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ON BOUNDEDNESS AND SMALL-ORTHOGONALITY CLASSES

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Dedicated to Jiří Adámek on the occasion of his sixtieth birthday

Abstract: A characterization of locally bounded categories and a criterion to identify small-orthogonality classes in these categories are given.

1. Introduction

In [11], P. Gabriel and F. Ulmer proved that in locally presentable categories the orthogonal subcategory $\mathcal{N}^\perp$ is reflective for any set $\mathcal{N}$ of morphisms. The key point of the proof is the fact that for any object of the base category there is some infinite regular cardinal α such that the object is α-small, where α-smallnes means α-presentability. In [10] and [15], P. Freyd and M. Kelly gave a generalization of this property for a wider range of categories, using a different concept of smallness for objects: boundedness. They showed that in a locally bounded category (as defined in [14] and [17]) the subcategory of all objects orthogonal to a set of morphisms is reflective. (In fact they went further: they proved that $\mathcal{N}^\perp$ is reflective for every class $\mathcal{N}$ which is the union of a set of morphisms with a class of epimorphisms.)

In a cocomplete category $\mathcal{A}$ an object A is said to be α-bounded if the hom-functor $\mathcal{A}(A,-)$ preserves α-directed unions. A locally bounded category (see [14]) is a complete and cocomplete category $\mathcal{A}$ with a proper factorization system $(\mathcal{E},\mathcal{M})$ and an $\mathcal{E}$-generator $\mathcal{G}$ such that (i) $\mathcal{A}$ has $\mathcal{E}$-cointersections and (ii) there is a regular cardinal α such that each object of $\mathcal{G}$ is α-bounded. We call these categories *locally α-bounded* when they are $\mathcal{E}$-cowellpowered and α is a regular cardinal which fits the condition (ii). Locally presentable categories and epi-reflective subcategories of the category of topological spaces are examples of locally bounded categories. We show that a cocomplete and cowellpowered category is locally bounded precisely when there is a regular cardinal α and a set $\mathcal{H}$ of α-bounded objects such

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that any object A of $\mathcal{A}$ is an α-directed union of objects of $\mathcal{H}$. This characterization will be useful in the study of small-orthogonality classes, that is, subcategories of the form $\mathcal{N}^\perp$ for $\mathcal{N}$ a set of morphisms.

In [13] the α-orthogonality classes of a locally α-presentable category were proved to be exactly the subcategories closed under limits and α-directed colimits, for all uncountable regular cardinals α. (Recall that, following [4], an α-orthogonality class is a subcategory of the form $\mathcal{N}^\perp$ for some set $\mathcal{N}$ of morphisms whose domains and codomains are α-presentable.) This characterization does not work for $\alpha = \aleph_0$, as it was shown in [20] and [12]. A description of the $\aleph_0$-orthogonality classes in locally finitely presentable categories in terms of closure properties was given in [5]: they are the subcategories $\mathcal{A}$ closed under products, directed colimits and $\mathcal{A}$-pure subobjects. In the context of locally bounded categories we shall adopt the terminology *α-orthogonality class* as expected: the meaning is as in [4], just replacing "presentable" by "bounded". The motive of this paper is to characterize the reflective subcategories of locally bounded categories which are small-orthogonality classes. In cowellpowered locally bounded categories a subcategory is a small-orthogonality class iff it is an α-orthogonality class for some α. We are going to restrict ourselves to reflective subcategories whose reflector preserves $\mathcal{M}$-monomorphisms. The properties of these reflectors were studied for instance by Ringel in [18]. We show that, in locally α-bounded categories, a reflective subcategory with an $\mathcal{M}$-preserving reflector is an α-orthogonality class iff it is closed under α-directed unions and α-$\mathcal{B}$-simple subobjects. (The notion of α-$\mathcal{B}$-simple morphism is parallel to the one of α-$\mathcal{B}$-pure morphism, used in [5]: If $\mathcal{B}$ is a subcategory of $\mathcal{A}$, a morphism $f : A \to B$ of $\mathcal{A}$ is said to be *α-$\mathcal{B}$-simple* provided that, if we have morphisms e, u and v such that $f \cdot u = v \cdot e$ and e is a $\mathcal{B}$-epimorphism, then there exists a morphism u' such that $u' \cdot e = u$.) For instance, the category $\mathbf{Top}_0$ of T_0-spaces is an $\aleph_0$-orthogonality class of $\mathbf{Top}$, but the category $\mathbf{Sob}$ of sober spaces is not an $\aleph_0$-orthogonality class of $\mathbf{Top}_0$; in both cases we have $\mathcal{M}$-preserving reflectors for $\mathcal{M} = \{\text{embeddings}\}$.

2. Properties of locally bounded categories

Let $\mathcal{A}$ be a category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ (where proper means that $\mathcal{E}$ and $\mathcal{M}$ consist of epimorphisms and monomorphisms, respectively). Recall that $\mathcal{E}$ and $\mathcal{M}$ determine each other: $\mathcal{E} = \mathcal{M}^\uparrow$ and $\mathcal{M} = \mathcal{E}^\downarrow$ ([10]).

A set $\mathcal{G}$ is said to be an $\mathcal{E}$*-generator* of $\mathcal{A}$ if for each object A there is some subset $\{G_i,\ i \in I\}$ of $\mathcal{G}$ and an $\mathcal{E}$-morphism $e : \coprod_{i\in I} G_i \to A$. (A detailed study of $\mathcal{E}$-generators is made in, e.g., [6] and [7].)

Let $m_i : A_i \to A,\ i \in I$, be a diagram in $\mathcal{A}$ with all $m_i \in \mathcal{M}$. The $\mathcal{M}$*-union* (or just *union*) of $(m_i)_{i\in I}$ is the supremum of $(m_i)_{i\in I}$, up to isomorphism, in the class of all $\mathcal{M}$-subobjects of A. It can be obtained through the $(\mathcal{E}, \mathcal{M})$-factorization of the canonical morphism $\coprod_{i\in I} A_i \to A$:

$$\begin{tikzcd}
A_i \arrow[r, "\nu_i"] \arrow[rrr, bend left, "m_i"] \arrow[drr, "n_i\in\mathcal{M}"'] & \coprod A_i \arrow[rr, dashed] \arrow[dr, "e\in\mathcal{E}"] & & A \\
& & B \arrow[ur, "m\in\mathcal{M}"'] &
\end{tikzcd}$$

We shall often write $\cup_{i\in I} m_i = m$ or $\cup_{i\in I} A_i = B$ for short.

Let α be an infinite regular cardinal. An object A is said to be α*-bounded* if the hom-functor $\mathcal{A}(A, -)$ preserves α-directed unions.

2.1. Definition (1) ([14], [17]) A category $\mathcal{A}$ is said to be *locally bounded* if it is cocomplete, has a proper factorization system $(\mathcal{E}, \mathcal{M})$, and there is an infinite regular cardinal α such that:

(i) $\mathcal{A}$ has $\mathcal{E}$-cointersections;

(ii) $\mathcal{A}$ has an $\mathcal{E}$-generator whose objects are all α-bounded.

(2) By a *locally α-bounded category with respect to $\mathcal{M}$* we shall mean a category under the conditions of (1), for a given α, which moreover is $\mathcal{E}$-cowellpowered. The mention to $\mathcal{M}$ will often be omitted.

2.2. Remark Every locally bounded category is complete. In [14] and [17], the authors include to be complete in the definition of locally bounded category. However the completeness comes for free, since any $\mathcal{E}$-cocomplete category with an $\mathcal{E}$-generator is complete. This follows from the fact that any such category is total (see [7]), that is, the Yoneda embedding $\mathcal{A} \hookrightarrow [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ has a left adjoint ([16]); and any total category is complete and $\mathcal{M}$-complete (see [7] and [8]).

2.3. Examples (1) Every locally presentable category is locally bounded with respect to monomorphisms, and also with respect to strong monomorphisms (see [10] and [2]).

(2) The category **Top** of topological spaces is locally $\aleph_0$-bounded with respect to strong monomorphisms (= embeddings). And every epi-reflective

subcategory of **Top** is locally $\aleph_0$-bounded with respect to embeddings. More generally, any $\mathcal{E}$-reflective subcategory $\mathcal{B}$ of a locally α-bounded category with respect to $\mathcal{M}$ is also locally α-bounded with respect to $\mathcal{M} \cap Mor(\mathcal{B})$ ([10], [2]).

(3) Any topological category over **Set** (see [3]) is locally $\aleph_0$-bounded with respect to strong monomorphisms.

(4) The category **Ban** of Banach spaces and linear contractions is locally $\aleph_1$-bounded ([14], [17]).

2.4. Remark The following properties are easily verified:

(i) In a locally bounded category for each object A there is an infinite regular cardinal α such that A is α-bounded ([10], 3.1.2).

(ii) In a cocomplete category if β and γ are regular cardinals such that $\beta \leq \gamma$, then any β-bounded object is also γ-bounded; consequently, the fulfillment of 2.1 for $\alpha = \beta$ ensures that it also holds for $\alpha = \gamma$.

2.5. Lemma *In a cocomplete category with a proper factorization system $(\mathcal{E}, \mathcal{M})$ any $\mathcal{E}$-quotient of an α-bounded object is α-bounded.*

Proof Let B be α-bounded, let $e : B \to E$ belong to $\mathcal{E}$ and let

$$C_i \xrightarrow{\quad n_i \quad} C \quad (i \in I)$$

be an α-directed $\mathcal{M}$-union, that is, $1_C = \cup_{i \in I} n_i$. Given $f : E \to C$, there are some i and some morphism $f' : B \to C_i$ such that $f \cdot e = n_i \cdot f'$. Then, since $n_i \in \mathcal{M}$ and $e \in \mathcal{M}^{\uparrow}$, there exists $f'' : E \to C_i$ such that $f = n_i \cdot f''$. $\square$

2.6. Remark The property stated in Lemma 2.5 contrasts to the case of α-presentability: a quotient of an α-presentable object is not necessarily α-presentable (see Remark 1.3 of [4]).

2.7. Lemma In a cocomplete category with a proper $(\mathcal{E}, \mathcal{M})$ factorization system:

(i) any α-small colimit of α-bounded objects is α-bounded;

(ii) any α-small union of α-bounded objects is α-bounded.

Proof (i) We are going to prove the statement for the particular case of coproducts. Then the result follows for colimits taking into account Lemma 2.5 and the fact that $\mathcal{M} \subseteq Mono$ implies that $RegEpi \subseteq \mathcal{E}$.

Let A_k $(k \in K)$ be an α-small set of α-bounded objects. Let $c_i : C_i \to C$ $(i \in I)$ be an α-directed union, and consider a morphism $d : \amalg_{k \in K} A_k \to C$.

Since every A_k is α-bounded, there are morphisms $f_k : A_k \to C_{i_k}$ such that $d \cdot \nu_k = c_{i_k} \cdot f_k$ for all k (where ν_k are the injections of the coproduct). Since K is α-small and I is α-directed, there is some $i \in I$ such that $i_k \leq i$, $k \in K$. Then, putting $g_k = (A_k \xrightarrow{f_k} C_{i_k} \longrightarrow C_i)$, we obtain $c_i \cdot g_k = d \cdot \nu_k$. Let $h : \amalg A_k \to C_i$ be the morphism determined by the morphisms g_k and the universality of the coproduct. Then we have $d = c_i \cdot h$.

(ii) Let $m_k : A_k \to A$ $(k \in K)$ be a union (not necessarily α-directed) with K α-small and all A_k α-bounded. Let $c_i : C_i \to C$ $(i \in I)$ be an α-directed union, and consider a morphism $f : A \to C$. Since $1_A = \cup_{k \in K} m_k$, the induced canonical morphism $e : \amalg A_k \to A$ belongs to $\mathcal{E}$. Put

$$d = f \cdot e$$

and let i and $h : \amalg A_k \to C_i$ be obtained as in (i). Then, we have the following commutative diagram:

$$\begin{tikzcd} \amalg A_k \arrow[r, "e"] \arrow[d, "h"'] & A = \cup A_k \arrow[d, "f"] \\ C_i \arrow[r, "c_i"] & C \end{tikzcd}$$

By the diagonal fill-in property, there exists a morphism $t : A \to C_i$ such that $c_i \cdot t = f$. □

2.8. Theorem *Let $\mathcal{A}$ be a cocomplete and $\mathcal{E}$-cowellpowered category with a proper factorization system $(\mathcal{E}, \mathcal{M})$. The following conditions are equivalent:*

(i) $\mathcal{A}$ is locally α-bounded with respect to $\mathcal{M}$.

(ii) There is a set $\mathcal{H}$ of α-bounded objects such that any object of $\mathcal{A}$ is an α-directed $\mathcal{M}$-union of objects of $\mathcal{H}$.

Proof (ii) ⇒ (i): It is clear that if $\mathcal{H}$ is a set as in (ii), then it is an $\mathcal{E}$-generator of $\mathcal{A}$. In fact, given $A \in \mathcal{A}$, let $H_i \xrightarrow{m_i} A$ $(i \in I)$ be an α-directed $\mathcal{M}$-union, with all H_i in $\mathcal{H}$. This exactly means that the induced canonical morphism $\amalg H_i \to A$ belongs to $\mathcal{E}$.

(i) ⇒ (ii): Let $\mathcal{G}$ be an $\mathcal{E}$-generator of $\mathcal{A}$ with all objects α-bounded. The class of objects

$$\mathcal{H} = \{ \mathcal{E}\text{-quotients of } \alpha\text{-small coproducts of objects of } \mathcal{G}\}$$

is essentially small, because $\mathcal{G}$ is small and $\mathcal{A}$ is $\mathcal{E}$-cowellpowered. Moreover, from 2.5 and 2.7, the objects of $\mathcal{H}$ are α-bounded. We show that $\mathcal{H}$ fulfils (ii).

Let $A \in \mathcal{A}$, and let

$$\{f_i : G_i \to A,\ i \in I\} = \bigcup_{G \in \mathcal{G}} \mathcal{A}(G, A)\,.$$

Let

$$\mathcal{J} = \{J \subseteq I \ :\ J \text{ is } \alpha\text{-small}\}.$$

Consider the following commutative diagram

$$\begin{tikzcd}
G_j \arrow[d, "\nu_j^J"'] \arrow[rrd, "f_j"] & & \\
G_J \arrow[r, "e_J"] \arrow[d, "\nu_J^K"'] & Q_J \arrow[r, "m_J"] \arrow[d, dashed, "d_J^K"] & A \\
G_K \arrow[r, "e_K"'] & Q_K \arrow[ru, "m_K"'] &
\end{tikzcd}$$

where:

- $G_J = \amalg_{j \in J} G_j$ and the morphisms ν_j^J are the corresponding injections;
- for each $J \subseteq K$, $\nu_J^K : G_J \to G_K$ is the obvious canonical morphism;
- $f_J : G_J \to A$ is the morphism determined by f_j, $j \in J$;
- $m_J \cdot e_J$ is the $(\mathcal{E}, \mathcal{M})$ factorization of $f_J : G_J \to A$;
- for each $J \subseteq K$, $d_J^K : Q_J \to Q_K$ is the morphism given by the diagonal fill-in property applied to the equality $(m_K \cdot e_K) \cdot \nu_J^K = m_J \cdot e_J$.

For $\mathcal{J}$ equipped with the inclusion order, both the diagrams

$$\big(\nu_J^K : G_J \to G_K\big)_{J \subseteq K,\, J,K \in \mathcal{J}} \qquad \text{and} \qquad \big(d_J^K : Q_J \to Q_K\big)_{J \subseteq K,\, J,K \in \mathcal{J}}$$

are α-directed. Moreover the colimit of the former one is $\amalg_{i \in I} G_i$. Let $\gamma_J : Q_J \to C = \operatorname{Colim} Q_J$ be the colimit cocone of the latter one. Then there is a morphism $e : \amalg_{i \in I} G_i \to C$ making the following diagram commutative.

$$\begin{tikzcd}
G_J \arrow[r, "e_J"] \arrow[d, "\nu_J"'] & Q_J \arrow[r, "m_J"] \arrow[d, "\gamma_J"] & A \\
\amalg_{i \in I} G_i \arrow[r, "e"'] & C \arrow[r, "e'"'] & \cup_{J \in \mathcal{J}} Q_J \arrow[u, "m'"']
\end{tikzcd}$$

The morphism e belongs to $\mathcal{E}$, since all e_J do. Let $m' \cdot e'$ be the $(\mathcal{E}, \mathcal{M})$ factorization of the canonical morphism from C to A determined by the morphisms m_J. By hypothesis, $m' \cdot (e' \cdot e) : \amalg_{i \in I} G_i \to A$ belongs to $\mathcal{E}$

(because $\mathcal{G}$ is an $\mathcal{E}$-generator). Consequently, m' lies in $\mathcal{E}$, and, since it also belongs to $\mathcal{M}$, is an isomorphism, that is, A is an union of the $\mathcal{M}$-subobjects

$$m_J : Q_J \to A, \ J \in \mathcal{J} \ . \qquad \square$$

2.9. Corollary *In $\mathcal{E}$-cowellpowered locally bounded categories for any infinite regular cardinal β the class of all β-bounded objects is essentially small.*

Proof Let $\mathcal{A}$ be locally α-bounded. Without loss of generality we assume that $\beta \geq \alpha$. Then $\mathcal{A}$ is also locally β-bounded and has a set $\mathcal{H}$ of β-bounded objects such that any object of $\mathcal{A}$ is an $\mathcal{M}$-union of objects of $\mathcal{H}$. Given a β-bounded object A let $m_i : H_i \to A \ \ (i \in I)$ be that existing union. The β-boundedness of A implies the equality $m_i \cdot t = 1_A$ for some $t : A \to H_i$. But then $A \simeq H_i$. $\square$

3. Small-orthogonality classes

In this section we study the following problem: When is a reflective subcategory[*] $\mathcal{B}$ of a locally bounded category $\mathcal{A}$ a *small-orthogonality class*, i.e., a category of the form $\mathcal{N}^{\perp}$, for $\mathcal{N}$ a set of morphisms? In this study we restrict ourselves to the particular case of the reflector $R : \mathcal{A} \to \mathcal{B}$ preserving $\mathcal{M}$-monomorphisms. More precisely, we characterize those reflective subcategories of a locally α-bounded category with an $\mathcal{M}$-preserving reflector which are of the form $\mathcal{N}^{\perp}$ with all morphisms of $\mathcal{N}$ having α-bounded domains and codomains.

In the case of locally presentable categories the subcategories of the form $\mathcal{N}^{\perp}$ for $\mathcal{N}$ a set of morphisms with α-presentable domains and codomains were characterized in [13] and [5] (see Introduction).

Along this section by an *α-orthogonality class* of a locally bounded category we shall mean a subcategory of the form $\mathcal{N}^{\perp}$ for some set $\mathcal{N}$ whose all morphisms have α-bounded domains and codomains. We borrow this terminology from [4] using boundedness in the place of presentability.

3.1. Remark Recall that, for $\mathcal{B}$ a subcategory of $\mathcal{A}$, a morphism $g : C \to D$ of $\mathcal{A}$ is said to be a $\mathcal{B}$-*epimorphism* if for any pair of morphisms a, $b : D \to B$ with $B \in \mathcal{B}$, the equality $a \cdot g = b \cdot g$ implies $a = b$.

Let $\mathcal{A} = \mathbf{Top}$. If $\mathcal{B} = \mathbf{Haus}$ the $\mathcal{B}$-epimorphisms are just the dense morphisms of $\mathbf{Top}$. If $\mathcal{B} = \mathbf{Top}_0$ the $\mathcal{B}$-epimorphisms are the b-dense morphisms, i.e., the continuous maps $f : X \to Y$ such that $\overline{\{y\}} \cap H \cap f(X) \neq \emptyset$ for each

[*]Along this paper all subcategories are assumed to be full and isomorphism-closed.

$y \in Y$ and each open set H of Y containing y. More generally, if $\mathcal{A}$ has equalizers and a proper factorization system $(\mathcal{E}, \mathcal{M})$, then for any subcategory $\mathcal{B}$ of $\mathcal{A}$ the $\mathcal{B}$-epimorphisms are the morphisms which are dense with respect to the regular closure operator induced in $\mathcal{A}$ by $\mathcal{B}$ ([9]).

If $\mathcal{B}$ is reflective in $\mathcal{A}$ it is easy to see that the $\mathcal{B}$-epimorphisms are just those morphisms of $\mathcal{A}$ whose image by the reflector is an epimorphism in $\mathcal{B}$.

3.2. Definition Let $\mathcal{A}$ be a locally bounded category and let $\mathcal{B}$ be a subcategory of $\mathcal{A}$. A morphism $f : A \to B$ of $\mathcal{A}$ is said to be *α-$\mathcal{B}$-simple* provided that in each commutative diagram

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "u"'] & D \arrow[d, "v"] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

with C and D α-bounded and g a $\mathcal{B}$-epimorphism, u factorizes through g, i.e., $u = u' \cdot g$ for some u'.

3.3. Remark It is easy to see that (compare with the properties of $\mathcal{B}$-pure morphisms in [5]):

(i) The composition of α-$\mathcal{B}$-simple morphisms is an α-$\mathcal{B}$-simple morphism.

(ii) If $f \cdot g$ is α-$\mathcal{B}$-simple than g is α-$\mathcal{B}$-simple.

(iii) Every γ-$\mathcal{B}$-simple morphism is α-$\mathcal{B}$-simple for $\gamma \geq \alpha$.

(iv) All α-$\mathcal{B}$-simple morphisms are monomorphisms; and every equalizer is an α-$\mathcal{B}$-simple morphism.

(v) If $\mathcal{B}$ is cogenerating in $\mathcal{A}$, then

$$StrongMono(\mathcal{A}) \subseteq \{\mathcal{B}\text{-}\alpha\text{-simple morphisms}\}.$$

It follows from the fact that, in this case, every $\mathcal{B}$-epimorphism is an epimorphism in $\mathcal{A}$.

3.4. Proposition *Let $\mathcal{A}$ be a locally α-bounded category with respect to $\mathcal{M}$. Then any α-orthogonality class of $\mathcal{A}$ is a reflective subcategory of $\mathcal{A}$ which is*

(i) closed under α-directed $\mathcal{M}$-unions;

(ii) locally α-bounded with respect to $\mathcal{M}' = \mathcal{M} \cap Mor(\mathcal{B})$;

(iii) closed under α-$\mathcal{B}$-simple subobjects.

Proof Let $\mathcal{B} = \mathcal{N}^{\perp}$ for $\mathcal{N}$ a set of morphisms in $\mathcal{A}$ with α-bounded domains and codomains. From [10], we know that $\mathcal{B}$ is reflective and has an $(\mathcal{E}', \mathcal{M}')$

proper factorization system, with $\mathcal{E}' = (\mathcal{M}')^{\uparrow}$, and moreover that the cowellpoweredness of $\mathcal{A}$ with respect to $\mathcal{E}$ implies the $\mathcal{E}'$-cowellpoweredness of $\mathcal{B}$. Let $R : \mathcal{A} \to \mathcal{B}$ be the reflector functor.

(i) Let

$$b_i : B_i \to Z \qquad (i \in I)$$

be an α-directed $\mathcal{M}$-union in $\mathcal{A}$ with all $B_i \in \mathcal{B}$. We want to show that $Z \in \mathcal{B} = \mathcal{N}^{\perp}$. Let $h : X \to Y$ be a morphism of $\mathcal{N}$ and let $f : X \to Z$. Since X is α-bounded there is some i and some $f' : X \to B_i$ such that $b_i \cdot f' = f$. The morphism f' factorizes through h, because $B_i \in \mathcal{B}$, then so does the morphism f. To show the uniqueness of the last factorization, let y, $y' : Y \to Z$ be such that $y \cdot h = y' \cdot h$. Since Y is α-bounded, we can find $k \in I$ and t, $t' : Y \to B_k$ such that $y = b_k \cdot t$ and $y' = b_k \cdot t'$. Now the equality $b_k \cdot t \cdot h = b_k \cdot t' \cdot h$, the orthogonality of B_k to h and the fact that $b_k \in \mathcal{M}$ imply that $t = t'$, thus $y = y'$.

(ii) Of course $\mathcal{B}$ is cocomplete. Moreover:

(a) If X is an α-bounded object of $\mathcal{A}$, then RX is an α-bounded object of $\mathcal{B}$. This is clear since, from (i), every α-directed $\mathcal{M}'$-union in $\mathcal{B}$ is an α-directed $\mathcal{M}$-union in $\mathcal{A}$.

(b) If $\mathcal{G}$ is an $\mathcal{E}$-generator of $\mathcal{A}$ then it is well-know that $R(\mathcal{G})$ is an $\mathcal{E}'$-generator of $\mathcal{B}$ ([10]). In fact, let $A \in \mathcal{B}$, and let $e : \amalg_{i \in I} G_i \to A$ be a morphism of $\mathcal{E}$ with all G_i in $\mathcal{G}$. Then the morphism $Re : \amalg_{i \in I} RG_i \to A$ belongs to $\mathcal{E}'$ since, as it is easily seen, $R(\mathcal{E}) \subseteq (\mathcal{M}')^{\uparrow}$.

(iii) Let $m : Z \to B$ be an α-$\mathcal{B}$-simple morphism with $B \in \mathcal{B}$. We want to show that $Z \in \mathcal{B}$. Let $h : X \to Y$ lay in $\mathcal{N}$. Given a morphism $f : X \to Z$, since $B \in \mathcal{N}^{\perp}$, we get f' such that $f' \cdot h = m \cdot f$. Because m is α-$\mathcal{B}$-simple, there is f'' such that $f'' \cdot h = f$. The uniqueness of f'' follows from the fact that $m \cdot f$ factors uniquely through h and m is a monomorphism. □

3.5. Remark Let $\mathcal{A}$ be a locally α-bounded category with respect to $\mathcal{M}$. Let $\mathcal{B}$ be a subcategory of $\mathcal{A}$ which is locally α-bounded with respect to $\mathcal{M} \cap Mor(\mathcal{B})$ and closed under limits and under α-directed $\mathcal{M}$-unions. Then $\mathcal{B}$ is reflective. In fact, the inclusion functor $\mathcal{B} \hookrightarrow \mathcal{A}$ fulfils the solution set condition: Given $A \in \mathcal{A}$, there is some regular cardinal $\lambda \geq \alpha$ such that A is λ-bounded in $\mathcal{A}$ and $\mathcal{B}$ is a locally λ-bounded category. Consequently, there is a set $\{B_i,\, i \in I\}$ of λ-bounded objects of $\mathcal{B}$ such that every object of $\mathcal{B}$ is a λ-directed $\mathcal{M} \cap Mor(\mathcal{B})$-union of B_i's. But, being closed in $\mathcal{A}$

under α-directed unions, $\mathcal{B}$ is also closed under λ-directed unions. Then, any morphism $g : A \to B$ with codomain in $\mathcal{B}$ factorizes through some of the objects B_i.

Next we want to characterize the reflective subcategories of a locally bounded category which are small-orthogonality classes. We restrict ourselves to reflective subcategories whose reflector preserves $\mathcal{M}$-monomorphisms. This kind of reflectors were studied by Ringel in [18]. $\mathbf{Top_0}$ and **Sob** are examples of subcategories of **Top** whose reflector preserves embeddings. Let $\mathbf{Sob}_\alpha$ denote the limit-closure in **Top** of the ordinal α regarded as a topological space with the Alexandrov topology. Both **Top** and $\mathbf{Top_0}$ have an {embeddings}-preserving reflector into $\mathbf{Sob}_\alpha$ (see [19]).

3.6. Theorem *Let $\mathcal{A}$ be a locally α-bounded category with respect to $\mathcal{M}$. Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}$ whose reflector preserves morphisms of $\mathcal{M}$. Then $\mathcal{B}$ is an α-orthogonality class in $\mathcal{A}$ iff it is closed under α-directed $\mathcal{M}$-unions and α-$\mathcal{B}$-simple objects.*

Proof The necessity was proved in 3.4.

In order to prove the sufficiency, we first show that the reflector $R : \mathcal{A} \to \mathcal{B}$ preserves α-directed $\mathcal{M}$-unions. Given an α-directed $\mathcal{M}$-union $m_i : X_i \to X$ $(i \in I)$, we have commutative diagrams

$$\begin{tikzcd}
X_i \arrow[rrr, "r_{X_i}"] \arrow[dr, "\nu_i"] \arrow[dd, "m_i"'] & & & RX_i \arrow[dl, "R\nu_i"'] \arrow[dd, "Rm_i"] \\
& \amalg_{i\in I} X_i \arrow[r, "r"] \arrow[dl, "e"] & \amalg_{i\in I} RX_i \arrow[dr, "Re"'] & \\
X \arrow[rrr, "r_X"'] & & & RX
\end{tikzcd}$$

where $e \in \mathcal{E}$. But, as it is easy to see, $R(\mathcal{E}) \subseteq \mathcal{E}' = (\mathcal{M}')^{\uparrow}$ for $\mathcal{M}' = \mathcal{M} \cap Mor(\mathcal{B})$. Then $Rmi : RX_i \to RX$ forms an $\mathcal{M}'$-union in $\mathcal{B}$.

To finish the proof, we show that, for

$$\mathcal{N} = \{h : X \to Y \text{ in } \mathcal{A},\ h \perp \mathcal{B},\ X, Y\ \alpha\text{-bounded}\},$$

$\mathcal{N}^\perp \subseteq \mathcal{B}$, and thus $\mathcal{B} = \mathcal{N}^\perp$. Let $X \in \mathcal{N}^\perp$. We show that the reflection $r_X : X \to RX$ of X in $\mathcal{B}$ is α-$\mathcal{B}$-simple; consequently, as $\mathcal{B}$ is closed under α-$\mathcal{B}$-subobjects, $X \in \mathcal{B}$. Let $f : Y \to Z$ be a $\mathcal{B}$-epimorphism with Y and Z α-bounded. Given morphisms $s : Y \to X$ and $t : Z \to RX$ such that

$t \cdot f = r_X \cdot s$, let $m_i : X_i \to X$ be an α-directed $\mathcal{M}$-union in $\mathcal{A}$ with all X_i α-bounded. Then there is some $i \in I$ and $s' : Y \to X_i$ such that $m_i \cdot s' = s$. The closedness of $\mathcal{B}$ under α-directed $\mathcal{M}$-unions and the fact that Z is α-bounded implies the existence of some $j \in I$ and a morphism $t' : Z \to RX_j$ such that $Rm_j \cdot t' = t$. Since I is α-directed, we can then find $k \in I$ and morphisms $\overline{s}$ and $\overline{t}$ such that the following diagram is commutative (the commutativity of the upper quadrilateral derives from the fact that Rm_k is monic):

$$\begin{tikzcd}
Y \arrow[rrr, "f"] \arrow[dr, "\overline{s}"] \arrow[dd, "s"'] & & & Z \arrow[dl, "\overline{t}"'] \arrow[dd, "t"] \\
& X_k \arrow[r, "r_{X_k}"] \arrow[dl, "m_k"] & RX_k \arrow[dr, "Rm_k"'] & \\
X \arrow[rrr, "r_X"'] & & & RX
\end{tikzcd}$$

Let $X_k \xrightarrow{f'} W \xleftarrow{s'} Z$ be the pushout of f along $\overline{s}$. Since $r_{X_k} \perp \mathcal{B}$, any morphism $g : X_k \to B$ with $B \in \mathcal{B}$ is factorizable through f'. Furthermore, as it is easily seen, the pushout of a $\mathcal{B}$-epimorphism is also a $\mathcal{B}$-epimorphism. Hence $f' \perp \mathcal{B}$. The domain of f' is α-bounded, and from Lemma 2.7, also its codomain is α-bounded, then $f' \in \mathcal{N}$. Hence there is a morphism $n : W \to X$ such that $n \cdot f' = m_k$. Therefore, $n \cdot s'$ is the needed diagonal morphism, since $(n \cdot s') \cdot f = n \cdot f' \cdot \overline{s} = m_k \cdot \overline{s} = s$. □

3.7. Examples (1) The category $\mathbf{Top}_0$ is an $\aleph_0$-orthogonality class in **Top**. In fact $\mathbf{Top}_0 = \{h\}^{\perp}$ where h is the map $h : \{0, 1\} \to \{0\}$, considering the two-elements set with the trivial topology.

(2) **Sob** is not an $\aleph_0$-orthogonality class in $\mathbf{Top}_0$. It follows from the above theorem taking into account that **Sob** is not closed under $\aleph_0$-**Sob**-simple subobjects.

To see that, we show that every **Sob**-epimorphism $e : X \to Y$ with X and Y finite is a surjection. (We recall that the **Sob**-epimorphisms of $\mathbf{Top}_0$ are the b-dense morphisms, see 3.1.) Let $y \in Y$, let $\{H_i,\, i \in I\}$ be the set of all open neighbourhoods of y, and put $H = \bigcap_{i \in I} H_i$. Since I is finite, H is an open containing y, and, then, $H \cap e(X) \cap \overline{\{y\}} \neq \emptyset$. Let y' be an element of that intersection. Thus $\overline{\{y'\}} \subseteq \overline{\{y\}}$. But for all H_i we have $y' \in H_i$, hence $\overline{\{y\}} = \overline{\{y'\}}$. Since $Y \in \mathbf{Top}_0$, we conclude that $y = y'$, then $y \in e(X)$.

As a consequence we have that

$$\{embeddings\} \subseteq \{\ \aleph_0\text{-}\mathbf{Sob}\text{-simple morphisms}\}.$$

But then, if **Sob** were closed under $\aleph_0$-**Sob**-subobjects, it would be also closed under embeddings, what is obviously false (since the reflections are embeddings).

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