix}$ satisfies (26) then, from Lemma 2, we obtain

$$(\det(\tilde{Y}_n))' = \frac{\operatorname{tr}(\mathcal{B}_n - C/2\,I)}{zA}\det(\tilde{Y}_n)\,.$$

From (8) we get $\det(\tilde{Y}_n) = 2h_n z^n/w$, thus last equation is equivalent to

$$\frac{w'}{w} = \frac{nA - \operatorname{tr}(\mathcal{B}_n - C/2\,I)}{zA}\,.$$

Using $\operatorname{tr}(\mathcal{B}_n) = nA$ (cf. (20)) in previous equation, it follows that

$$\frac{w'}{w} = \frac{C}{zA},$$

and we conclude that μ is semi-classical and w is given by (25). ∎

5. Solutions of the Sylvester differential equations

In this section we solve the Sylvester differential equations (11), $zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, $\forall n \in \mathbb{N}$. In what comes next, we use a particular case of a result on matrix Riccati equations, known as Radon's Lemma (see [1]).

Theorem 5. *Let F satisfy $zAF' = BF^2 + CF + D$ and $\{Y_n\}$ be the corresponding sequence given in (4). If $\mathcal{P}_n$ and $\mathcal{L}$ ($\mathcal{L}$ invertible) satisfy, $\forall n \in \mathbb{N}$,*

$$\begin{cases} zA(z)\mathcal{L}'(z) = \mathcal{C}(z)\mathcal{L}(z) \\ \mathcal{L}(z_0) = I \end{cases} \tag{32}$$

and

$$\begin{cases} zA(z)\mathcal{P}_n'(z) = \mathcal{B}_n(z)\mathcal{P}_n(z) \\ \mathcal{P}_n(z_0) = Y_n(z_0) \end{cases} \tag{33}$$

where $\mathcal{B}_n$ *and* $\mathcal{C}$ *are given by (13) and (14), respectively, then,* $\forall n \in \mathbb{N}$,

$$Y_n = \mathcal{P}_n \, \mathcal{L}^{-1} \,. \tag{34}$$

Proof: To $zAF' = BF^2 + CF + D$ we associate (11), $zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C}$, with $\mathcal{B}_n$ and $\mathcal{C}$ given by (13) and (14), respectively (see Theorem 3).

Let $\mathcal{L}$ and $\mathcal{P}_n$ satisfy (32) and (33). Then, since

$$zA(\mathcal{P}_n \mathcal{L}^{-1})' = zA\mathcal{P}_n' \mathcal{L}^{-1} + zA\mathcal{P}_n (\mathcal{L}^{-1})'$$

and $(\mathcal{L}^{-1})' = -\mathcal{L}^{-1}\mathcal{L}'\mathcal{L}^{-1}$, using (33) we get

$$zA(\mathcal{P}_n \mathcal{L}^{-1})' = \mathcal{B}_n \mathcal{P}_n \mathcal{L}^{-1} - zA\mathcal{P}_n \mathcal{L}^{-1}\mathcal{L}'\mathcal{L}^{-1} \,.$$

Using (32) it follows that

$$zA(\mathcal{P}_n \mathcal{L}^{-1})' = \mathcal{B}_n \mathcal{P}_n \mathcal{L}^{-1} - \mathcal{P}_n \mathcal{L}^{-1}\mathcal{C}\mathcal{L}\mathcal{L}^{-1} \,,$$

i.e., $Y_n = \mathcal{P}_n \mathcal{L}^{-1}$ satisfies

$$zAY_n' = \mathcal{B}_n Y_n - Y_n \mathcal{C} \,.$$

Thus, the assertion follows. ■

Remark . The solution of (32) is given by $\mathcal{L}(z) = L(z)L^0$, with L a fundamental matrix of the differential system (32) satisfying $zAL' = \mathcal{C}L$, and $L^0 = L(z_0)^{-1}$. The solution of (33) is given by $\mathcal{P}_n(z) = P_n(z)P_n^0$, with P_n a fundamental matrix of (33) satisfying $zAP_n' = \mathcal{B}_n P_n$, and P_n^0 satisfying $P_n(z_0)P_n^0 = Y_n(z_0)$, i.e., $P_n^0 = (P_n(z_0))^{-1}Y_n(z_0)$. Then, if we substitute $\mathcal{L}$ and $\mathcal{P}_n$, given as above, in (34), the solution of the Sylvester differential equations (11) becomes

$$Y_n(z) = P_n(z)E_n L^{-1}(z) \tag{35}$$

with

$$E_n = (P_n(z_0))^{-1}Y_n(z_0)L(z_0) \,. \tag{36}$$

5.1. Solution of (32). We search for a matrix L of order 2 satisfying $zA(z)L'(z) = \mathcal{C}(z)L(z)$, with $\mathcal{C}$ given in (14).

Lemma 3. *Let* L *be a fundamental matrix of solutions of (32). Then,* $\det(L(z)) = \det(L(z_0))$.

Proof: From Lemma 2 (cf. (24)) we have

$$(\det(L))' = \frac{\operatorname{tr}(\mathcal{C})}{zA} \det(L)\,.$$

Since $\operatorname{tr}(\mathcal{C}) = 0$, it follows that $(\det(L))' = 0$, i.e.,

$$\det(L) = c,\ c \in \mathbb{C}\,.$$

Thus, $\det(L(z)) = \det(L(z_0))$, for some $z_0 \in \mathbb{C}$. ■

Lemma 4. *Let $\mathcal{C}$ be the matrix defined by (14). Then,*

(a) $\mathcal{C}^2 = \beta\, I, \quad \beta = (C/2)^2 - BD$ *;*

(b) *The eigenvalues of $\mathcal{C}$ are $\pm\sqrt{\beta}$;*

(c) *The eigenspace corresponding to $\sqrt{\beta}$ is $V_{\sqrt{\beta}} = \operatorname{span}\{\begin{bmatrix} D \\ C/2 - \sqrt{\beta} \end{bmatrix}\}$ and the eigenspace corresponding to $-\sqrt{\beta}$ is $V_{-\sqrt{\beta}} = \operatorname{span}\{\begin{bmatrix} D \\ C/2 + \sqrt{\beta} \end{bmatrix}\}$.*

In what follows, L_1, L_2 are column vectors of size 2.

Lemma 5. *Let $L = [L_1\ L_2]$ be a fundamental matrix of (32). Then,*

$$zAL_1' = \sqrt{\beta} L_1 + zAc_1 V_{-\sqrt{\beta}}\,, \tag{37}$$

$$zAL_2' = -\sqrt{\beta} L_2 + zAc_2 V_{\sqrt{\beta}}\,, \tag{38}$$

with c_1, c_2 functions.

Proof: From (32) it follows that

$$(\mathcal{C} + \sqrt{\beta}\ I)\left(L_1' - \frac{\sqrt{\beta}}{zA} L_1\right) = 0_{2\times 1}\,, \tag{39}$$

$$(\mathcal{C} - \sqrt{\beta}\ I)\left(L_2' + \frac{\sqrt{\beta}}{zA} L_2\right) = 0_{2\times 1}\,. \tag{40}$$

Since the eigenvalues of $\mathcal{C}$ are $\pm\sqrt{\beta}$, and the corresponding eigenvectors are $V_{\sqrt{\beta}}$ and $V_{\sqrt{-\beta}}$, from (39) and (40) we obtain, respectively,

$$L_1' - \frac{\sqrt{\beta}}{zA} L_1 = c_1(z)\, V_{-\sqrt{\beta}}$$

$$L_2' + \frac{\sqrt{\beta}}{zA} L_2 = c_2(z)\, V_{\sqrt{\beta}}$$

where c_1, c_2 are functions. Thus, (37) and (38) follow. ■

5.2. Solution of (33). We search for matrices P_n of order two satisfying, for each $n \in \mathbb{N}$,

$$zAP_n' = \mathcal{B}_n P_n \,. \tag{41}$$

Hereafter we will consider $z_1 \in \mathbb{C}$ and $\tilde{C}$ be an analytic function such that $\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt$ is defined (in suitable domains).

Lemma 6. *$\tilde{P}_n$ is a solution of*

$$zA\tilde{P}_n' = (\mathcal{B}_n - \tilde{C}/2\, I)\tilde{P}_n \tag{42}$$

if, and only if, $P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n$ is a solution of (41).

Proof: Let $\tilde{P}_n$ be a solution of (42). We have that

$$zA(e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n)' = \frac{\tilde{C}}{2} e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n + zA\tilde{P}_n'\, e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt} \,.$$

Since $\tilde{P}_n$ satisfies (42), we obtain

$$zA(e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt}\, \tilde{P}_n)' = \mathcal{B}_n \tilde{P}_n e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt}$$

thus $P_n = \tilde{P}_n e^{\int_{t_1}^{z} \frac{\tilde{C}/2}{tA} dt}$ is a solution of (41). Analogously one can see that the converse holds. ∎

Taking into account previous lemma, we will solve (41) searching for a solution $\{P_n\}$ given by $P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt}\; \tilde{P}_n \,, n \in \mathbb{N}$. Furthermore, taking into account Theorem 4, we will consider $\tilde{C}$ as a polynomial and $\tilde{P}_n = \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ (\tilde{\phi}_n)^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix}$, $\forall n \in \mathbb{N}$, where $\{\tilde{\phi}_n\}$ is a MOPS on the unit circle, orthogonal with respect to a measure $\tilde{\mu}$ with weight function

$$\tilde{w} = K\, e^{\int_{z_1}^{z} \frac{\tilde{C}}{tA} dt} ,\ K \in \mathbb{C} \,, \tag{43}$$

and $\{\tilde{Q}_n\}$ is the corresponding sequence of functions of the second kind. Hence,

$$P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt} \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ (\tilde{\phi}_n)^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix} , n \in \mathbb{N} \,. \tag{44}$$

Lemma 7. *Let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$ and $\{\phi_n\}$ the corresponding MOPS. For all $n \in \mathbb{N}$, let P_n be a fundamental* *matrix of the corresponding differential system (33). If P_n is given by (44), then the following equations hold:*

$$P_n = \tilde{\mathcal{A}}_n P_{n-1}, \quad \tilde{\mathcal{A}}_n = \begin{bmatrix} z & \tilde{a}_n \\ \bar{\tilde{a}}_n z & 1 \end{bmatrix}, \; n \in \mathbb{N}, \tag{45}$$

$$zA\tilde{\mathcal{A}}_n' = \mathcal{B}_n \tilde{\mathcal{A}}_n - \tilde{\mathcal{A}}_n \mathcal{B}_{n-1}, \; n \geq 2. \tag{46}$$

Proof: To establish (45) we recall that $\{\tilde{P}_n\}$ satisfies the recurrence relations in the matrix form (see Theorem 1)

$$\tilde{P}_n = \tilde{\mathcal{A}}_n \tilde{P}_{n-1}, \quad \tilde{\mathcal{A}}_n = \begin{bmatrix} z & \tilde{a}_n \\ \bar{\tilde{a}}_n z & 1 \end{bmatrix}, \; n \in \mathbb{N},$$

with $\tilde{a}_n = \tilde{\phi}_n(0)$. Thus P_n given by (44) satisfies (45), $\forall n \in \mathbb{N}$.

We now establish (46).

Since P_n satisfies $zAP_n' = \mathcal{B}_n P_n$, then by substituting $P_n = \tilde{\mathcal{A}}_n P_{n-1}$ in previous equation, there follows

$$zA\tilde{\mathcal{A}}_n' P_{n-1} + \tilde{\mathcal{A}}_n zAP_{n-1}' = \mathcal{B}_n \tilde{\mathcal{A}}_n P_{n-1}, \; n \geq 2.$$

Using $zAP_{n-1}' = \mathcal{B}_{n-1} P_{n-1}$ in last equation we get

$$zA\tilde{\mathcal{A}}_n' P_{n-1} + \tilde{\mathcal{A}}_n \mathcal{B}_{n-1} P_{n-1} = \mathcal{B}_n \tilde{\mathcal{A}}_n P_{n-1}.$$

Thus,

$$(zA\tilde{\mathcal{A}}_n' + \tilde{\mathcal{A}}_n \mathcal{B}_{n-1}) P_{n-1} = \mathcal{B}_n \tilde{\mathcal{A}}_n P_{n-1}.$$

Since P_n is regular ($\det(P_n) \neq 0$, $\forall n \in \mathbb{N}$, $\forall z \neq 0$) then

$$zA\tilde{\mathcal{A}}_n' + \tilde{\mathcal{A}}_n \mathcal{B}_{n-1} = \mathcal{B}_n \tilde{\mathcal{A}}_n$$

follows, and we obtain (46). ■

Remark . From (19) and (46) we get the equations

$$zA(\mathcal{A}_n - \tilde{\mathcal{A}}_n)' = \mathcal{B}_n(\mathcal{A}_n - \tilde{\mathcal{A}}_n) - (\mathcal{A}_n - \tilde{\mathcal{A}}_n)\mathcal{B}_{n-1}, \; n \geq 2.$$

Hence,

$$\begin{cases} \overline{\lambda}_n \Theta_{n,1} = \lambda_n \Theta_{n-1,2} \\ \lambda_n l_{n,1} = \lambda_n l_{n-1,2} \\ \overline{\lambda}_n \Theta_{n-1,1} = \lambda_n \Theta_{n,2} \\ \overline{\lambda}_n l_{n,2} - \overline{\lambda}_n l_{n-1,1} = \overline{\lambda}_n zA \end{cases} \tag{47}$$

where $\lambda_n = a_n - \tilde{a}_n, \; a_n = \phi_n(0), \; \tilde{a}_n = \tilde{\phi}_n(0), \; \forall n \in \mathbb{N}.$

Hereafter we will denote linear fractional transformations $T(F) = \dfrac{a+bF}{c+dF}$ by $T_{(a,b;c,d)}(F)$.

Theorem 6. *Let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$, $\{\phi_n\}$ the corresponding MOPS, and for all $n \in \mathbb{N}$, let P_n be a fundamental matrix of the corresponding differential system (33), given by (44). Let $\tilde{F}$ be the Carathéodory function associated with $\{\tilde{\phi}_n\}$ given in (44). Then, there exists a unique linear fractional transformation, $T_{(a,b;c,d)}$, with $a, b, c, d \in \mathbb{P}$ and $ad - bc \not\equiv 0$, such that $F = T_{(a,b;c,d)}(\tilde{F})$.*

Proof: To prove that F is a linear fractional transformation of $\tilde{F}$, we begin by establishing that the reflection coefficients of $\{\phi_n\}$ and $\{\tilde{\phi}_n\}$, i.e., $a_n = \phi_n(0)$ and $\tilde{a}_n = \tilde{\phi}_n(0)$, differ only in a finite number of indexes.

Let us write $\lambda_n = a_n - \tilde{a}_n$, $\forall n \in \mathbb{N}$. First we establish that $\mathcal{Z} = \{n \in \mathbb{N} : \lambda_n \neq 0\}$ is a finite set. In fact, if $\mathcal{Z}$ was not finite, for example, $\mathcal{Z} \equiv \mathbb{N}$, then $\lambda_n \neq 0$, $\forall n \in \mathbb{N}$. But from (47) we would obtain

$$l_{n,1} = l_{n-1,2}, \ \forall n \in \mathbb{N} .$$

Substituting in (23), we would obtain

$$\Theta_{n,1} = z\Theta_{n-1,1}, \ \forall n \in \mathbb{N} ,$$

hence

$$\Theta_{n,1} = z^n \Theta_{1,1}, \forall n \in \mathbb{N} .$$

But this is a contradiction with the fact that $\deg(\Theta_n)$ is bounded. Therefore, $\mathcal{Z} \not\equiv \mathbb{N}$. On the other hand, if we consider, without loss of generality, the case

$$\begin{cases} a_n = \tilde{a}_n \,, & n = 1, 2, \ldots, n_0 \,, \\ a_n \neq \tilde{a}_n \,, & n \geq n_0 \,, \end{cases}$$

then we will obtain the same conclusion.

To conclude that F is a rational transformation of $\tilde{F}$ of the referred type, we take into account its representation in continued fraction given in Theorem 2. To establish the uniqueness of $T_{(a,b;c,d)}$ we remind that the inverse of $T_{(a,b;c,d)}$, $ad - bc \neq 0$, is given by $T_{(a,-c;-b,d)}$. Therefore, if T_1 and T_2 are two linear fractional transformations such that $T_1(\tilde{F}) = T_2(\tilde{F})$, then the composition $T_2^{-1} \circ T_1$ satisfies $(T_2^{-1} \circ T_1)(\tilde{F}) = \tilde{F}$, and thus we obtain $T_2^{-1} \circ T_1 = id$, i.e., $T_1 = T_2$. Thus, the uniqueness of T is established. ■

5.3. Determination of the polynomial $\tilde{C}$. In what follows we determine the polynomial $\tilde{C}$ which defines $\{P_n\}$ given in (44).

Lemma 8. *Under the conditions of previous theorem, let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$, let $\tilde{C}$ be a polynomial which defines a weight $\tilde{w}$ given by (43), and $\tilde{F}$ the Carathéodory function associated with $\tilde{w}$. Let $T_{(\alpha_1,-\beta_1;-\alpha_2,\beta_2)}$, $\alpha_i, \beta_i \in \mathbb{P}$, $i = 1, 2$, $\alpha_1\beta_2 - \alpha_2\beta_1 \not\equiv 0$, such that $F = T(\tilde{F})$. Let us consider the first order linear differential equation for $\tilde{F}$,*

$$zA\tilde{F}' = \tilde{C}\tilde{F} + \tilde{D}\,,\ \tilde{D} \in \mathbb{P}\,. \tag{48}$$

Then, the following relations hold:

$$B = (\alpha_2\beta_2' - \alpha_2'\beta_2)zA + \alpha_2\beta_2\tilde{C} + \beta_2^2\tilde{D}\,, \tag{49}$$

$$C = (\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)zA + (\alpha_1\beta_2 + \alpha_2\beta_1)\tilde{C} + 2\beta_1\beta_2\tilde{D}\,, \tag{50}$$

$$D = (\alpha_1\beta_1' - \alpha_1'\beta_1)zA + \alpha_1\beta_1\tilde{C} + \beta_1^2\tilde{D}\,, \tag{51}$$

where we have considered, without lost of generality, $\alpha_2\beta_1 - \alpha_1\beta_2 = 1$.

Proof: Since $\tilde{w}'/\tilde{w} = \tilde{C}/(zA)$ (cf. (43)), then $\tilde{w}$ is semi-classical. Therefore, (48) is a consequence of Lemma 1.

Let us write $F = \dfrac{\alpha_1 - \beta_1\tilde{F}}{-\alpha_2 + \beta_2\tilde{F}}$, i.e., $\tilde{F} = \dfrac{\alpha_1 + \alpha_2 F}{\beta_1 + \beta_2 F}$. Using $\tilde{F} = \dfrac{\alpha_1 + \alpha_2 F}{\beta_1 + \beta_2 F}$ in (48), it follows that

$$zA(\alpha_2\beta_1 - \alpha_1\beta_2)F' = B_2F^2 + C_2F + D_2\,, \tag{52}$$

with

$$\begin{aligned}
B_2 &= (\alpha_2\beta_2' - \alpha_2'\beta_2)zA + \alpha_2\beta_2\tilde{C} + \beta_2^2\tilde{D}\,,\\
C_2 &= (\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)zA + (\alpha_1\beta_2 + \alpha_2\beta_1)\tilde{C} + 2\beta_1\beta_2\tilde{D}\,,\\
D_2 &= (\alpha_1\beta_1' - \alpha_1'\beta_1)zA + \alpha_1\beta_1\tilde{C} + \beta_1^2\tilde{D}\,.
\end{aligned}$$

Hence, F satisfies $zAF' = BF^2 + CF + D$ and (52), thus it follows that

$$\frac{zA(\alpha_2\beta_1 - \alpha_1\beta_2)}{zA} = \frac{B_2}{B} = \frac{C_2}{C} = \frac{D_2}{D}\,.$$

Therefore, if $\alpha_2\beta_1 - \alpha_1\beta_2 = 1$, then

$$B = B_2,\ C = C_2,\ D = D_2,$$

and (49)-(51) follow. ∎

According with Theorem 6, for each polynomial $\tilde{C}$ defining a weight $\tilde{w}$ by (43) and $\{P_n\}$ as in (44), there exists a unique linear fractional transformation T such that $F = T(\tilde{F})$, with $\tilde{F}$ the Carathéodory function associated with $\tilde{w}$. In this issue, we pose the question: being $\tilde{C}_1$ and $\tilde{C}_2$ polynomials (defining weights of the same type as in (43)) and $\tilde{F}_1$, $\tilde{F}_2$ the corresponding Carathéodory functions such that F is a linear fractional transformation of $\tilde{F}_i\,,\ i = 1, 2$, to obtain relations between $\tilde{C}_1$ e $\tilde{C}_2$. Next lemma gives us an answer.

Lemma 9. *Under the same conditions of previous lemma, let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D$. Let $\tilde{C}_1, \tilde{C}_2$ be polinomials defining semi-classical weights of the type (43), and let F_1 and F_2 be the corresponding Carathéodory functions, non rational, satisfying*

$$zAF_1' = \tilde{C}_1 F_1 + \tilde{D}_1\,, \tag{53}$$

$$zAF_2' = \tilde{C}_2 F_2 + \tilde{D}_2\,. \tag{54}$$

Let $T_1 = T_{(\alpha_1,-\beta_1;-\alpha_2,\beta_2)}\,,\ T_2 = T_{(\gamma_1,-\eta_1;-\gamma_2,\eta_2)}$ be the transformations such that $T_1(F_1) = F$, $T_2(F_2) = F$. If we assume, without loss of generality, that $\alpha_2\beta_1 - \alpha_1\beta_2 = 1,\ \gamma_2\eta_1 - \gamma_1\eta_2 = 1$, then the following relations take place:

$$\begin{aligned}(\alpha_2\beta_2' - \alpha_2'\beta_2)zA + \alpha_2\beta_2\tilde{C}_1 + \beta_2^2\tilde{D}_1& \\ = (\gamma_2\eta_2' - \gamma_2'\eta_2)zA &+ \gamma_2\eta_2\tilde{C}_2 + \eta_2^2\tilde{D}_2\,,\end{aligned} \tag{55}$$

$$\begin{aligned}&(\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)zA + (\alpha_1\beta_2 + \alpha_2\beta_1)\tilde{C}_1 + 2\beta_1\beta_2\tilde{D}_1 \\ &\quad = (\gamma_2\eta_1' + \gamma_1\eta_2' - \gamma_2'\eta_1 - \gamma_1'\eta_2)zA + (\gamma_1\eta_2 + \gamma_2\eta_1)\tilde{C}_2 + 2\eta_1\eta_2\tilde{D}_2\,,\end{aligned} \tag{56}$$

$$\begin{aligned}(\alpha_1\beta_1' - \alpha_1'\beta_1)zA + \alpha_1\beta_1\tilde{C}_1 + \beta_1^2\tilde{D}_1& \\ = (\gamma_1\eta_1' - \gamma_1'\eta_1)zA &+ \gamma_1\eta_1\tilde{C}_2 + \eta_1^2\tilde{D}_2\,.\end{aligned} \tag{57}$$

Proof: Since $F = T_1(F_1)$ with F_1 satisfying (53), from previous lemma we obtain

$$\begin{aligned}B &= (\alpha_2\beta_2' - \alpha_2'\beta_2)zA + \alpha_2\beta_2\tilde{C}_1 + \beta_2^2\tilde{D}_1\,,\\ C &= (\alpha_2\beta_1' + \alpha_1\beta_2' - \alpha_2'\beta_1 - \alpha_1'\beta_2)zA + (\alpha_1\beta_2 + \alpha_2\beta_1)\tilde{C}_1 + 2\beta_1\beta_2\tilde{D}_1\,,\\ D &= (\alpha_1\beta_1' - \alpha_1'\beta_1)zA + \alpha_1\beta_1\tilde{C}_1 + \beta_1^2\tilde{D}_1\,.\end{aligned}$$

Also, since $F = T_2(F_2)$ with F_2 satisfying (54), from previous lema we obtain

$$B = (\gamma_2\eta_2' - \gamma_2'\eta_2)zA + \gamma_2\eta_2\tilde{C}_2 + \eta_2^2\tilde{D}_2\,,$$
$$C = (\gamma_2\eta_1' + \gamma_1\eta_2' - \gamma_2'\eta_1 - \gamma_1'\eta_2)zA + (\gamma_1\eta_2 + \gamma_2\eta_1)\tilde{C}_2 + 2\eta_1\eta_2\tilde{D}_2\,,$$
$$D = (\gamma_1\eta_1' - \gamma_1'\eta_1)zA + \gamma_1\eta_1\tilde{C}_2 + \eta_1^2\tilde{D}_2\,.$$

Therefore, (55)-(57) follow. ∎

We now state the main result of this section, a representation formulae for $\{Y_n\}$, defined in (4), associated with a Caratéodory function F that satisfies $zAF' = BF^2 + CF + D$.

Theorem 7. *Let F be a Carathéodory function satisfying $zAF' = BF^2 + CF + D\,,\ A, B, C, D \in \mathbb{P}$, and let $\{Y_n\}$ be the corresponding sequence given by (4). Then, there exists a polynomial $\tilde{C}$ (defined by Lemmas 8 and 9), and a weight $\tilde{w}(z) = K\, e^{\int_{z_1}^{z} \frac{\tilde{C}}{tA} dt}\,,\ \ K \in \mathbb{C}\,,$ such that*

$$Y_n = \begin{bmatrix} \sqrt{\tilde{w}}\tilde{\phi}_n & -\tilde{Q}_n/\sqrt{\tilde{w}} \\ \sqrt{\tilde{w}}\tilde{\phi}_n^* & \tilde{Q}_n^*/\sqrt{\tilde{w}} \end{bmatrix} E_n\, L^{-1}\,,\ \ n \in \mathbb{N}\,,$$

where $\{\tilde{\phi}_n\}$ is the MOPS with respect to $\tilde{w}$, $\{\tilde{Q}_n\}$ is the sequence of functions of the second kind associated with $\{\tilde{\phi}_n\}$, E_n are the matrices defined in (36), and L is a fundamental matrix of (32).

Proof: These equations are a direct application of Theorem 6, namely formulae (35). ∎

6. Example

Let us consider the sequence of Jacobi orthogonal polynomials on the unit circle, $\{\phi_n\}$, with parameters $\alpha = \beta$, $\tilde{F}$ the corresponding Carathéodory function. Let $\{\Omega_n\}$ be the sequence of associated polynomials of the second kind and F be the corresponding Carathéodory function. F satisfies (see [3])

$$z(z^2-1)F'(z) = -2\alpha c_0(z^2-1)F^2(z) - 2\alpha(z^2+1)F(z)\,,$$

where c_0 is the moment of order zero of the Jacobi measure on the unit circle.

Taking into account Theorem 6, firstly we will solve the following differential systems:

$$z(z^2-1)L'(z) = \begin{bmatrix} -\alpha(z^2+1) & 0 \\ -2\alpha c_0(z^2-1) & \alpha(z^2+1) \end{bmatrix} L(z)\,, \tag{58}$$

$$z(z^2-1)P_n'(z) = \mathcal{B}_n(z)P_n(z)\,. \tag{59}$$

In what follows we consider a complex domain G such that $\{0, 1, -1\} \subsetneq G$, and a z_0 in G.

Lemma 10. *The fundamental matrix of solutions of (58) is given by*

$$L(z) = z^{-\alpha}(z^2-1)^{\alpha} \\ \times \begin{bmatrix} z^{2\alpha}(z^2-1)^{-2\alpha} & z^{2\alpha}(z^2-1)^{-2\alpha} \\ 1 - 2\alpha c_0 \int_{z_1}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha} dt & 1 - 2\alpha c_0 \int_{z_2}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha} dt \end{bmatrix}$$

with $z_1 \neq z_2$.

Now we obtain a solution of (59). Takin into account Theorem 4, henceforth we will consider $\tilde{C}$ as polynomial and we will solve (59) searching for a solution $\tilde{P}_n$ given by (44), $P_n = e^{\int_{z_1}^{z} \frac{\tilde{C}/2}{tA} dt} \begin{bmatrix} \tilde{\phi}_n & -\tilde{Q}_n/\tilde{w} \\ \tilde{\phi}_n^* & \tilde{Q}_n^*/\tilde{w} \end{bmatrix}, \forall n \in \mathbb{N}$, with $A = z^2 - 1$, $\{\tilde{\phi}_n\}$ the MOPS with respect to $\tilde{w}$, $\{\tilde{Q}_n\}$ the corresponding sequence of functions of the second kind, and $\tilde{w} = K e^{\int_{z_1}^{z} \frac{\tilde{C}}{tA} dt}$.

On the other hand, F is a linear fractional transformation of $\tilde{F}$ given by $F = 1/\tilde{F}$ (see, for example, [15, 16]), with $\tilde{F}$ satisfying (see [17])

$$z(z^2-1)\tilde{F}' = 2\alpha(z^2+1)\tilde{F} + 2\alpha c_0(z^2-1)\,.$$

Therefore, by Lemma 8, $\tilde{C} = 2\alpha(z^2+1)$ follows, and consequently we obtain $\tilde{w} = \left((z^2-1)/z\right)^{2\alpha}$.

From Theorem 8, the following representation for $Y_n = \begin{bmatrix} \phi_n & -\Omega_n \\ \phi_n^* & \Omega_n^* \end{bmatrix}$ holds, $\forall n \in \mathbb{N}$:

$$Y_n\, K = \begin{bmatrix} \tilde{\phi}_n & -\left((z^2-1)/z\right)^{-2\alpha} \tilde{Q}_n \\ (\tilde{\phi}_n)^* & \left((z^2-1)/z\right)^{-2\alpha} (\tilde{Q}_n)^* \end{bmatrix} E_n \\ \times \begin{bmatrix} 1 - 2\alpha c_0 \int_{z_2}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha} dt & -z^{2\alpha}(z^2-1)^{-2\alpha} \\ -1 + 2\alpha c_0 \int_{z_1}^{z} t^{2\alpha-1}(t^2-1)^{-2\alpha} dt & z^{2\alpha}(z^2-1)^{-2\alpha} \end{bmatrix},$$

where $K = 2\alpha c_0 \displaystyle\int_{z_1}^{z_2} t^{2\alpha-1}(t^2-1)^{-2\alpha}\, dt\,,\;\; E_n = (P_n(z_0))^{-1} Y_n(z_0) L(z_0)$.

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