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COVARIANT DIFFERENTIATION UNDER ROLLING MAPS

FÁTIMA SILVA LEITE AND KRZYSZTOF ANDRZEJ KRAKOWSKI

Abstract: We study properties of covariant derivatives of vector fields along curves on Riemannian manifolds mapped by rolling the manifolds without slip or twist. A natural definition of Riemannian polynomials arises from these properties.

1. Introduction

By the Whitney's Theorem [1] every Riemannian manifold can be embedded in Euclidean space $\mathbb{R}^n$, with large enough n. This paper studies two manifolds embedded in the same Euclidean space, with one manifold rolling on the other without slip or twist. The operation of rolling reveals geometric properties of the embedded manifolds that have been successfully used in deriving interpolating curves [3] and in optimal control problems [4]. Cases of rolling symmetric spaces are studied in [3] and [2].

Here we assume the definition of a rolling map as in Sharpe [5] and derive interesting properties of covariant derivatives of vector fields along rolling curves. As a consequence, Riemannian polynomials are shown to be invariant under the operation of rolling without twist or slip. This generalizes the well known result that a rolling curve is a geodesic if and only if its development is also a geodesic [5].

The paper is organized as follows. The definition of a rolling map, as in Sharpe [5], appears in Section 2. Here we also unreveal some properties of rolling without slip or twist and present the intuitive example of the unit sphere rolling on a hyperplane. Main results related to covariant derivatives are presented in Section 3. Section 4 concludes this note.

2. Rolling Manifolds

The operation of rolling manifolds is defined in [5] as an isometry in the ambient Euclidean space. We shall confine ourselves with the case where the rolling is an element of the Euclidean group, *cf.* [3].

Consider a manifold $\mathbf{M}_1$ rolling on a manifold $\mathbf{M}_0$, where both $\mathbf{M}_1$ and $\mathbf{M}_0$ are isometrically embedded in $\mathbb{R}^n$, i.e., $\mathbf{M}_1 \overset{\imath}{\hookrightarrow} \mathbb{R}^n$ with the pullback metric

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on $\mathbf{M}_1$ given by $g \stackrel{\text{def}}{=} \imath^*\bar{g}$, where $\bar{g}$ is the standard Euclidean metric (and similarly for $\mathbf{M}_0$). For the definition of rolling we consider one-parameter families $h\colon I \to \mathbb{SE}(n)$, where $I \subset \mathbb{R}$ is an interval of the real line and $h(t)\colon \mathbb{R}^n \to \mathbb{R}^n$ is an action of the Euclidean group $\mathbb{SE}(n) = \mathbb{SO}(n) \ltimes \mathbb{R}^n$ in the ambient space. The elements of $\mathbb{SE}(n)$ will be represented by pairs (R, s), acting on points in $\mathbb{R}^n$ accordingly, i.e., $(R, s)\,(p) = R\,p + s$, with the group operations

$$(R_2, s_2) \circ (R_1, s_1) \stackrel{\text{def}}{=} (R_2R_1, R_2s_1 + s_2) \quad \text{and} \quad (R, s)^{-1} \stackrel{\text{def}}{=} \left(R^{\mathrm{T}}, -R^{\mathrm{T}}s\right).$$

Definition 1. *The map $h\colon I \to \mathbb{SE}(n)$ is called a rolling of $\mathbf{M}_1$ on $\mathbf{M}_0$ without slip or twist if h satisfies the following properties:*

rolling: *there exists a piecewise smooth curve $\sigma_1\colon I \to \mathbf{M}_1$ (called the rolling curve) such that, for all $t \in I$,*

(i) $\sigma_0(t) \stackrel{\text{def}}{=} h(t)(\sigma_1(t)) \in \mathbf{M}_0$;

(ii) $\mathcal{T}_{h(t)(\sigma_1(t))}\left(h(t)(\mathbf{M}_1)\right) = \mathcal{T}_{h(t)(\sigma_1(t))}(M_0)$.

The curve $\sigma_0\colon I \to \mathbf{M}_0$ is called the development of σ_1 on $\mathbf{M}_0$.

no-slip: $\left(\dot{h}(t)\,h^{-1}(t)\right)(\sigma_0(t)) = 0$, *for all $t \in I$.*

no-twist: *a pair of conditions, for all $t \in I$,*

(i) the tangential part $\left(\dot{h}(t)\,h^{-1}(t)\right)_*(\mathcal{T}_{\sigma_0(t)}M_0) \subset \left(\mathcal{T}_{\sigma_0(t)}M_0\right)^{\perp}$;

(ii) the normal part $\left(\dot{h}(t)\,h^{-1}(t)\right)_*(\mathcal{T}_{\sigma_0(t)}M_0)^{\perp} \subset \mathcal{T}_{\sigma_0(t)}M_0$.

Remark 2. In the definition above, $\dot{h}(t)\,h^{-1}(t)$ is understood as a mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, mapping p to $\dot{R}(t)\,R^{-1}(t)\,(p - s(t)) + \dot{s}(t)$, and $\left(\dot{h}(t)\,h^{-1}(t)\right)_*$ is the corresponding push-forward mapping.

A straightforward calculation then shows that the no-slip condition has an equivalent formulation as

$$\dot{R}(t)\,R^{-1}(t)(\sigma_0(t) - s(t)) = -\dot{s}(t), \tag{1}$$

and, similarly, the tangential and normal parts of the no-twist condition may be rewritten, respectively, as

$$\dot{R}(t)\,R^{-1}(t)(\mathcal{T}_{\sigma_0(t)}M_0) \subset \left(\mathcal{T}_{\sigma_0(t)}M_0\right)^{\perp}; \tag{2}$$

$$\dot{R}(t)\, R^{-1}(t)(\mathcal{T}_{\sigma_0(t)}M_0)^{\perp} \subset \mathcal{T}_{\sigma_0(t)}M_0. \tag{3}$$

Remark 3. In physics, the term ”no-slip” is used in the context of rolling surfaces as the property that both surfaces have the same velocity at the point of contact. In our context, this translates as

$$\dot{\sigma}_0(t) = h(t)\dot{\sigma}_1(t).$$

Taking derivatives on both sides of the equality $\sigma_0(t) = h(t)(\sigma_1(t))$, it is easy to conclude that

$$\dot{\sigma}_0(t) = h(t)\dot{\sigma}_1(t) \Leftrightarrow \left(\dot{h}(t)\, h^{-1}(t)\right)(\sigma_0(t)) = 0,$$

which explains the name given to the last condition in the definition of rolling.

We end this section with two parcial results and an illustrative example, the unit sphere S^n rolling on a hyperplane of $\mathbb{R}^{n+1}$.

Lemma 4. *If* $\left(\mathcal{T}_{\sigma_0(t)}\mathbf{M}_0\right)^{\perp}$ *is one dimensional, then the condition (ii) of the no-twist is always satisfied.*

Proof: Since $R \in \mathbb{SO}(n)$, $A(t) = \dot{R}(t)\, R^{-1}(t) \in \mathfrak{so}(n)$, i.e., $A(t)$ is skew-symmetric. Consequently, $\langle A(t)v, v\rangle = 0$, for any v in the normal subspace $\left(\mathcal{T}_{\sigma_0(t)}\mathbf{M}_0\right)^{\perp}$. Hence $A(t)v \in \mathcal{T}_{\sigma_0(t)}\mathbf{M}_0$. *Q.E.D.*

Lemma 5. *If* $h(t) = (R(t), s(t)) \in \mathbb{SE}(n)$ *satisfies* $h(t)\, p_0 = p_0$*, for some* $p_0 \in \mathbf{M}_1$*, and* $s(t)$ *is orthogonal to* p_0*, then* $s(t) = 0$*,* $R(t)\, p_0 = p_0$*, and, consequently,* h *is not a rolling map.*

Proof: Since $h(t)$ fixes the point p_0, we have

$$h(t)\, p_0 = R(t)\, p_0 + s(t) = p_0, \quad \text{hence} \quad R(t)\, p_0 = p_0 - s(t).$$

Taking squares of both sides of the above equality and using the orthogonality condition between $s(t)$ and p_0, it follows by the Pythagoras theorem that

$$\|p_0\|^2 = \|R(t)\, p_0\|^2 = \|p_0 - s(t)\|^2 = \|p_0\|^2 + \|s(t)\|^2 .$$

Consequently, $s(t) = 0$ and $R(t)\, p_0 = p_0$. *Q.E.D.*

Example 6 (the unit sphere)**.** Consider the unit sphere S^n rolling on the affine tangent space at a point $p_0 \in S^n$, cf [3]. Now, $\mathbf{M}_1 = \mathbf{S}^n \subset \mathbb{R}^{n+1}$ and $\mathbf{M}_0$ is defined as

$$M_0 = \left\{\, x \in \mathbb{R}^{n+1} \ : \ x = p_0 + \Omega\, p_0 \quad \text{and} \quad \Omega \in \mathfrak{so}(n+1) \,\right\}.$$

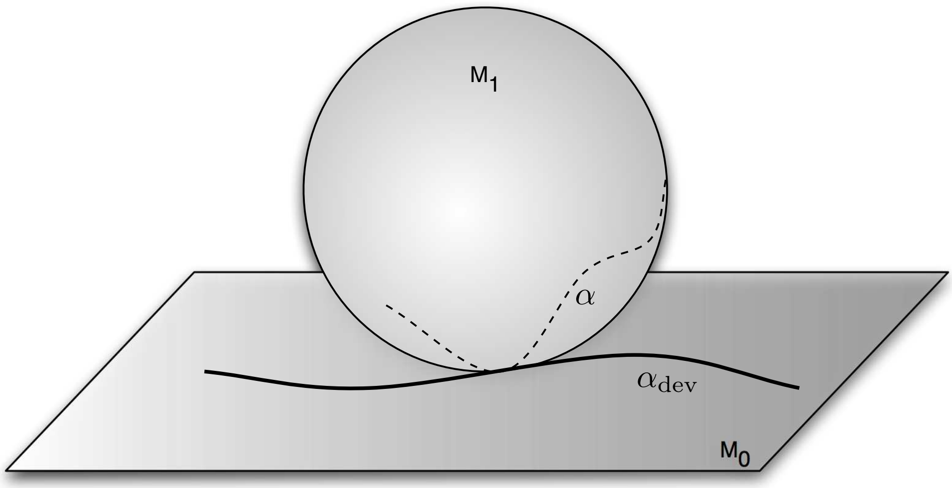

FIGURE 1. Rolling the unit sphere

Let $h = (R, s)$ be a rolling map satisfying $h(0) = (I, 0)$ and σ_1 the rolling curve starting at point p_0, i.e., $\sigma_1(0) = p_0$. (In Figure 1., α is the rolling curve σ_1 and α_{dev} stands for its development σ_0).

Rolling: Since $h(t)$ sends the origin to $0+s(t)$, $s(t)$ gives the coordinates of the center of the sphere $h(t)S^n$ which is tangent to M_0 at the point $\sigma_0(t)$ of the developed curve. This follows from condition ii) of rolling. Consequently, $\sigma_0(t) = p_0 + s(t)$. If, in addition, we impose condition i) of rolling, it follows that the rolling curve is given by $\sigma_1(t) = R^{\mathrm{T}}(t)\, p_0$.

No-slip: Also, since $\sigma_0(t) = p_0 + s(t)$, the non-slip condition (1), for the sphere, is equivalent to

$$\dot{R}(t)R^{\mathrm{T}}(t)\, p_0 = -\dot{s}(t). \tag{4}$$

Note that $A(t) = \dot{R}(t)R^{\mathrm{T}}(t) \in \mathfrak{so}(n+1)$.

It is convenient to change the basis with $Q \in O(n+1)$ such that $Q\, p_0 = -e_{n+1}$ (the south pole). Under this change of basis, the no-slip condition $A(t)\, p_0 = -\dot{s}(t)$ is equivalent to $Q\, A(t)\, Q^{\mathrm{T}}\, e_{n+1} = Q\, \dot{s}(t)$. From here we can conclude that $QA(t)Q^{\mathrm{T}}$ has the following block structure, for some $n \times n$ skew-symmetric matrix Ω.

$$Q\, A(t)\, Q^{\mathrm{T}} = \left[\begin{array}{c|c} \Omega(t) & Q\, \dot{s}(t) \\ \hline -(Q\, \dot{s}(t))^{\mathrm{T}} & 0 \end{array}\right].$$

No-twist: Since $(\mathcal{T}_{\sigma_0(t)}M_0)^\perp$ is spanned by p_0, Lemma 4 applies and the normal part is satisfied. Now, we show that the tangencial part holds only if the skew-symmetric matrix $A(t) = \dot{R}(t)R^{\mathrm{T}}(t)$ is given by

$$A(t) = p_0\,(\dot{s}(t))^{\mathrm{T}} - \dot{s}(t)\,(p_0)^{\mathrm{T}}.$$

Under the above change of basis, the tangencial part of the no-twist condition implies that the diagonal blocks of $Q\,A(t)\,Q^{\mathrm{T}}$ are zero and, consequently the rank two matrix $Q\,A(t)\,Q^{\mathrm{T}}$ can be written as $(Q\,\dot{s}(t))\,e_{n+1}^{\mathrm{T}} - e_{n+1}\,(Q\,\dot{s}(t))^{\mathrm{T}}$. So,

$$\begin{aligned}
Q\,A\,Q^{\mathrm{T}} &= Q\,\dot{s}\,e_{n+1}^{\mathrm{T}} - e_{n+1}\,(Q\,\dot{s})^{\mathrm{T}} \\
&= Q\,\dot{s}\,e_{n+1}^{\mathrm{T}}\,Q\,Q^{\mathrm{T}} - Q\,Q^{\mathrm{T}}\,e_{n+1}\,\dot{s}^{\mathrm{T}}\,Q^{\mathrm{T}} \\
&= -Q\,\dot{s}\,p_0^{\mathrm{T}}\,Q^{\mathrm{T}} + Q\,p_0\,\dot{s}^{\mathrm{T}}\,Q^{\mathrm{T}} \\
&= Q\,(p_0\,\dot{s}^{\mathrm{T}} - \dot{s}\,p_0^{\mathrm{T}})\,Q^{\mathrm{T}}.
\end{aligned}$$

This is in accordance with what can be found in the vast literature about the rolling sphere.

3. Covariant Differentiation and Rolling

In this section we consider a manifold $\mathbf{M}_1$ rolling on a hyperplane $\mathbf{M}_0 \simeq \mathbb{R}^m$, with $m = n-1$.* Both $\mathbf{M}_1$ and $\mathbf{M}_0$ are isometrically embedded in $\mathbb{R}^n$, i.e., $\mathbf{M}_1 \overset{\imath}{\hookrightarrow} \mathbb{R}^n$ with the pullback metric on $\mathbf{M}_1$ given by $g \overset{\text{def}}{=} \imath^*\bar{g}$, where $\bar{g}$ is the standard Euclidean metric. Because the embedding is isometric the covariant derivative ∇ on $\mathbf{M}_1$ satisfies $\nabla_X Y = \pi^\top\left(\overline{\nabla}_X Y\right)$, where $\overline{\nabla}$ is the standard Euclidean connection defined by $\overline{\nabla}_X\left(Y^j\partial_j\right) \overset{\text{def}}{=} \left(XY^j\right)\partial_j$ and $\pi^\top\colon \mathcal{T}_p\mathbb{R}^n \to \mathcal{T}_p\mathbf{M}_1$ is the orthogonal projection. The development curve σ_0 on $\mathbf{M}_0$ is the image under the rolling without slip or twist of the rolling curve σ_1 on $\mathbf{M}_1$, i.e.,

$$\sigma_0\colon I \to \mathbf{M}_0 \quad \text{and} \quad \sigma_1\colon I \to \mathbf{M}_1, \quad \text{where} \quad h(t)(\sigma_1(t)) = \sigma_0(t).$$

Let $v_1\colon I \to \mathcal{T}\mathbf{M}_1$ be a vector field along σ_1, where $v_1(t) \in \mathcal{T}_{\sigma_1(t)}\mathbf{M}_1$. Then there exists a corresponding vector field v_0 along σ_0, given by

$$v_0(t) = h_*(t)(v_1(t)) = R(t)(v_1(t)). \tag{5}$$

*Note that the roles of $\mathbf{M}_1$ and $\mathbf{M}_0$ in [5, Proposition 3.7] are reversed but the above follows the convention in the definition of rolling.

Differentiating (5) with respect to t yields

$$\dot{v}_0(t) = \dot{R}(t)(v_1(t)) + R(t)(\dot{v}_1(t)) = \dot{R}(t)\left(R^{-1}(t)(v_0(t))\right) + R(t)(\dot{v}_1(t)). \quad (6)$$

Since $\mathbf{M}_0 \simeq \mathbb{R}^m$, then the covariant derivative $\overline{\nabla}^0$ in $\mathbf{M}_0$ is simply the usual derivative, i.e.,

$$\overline{\nabla}^0_{\dot{\sigma}_0(t)} v_0(t) = \dot{v}_0(t).$$

Remark 7. We recall that, since the ambient space $\mathbb{R}^n$ may be decomposed as a direct sum of a vector subspace and its orthogonal complement, then for any manifold M embedded in $\mathbb{R}^n$ and any point $p \in M$ the following decomposition holds

$$\mathbb{R}^n = \mathcal{T}_p\mathbf{M} \oplus (\mathcal{T}_p\mathbf{M})^\perp.$$

We have the following three results related to covariant derivatives on $\mathbf{M}_0$ and $\mathbf{M}_1$ along the curves σ_0 and σ_1 respectively.

Lemma 8. *Let $v_1 \colon I \to \mathcal{T}_{\sigma_1(t)}\mathbf{M}_1$ be a vector field along $\sigma_1 \colon I \to \mathbf{M}_1$ and v_0 be the corresponding vector field along the development curve $\sigma_0 \colon I \to \mathbf{M}_0$. Then, the covariant derivative of v_1 (along σ_1) is given by*

$$\nabla_{\dot{\sigma}_1(t)} v_1(t) = R^{-1}(t)(\dot{v}_0(t)) \in \mathcal{T}_{\sigma_1(t)}\mathbf{M}_1. \quad (7)$$

Proof: By (6) it follows that $R(t)(\dot{v}_1(t))$ uniquely decomposes into the sum of two orthogonal vectors in the ambient space $\mathbb{R}^n$

$$R(t)(\dot{v}_1(t)) = \dot{v}_0(t) - \dot{R}(t)\left(R^{-1}(t)(v_0(t))\right),$$

where $\dot{v}_0(t) \in \mathcal{T}_{\sigma_0(t)}\mathbf{M}_0$ and $\dot{R}(t)\left(R^{-1}(t)(v_0(t))\right) \in \left(\mathcal{T}_{\sigma_0(t)}\mathbf{M}_0\right)^\perp$. Applying the push-forward of the inverse of $h(t)$ to both sides of the last equality yields

$$h_*^{-1}(t)\left(R(t)(\dot{v}_1(t))\right) = \dot{v}_1(t) = R^{-1}(t)\left(\dot{v}_0(t)\right) - R^{-1}(t)\left(\dot{R}(t)\left(R^{-1}(t)(v_0(t))\right)\right).$$

Since R preserves the metric $\bar{g}$ of the ambient space and the metric g on $\mathbf{M}_1$ is induced by $\bar{g}$, orthogonality of vectors is also preserved by R. From the "rolling" condition we have $\mathcal{T}_{\sigma_0(t)}\mathbf{M}_0 = \mathcal{T}_{h(t)(\sigma_1(t))}(h(t)\mathbf{M}_1)$. Therefore, the above equality is the unique decomposition of $\dot{v}_1(t)$ into the sum of the two orthogonal vectors at $\sigma_1(t)$ in $\mathbb{R}^n$, namely

$$R^{-1}(t)\left(\dot{v}_0(t)\right) \in \mathcal{T}_{\sigma_1(t)}\mathbf{M}_1 \quad \text{and} \quad R^{-1}(t)\left(\dot{R}(t)\left(R^{-1}(t)(v_0(t))\right)\right) \in \left(\mathcal{T}_{\sigma_1(t)}\mathbf{M}_1\right)^\perp.$$

Finally, because $\nabla_{\dot{\sigma}_1(t)} v_1(t) = \pi^\top_{\sigma_1(t)}(\dot{v}_1(t))$, the result follows. (The notation $\pi^\top_{\sigma_1(t)}$ stands for the orthogonal projection onto $\mathcal{T}_{\sigma_1(t)}\mathbf{M}_1$. *Q.E.D.*

Theorem 9. *Let* $v_1\colon I \to \mathcal{T}_{\sigma_1(t)}\mathbf{M}_1$ *be a vector field along the rolling curve* $\sigma_1\colon I \to \mathbf{M}_1$ *and* v_0 *be the corresponding vector field along the development curve* $\sigma_0\colon I \to \mathbf{M}_0$. *Then, for any* $k \geq 1$, *the following holds:*

$$\nabla^k_{\dot\sigma_1(t)} v_1(t) = R^{-1}(t)\left(v_0^{(k)}(t)\right) \in \mathcal{T}_{\sigma_1(t)}\mathbf{M}_1. \tag{8}$$

Proof: Induction. When $k = 1$ the statement is true by Lemma 8. Suppose that (8) holds for k. We prove the theorem by showing that (8) holds also for $k + 1$.

$$\begin{aligned}
\nabla^{k+1}_{\dot\sigma_1(t)} v_1(t) &= \pi^{\top}_{\sigma_1(t)}\left(\frac{d}{dt}\left(\nabla^k_{\dot\sigma_1(t)} v_1(t)\right)\right) = \pi^{\top}_{\sigma_1(t)}\left(\frac{d}{dt}\left(R^{-1}(t)\left(v_0^{(k)}(t)\right)\right)\right) \\
&= \pi^{\top}_{\sigma_1(t)}\left(\dot R^{-1}(t)\left(v_0^{(k)}(t)\right) + R^{-1}(t)\left(v_0^{(k+1)}(t)\right)\right) \\
&= \pi^{\top}_{\sigma_1(t)}\left(-\left(R^{-1}(t)\,\dot R(t)\,R^{-1}(t)\right)\left(v_0^{(k)}(t)\right) + R^{-1}(t)\left(v_0^{(k+1)}(t)\right)\right) \\
&= R^{-1}(t)\left(v_0^{(k+1)}(t)\right),
\end{aligned}$$

where we used the fact that $R(t) \in \mathbb{SO}(n)$ and

$$\left(R^{-1}(t)\,\dot R(t)\,R^{-1}(t)\right)\left(v_0^{(k)}(t)\right) = \left(\dot h(t)\,h^{-1}(t)\right)_*\left(v_0^{(k)}(t)\right) \in \left(\mathcal{T}_{\sigma_1(t)}\mathbf{M}_1\right)^{\perp}.$$

Q.E.D.

It is well known that σ_0 is a geodesic in M_0 if and only if σ_1 is a geodesic in M_1 (see, for instance, [5]). As a consequence of the previous theorem we can now compare polynomial curves in Euclidean spaces with their counterparts on any manifold M, also generalizing what has already been done for some particular manifolds in [3]. We first define Riemannian polynomials of degree k on M as curves $t \mapsto \sigma(t)$ which satisfy

$$\nabla^k_{\dot\sigma(t)}\dot\sigma(t) = 0.$$

Corollary 10. *The developed curve* σ_0 *is a polynomial of degree* k *in* M_0 *if and only if the rolling curve* σ_1 *is a Riemannian polynomial of degree* k *in* M_1.

Proof: This follows from the theorem if assuming that v_1 is the velocity vector field along σ_1 and, consequently, v_0 is the velocity vector field along σ_0. The identity in the theorem may now be written as

$$\nabla^k_{\dot\sigma(t)}\dot\sigma(t) = R^{-1}(t)\left(\sigma_0^{(k+1)}(t)\right)$$

and the result is proved. *Q.E.D.*

4. Conclusion

The operation of rolling manifolds studied in [5] and [3] gives an insight into geometric properties of the manifolds involved. In particular, we have shown the relationship between covariant derivative along a curve in one manifold and covariant derivative along the development curve in another manifold. As a consequence, a natural definition of Riemannian polynomial curves on manifolds arises. In this paper, we extend to general Riemannian manifolds some of the results presented in [3] for the spheres, rotation groups and Grassmann manifolds.

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References

[1] Victor Guillemin and Alan Pollack. *Differential Topology.* Prentice-Hall, New Jersey, 1974.

[2] Knut Hüper, Martin Kleinsteuber, and Fátima Silva Leite. Rolling stiefel manifolds. *International Journal of Systems Science*, To appear.

[3] Knut Hüper and Fátima Silva Leite. On the geometry of rolling and interpolation curves on S^n, SO_n and Graßmann manifolds. *Journal of Dynamical and Control Systems*, 13(4):467–502, October 2007.

[4] Velimir Jurdjevic. *Geometric Control Theory.* Number 51 in Cambridge Studies in Advanced Mathematics. Cambridge University Press, New York, 1997.

[5] Richard W. Sharpe. *Differential Geometry: Cartan's Generalization of Klein's Erlangen Program.* Number 166 in Graduate Texts in Mathematics. Springer-Verlag, New York, 1997.

Fátima Silva Leite
Department of Mathematics, University of Coimbra, Apartado 3008, 3001-454 Coimbra, Institute of Systems and Robotics, University of Coimbra - Pólo II, 3030-290 Coimbra, Portugal

E-mail address: fleite@mat.uc.pt

Krzysztof Andrzej Krakowski
School of Science and Technology, University of New England, Armidale NSW, 2351, Australia

E-mail address: kris@mcs.une.edu.au